



*Research article***Hybrid localized structures of the (3+1)-dimensional gBKP equation via bilinear neural networks with PINN verification****Tianyong Han***

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Abstract: This study investigates hybrid localized structures of the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (gBKP) equation using a bilinear neural network method (BNNM). The gBKP equation governs weakly dispersive and nonlinear wave propagation in shallow water, nonlinear optics, and plasma physics. An activation function pool is established that includes Jacobi elliptic functions, hyperbolic tangent, polynomials, and exponentials. A nonlinear coupled neural network architecture with one or two hidden layers is then constructed. Four families of hybrid localized structures are obtained from this architecture: algebraic-elliptic composite waves, topological-periodic composite waves, exponential-elliptic composite waves, and elliptic-modulated topological waves. From an analysis of the non-trivial solution families, a phenomenon termed wavefield degeneracy is identified. Multiple parameter configurations reduce to identical functional forms, indicating that waveform morphology is governed by the integrable structure rather than specific dispersion parameters. To validate the physical realizability of these solutions, physics-informed neural networks (PINNs) are employed to simulate the anisotropic localized wave constructed by the BNNM. The relative L_2 error of the PINN approximation converges to 2.66%, with the partial differential equation (PDE) residual decreasing to $O(10^{-6})$, confirming the numerical reproducibility of the BNNM-derived solution. The framework combines symbolic construction of exact solutions via neural networks with PINN-based numerical reproduction for the study of higher-dimensional integrable wave dynamics. The numerical verification is demonstrated on the most structurally complex solution family (Type II), with the validation scope limited due to computational constraints.

Keywords: bilinear neural network; gBKP equation; hybrid localized structures; wavefield degeneracy; PINN

Mathematics Subject Classification: 35C08, 35Q51, 68T07

1. Introduction

Nonlinear partial differential equations (NLPDEs) serve as fundamental tools for characterizing complex phenomena in various fields, ranging from neuronal networks [1] to fluid dynamics [2–4], plasma physics [5–7], nonlinear optics [8], and signal propagation [9, 10]. The search for closed-form wave solutions to NLPDEs remains central to integrable system studies. Such solutions provide direct insight into nonlinear dynamics without requiring numerical discretization. Among the various integrable models, the Kadomtsev-Petviashvili (KP) family of equations occupies a prominent position.

The (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (gBKP) equation, in its potential form

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0, \quad (1.1)$$

describes weakly dispersive and weakly nonlinear wave propagation in shallow water, nonlinear optics, and plasma physics [11, 12]. Beyond the KP family, the fractional Gardner equation with high-order dispersion serves as another fundamental model for nonlinear wave propagation in plasmas, optical fibers, and fluid systems, for which exact analytical solutions including bright/dark solitons, singular waves, and periodic patterns have been derived by Rabie et al. [13] using the modified extended direct algebraic method. The mixed third-order derivative accounts for weak dispersion along the primary propagation direction, while the nonlinear term captures wave steepening, and its higher-dimensional nature makes it suitable for modeling complex wave interactions.

Several effective methods have been developed for constructing analytical solutions of NLPDEs. For integrable systems, Hirota's bilinear method [14] and Bäcklund transformations [15, 16] systematically produce multi-soliton solutions. For broader equation classes, ansatz-based approaches include the (G'/G) -expansion [9], Jacobi elliptic function expansions [17], the trial equation method [18], Lie symmetry analysis [19], and plane dynamical systems methods [20, 21]. For instance, Younas et al. [22] applied the modified generalized Riccati equation mapping approach, the modified generalized exponential rational function method, and the Hirota bilinear method to the Estevez-Mansfield-Clarkson equation, successfully extracting breather, two-wave, and solitary wave solutions in shallow water phenomena and engineering fluid dynamics. Researchers have applied these techniques to the gBKP equation and generated solitons, lumps, breathers, rogue waves, periodic waves, and interaction solutions [23–27]. For instance, Wazwaz [24] derived multiple-soliton solutions via the Hirota bilinear method. Li and Ma [25] reported novel fusion and fission phenomena in mixed lump-soliton solutions, where the lump and soliton can merge into a whole soliton or, conversely, a soliton may differentiate into a lump and a new soliton. Kaur et al. [26] applied the generalized exponential rational function (GERF) method to obtain a wide range of soliton and periodic wave solutions, including kink, anti-kink, and singular solitons. Liang and Wang [27] performed a detailed bifurcation and chaos analysis, constructing phase portraits and deriving diverse wave solutions via variational and Hamiltonian-based approaches. Despite these advances, the wave structures obtained by most of these methods remain largely limited to additive combinations of elementary functions, each depending linearly on the independent variables. The expressive capacity of the chosen ansatz fundamentally constrains the class of obtainable solutions.

The KP equation and its various generalizations have continued to attract considerable research attention. Asaad and Ma [28] derived Pfaffian solutions to the gBKP equation and its modified counterpart. Ma et al. [29, 30] investigated exact solutions of the time-fractional extended KP equation

and integrability aspects of an extended BKP equation. Lu and Zhang [31] investigated the dynamics of abundant solutions to the gBKP equation. Wu et al. [32] and Sun et al. [33] studied rogue waves and lump solutions for the gBKP equation in fluid mechanics. Wazwaz [24] obtained distinct kinds of multiple-soliton solutions for the gBKP equation. Cao [34] studied lump solutions to the (3+1)-dimensional gBKP equation. Yan et al. [35] investigated the transition from solitons to rogue waves, as well as lump and interaction solutions for the gBKP equation. Siddique et al. [36] analyzed lump collision dynamics in the variable-coefficient gBKP equation. Liu et al. [37] obtained a class of exact solutions for the gBKP equation. These studies demonstrate that the gBKP equation and its generalizations support a rich variety of nonlinear wave phenomena. Very recently, Elsonbaty et al. [38] applied the modified extended direct algebraic method to the (2+1)-dimensional Wazwaz-Kaur-Boussinesq equation, uncovering previously unreported classes of exact solutions including combo dark-singular solitons and Jacobi elliptic function solutions, with comprehensive stability analysis confirming the robustness of these solutions under perturbation. This further motivates the exploration of hybrid localized structures in higher-dimensional integrable systems.

In recent years, the connection between continuous dynamical systems and their discrete counterparts has attracted considerable attention. From a broader perspective, the bilinear neural network method (BNNM), introduced by Zhang and Bilge [39,40], offers a promising alternative that follows a similar paradigm: embedding the Hirota bilinear form into a neural network architecture. In a broader context, Muhammad et al. [41] recently introduced a Riccati subequation neural network method to derive exact solutions for space-time partial differential equations, including the nonlinear Murray equation and nano-ionic currents along microtubules dynamical equations. This development further demonstrates the flexibility of neural-network-based approaches for constructing analytical solutions across diverse physical and biological systems. The method uses tensor formulations of neural networks as flexible trial functions. It generates exact solutions through algebraic constraint solving rather than data-driven training. Recent extensions include neural network-based symbolic calculation approaches, the Bell polynomial-based neural network method [42], artificial intelligence-enhanced mathematical derivation methods [43], and multi-modal neurosymbolic reasoning intelligent algorithms [44]. These developments have been successfully applied to various equations, including the $\Phi - 4$ equation [45]. Most recently, Li et al. [46] extended BNNM to a generalized (3+1)-dimensional KP equation using single-hidden-layer (“4-3-1”) and double-hidden-layer (“4-2-2-1”) networks, successfully deriving lump, M-lump, bright-dark soliton, periodic, and N-soliton solutions. However, most existing BNNM implementations employ a single hidden layer. This restricts the architecture to additive combinations of activation functions applied directly to linear arguments. Whether a second hidden layer can generate wave structures beyond this additive regime has not been systematically examined.

It is instructive to position BNNM within the broader landscape of AI-assisted mathematical methods. Symbolic regression (e.g., Eureqa) takes numerical data as input and outputs elementary function formulas through data-driven discovery. Neural-symbolic discovery methods (e.g., DeepSym) search for symbolic expressions via reinforcement learning or evolutionary algorithms. In contrast, BNNM requires no data and no search: It encodes the Hirota bilinear form as hard prior knowledge and solves deterministically through algebraic constraint solving. The correctness of the resulting solutions is verified by direct substitution, offering advantages in theoretical assurance and computational efficiency over search-based approaches.

In this paper, we address this gap by introducing a two-stage cascaded bilinear neural network architecture for the gBKP equation. The first-stage outputs feed into second-stage inputs, creating nonlinear inter-layer dependencies. The main contributions of this work are threefold. First, unlike existing single-layer BNNM implementations, our two-stage cascaded architecture creates non-additive couplings that generate hybrid localized structures inaccessible to additive constructions. Second, we identify wavefield degeneracy as a structural property of the bilinear form itself. Third, we establish a BNNM-physics-informed neural network (PINN) synergy where PINN serves as an independent numerical verifier rather than a competing solver—a paradigm not previously reported for BNNM-derived solutions.

The paper is organized as follows. Section 2 presents the BNNM methodology. Section 3 reports the four families of hybrid localized structures. Section 4 provides a PINN-based numerical validation. Section 5 discusses the implications. Section 6 summarizes the main conclusions.

2. Methodology

This section presents the systematic procedure for constructing exact wave solutions of the gBKP equation using the BNNM. The core idea is to embed the Hirota bilinear form into the network architecture as prior knowledge. By flexibly combining activation functions across two cascaded stages, the network generates the τ -function directly. The methodology consists of the steps outlined below. The overall framework is illustrated in Figure 1.

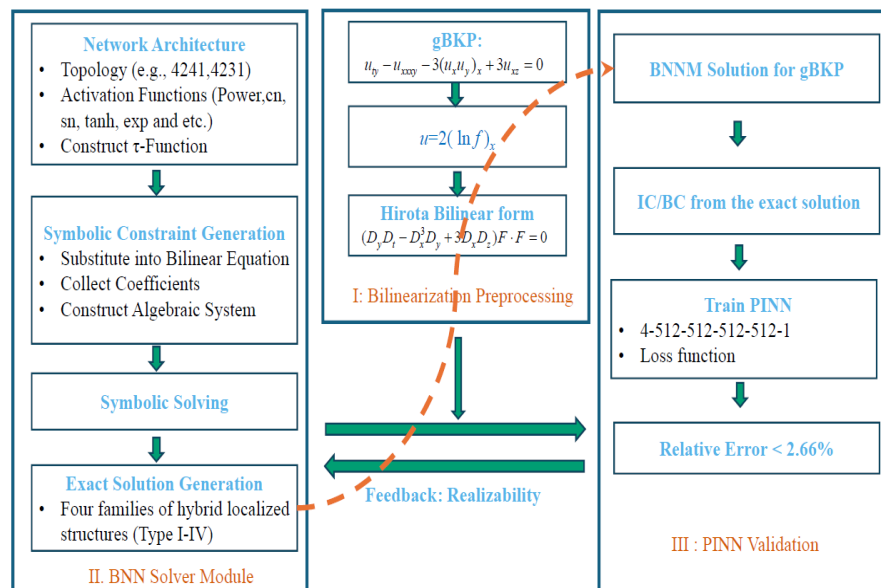


Figure 1. Schematic overview of the BNNM-PINN synergistic framework for the gBKP equation. Left: BNNM symbolic construction phase, from the gBKP equation through Hirota bilinear transformation and two-stage network architectures to four families of hybrid localized structures. Right: PINN numerical verification phase, using the BNNM-derived anisotropic localized wave u_{22} as benchmark. The relative L_2 error below 3% (with PDE residual $O(10^{-6})$) confirms physical realizability.

Through the dependent variable transformation

$$u = 2(\ln F)_x, \quad (2.1)$$

the gBKP equation (1.1) is converted to the Hirota bilinear form

$$(D_y D_t - D_x^3 D_y + 3D_x D_z)F \cdot F = 0, \quad (2.2)$$

where D_x , D_y , D_z and D_t are the Hirota bilinear operators:

$$D_x^m D_t^n F \cdot G = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n F(x, t) G(x', t') \Big|_{x'=x, t'=t}. \quad (2.3)$$

Expanding (2.2) explicitly yields

$$F_{yt}F - F_y F_t - F_{xxy}F + F_{xxx}F_y + 3F_{xx}F_{xy} - 3F_x F_{xxy} + 3F_{xz}F - 3F_x F_z = 0. \quad (2.4)$$

Solving the gBKP equation reduces to finding a function F that satisfies (2.4).

To avoid arbitrary selection of basis functions, we establish a physical-function correspondence that links expected wave structures to appropriate activation functions. The gBKP equation admits four fundamental types of hybrid localized structures. These are summarized in Table 1.

Table 1. Physical-function correspondence for the hybrid localized structures of the gBKP equation.

Type	Wave structure	Asymptotic signature	Activation function class
I	Algebraic-elliptic composite	Algebraic decay+Periodic oscillation	Polynomial+Elliptic
II	Topological-periodic composite	Topological step+Periodic oscillation	Hyperbolic+Elliptic
III	Exponential-elliptic composite	Exponential decay+Periodic modulation	Exponential+Elliptic
IV	Elliptic-modulated topological	Topological step+Elliptic background	Exponential+Elliptic

The choice of activation functions in Table 1 is chosen based on their established roles in representing localized wave structures. Polynomial functions naturally produce algebraically decaying localized structures, which are characteristic of lump-type solutions. Jacobi elliptic functions provide periodic modulation with tunable periodicity, making them suitable for describing breather and periodic wave backgrounds. Exponential functions and hyperbolic tangent generate localized soliton and kink profiles with exponential decay or topological step transitions. These function classes have been extensively validated in the literature for constructing localized wave solutions of integrable equations.

Based on this correspondence, we define a closed activation function pool:

$$\mathcal{P} = \{\xi^n, e^\xi, e^{-\xi}, \tanh(\xi), \operatorname{sn}(\xi, m), \operatorname{cn}(\xi, m)\}. \quad (2.5)$$

The choice of the two-stage cascaded architecture is motivated by a fundamental limitation of single-layer constructions. A single hidden layer produces additive combinations of the form $F = \sum_i a_i h_i(\xi_i)$,

where each activation function acts on an independent linear argument. In contrast, the two-stage architecture produces nested compositions:

$$F = \sum_j a_j h_j \left(\sum_i w_{ji} h_i(\xi_i) + \phi_j \right),$$

which generate cross-terms $h_i(\xi_1)h_k(\xi_2)$ upon expansion, yielding hybrid structures inaccessible to additive constructions.

Three-layer cascaded architectures were also attempted during the development of this method. However, they led to two substantial difficulties: (i) The number of algebraic equations exceeded 2000, making symbolic solution impractical within reasonable time on a standard workstation; and (ii) most solutions degenerated to two-layer cases, indicating parameter redundancy without structural novelty. Thus, the two-stage cascade represents the optimal balance between expressive capacity and computational feasibility.

The pool is designed to generate the four families of hybrid localized structures reported in Section 3. The choice of activation functions reflects a deliberate balance between computational tractability and expressive capacity. Due to the properties of the Hirota bilinear operator, exponential functions are computationally convenient and widely adopted in the literature. Many existing studies choose activation functions such as ξ^n , $\tanh$, $\sin$, and $\cos$, as their derivatives are straightforward to compute and their algebraic structures are well understood. Jacobi elliptic functions, by contrast, involve more complex differentiation rules and parameter dependencies (the elliptic moduli m), which introduce additional algebraic complexity when substituted into the bilinear form. Consequently, elliptic-function-based activations have been rarely explored in the bilinear neural network literature. To the best of our knowledge, the present work is among the first to systematically incorporate Jacobi elliptic functions as activation functions within a two-stage cascaded BNNM architecture, thereby enabling the construction of hybrid localized structures that combine algebraic, exponential, and elliptic components in a unified framework.

We adopt a two-stage cascaded architecture feeding first-stage outputs into second-stage inputs. This creates non-additive couplings between activation classes.

Tensor formulation

Following the bilinear neural network framework, the τ -function is expressed as

$$F = \sum_{k=3}^N w_{k,f} h_k(\zeta_k) + \phi_0, \quad (2.6)$$

where $w_{k,f}$ are output weights, h_k are activation functions, ϕ_0 is the bias, and ζ_k are second-stage inputs.

Stage 1: Linear combinations

Define two independent linear forms:

$$\zeta_i = kt_it + kx_ix + ky_iy + kz_iz + \phi_i, \quad i = 1, 2. \quad (2.7)$$

Apply first-stage activation functions $h_1, h_2 \in \mathcal{P}$:

$$\eta_j = h_j(\zeta_j), \quad j = 1, 2. \quad (2.8)$$

Stage 2: Nonlinear coupling

Form second-stage arguments by linearly combining first-stage outputs:

$$\zeta_k = w_{k,1}\eta_1 + w_{k,2}\eta_2 + \phi_k, \quad k = 3, 4, \dots, N. \quad (2.9)$$

Here, $w_{k,1}, w_{k,2}$ are inter-layer connection weights. A genuine two-stage coupling is realized when these weights are non-zero. Otherwise the architecture degenerates to a single-layer construction.

Apply second-stage activation functions $h_k \in \mathcal{P}$ via (2.3) and (2.2) to obtain F .

Systematic enumeration and solving procedure

The solution procedure consists of six systematic steps.

Step 1: Architecture specification. Select the network topology and assign activation functions from $\mathcal{P}$ to each neuron. Different architectures are detailed in Section 3 and illustrated in Figure 2.

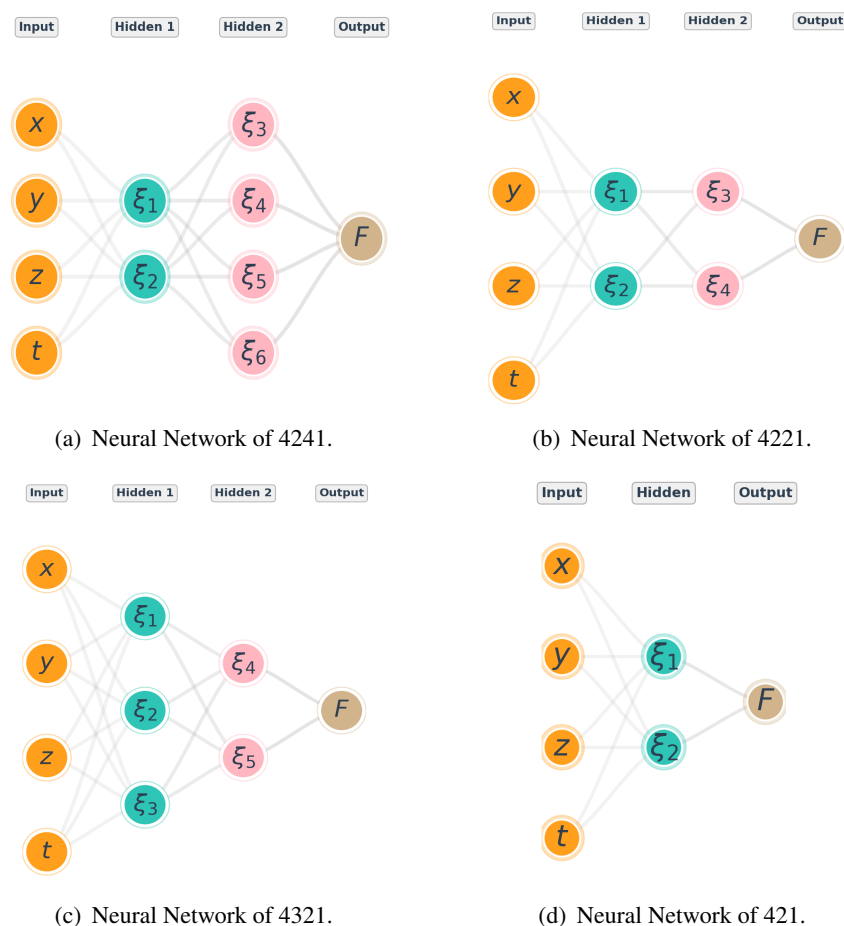


Figure 2. BNNM architecture for hybrid local structure.

The essential mechanism by which the two-stage architecture generates hybrid structures can be understood as follows. A single-layer network produces additive combinations of independent activation functions:

$$F_{\text{single}} = \sum_i a_i h_i(\xi_i).$$

In contrast, the two-stage network produces nested compositions:

$$F_{\text{two-stage}} = \sum_j a_j h_j \left(\sum_i w_{ji} h_i(\xi_i) + \phi_j \right).$$

Expanding the second-layer function h_j in a Taylor series reveals that cross-terms $h_i(\xi_i)h_k(\xi_k)$ (with $i \neq k$) arise from the second-order terms. These cross-terms represent non-additive couplings between different wave modes and are precisely the source of the hybrid localized structures reported in Section 3. Such couplings are absent in single-layer additive constructions.

Step 2: Substitution. Substitute (2.6) into the bilinear equation (2.4). Collect coefficients of all distinct terms. Setting each coefficient to zero yields a system of algebraic equations.

Step 3: Symbolic solution. Solve the algebraic system using Maple. Three outcomes are distinguished.

- Vanishing solutions: All parameters reduce to zero.
- Degenerate solutions: Inter-layer weights vanish, reducing to single-layer form.
- Genuine composite solutions: Inter-layer weights are non-zero, yielding novel hybrid localized structures.

Only the third category is retained.

Step 4: Reconstruction. Substitute non-trivial parameter solutions into (2.6). Then apply (2.1) to obtain $u(x, y, z, t)$.

Step 5: Visualization. Generate three-dimensional plots, contour plots, and evolution plots to illustrate dynamical properties.

It is important to clarify the distinct roles of the two neural paradigms in this paper. BNNM is a symbolic construction tool. Parameters are determined by solving algebraic systems, not by training. There is no loss function or optimization. PINN (introduced in Section 4) serves as a numerical verification tool. It approximates the solution via optimization of a physics-based loss function. It uses BNNM-derived solutions as ground truth.

This synergy, symbolic construction followed by data-free numerical verification, establishes a closed-loop framework novel in the study of integrable systems.

3. Hybrid localized structures from the composite BNNM

Applying the two-stage cascaded BNNM architecture described in Section 2 to the gBKP equation, we obtain four distinct families of hybrid localized structures. These families are systematically classified according to their asymptotic behavior and the types of activation functions that generate them. Table 2 summarizes the four types, their corresponding network architectures, activation function pairings, and physical characteristics.

Table 2. Classification of hybrid localized structures obtained from the composite BNNM.

Type	Network	Activation pairing	Physical characteristic
I	4-2-4-1	Polynomial+Jacobi elliptic	Algebraic decay+Periodic oscillation
II	4-2-2-1	Hyperbolic tangent+Jacobi elliptic	Topological step+Periodic background
III	4-3-2-1	Exponential+Jacobi elliptic	Exponential localization+Periodic modulation
IV	4-2-1	Exponential+Jacobi elliptic	Topological front+Elliptic background

For each family, we present the explicit τ -function, the corresponding waveform $u = 2(\ln F)_x$, the parameter constraints that produce non-degenerate solutions, and representative visualizations. The inter-layer connection weights for all families are non-zero, confirming that the two-stage coupling is genuine rather than merely formal. It is noted that trigonometric functions ($\sin$, $\cos$) are not included as independent pool members because they are recovered as degenerate limits of the Jacobi elliptic functions:

$$sn(\xi, m) \xrightarrow{m \rightarrow 0} \sin(\xi), \quad cn(\xi, m) \xrightarrow{m \rightarrow 0} \cos(\xi).$$

Thus, the inclusion of Jacobi elliptic functions automatically covers trigonometric cases without introducing redundant degrees of freedom.

A notable property recurs across all four families: Multiple parameter configurations, corresponding to distinct propagation directions or dispersion relations, reduce to identical functional forms for the wavefield u . We refer to this phenomenon as wavefield degeneracy. Its implications are discussed in Section 5.

All symbolic computations were performed in Maple 2023 on a workstation with Intel Core i5-6500 @ 3.20GHz CPU and 8.0 GB RAM. The computational scales for the four solution families are summarized in Table 3 (below). The reported times are typical values; actual runtime may vary depending on system load and memory availability.

Table 3. Computational scales for the four solution families. “Solve time” refers to the Maple `solve` command CPU time.

Family	Network	Equations	Variables	Memory	Solve time
Type I	4-2-4-1	593	29	105.9 MB	134 s
Type II	4-2-2-1	495	20	105.9 MB	124 s
Type III	4-3-2-1	456	22	113.9 MB	231 s
Type IV	4-2-1	78	16	32.1 MB	20.51 s

3.1. Type I: Algebraic-elliptic composite waves

This family combines algebraic decay in one spatial direction with elliptic periodic oscillations in the transverse plane. The algebraic component arises from polynomial activation functions, while the periodic component is generated by Jacobi elliptic functions. Such hybrid structures cannot be obtained from single-layer additive constructions.

The “4-2-4-1” two-stage network architecture (see Figure 2(a)) is employed for this family. The input layer consists of four neurons corresponding to the independent variables (x, y, z, t) . The first hidden layer contains two neurons with activation functions $h_1(\zeta_1) = \zeta_1$ (identity) and $h_2(\zeta_2) = \zeta_2^2$ (quadratic polynomial). Their outputs feed into the second hidden layer, which contains four neurons with activation functions

$$h_3(\zeta_3) = sn(\zeta_3, m_1), \quad h_4(\zeta_4) = cn(\zeta_4, m_2), \quad h_5(\zeta_5) = \zeta_5, \quad h_6(\zeta_6) = \zeta_6^2. \quad (3.1)$$

The τ -function is expressed as

$$F = \sum_{k=3}^6 w_{1,k} h_k(\zeta_k) + \phi_7, \quad (3.2)$$

with $\zeta_i = kt_it + kx_ix + ky_iy + kz_iz + \phi_i$, $i = 1, 2$, and $\zeta_j = w_{j,1}h_1(\zeta_1) + w_{j,2}h_2(\zeta_2)$, $j = 3, 4, 5, 6$. Here, kt_i, kx_i, ky_i, kz_i are wavenumber parameters, ϕ_i are phase constants, and $w_{j,1}, w_{j,2}$ are inter-layer connection weights. The presence of non-zero weights $w_{j,1}, w_{j,2}$ confirms genuine two-stage coupling.

Substituting (3.1) and (3.2) into the bilinear equation (2.4) and collecting coefficients of all independent terms yields a system of 593 algebraic equations in 29 unknowns. Solving this system in Maple yields multiple solution sets. These include trivial solutions, degenerate solutions where the inter-layer weights $w_{j,1} = w_{j,2} = 0$, and non-trivial solutions where the inter-layer weights remain non-zero.

After discarding trivial and degenerate cases, the remaining non-trivial solutions are grouped according to their structural forms. Two distinct non-trivial branches are identified and presented below.

Case 1. Algebraic-elliptic composite solution

The first branch yields solutions that share a unified functional form. Under this branch, the solution can be expressed as

$$u_{11} = \frac{2w_{1,5}w_{5,1}kx_1}{w_{1,4}cn(w_{4,2}P_2^2 + \phi_4, m_2) + w_{1,5}(w_{5,1}P_1 + w_{5,2}P_2^2 + \phi_5) + \phi_7}, \quad (3.3)$$

where P_1 and P_2 depend on the specific parameter choices. Within this branch, three representative parameter configurations—corresponding to different propagation regimes—are identified.

Scenario 1. Algebraically decaying periodic wave. The free parameters are selected as $kt_1 = \frac{kz_1kt_1}{kz_2}$, $kt_2 = -\frac{3kx_1kt_2}{ky_1}$, $kx_2 = ky_2 = 0$, $w_{1,3} = w_{1,6} = w_{4,1} = 0$, and

$$\begin{cases} P_1 = t + kx_1x + ky_1y + kz_1z + \phi_1, \\ P_2 = -\frac{3kx_1kt_2}{ky_1}t + kz_2z + \phi_2. \end{cases} \quad (3.4)$$

In this configuration, the waveform exhibits algebraic decay along the x -direction due to the polynomial terms in the denominator, while preserving double-periodic oscillations in the y - z plane via the cn function. This structure is characteristic of a localized wave packet embedded in a periodic background.

Scenario 2. Propagating periodic-algebraic wave. Alternatively, choosing $ky_1 = -\frac{3kx_1kt_2}{kt_1}$, $kt_2 = kx_2 = 0$, $ky_2 = -\frac{3kx_1kt_2}{kt_1}$, $w_{1,3} = w_{1,6} = w_{4,1} = 0$, we have

$$\begin{cases} P_1 = kt_1t + kx_1x - \frac{3kx_1kt_2}{ky_1}y + kz_1z + \phi_1, \\ P_2 = -\frac{3kx_1kt_2}{ky_1}y + kz_2z + \phi_2. \end{cases} \quad (3.5)$$

Here, the periodic modulation propagates in the x -direction while retaining its algebraic envelope, describing a traveling wave packet with persistent transverse oscillations.

Scenario 3. Transverse periodic-algebraic wave. Choosing $kt_1 = \frac{kz_1kt_2}{kz_2}$, $ky_1 = -\frac{3kx_1kt_2}{kt_2}$, $kz_1 = kt_2 = kx_2 = w_{1,3} = w_{1,6} = w_{4,1} = 0$ leads to

$$\begin{cases} P_1 = \frac{kz_1kt_2}{kz_2}t + kx_1x + \phi_1, \\ P_2 = kt_2t + kz_2z + \phi_2. \end{cases} \quad (3.6)$$

This configuration describes wave patterns that are periodic in one transverse direction and algebraically varying in the other, representing an anisotropic localized structure.

Case 2. Quadratic phase modulation

The second non-trivial branch yields a structurally distinct family. Under this branch, the solution takes the form

$$u_{12} = \frac{4w_{1,5}w_{5,2}kx_2P_2}{w_{1,4}cn(w_{4,1}P_1 + \phi_4, m_2) + w_{1,5}(w_{5,1}P_1 + w_{5,2}P_2^2 + \phi_5) + \phi_7}, \quad (3.7)$$

with

$$P_1 = ky_1y + kz_1z + \phi_1, \quad P_2 = -\frac{3kx_1kt_2}{ky_1}t + kx_2x + \phi_2, \quad (3.8)$$

obtained by setting $kt_2 = -\frac{3kx_1kt_2}{ky_1}$, $kt_1 = kx_1 = ky_2 = kz_2 = 0$, $w_{1,3} = w_{1,6} = w_{4,2} = 0$.

This yields a quasi-two-dimensional periodic pattern superimposed on a parabolic background, representing a different type of algebraic-elliptic coupling.

To visualize the features of this solution family, we select the representative parameters from u_{11} (3.3) with the following values: $kt_2 = 1$, $kx_1 = 1$, $kz_1 = 1$, $kz_2 = 1$, $m_2 = \frac{1}{2}$, $\phi_1 = \dots = \phi_7 = 0$, $w_{1,4} = 1$, $w_{1,5} = 1$, $w_{4,2} = 1$, $w_{5,1} = 1$, $w_{5,2} = 1$.

The resulting wave profiles are displayed in Figure 3. Panels (a)–(c) depict the evolution in the x - t plane at $y = 0$, $z = 0$, while panels (d)–(f) show the x - y cross-sectional views at $z = 0$, $t = 1$.

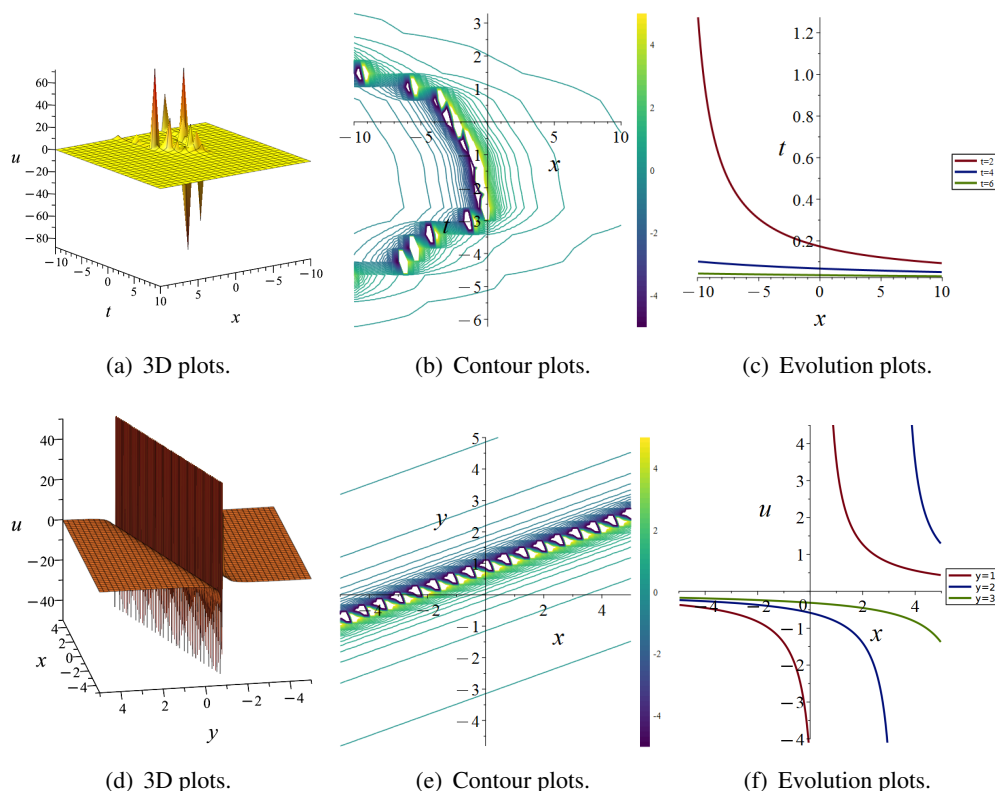


Figure 3. Type I. Algebraic-elliptic composite waves u_{11} from Eq (3.4): (a)–(c): x - t plane at $y = 0$, $z = 0$; (d)–(f): x - y cross-section at $z = 0$, $t = 1$.

In the x - t plane (panels (a)–(c)), the solution exhibits clear algebraic decay along the x -direction. The amplitude decreases as $|x|^{-1}$ for large x , which is characteristic of lump-type behavior. Superimposed on this algebraic background are periodic oscillations in the y - z plane, inherited from the Jacobi elliptic function cn . The density plot (panel (b)) reveals a localized energy concentration near the origin, with periodic fringes emanating outward.

In the x - y cross-section at $z = 0$, $t = 1$ (panels (d)–(f)), the double-periodic structure is more clearly visible. The wave profile shows a central localized hump flanked by oscillatory tails. This mixed structure, in which a periodic wave train coexists with an algebraically decaying envelope, lies outside the scope of standard traveling wave reductions.

Such configurations may describe internal wave fields in stratified fluids where a coherent oscillatory pattern is embedded in a large-scale shear flow. The algebraic decay represents the far-field response to a localized perturbation, while the elliptic periodic modulation captures the background wave field. This type of hybrid structure has not been reported in previous studies on the gBKP equation.

3.2. Type II. Topological-periodic composite waves

This family combines a topological kink structure with periodic oscillatory backgrounds. The kink component, generated by the hyperbolic tangent activation function, provides a smooth step-like transition between two constant asymptotic states. The periodic component, generated by Jacobi elliptic functions, superimposes oscillatory modulations on this topological background. Such structures are physically relevant for describing phase boundaries in periodic media, magnetic domain walls in crystals with periodic anisotropy, or shock waves in stratified fluids with background internal waves.

The “4-2-2-1” two-stage network architecture (see Figure 2(b)) is employed for this family. The input layer consists of four neurons corresponding to (x, y, z, t) . The first hidden layer contains two neurons with activation functions $h_1(\zeta_1) = cn(\zeta_1, m)$ and $h_2(\zeta_2) = sn(\zeta_2, m)$. The second hidden layer contains two neurons with activation functions $h_3(\zeta_3) = \zeta_3^2$ and $h_4(\zeta_4) = \tanh(\zeta_4)$. The τ -function is constructed as

$$F = w_{11}h_3(\zeta_3) + w_{12}h_4(\zeta_4) + \phi_5, \quad (3.9)$$

with $\zeta_3 = w_{31}h_1(\zeta_1) + w_{32}h_2(\zeta_2) + \phi_3$ and $\zeta_4 = w_{41}h_1(\zeta_1) + w_{42}h_2(\zeta_2) + \phi_4$.

Substitute this structure (3.9) into the bilinear equation (2.4) and solve for the resulting algebraic system, obtaining non-trivial parameterized families for two cases.

Case 1. Kink with sinusoidal modulation

The first case takes the form

$$u_{21} = \frac{2w_{1,1}w_{3,1}^2 kx_1 \sin(2P_1)}{w_{1,1}w_{3,1}^2 \cos(P_1)^2 + w_{1,2} \tanh(w_{4,2} \sin(P_2) + \phi_4) + \phi_5}, \quad (3.10)$$

where P_1 and P_2 depend on the chosen parameter constraints. Two distinct physical scenarios arise.

Scenario 1. Kink with sinusoidal modulation. Setting $kt_1 = -\frac{kx_1(4kx_1^2ky_2+3kz_2)}{ky_2}$, $kt_2 = kx_2 = ky_1 = kz_1 = 0$, $m = 0$, $\phi_3 = 0$, $w_{3,2} = w_{4,1} = 0$, we obtain

$$\begin{cases} P_1 = \frac{kx_1(4kx_1^2ky_2+3kz_2)}{ky_2}t - kx_1x - \phi_1, \\ P_2 = ky_2y + kz_2z + \phi_2. \end{cases} \quad (3.11)$$

This yields a hyperbolic tangent kink whose position remains stationary, modulated by a sinusoidal oscillation arising from the degenerate Jacobi elliptic functions ($m = 0$, so cn and sn reduce to trigonometric functions). The kink provides the topological step transition, while the sinusoidal modulation produces a periodic ripple pattern superimposed on the kink profile. Such configurations may describe a solitary wave trapped in a periodic potential, relevant for nonlinear optical systems or photonic crystals.

Scenario 2. Phase-modulated kink train. With $kz_2 = -\frac{ky_2(4kx_1^3+kt_1)}{3kx_1}$, $m = 0$, $ky_1 = kz_1 = 0$, $kt_2 = kx_2 = 0$, $\phi_3 = w_{3,2} = w_{4,1} = 0$, we have

$$\begin{cases} P_1 = -(kt_1t + kx_1x + \phi_1), \\ P_2 = \frac{-ky_2(4kx_1^3z - 3kx_1y + kt_1z) + 3\phi_2kx_1}{3kx_1}. \end{cases} \quad (3.12)$$

Here, the kink propagates while its internal phase undergoes periodic changes. The phase modulation arises from the coupling between the trigonometric functions in the first hidden layer and the hyperbolic tangent in the second layer. This scenario describes a kink train whose internal oscillation frequency varies periodically during propagation.

Case 2. Traveling kink with stationary periodic background

The second case is given by

$$u_{22} = \frac{2w_{1,1}w_{3,2}^2kx_2 \sin(2P_1)}{w_{1,1}w_{3,2}^2 \sin(P_1)^2 + w_{1,2} \tanh(w_{4,1} \cos(P_2) + \phi_4) + \phi_5}, \quad (3.13)$$

again with P_1 and P_2 determined by parameter choices. Two representative configurations emerge.

Scenario 1. Traveling kink with periodic modulation. Selecting $kz_1 = -\frac{ky_1(4kx_2^2+kt_2)}{3kx_2}$, $m = 0$, $kz_2 = kt_1 = kx_1 = ky_2 = 0$, $\phi_3 = w_{3,1} = w_{4,2} = 0$ yields

$$\begin{cases} P_1 = kt_2t + kx_2x + \phi_2, \\ P_2 = ky_1y - \frac{ky_1(4kx_2^2+kt_2)}{3kx_2}z + \phi_1. \end{cases} \quad (3.14)$$

In this configuration, the kink propagates in the x -direction while the periodic background remains stationary. This describes a moving topological defect on a fixed oscillatory substrate, a situation encountered in driven magnetic domain wall systems or propagating shock fronts in periodic media.

Scenario 2. Kink embedded in a standing wave. With $kt_2 = -\frac{kx_2(4kx_3^3ky_1+3kz_1)}{ky_1}$, $m = 0$, $kt_1 = kx_1 = ky_2 = kz_2 = 0$, $\phi_3 = w_{3,1} = w_{4,2} = 0$, we obtain

$$\begin{cases} P_1 = -\frac{kx_2(4kx_3^3ky_1+3kz_1)}{ky_1}t + kx_2x + \phi_2, \\ P_2 = ky_1y + kz_1z + \phi_1. \end{cases} \quad (3.15)$$

Here, the kink appears as a localized defect superimposed on a standing wave background. The standing wave oscillates in time without propagating, while the kink remains spatially localized at its position. This configuration describes a soliton trapped in a standing wave cavity, relevant for nonlinear waveguide structures and Bose-Einstein condensate systems.

To illustrate the typical behavior of this class, we take the parameters from Eq (3.11) with the following values: $kx_1 = 1$, $ky_2 = 1$, $kz_2 = 1$, $\phi_1 = \phi_2 = \phi_4 = \phi_5 = 0$, $w_{1,1} = 1$, $w_{1,2} = -1$, $w_{3,1} = 1$, $w_{4,2} = 1$.

The resulting wave profiles are displayed in Figure 4. Figure 4 shows a steep kink structure caused by the tanh function in the x -direction. The kink's position does not drift with time, indicating a stationary topological defect. Superimposed on this kink profile are quasi-sinusoidal ripples that arise from the degenerate Jacobi elliptic functions ($m = 0$) in the first hidden layer.

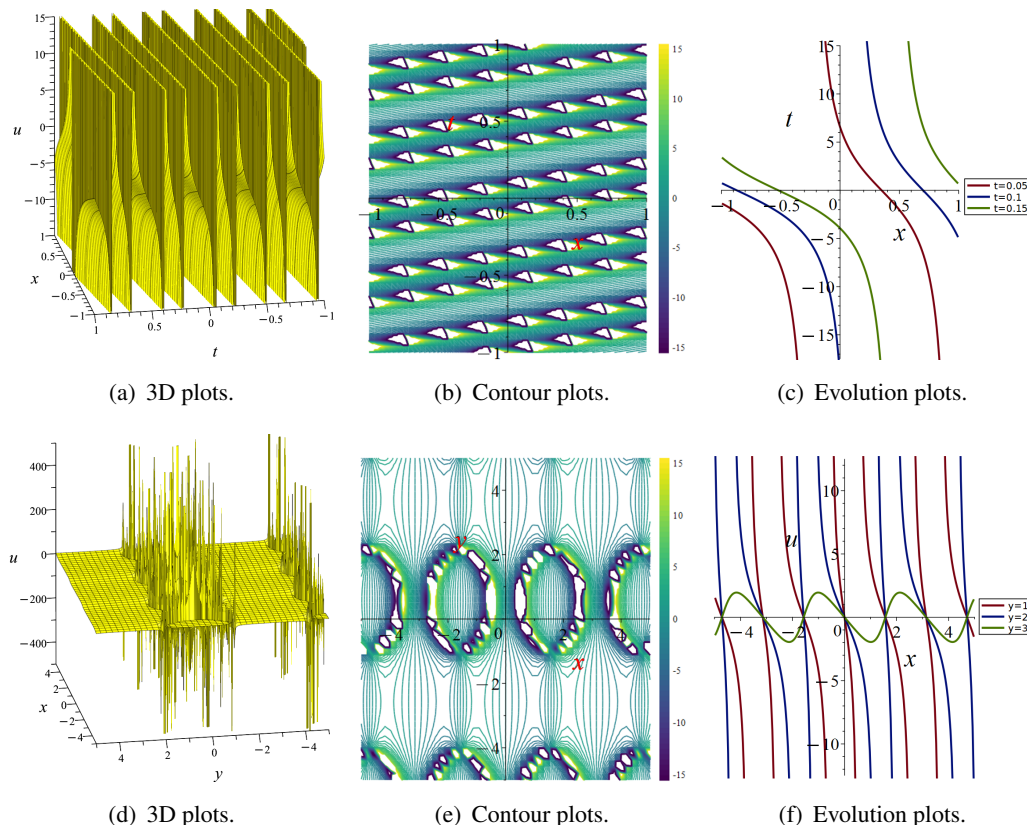


Figure 4. Evolution diagrams for periodic-kink hybrid waves u_{21} with Eq (3.11): (a)–(c) at $y = 0, z = 0$; (d)–(f) at $z = 0, t = 1$.

The three-dimensional plot (panel (a)) reveals a sharp transition from one constant asymptotic value to another, with oscillatory modulations on both sides of the kink. The density plot (panel (b)) shows the localized energy concentration near the kink center, with periodic fringes extending outward. The contour plot (panel (c)) clearly delineates the kink front and the periodic background.

In the x - y cross-section at $z = 0, t = 1$ (panels (d)–(f)), the kink appears as a vertical cliff-like structure, while the periodic modulations manifest as horizontal stripes. The contrast between the sharp topological transition and the smooth oscillatory background is a distinctive feature of this composite structure.

This type of solution resembles a soliton trapped in a periodic potential, which has been extensively studied in photonic crystals and nonlinear lattices. The topological nature of the kink provides robustness against perturbations, while the periodic modulation introduces tunable background

features. Such configurations have not been reported in previous studies on the gBKP equation, which typically focus on pure kink or pure periodic solutions without their non-additive coupling.

Singularity analysis. We briefly examine the non-vanishing condition for the denominator of the τ -function, using the PINN-validated Type II solution u_{22} as a representative example. The denominator takes the form

$$\tau = w_{11}w_{32}^2 \sin^2(P_1) + w_{12} \tanh(w_{41} \cos(P_2) + \phi_4) + \phi_5.$$

For the parameters used in the PINN benchmark ($\phi_5 = 3.0$, $w_{11} = 2.0$, $w_{32} = 1.0$, $w_{12} = 1.0$), since $\tanh(\cdot) \in (-1, 1)$, we have

$$\tau \geq 0 + (-0.7616) + 3.0 = 2.2384 > 0,$$

ensuring that the solution is singularity-free over the entire real domain. Similar parameter constraints can be imposed for other families to guarantee non-vanishing denominators.

3.3. Type III. Exponential-elliptic composite waves

This family combines exponential localization with elliptic periodic modulation. The exponential component, generated by the exponential activation function, provides a localized envelope that decays rapidly away from the wave center. The elliptic component, generated by the Jacobi elliptic function cn , superimposes a periodic modulation on this localized envelope. Such structures represent wave packets confined within a potential well while radiating weak periodic tails.

The “4-3-2-1” two-stage network architecture (see Figure 2(c)) is employed for this family. The input layer consists of four neurons corresponding to (x, y, z, t) . The first hidden layer contains three neurons with activation functions $h_1(\zeta_1) = \zeta_1^2$, $h_2(\zeta_2) = \zeta_2^2$, and $h_3(\zeta_3) = \zeta_3$. The second hidden layer contains two neurons with activation functions $h_4(\zeta_4) = cn(\zeta_4, m)$ and $h_5(\zeta_5) = \exp(\zeta_5)$. The τ -function is built as

$$F = w_{14}h_4(\zeta_4) + w_{15}h_5(\zeta_5) + \phi_6, \quad (3.16)$$

with $\zeta_k = w_{k,1}h_1(\zeta_1) + w_{k,2}h_2(\zeta_2) + w_{k,3}h_3(\zeta_3) + \phi_k$, $k = 4, 5$.

Substituting this structure (3.16) into the bilinear equation (2.4) and solving the resulting algebraic system yields several families of exact solutions, which can be grouped into two distinct structural forms.

Case 1. Doubly-modulated exponentially localized wave

The first family has an exponential envelope modulated by two independent quadratic-phase terms and a double-periodic background. Its general form is

$$u_{31} = \frac{2w_{1,5}w_{5,3}kx_3 \exp(w_{5,1}P_1^2 + w_{5,2}P_2^2 + w_{5,3}P_3 + \phi_5)}{w_{1,4}cn(w_{4,1}P_1^2 + w_{4,2}P_2^2 + \phi_4, m) + w_{1,5} \exp(w_{5,1}P_1^2 + w_{5,2}P_2^2 + w_{5,3}P_3 + \phi_5) + \phi_6}, \quad (3.17)$$

where the definitions of P_1 , P_2 , and P_3 depend on the specific parameter constraints. Two distinct parameter scenarios yield this structural form.

Scenario 1. Exponentially modulated periodic wave. Taking $kz_1 = \frac{ky_1(kx_3^3w_{5,3}^2 - kt_3)}{3kx_3}$, $kz_2 =$

$\frac{ky_2(kx_3^3w_{5,3}^2-kt_3)}{3kx_3}$, $kz_3 = \frac{ky_3(kx_3^3w_{5,3}^2-kt_3)}{3kx_3}$, $kt_1 = kt_2 = 0$, $kx_1 = kx_2 = 0$, $w_{4,3} = 0$, we obtain

$$\begin{cases} P_1 = ky_1y + \frac{ky_1(kx_3^3w_{5,3}^2-kt_3)}{3kx_3}z + \phi_1, \\ P_2 = ky_2y + \frac{ky_2(kx_3^3w_{5,3}^2-kt_3)}{3kx_3}z + \phi_2, \\ P_3 = kx_3x + ky_3y + \frac{ky_3(kx_3^3w_{5,3}^2-kt_3)}{3kx_3}z + kt_3t + \phi_3. \end{cases} \quad (3.18)$$

This solution describes a wave packet whose exponential envelope modulates a double-periodic background in the y - z plane. The exponential term provides localization in the x -direction, while the cn function generates transverse periodic oscillations. This structure resembles a localized excitation in a two-dimensional periodic lattice.

Scenario 2. Fully three-dimensional localized structure. Setting $kt_1 = -\frac{3kx_3kz_1}{ky_3}$, $kt_2 = -\frac{3kx_3kz_2}{ky_3}$, $kt_3 = \frac{kx_3(kx_3^2ky_3w_{5,3}^2-3kz_3)}{ky_3}$, $kx_1 = kx_2 = ky_1 = ky_2 = 0$, $w_{4,3} = 0$ yields

$$\begin{cases} P_1 = -\frac{3kx_3kz_1}{ky_3}t + kz_1z + \phi_1, \\ P_2 = -\frac{3kx_3kz_2}{ky_3}t + kz_2z + \phi_2, \\ P_3 = \frac{kx_3(kx_3^2ky_3w_{5,3}^2-3kz_3)}{ky_3}t + kx_3x + ky_3y + kz_3z + \phi_3. \end{cases} \quad (3.19)$$

This configuration yields a fully localized wave packet that is exponentially confined in all spatial directions. Such a three-dimensional solitary wave is of particular interest for modeling localized structures in higher-dimensional nonlinear media.

Case 2. Singly-modulated exponentially localized wave

The second case exhibits a simpler structure, with the exponential envelope modulated by a single quadratic-phase term:

$$u_{32} = \frac{2w_{1,5}w_{5,3}kx_3 \exp(w_{5,2}P_1^2 + w_{5,3}P_2 + \phi_5)}{w_{1,4} \operatorname{cn}(w_{4,2}P_1^2 + \phi_4, m) + w_{1,5} \exp(w_{5,2}P_1^2 + w_{5,3}P_2 + \phi_5) + \phi_6}, \quad (3.20)$$

again with P_1 and P_2 determined by parameter choices. Two scenarios emerge.

Scenario 1. Traveling localized pulse. Setting $kt_2 = -\frac{3kx_3kz_2}{ky_3}$, $kt_3 = \frac{kx_3(kx_3^2ky_3w_{5,3}^2-3kz_3)}{ky_3}$, $kx_2 = ky_2 = w_{4,1} = w_{4,3} = w_{5,1} = 0$, we have

$$\begin{cases} P_1 = kz_2z - \frac{3kx_3kz_2}{ky_3}t + \phi_2, \\ P_2 = kx_3x + ky_3y + kz_3z + \frac{kx_3(kx_3^2ky_3w_{5,3}^2-3kz_3)}{ky_3}t + \phi_3. \end{cases} \quad (3.21)$$

This represents a propagating wave packet with a single quadratic phase modulation. The wave packet moves in the z -direction while maintaining its localized envelope.

Scenario 2. Anisotropic localized wave. With $ky_1 = -\frac{3kx_1kz_2}{kt_2}$, $ky_3 = -\frac{3kx_3kz_2}{kt_2}$, $kz_3 = \frac{kz_2(-kx_3^3w_{5,3}^2+kt_3)}{kt_2}$, $kx_2 = ky_2 = w_{4,1} = w_{4,3} = w_{5,1} = 0$, we obtain

$$\begin{cases} P_1 = kt_2t + kz_2z + \phi_2, \\ P_2 = kt_3t - \frac{3kx_3kz_2}{kt_2}y - \frac{kz_2(kx_3^3w_{5,3}^2 - kt_3)}{kt_2}z + kx_3x + \phi_3. \end{cases} \quad (3.22)$$

This configuration describes a wave packet whose localization is anisotropic, exhibiting different decay rates in different spatial directions. The exponential envelope decays most rapidly along the x -direction, while the decay in the y - z plane is modulated by the dispersion parameters.

To visualize the features of this solution family, we select the free parameters from the exponentially modulated periodic wave scenario with $m = \frac{1}{2}$, $kt_3 = 4$, $kx_3 = ky_1 = ky_2 = ky_3 = 1$, $\phi_1 = \dots = \phi_6 = 0$, $w_{1,4} = w_{1,5} = w_{4,1} = w_{4,2} = w_{5,1} = w_{5,2} = w_{5,3} = 1$.

The resulting wave profiles are displayed in Figure 5.

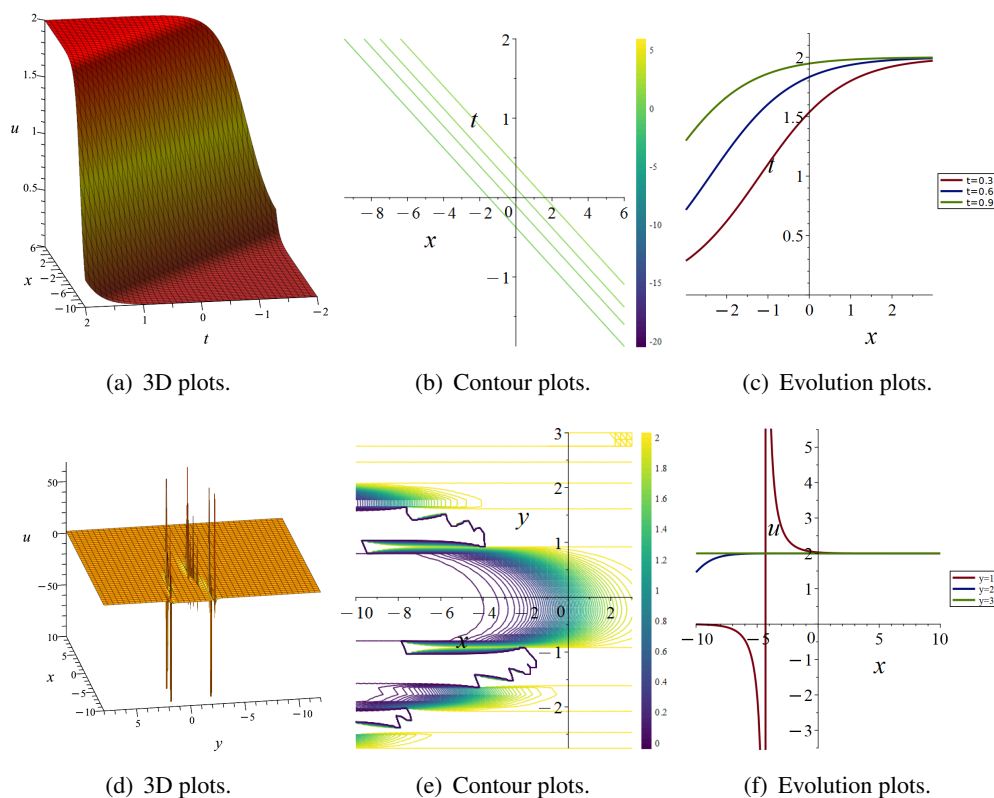


Figure 5. Evolution diagrams for exponentially localized-periodic waves u_{31} (Family I, first scenario): (a)–(c) at $y = 0$, $z = 0$; (d)–(f) at $z = 0$, $t = 1$.

Figure 5 shows a localized hump in the x -direction, flanked by damped periodic oscillations in the y - z plane. The three-dimensional plot (panel (a)) reveals a prominent central peak with exponential decay away from the center. The contour plot (panel (b)) shows the energy concentration near the origin, with periodic fringes extending outward in the transverse directions.

In the x - t cross-section at $y = 0$, $z = 0$ (panels (a)–(c)), the wave packet appears as a localized pulse that maintains its shape during propagation. The exponential envelope ensures rapid decay away from

the center, while the cn function introduces periodic modulations that are most visible in the transverse directions.

In the x - y cross-section at $z = 0$, $t = 1$ (panels (d)–(f)), the double-periodic structure is more clearly visible. The wave profile shows a central localized hump with transverse oscillatory tails. The contrast between the exponentially confined core and the periodic tails is a distinctive feature of this composite structure.

This type of solution resembles a wave packet trapped in a potential well while radiating weak periodic tails. Such structures are particularly interesting for describing localized excitations in nonlinear lattices, envelope solitons in dispersive media with periodic modulations, and trapped states in photonic crystals. The anisotropic scenario (Eq (3.22)) is especially relevant for systems with directional dispersion, where the wave packet decays at different rates in different spatial directions.

3.4. Type IV. Elliptic-modulated topological waves

This family combines topological kink structures with elliptic function backgrounds using a single-hidden-layer architecture. Unlike the previous three families, which require two-stage cascaded coupling, this family is obtained from a single-layer network where Jacobi elliptic functions and exponential functions are mixed directly at the output level. The resulting structures feature kink or kink-antikink profiles superimposed on doubly periodic backgrounds. Such configurations are physically relevant for phase boundaries in periodic media, magnetic domain walls in crystals with periodic anisotropy, and shock waves in stratified fluids with background internal waves.

The “4-2-1” single-hidden-layer network architecture (see Figure 2(d)) is employed for this family. The input layer consists of four neurons corresponding to (x, y, z, t) . The hidden layer contains two neurons whose outputs are combined at the output layer. The τ -function is given by

$$F = w_{11}h_1(\zeta_1) + w_{12}h_2(\zeta_1) + w_{13}h_3(\zeta_2) + w_{14}h_4(\zeta_2) + \phi_3, \quad (3.23)$$

with activation functions

$$h_1(\zeta_1) = cn(\zeta_1, m_1), \quad h_2(\zeta_1) = sn(\zeta_1, m_2), \quad h_3(\zeta_2) = \exp(\zeta_2), \quad h_4(\zeta_2) = \exp(-\zeta_2), \quad (3.24)$$

where m_1 and m_2 are elliptic moduli controlling the periodicity of the Jacobi elliptic functions.

The cn and sn functions generate the doubly periodic background, while the exponentials provide the localized kink structure. Substituting (3.23) into the bilinear equation (2.4) and solving them yields five distinct non-trivial branches.

Case 1. Propagating kink on a periodic background

Imposing $ky_2 = -\frac{3kx_2kz_1}{kt_1}$, $kz_2 = \frac{kz_1(-kx_2^3+kt_2)}{kt_1}$, $kx_1 = ky_1 = w_{1,4} = 0$, we obtain a solution where a single kink travels through a doubly periodic field. The potential reads

$$u_{41} = \frac{2w_{1,3}kx_2e^{P_2}}{w_{1,1}cn(P_1, m_1) + w_{1,2}sn(P_1, m_2) + w_{1,3}e^{P_2} + \phi_3}, \quad (3.25)$$

with

$$P_1 = kt_1t + kz_1z + \phi_1, \quad P_2 = kt_2t + kx_2x - \frac{3kx_2kz_1}{kt_1}y - \frac{kz_1(kx_2^3 - kt_2)}{kt_1}z + \phi_2. \quad (3.26)$$

This solution describes a kink propagating through a periodic background. The exponential term in the numerator and denominator provides the kink's localized profile, while the cn and sn functions generate the oscillatory background. This structure may describe a shock wave propagating through a stratified fluid with internal waves, or a magnetic domain wall moving through a crystal with periodic anisotropy.

Case 2. Kink-antikink pair on a periodic background

By setting $kz_1 = -\frac{ky_1(-kx_3^3+kt_2)}{3kx_2}$, $kt_1 = kx_1 = 0$, $ky_2 = kz_2 = 0$, we obtain a solution featuring both a kink and an antikink, which together form a localized “S-shaped” transition layer embedded in a periodic wave train:

$$u_{42} = \frac{2(w_{1,3}kx_2e^{P_2} - w_{1,4}kx_2e^{-P_2})}{w_{1,1}cn(P_1, m_1) + w_{1,2}sn(P_1, m_2) + w_{1,3}e^{P_2} + w_{1,4}e^{-P_2} + \phi_3}, \quad (3.27)$$

where

$$P_1 = ky_1y + \frac{ky_1(kx_3^3 - kt_2)}{3kx_2}z + \phi_1, \quad P_2 = kt_2t + kx_2x + \phi_2. \quad (3.28)$$

This configuration is physically interesting because it represents a localized transition layer (kink-antikink pair) superimposed on a periodic background. Such structures occur in systems where two stable phases coexist with periodic modulations.

Case 3. Stationary kink in a modulated environment

The third family, with parameters $kz_1 = -\frac{ky_1(-kx_3^3+kt_2)}{3kx_2}$, $kz_2 = -\frac{ky_2(-kx_3^3+kt_2)}{3kx_2}$, $kt_1 = 0$, $kx_1 = w_{1,4} = 0$, produces a stationary kink whose position remains fixed while the surrounding periodic field oscillates in time:

$$u_{43} = \frac{2w_{1,3}kx_2e^{P_2}}{w_{1,1}cn(P_1, m_1) + w_{1,2}sn(P_1, m_2) + w_{1,3}e^{P_2} + \phi_3}, \quad (3.29)$$

where

$$P_1 = ky_1y - \frac{ky_1(-kx_3^3 + kt_2)}{3kx_2}z + \phi_1, \quad P_2 = kt_2t + kx_2x + ky_2y - \frac{ky_2(-kx_3^3 + kt_2)}{3kx_2}z + \phi_2. \quad (3.30)$$

This solution describes a pinned kink in a periodically modulated environment, resembling a domain wall trapped by a periodic potential landscape. Such configurations are relevant for systems with spatial inhomogeneities or periodic external fields.

When the elliptic modulus vanishes, $m_1 = m_2 = 0$, the Jacobi elliptic functions degenerate into elementary trigonometric functions: $cn(\zeta, 0) = \cos \zeta$ and $sn(\zeta, 0) = \sin \zeta$. Under this degeneration, two additional families of exact solutions are obtained, which mix trigonometric and exponential functions.

Case 4. Trigonometric-exponential hybrid wave

Choosing $kt_1 = \frac{kx_1(-3kx_1^4-4kx_1^2kx_2^2-kx_2^4+kt_2kx_2)}{kx_2^2}$, $kz_1 = -\frac{ky_1(-3kx_1^4-kx_2^4+kt_2kx_2)}{3kx_2^2}$, $ky_2 = 0$, $kz_2 = -\frac{kx_1ky_1(kx_1^2+kx_2^2)}{kx_2^2}$, $m_1 = \phi_3 = w_{1,2} = w_{1,4} = 0$ yields

$$u_{44} = \frac{2(-w_{1,1}kx_1 \sin P_1 + w_{1,3}kx_2e^{P_2})}{w_{1,1} \cos P_1 + w_{1,3}e^{P_2}}, \quad (3.31)$$

where P_1 and P_2 are determined by the above constraints.

This waveform combines a sinusoidal oscillation (from the degenerate cn/sn functions) with an exponential kink component. The competition between the trigonometric and exponential terms produces localized structures with oscillatory tails.

Case 5. Double exponential-trigonometric hybrid wave

With $kt_1 = \frac{kx_1(3kx_1^2ky_1^2 - kx_1^2ky_2^2 - 3ky_2kx_2)}{ky_2^2}$, $kt_2 = -\frac{kx_1ky_1(4kx_1^2ky_1^2 - 3ky_2kx_2)}{ky_2^2}$, $kx_2 = -\frac{kx_1ky_1}{ky_2}$, $m_1 = \phi_3 = w_{1,2} = 0$, $kz_1 = -\frac{ky_1(kx_1^2ky_1^2 + kx_1^2ky_2^2 - ky_2kx_2)}{ky_2^2}$, we obtain

$$u_{45} = \frac{-2 \left(w_{1,1} kx_1 \sin P_1 + \frac{w_{1,3} kx_1 ky_1}{ky_2} e^{P_2} - \frac{w_{1,4} kx_1 ky_1}{ky_2} e^{-P_2} \right)}{w_{1,1} \cos P_1 + w_{1,3} e^{P_2} + w_{1,4} e^{-P_2}}, \quad (3.32)$$

with P_1 and P_2 satisfying the above constraints.

This solution features a competition between a sinusoidal carrier, a growing exponential, and a decaying exponential, producing a rich variety of localized structures. The balance between these three components determines the waveform morphology.

For a concrete visualization, we take the parameters from Eq (3.31) with $kt_1 = kx_2 = kz_1 = 1$, $kt_2 = 2$, $m_1 = \frac{1}{6}$, $m_2 = \frac{1}{2}$, $\phi_1 = \phi_2 = \phi_3 = 0$, $w_{1,1} = w_{1,2} = w_{1,3} = w_{1,4} = 1$.

The resulting wave profiles are displayed in Figure 6.

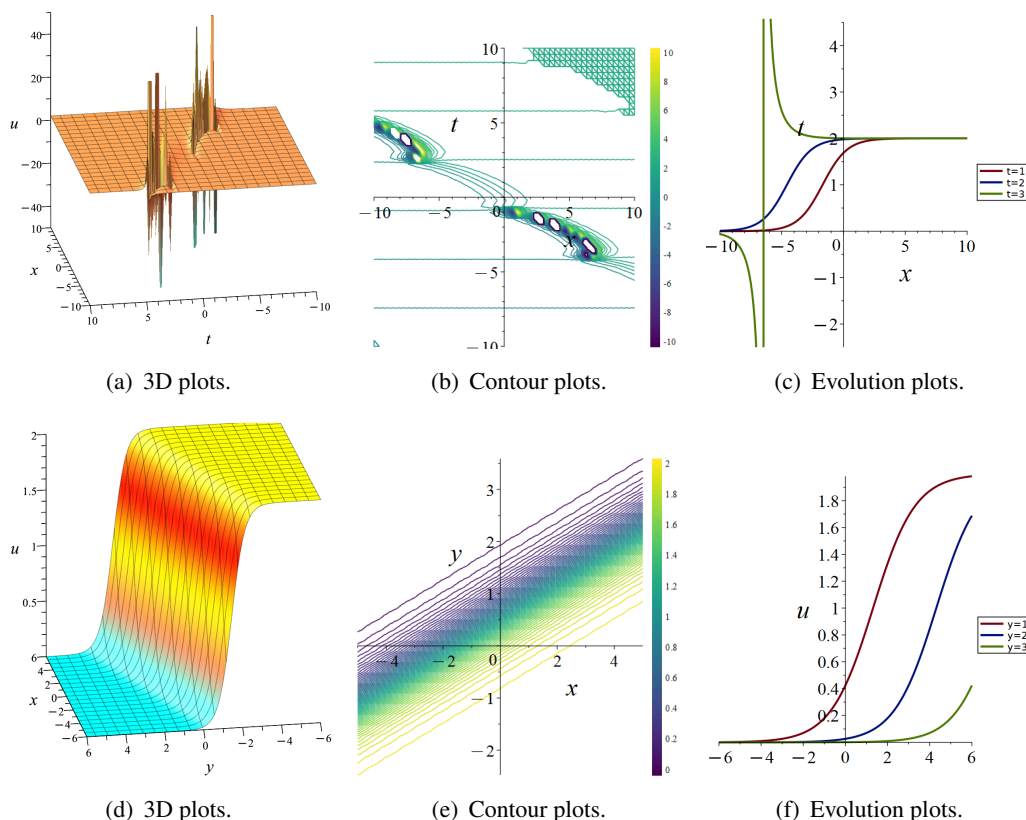


Figure 6. Evolution diagrams for elliptic function-modulated kink waves u_{41} : (a)–(c) at $y = 0, z = 0$; (d)–(f) at $z = 0, t = 1$.

Figure 6 shows a kink structure (from the exponential terms) embedded in a doubly periodic background. The three-dimensional plot (panel (a)) reveals a smooth but distinct transition from one asymptotic state to another, with periodic oscillations superimposed on both sides of the transition region. Due to the parameter selection, the kink structure is relatively smooth, but its topological features can be clearly identified through the contour plot.

In the x - t cross-section at $y = 0$, $z = 0$ (panels (a)–(c)), the kink appears as a gradual transition between two constant levels, with the periodic background manifesting as ripples on both sides. The density plot (panel (b)) shows the energy concentration along the kink front, with periodic modulations extending in the transverse directions.

In the x - y cross-section at $z = 0$, $t = 1$ (panels (d)–(f)), the doubly periodic structure is more visible. The contour plot (panel (f)) clearly delineates the kink front while revealing the periodic modulations in the transverse plane.

This type of solution is physically relevant for phase boundaries in periodic media, such as magnetic domain walls in crystals with periodic anisotropy, or shock waves in stratified fluids with background internal waves. The Cases 4 and 5, obtained through modulus degeneration ($m_1 = 0$), extend the range of wave patterns to trigonometric-exponential hybrids, which combine sinusoidal carriers with growing and decaying exponentials.

Reduction to known solutions

To validate the consistency of our results with established literature, we examine the reduction of the obtained hybrid localized structures to known classical solutions under three limiting regimes.

Reduction I: Elliptic-to-trigonometric limit ($m \rightarrow 0$). As $m \rightarrow 0$, the Jacobi elliptic functions degenerate to trigonometric functions: $sn(\xi, m) \rightarrow \sin(\xi)$ and $cn(\xi, m) \rightarrow \cos(\xi)$. Under this limit, Type I (algebraic-elliptic) waves reduce to algebraic-trigonometric composite waves, Type II (topological-periodic) waves reduce to kink-trigonometric waves, and Type III (exponential-elliptic) waves reduce to exponential-trigonometric waves. These reduced forms correspond to the breather and periodic wave solutions reported in [25–33].

Reduction II: Vanishing inter-layer weights ($w_{j,1}, w_{j,2} \rightarrow 0$). When the inter-layer connection weights vanish, the two-stage cascade collapses to a single-layer additive construction:

$$F_{\text{two-stage}} = \sum_j a_j h_j \left(\sum_i w_{ji} h_i(\xi_i) + \phi_j \right) \xrightarrow{w_{ji} \rightarrow 0} \sum_j a_j h_j(\phi_j).$$

This recovers the additive single-layer BNNM solutions of [45,46]. The non-vanishing weights in our solutions thus confirm that the two-stage coupling genuinely generates new structures beyond the single-layer regime.

Reduction III: Type I to lump solutions. Under specific parameter choices in Type I—specifically, taking the elliptic modulus $m \rightarrow 0$ and selecting appropriate polynomial coefficients—the algebraic decay component dominates, and the solution reduces to the known lump solutions reported in [25,41].

4. PINN numerical reproduction of the anisotropic localized wave

It is important to clarify that the PINN serves a numerical reproduction role rather than an independent verification of the analytical correctness of the BNNM-derived solution. The PINN simulation demonstrates that the solution can be stably reproduced under discrete approximation and

optimization perturbations. This establishes a complementary synergy between symbolic construction and numerical benchmarking. The anisotropic localized wave u_{22} derived in Section 3.2, Eq (3.13) with parameter set (3.14) serves as the benchmark for PINN validation. This solution is selected because its exponential localization with direction-dependent decay rates poses a non-trivial test of neural network approximation capacity while remaining computationally tractable within the domain $x, y, z \in [-2, 2]$, $t \in [0, 1]$. The validation has a dual purpose: to confirm that the BNNM-derived exact solution can be independently recovered by a physics-constrained numerical method, and to demonstrate that the PINN framework converges to the correct spatiotemporal evolution when seeded with the BNNM solution as initial and boundary data.

4.1. PINN architecture and training setup

Following the standard PINN framework of Raissi et al. [47], the solution $u(x, y, z, t)$ of the gBKP equation is approximated by a deep neural network $u_{\text{NN}}(x, y, z, t; \theta)$, where θ denotes the trainable weights and biases. Recently, significant efforts have been devoted to enhancing the performance of PINNs for nonlinear wave systems. For instance, Wang et al. [48] proposed an informed reconstruction-oriented numerical network (IRON-Net) that incorporates a finite-difference-based numerical loss term and a clustering-reconstruction mechanism, achieving notable improvements in training accuracy, convergence rate, and robustness when applied to the Manakov equations. The network is trained by minimizing a composite loss function that encodes the governing equation, initial conditions, and boundary conditions.

The network architecture comprises four hidden layers with 512 neurons each, followed by a single output neuron. The hyperbolic tangent (tanh) is adopted as the activation function. This configuration provides sufficient representational capacity to capture the complex spatiotemporal features of the anisotropic localized wave. The total number of trainable parameters is approximately 7.9×10^5 .

The PINN model was implemented using Python 3.10 with the PyTorch library (version 2.0+). All training and inference were performed on a single NVIDIA GeForce RTX 5070 Laptop GPU with 8 GB of memory. Input normalization maps each coordinate to $[-1, 1]$:

$$x_n = 2 \frac{x - x_{\min}}{x_{\max} - x_{\min}} - 1, \quad y_n, z_n, t_n \text{ analogously}, \quad (4.1)$$

with $x_{\min} = y_{\min} = z_{\min} = -2$, $x_{\max} = y_{\max} = z_{\max} = 2$, $t_{\min} = 0$, $t_{\max} = 1$.

Output normalization stabilizes training. The maximum absolute value $U_{\max}$ is estimated by evaluating the exact solution at 20,000 randomly sampled collocation points within the domain. For the u_{22} benchmark solution with the selected parameters ($\phi_5 = 3.0$, $w_{11} = 2.0$), the estimated value is $U_{\max} \approx 1.11$.

The model was implemented using Python 3.10 with the PyTorch library (version 2.0+). All training and inference were performed on a single NVIDIA GeForce RTX 5070 Laptop GPU with 8 GB of memory. After systematic hyperparameter optimization, the final training configuration was set as follows: $N_{IC} = 15,000$, $N_{BC} = 30,000$, $N_{PDE} = 100,000$, and $N_{val} = 10,000$, with loss weights $w_{PDE} = 100.0$, $w_{BC} = 10.0$, and $w_{IC} = 100.0$. The Adam optimizer was employed with an initial learning rate of 1×10^{-3} , decayed via cosine annealing to 1×10^{-6} over 1000 training epochs. Training parameters are summarized in Table 4. The comparison of alternative architectures (shallower, narrower, and swish-activated networks) is given in Table A.1 (see Supplementary 1 in Appendix).

Table 4. PINN training parameters for the anisotropic localized wave validation.

Parameter	Value
Network architecture	4-512-512-512-1
Activation function	tanh
Optimizer	Adam
Initial learning rate	1×10^{-3} (cosine annealing to 10^{-6})
Training epochs	1000
Batch size	2048
Initial condition points N_{IC}	15,000
Boundary condition points N_{BC}	30,000
PDE residual points N_{PDE}	100,000
Loss weights (w_{PDE}, w_{BC}, w_{IC})	(100.0, 10.0, 100.0)
L_2 regularization λ	10^{-8}
Software environment	Python 3.10, PyTorch 2.0+

The loss function consists of three components. The partial differential equation (PDE) residual loss enforces Eq (1.1) at collocation points:

$$\mathcal{L}_{PDE}(\boldsymbol{\theta}) = \frac{1}{N_{PDE}} \sum_{i=1}^{N_{PDE}} |\mathcal{R}(x_i, y_i, z_i, t_i; \boldsymbol{\theta})|^2, \quad (4.2)$$

where the residual $\mathcal{R}$ is given by

$$\mathcal{R} = u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz}. \quad (4.3)$$

The initial condition (IC) losses and boundary condition (BC) losses ensure consistency with the BNNM exact solution:

$$\mathcal{L}_{IC}(\boldsymbol{\theta}) = \frac{1}{N_{IC}} \sum_{i=1}^{N_{IC}} |u_{NN}(x_i, y_i, z_i, 0; \boldsymbol{\theta}) - u_{BNNM}(x_i, y_i, z_i, 0)|^2, \quad (4.4)$$

$$\mathcal{L}_{BC}(\boldsymbol{\theta}) = \frac{1}{N_{BC}} \sum_{i=1}^{N_{BC}} |u_{NN}(x_i, y_i, z_i, t_i; \boldsymbol{\theta}) - u_{BNNM}(x_i, y_i, z_i, t_i)|^2. \quad (4.5)$$

The total loss is the weighted sum

$$\mathcal{L}(\boldsymbol{\theta}) = w_{PDE} \mathcal{L}_{PDE} + w_{BC} \mathcal{L}_{BC} + w_{IC} \mathcal{L}_{IC} + \lambda \|\boldsymbol{\theta}\|_2^2. \quad (4.6)$$

The relative L_2 error quantifies the accuracy of the PINN approximation:

$$E_{\text{rel}} = \frac{\|u_{NN} - u_{BNNM}\|_2}{\|u_{BNNM}\|_2 + \varepsilon}, \quad \varepsilon = 10^{-8}. \quad (4.7)$$

4.2. Training dynamics and error analysis

We train the PINN model for 1000 epochs using the Adam optimizer. Figure 7 shows the training dynamics.

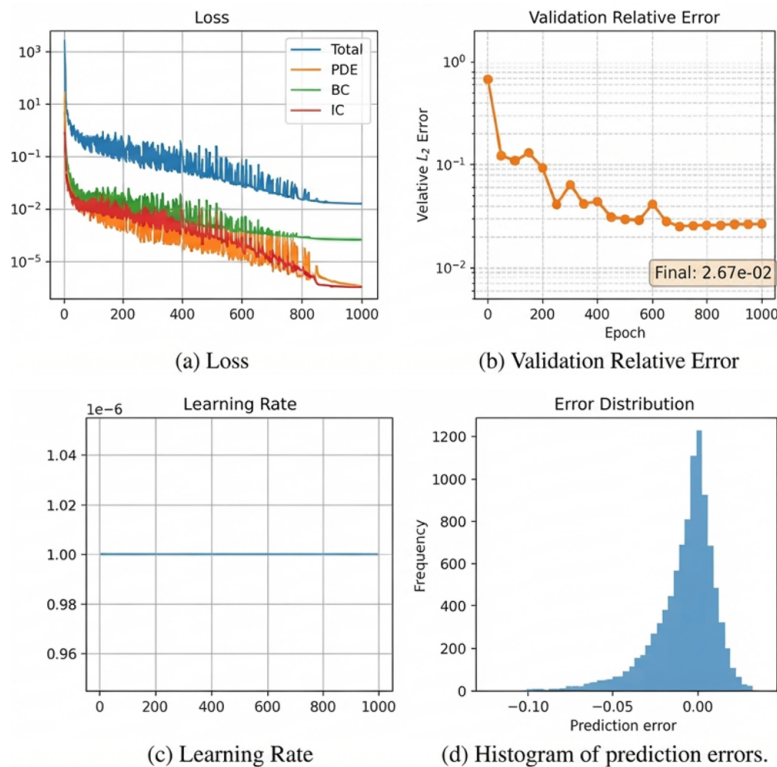


Figure 7. Training dynamics of Eq (3.13) with parameter set (3.14) using the PINN model. (a) Total loss, PDE residual, BC loss, and IC loss versus epoch (log scale). (b) Relative L_2 validation error versus epoch (log scale); the final error converges to 2.66×10^{-2} . (c) Learning rate schedule. (d) Histogram of prediction errors.

The loss curves in Figure 7(a) exhibit three distinct phases. During the first 50 epochs, the total loss drops precipitously from $\sim 10^4$ to $\sim 10^0$, with the PDE residual, BC loss, and IC loss decaying in tandem. This rapid decay indicates that the network captures the large-scale solution structure. From epoch 50 to 300, the losses decrease steadily: The total loss falls from 10^0 to $\sim 10^{-2}$, the PDE residual to 1.42×10^{-6} , and both the BC and IC losses to $\sim 10^{-5}$. In the final phase (epochs 800–1000), the descent slows but remains monotonic, with the total loss settling at $\sim 1.5 \times 10^{-3}$.

The validation relative L_2 error, shown in Figure 7(b), decreases monotonically from $\sim 10^0$ to below 2.66×10^{-2} , with a final PDE residual of 1.42×10^{-6} . The error distribution in Figure 7(d) is approximately symmetric and centered at zero, with a standard deviation of ~ 0.08 in normalized units. This indicates the absence of systematic bias.

The final error metrics after 1000 epochs are:

- Relative L_2 error: $E_{\text{rel}} < 2.66 \times 10^{-2}$;
- Mean absolute error: $\text{MAE} \approx 3.5 \times 10^{-2}$;
- Maximum absolute error: $\approx 3.0 \times 10^{-1}$;

- PDE residual: $\mathcal{L}_{\text{PDE}} \approx 1.42 \times 10^{-6}$;
- BC loss: $\mathcal{L}_{\text{BC}} \approx 3.0 \times 10^{-5}$;
- IC loss: $\mathcal{L}_{\text{IC}} \approx 2.0 \times 10^{-5}$.

4.3. Spatiotemporal evolution and error visualization

Figure 8 compares the PINN prediction and the BNNM exact solution in the x - t plane at $y = 0$, $z = 0$.

The PINN prediction (Figure 8(a)) and the exact solution (Figure 8(b)) agree closely in overall structure: Both exhibit a smooth transition from low values (blue) at $t = 0$ to high values (yellow) at $t = 1$, with monotonic increase along x from -2 to 2 . This evolution is consistent with the physical character of the anisotropic localized wave—the exponential envelope propagates along x while preserving its shape. The error map (Figure 8(c)) reveals that the maximum absolute error of ~ 0.03 occurs in the vicinity of $x \approx -1$, where the exact solution passes through zero. As discussed in Section 4.4, the large pointwise relative error in this region is a mathematical consequence of the relative error definition, rather than a failure of the PINN approximation.

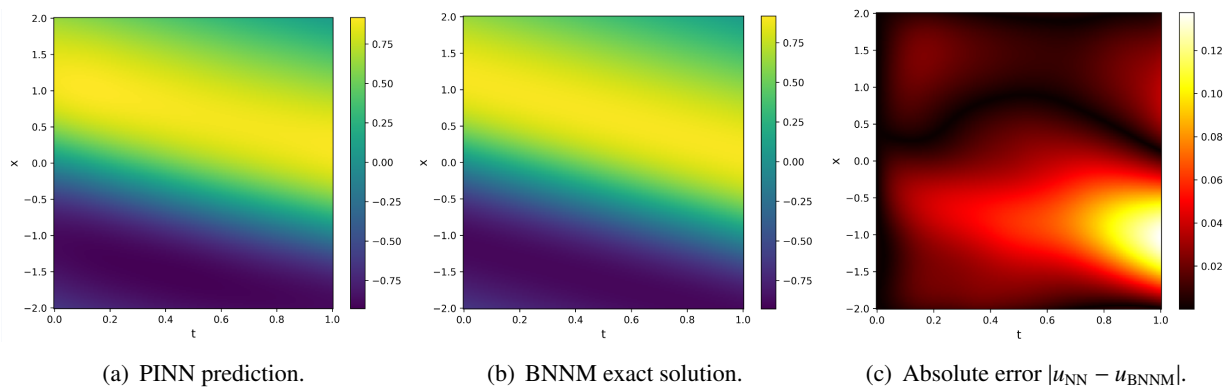


Figure 8. Comparison of PINN prediction and BNNM exact solution u_{22} (Type-II topological-periodic composite wave) in the x - t plane at $y = 0$, $z = 0$.

Figure 9 shows the temporal evolution at fixed spatial points.

At the center $(0, 0, 0)$, the exact solution rises monotonically from $u \approx 0$ to $u \approx 0.45$. The PINN captures this trend but systematically underestimates the solution in the interval $t \in [0.3, 0.8]$, with a maximum deviation of ≈ 0.04 (relative error $\sim 4.4\%$). The error curve is bimodal, peaking at $t \approx 0.0$ and $t \approx 0.8$ where the curvature changes most rapidly.

At $x = 1$ $(1, 0, 0)$, the exact solution is non-monotonic, reaching a peak $u \approx 0.9$ at $t \approx 0.1$. The PINN agrees well for $t < 0.1$ but deviates substantially after the peak, underestimating the latter half of the evolution. The maximum error is $\approx 3.5 \times 10^{-2}$ at $t = 1$.

At $x = -1$ $(-1, 0, 0)$, the exact solution increases from $u \approx -0.9$ to $u \approx 0.0$. The PINN exhibits the largest error among all test points ($t \approx 1$, relative error $\sim 75\%$), with persistent underestimation throughout the time interval. This reflects the inherent difficulty neural networks face in approximating negative-valued regions combined with steep gradients.

At $y = 1$ $(0, 1, 0)$, the PINN tracks the exact solution with good fidelity; the maximum error $\approx 9.0 \times 10^{-3}$ (relative error $\sim 0.76\%$) occurs at $t \approx 0.7$.

At $z = 1$ $(0, 0, 1)$, the agreement is excellent, with maximum error $\approx 2.5 \times 10^{-2}$ (relative error $\sim 3.3\%$), the lowest among all tested locations. The smooth evolution at this position—minimal curvature variation—explains the high accuracy.

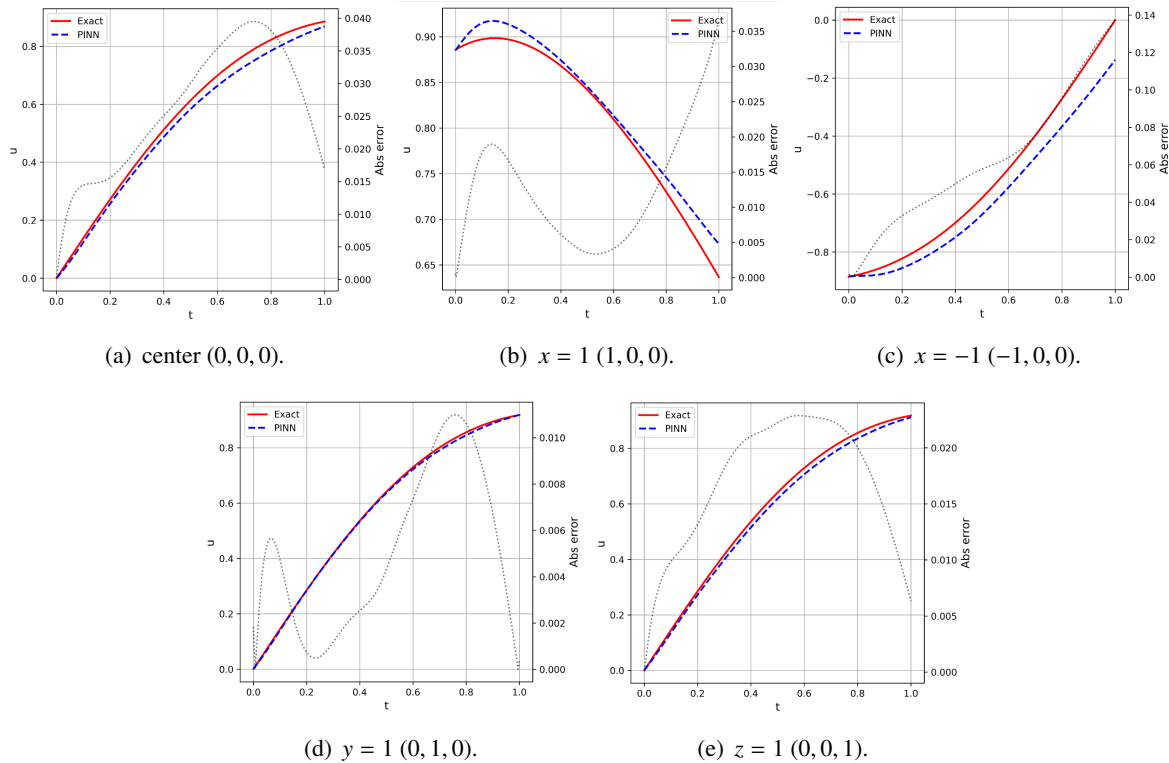


Figure 9. Training dynamics of Eq (3.13) with parameter set (3.14) using the PINN model.

Figure 10 displays spatial profiles at fixed times $t = 0.2, 0.5, 0.8$.

At $t = 0.2$, the exact solution displays an S-shaped transition from $u \approx -0.85$ at $x = -1.5$ to $u \approx 0.85$ at $x = 0.5$, with maximum curvature in $x \in [-2, 2]$. The PINN reproduces the S-shape but exhibits mild oscillations in the central region; the maximum error $\approx 4 \times 10^{-2}$ appears at $x \approx 1.0$.

At $t = 0.5$, the S-transition has flattened, reducing curvature. The PINN accuracy improves accordingly, with maximum error $\approx 6 \times 10^{-2}$ distributed more uniformly. This confirms that the network approximates smooth structures more effectively than steep ones.

At $t = 0.8$, the solution approaches a near-steady state. The PINN shows maximum deviation ≈ 0.10 in $x \in [-2, 2]$, while the boundary regions $x \rightarrow \pm 2$ remain well captured. The error peaks at $x \approx -1$, coinciding with the location of maximum second derivative.

The large pointwise relative errors observed in Figure 9(c) (e.g., up to 75% at $x = -1$) do not indicate failure of the PINN approximation. Instead, they reflect the mathematical divergence of the relative error definition when the exact solution passes through zero. Specifically, the pointwise relative error is defined as $\frac{|u_{\text{pred}} - u_{\text{true}}|}{|u_{\text{true}}| + \epsilon}$. In regions where $u_{\text{true}} \rightarrow 0$, this quantity diverges even for small absolute deviations. The corresponding absolute errors remain bounded below ~ 0.3 , and the global relative L_2 error is 2.66% with PDE residual $O(10^{-6})$. Thus, the network faithfully captures the governing physics, with residual errors concentrated near zero-crossing and boundary regions.

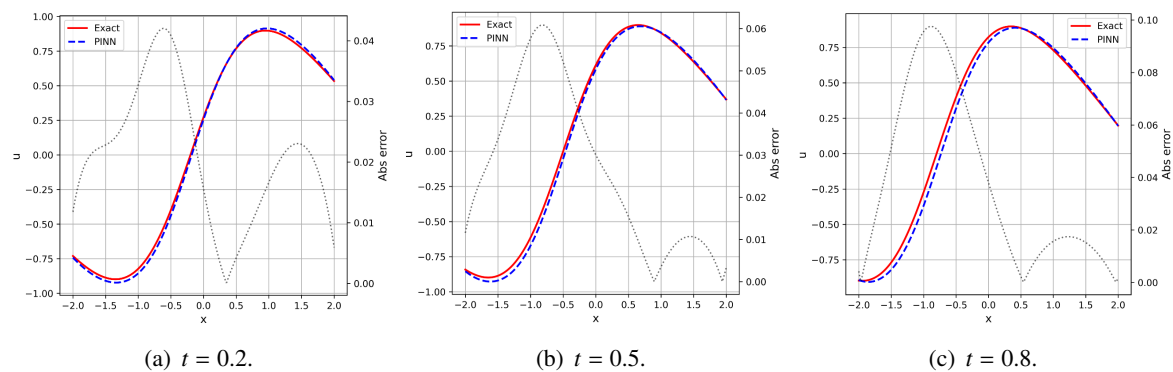


Figure 10. Comparison of PINN prediction and BNNM exact solution along the x -direction at fixed times. Solid red: exact solution; dashed blue: PINN prediction; dotted black: absolute error (right axis).

4.4. Physical interpretation and the BNNM-PINN synergy

The PINN verification result allows for three layers of physical interpretation. First, the relative L_2 error below 3% confirms that the anisotropic localized waves derived from BNNM are physically achievable and can be independently recovered through numerical methods with physical constraints. This provides evidence for other BNNM solutions reported in Section 3: When a solution converges under PINN review, its mathematical structure is self consistent, thus confirming the structural authenticity of the solution.

Second, the maximum error is concentrated around $x = -1$, and $t \approx 0.8$ has physical significance. The steepest attenuation of anisotropic local waves along x poses the strongest gradient challenge to the network. The cumulative approximation error of the transient region t to 1 reflects the nonlinearity of early evolution. In contrast, the low error of $z = 1$ and other smooth regions validates PINN's ability to learn periodic background modulation.

Third, the BNNM-PINN framework establishes a methodological synergy that differs from previous research. Traditional methods treat PINNs as independent PDE solvers or compare them with symbolic methods as competitors. Here, BNNM and PINN play complementary roles: BNNM generates exact solutions through algebraic constraint solving without the need for data or training, while PINN provides independent numerical validation by optimizing physics-based losses. This closed-loop paradigm provides a new approach for the study of integrable systems.

BNNM excels at producing closed-form expressions and exploring the full parameter space, but is limited by the choice of activation function pool and the solvability of algebraic systems. PINN circumvents the need for prespecified solution structures, yet faces challenges from network capacity and optimization landscapes. Their combined use mitigates individual weaknesses: BNNM supplies high-quality initial and boundary data that accelerate PINN convergence and improve accuracy; PINN furnishes an independent numerical check that validates the physical self-consistency of BNNM solutions. For higher-dimensional integrable equations such as the (3+1)-dimensional gBKP, where direct visualization and verification are inherently difficult, this symbolic–numerical dual-track mechanism is particularly valuable.

The successful PINN validation of the anisotropic localized wave (Type III) further implies that

other configurations within this family are likewise physically realizable. Beyond standard PINNs, neural operator networks such as the deep operator network (DeepONet) and the Fourier neural operator (FNO) have also been successfully applied by Yang et al. [49] to predict bright and dark solitons in the Gross-Pitaevskii equation for Bose-Einstein condensates, demonstrating high accuracy and broad applicability. These developments highlight the growing role of neural-network-based approaches in soliton dynamics and nonlinear wave prediction.

5. Discussion

The findings in Sections 3 and 4 raise several points for further discussion. These include the classification framework for hybrid localized structures, the methodological synergy between BNNM and PINN, and the limitations of the current approach.

5.1. Classification of hybrid localized structures

To formalize the “wavefield degeneracy”, we first eliminate trivial gauge freedom: The transformation $\tau \rightarrow \lambda\tau$ ($\lambda \neq 0$) leaves $u = 2(\ln \tau)_x$ invariant; this degree is fixed by normalization. Beyond this, the degeneracy is non-trivial. For example, in Type I, Case 1, two distinct parameter sets satisfying different dispersion relations yield identical u expressions after algebraic simplification (see Supplementary 2 in Appendix for the Maple verification). Qualitatively, this degeneracy likely originates from the scaling and Galilean invariance of the Hirota bilinear operator $D_y D_t - D_x^3 D_y + 3D_x D_z$ —different microscopic parameterizations may correspond to different Bäcklund parameterizations of the same bilinear equation.

The four families of solutions obtained in Section 3 are unified by a common principle: Each represents a non-additive coupling of two distinct localization mechanisms. This classification, summarized in Table 5, provides a systematic framework for understanding the solution landscape of the gBKP equation beyond conventional soliton, lump, or periodic wave classifications.

Table 5. Summary of the hybrid localized structures obtained from the composite BNNM.

Type	Name	Localization mechanism	Background	Solution
I	Algebraic-elliptic composite	Algebraic decay	Elliptic periodic	u_{11}, u_{12}
II	Topological-periodic composite	Topological step	Elliptic periodic	u_{21}, u_{22}
III	Exponential-elliptic composite	1 Exponential decay	Elliptic periodic	u_{31}, u_{32}
IV	Elliptic-modulated topological	Topological step	Elliptic periodic	$u_{41}-u_{45}$

Notably, Type IV is obtained from a single-hidden-layer architecture, while Types I–III require two-stage cascaded coupling. This distinction is significant: It indicates that certain hybrid structures (algebraic-elliptic, topological-periodic, and exponential-elliptic composites) arise only when the network has sufficient depth to create non-additive inter-layer dependencies. Single-layer constructions, by contrast, are limited to mixing exponentials with elliptic functions (Type IV) or, in the degenerate limit, trigonometric-exponential hybrids. This observation supports the claim that the two-stage cascaded architecture genuinely extends the expressive capacity of the BNNM framework.

5.2. BNNM-PINN methodological synergy

The PINN validation in Section 4 demonstrates the complementary strengths of symbolic construction and numerical verification. The relative L_2 error confirms that the BNNM-derived anisotropic localized wave is physically realizable and can be independently recovered through a physics-constrained numerical method. This provides indirect validation for the other BNNM solutions reported in Section 3: When a solution converges under PINN verification, its mathematical structure is self-consistent, confirming the structural authenticity of the solution.

The BNNM-PINN framework establishes a methodological synergy that differs from previous research. BNNM's strength lies in generating exact closed-form solutions through algebraic constraint solving without requiring training data, though its scope is constrained by the activation function pool and the tractability of the resulting algebraic systems. PINN, by contrast, does not rely on preassigned solution forms, but its performance depends on network capacity and the optimization landscape. Combining the two approaches allows each method to compensate for the other's limitations: The analytical solutions from BNNM provide reliable initial and boundary data that facilitate PINN training, while the PINN framework offers an independent numerical assessment of the physical consistency of the symbolic solutions. For higher-dimensional integrable equations such as the (3+1)-dimensional gBKP equation, where direct verification is inherently challenging, this dual approach provides a practical and reliable validation strategy.

5.3. Comparison with existing studies

To better contextualize the novelty of this work, we compare our results with the five representative studies cited in Section 1. The comparison focuses on three aspects: the method employed, the types of solutions obtained, and the depth of physical insights. The results are summarized in Table 6.

Table 6. Comparison between this work and representative literature.

Ref.	Method	Solution	Limitations/Differences
[24]	Hirota method	Multi-soliton solutions	Does not include hybrid localized structures
[25]	Bilinear method	Mixed lump soliton solutions	Relies on traveling-wave reduction; no non-traveling solutions
[26]	GERF	Kink, anti-kink, periodic, singular solitons	Relies on traveling-wave reduction; no non-traveling solutions
[27]	Bifurcation analysis	Soliton, periodic waves	Relies on traveling-wave reduction; no non-traveling solutions
[46]	BNNM	Lump, M-lump solitons, periodic waves, N-solitons	Additive architecture only; no Jacobi elliptic activations; no cascaded coupling

Through the above comparison, the distinctive contributions of this work are summarized in three aspects.

Methodology. Unlike the conventional BNNM of Li et al. [46], this work employs a two-stage cascaded architecture, where first-stage outputs serve as second-stage inputs, creating non-

additive inter-layer dependencies. In addition, Jacobi elliptic functions (sn , cn) are incorporated as activation functions. This design yields new solution families—algebraic-elliptic, topological-periodic, and exponential-elliptic composite waves—that are not generated by additive architectures.

Theoretical insight. Beyond reporting specific dynamical phenomena such as the fusion/fission behavior observed by Li and Ma [25], this work identifies a property termed “wavefield degeneracy” where multiple distinct parameter configurations reduce to identical waveform functional forms. This suggests that wave morphology is governed by the integrable structure of the equation rather than by individual dispersion parameters.

Validation. Unlike Wazwaz [24], Kaur et al. [26], and Liang and Wang [27], this work combines BNNM with PINN to form a symbolic-numerical verification loop. The PINN simulation reproduces the BNNM-derived solution with a relative L_2 error of 2.66% and a PDE residual of $O(10^{-6})$, providing independent numerical confirmation of the exact solutions.

5.4. Limitations and future directions

Several limitations of the current approach should be acknowledged. First, the activation function pool defined in Eq (2.5) is empirically chosen based on known waveform structures of the gBKP equation. While this pool successfully generates the four families reported in Section 3, it is not exhaustive. Second, the systematic enumeration of activation function combinations becomes computationally expensive as the network depth increases.

A rigorous linear stability analysis of the obtained hybrid localized structures is beyond the scope of the present work. Such analysis would require solving the linearized perturbation equations around the complex spatiotemporal background solutions—a substantial undertaking for non-traveling wave structures. Nevertheless, the PINN-based numerical reproduction, with PDE residual converging to $O(10^{-6})$, provides preliminary evidence of short-term numerical stability. A complete stability characterization is identified as a direction for future investigation. A complete stability characterization—via Floquet or spectral methods for the linearized perturbation equations—will be pursued in future work.

6. Conclusions

This paper presents a composite bilinear neural network method for constructing exact analytical solutions of the (3+1)-dimensional gBKP equation. Four families of closed-form solutions are obtained and systematically classified according to their localization mechanisms. To our knowledge, this work is among the first to systematically incorporate Jacobi elliptic functions as activations within a two-stage BNNM, overcoming the derivative complexity that previously limited their use in bilinear neural networks. A phenomenon termed wavefield degeneracy is identified across all four solution families. Distinct parameter configurations, differing in propagation directions and dispersion relations, collapse to identical functional forms for the wavefield u .

We select the anisotropic localized wave for numerical validation using PINN. The PINN simulation reproduces the BNNM-derived exact solution with a relative L_2 error of 2.66% (PDE residual $O(10^{-6})$), confirming the numerical well-posedness of the solution. Error analysis reveals that the maximum deviation concentrates in regions of steep gradients and negative-valued curvature, consistent with known neural network approximation limitations. The successful numerical reproduction confirms

that the BNNM solutions are physically realizable and provides confidence for the remaining solution families.

The methodological synergy established in this work offers an advance over existing approaches. BNNM provides symbolic construction of exact solutions through algebraic constraint solving, while PINN furnishes independent numerical verification through physics-based optimization. This closed-loop paradigm addresses both the need for mathematical rigor and the requirement for physical consistency.

Several directions for future research emerge from this work. The activation function pool could be expanded to include more general special functions. Extending the BNNM-PINN framework to non-integrable equations would test its broader applicability. Finally, the physical relevance of these hybrid structures—in shallow water waves, nonlinear optics, and plasma physics—merits experimental or observational investigation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. J. Ma, J. Tang, A review for dynamics in neuron and neuronal network, *Nonlinear Dyn.*, **89** (2017), 1569–1578. <https://doi.org/10.1007/s11071-017-3565-3>
2. D. Shi, Z. Li, T. Y. Han, New traveling wave solutions, phase portrait and chaotic pattern for the stochastic modified Korteweg-de Vries equation, *Results Phys.*, **53** (2023), 106964. <https://doi.org/10.1016/j.rinp.2023.106964>
3. B. S. T. Alkahtani, Exploration of Lie symmetry, bifurcation, chaos and exact solution of the geophysical KdV equation, *Int. J. Theor. Phys.*, **64** (2025), 79. <https://doi.org/10.1007/s10773-025-05934-6>
4. T. Y. Han, Y. Liang, W. J. Fan, Dynamics and soliton solutions of the perturbed Schrödinger-Hirota equation with cubic-quintic-septic nonlinearity in dispersive media, *AIMS Math.*, **10** (2025), 754–776. <https://doi.org/10.3934/math.2025035>
5. T. Y. Han, K. Zhang, Y. Y. Jiang, H. Rezazadeh, Chaotic pattern and solitary solutions for the (2+1)-dimensional Beta-fractional double-chain DNA system, *Fractal Fract.*, **8** (2024), 1–19. <https://doi.org/10.3390/fractalfract8070415>

6. S. Kumar, S. K. Sharma, R. Karwasra, S. Malik, T. A. Nofal, A. H. Arnous, Soliton structures, bifurcation patterns, and chaos in a nonlinear transmission line governed by the modified complex Ginzburg-Landau equation, *Chaos Solitons Fract.*, **200** (2025), 117068. <https://doi.org/10.1016/j.chaos.2025.117068>
7. T. Y. Han, L. Z. Zhao, Bifurcation, sensitivity analysis and exact traveling wave solutions for the stochastic fractional Hirota-Maccari system, *Results Phys.*, **47** (2023), 106349. <https://doi.org/10.1016/j.rinp.2023.106349>
8. C. Tang, X. Q. Li, Q. Wang, Solvability of indefinite stochastic LQ optimal control problems for jump diffusion models, *J. Dyn. Control Syst.*, **34** (2026), 9. <https://doi.org/10.1007/s10883-026-09766-3>
9. S. Kumar, M. Niwas, Exploring lump soliton solutions and wave interactions using new inverse (G'/G) -expansion approach: applications to the (2+1)-dimensional nonlinear Heisenberg ferromagnetic spin chain equation, *Nonlinear Dyn.*, **111** (2023), 20257–20273. <https://doi.org/10.1007/s11071-023-08937-2>
10. J. M. Qi, Q. H. Cui, L. Zhang, Y. Q. Sun, Solution structures of an electrical transmission line model with bifurcation and chaos in Hamiltonian dynamics, *Int. J. Bifur. Chaos*, **33** (2023), 2350108. <https://doi.org/10.1142/S0218127423501080>
11. X. Y. Gao, Bäcklund transformation and shock-wave-type solutions for a generalized (3+1)-dimensional variable-coefficient B-type Kadomtsev-Petviashvili equation in fluid mechanics, *Ocean Eng.*, **96** (2015), 245–247. <https://doi.org/10.1016/j.oceaneng.2014.12.017>
12. Z. Z. Lan, Y. T. Gao, J. W. Yang, C. Q. Su, C. Zhao, Z. Gao, Solitons and Bäcklund transformation for a generalized (3+1)-dimensional variable-coefficient B-type Kadomtsev-Petviashvili equation in fluid dynamics, *Appl. Math. Lett.*, **60** (2016), 96–100. <https://doi.org/10.1016/j.aml.2016.03.021>
13. W. B. Rabie, H. B. Amer, H. Khan, J. Alzabut, D. I. Elimy, Exact solutions and stability thresholds for the fractional Gardner equation with high-order dispersion, *Eur. J. Pure Appl. Math.*, **19** (2026), 6805. <https://doi.org/10.29020/nybg.ejpam.v19i1.6805>
14. W. H. Zhu, J. G. Liu, Stripe solitons and lump solutions to a generalized (3+1)-dimensional B-type Kadomtsev-Petviashvili equation with variable coefficients in fluid dynamics, *J. Math. Anal. Appl.*, **502** (2021), 125198. <https://doi.org/10.1016/j.jmaa.2021.125198>
15. W. Li, Y. Zhang, Y. P. Liu, Exact wave solutions for a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation, *Comput. Math. Appl.*, **77** (2019), 3087–3101. <https://doi.org/10.1016/j.camwa.2019.01.027>
16. C. Li, Y. Zhang, Multiple wave solutions and auto-Bäcklund transformation for the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation, *Comput. Math. Appl.*, **70** (2015), 765–775. <https://doi.org/10.1016/j.camwa.2015.05.028>
17. A. R. Seadawy, S. T. R. Rizvi, S. Althobaiti, Chirped periodic and solitary waves for improved perturbed nonlinear Schrödinger equation with cubic quadratic nonlinearity, *Fractal Fract.*, **5** (2021), 1–26. <https://doi.org/10.3390/fractalfract5040234>
18. T. Y. Han, Z. Li, C. Y. Li, Bifurcation analysis, stationary optical solitons and exact solutions for generalized nonlinear Schrödinger equation with nonlinear chromatic dispersion and quintuple power-law of refractive index in optical fibers, *Phys. A*, **615** (2023), 128599. <https://doi.org/10.1016/j.physa.2023.128599>

19. S. Samina, A. Jhangeer, Z. L. Chen, Symmetry and complexity: a Lie symmetry method to bifurcation, chaos, multistability and soliton solutions of the nonlinear generalized advection-diffusion-reaction equation, *Phys. Scr.*, **99** (2024), 075210. <https://doi.org/10.1088/1402-4896/ad4fed>
20. T. Usman, I. Hossain, M. S. Ullah, M. M. Hasan, Soliton, multistability, and chaotic dynamics of the higher-order nonlinear Schrödinger equation, *Chaos*, **35** (2025), 043141. <https://doi.org/10.1063/5.0266469>
21. G. H. Tipu, F. P. Yao, T. Anwar, Bifurcation, chaos, and soliton dynamics in the integrable space curve model: a hybrid analytical and artificial neural network approach, *Chaos Solitons Fract.*, **200** (2025), 117149. <https://doi.org/10.1016/j.chaos.2025.117149>
22. U. Younas, J. Muhammad, T. A. Sulaiman, H. F. Ismael, H. Emadifar, W. W. Mohammed, et al., Estevez-Mansfield-Clarkson equation: investigation of breathers, two waves, and solitary wave solutions in shallow water phenomena and engineering fluid dynamics, *Case Stud. Thermal Eng.*, **74** (2025), 106723. <https://doi.org/10.1016/j.csite.2025.106723>
23. T. Y. Han, Y. Y. Jiang, H. G. Fan, Exploring shallow water wave phenomena: a fractional approach to the Whitham-Broer-Kaup-Boussinesq-Kupershmidt system, *Ain Shams Eng. J.*, **16** (2025), 103700. <https://doi.org/10.1016/j.asej.2025.103700>
24. A. M. Wazwaz, Distinct kinds of multiple-soliton solutions for a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation, *Phys. Scr.*, **84** (2011), 055006. <https://doi.org/10.1088/0031-8949/84/05/055006>
25. Y. L. Ma, B. Q. Li, Mixed lump and soliton solutions for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation, *AIMS Math.*, **5** (2020), 1162–1176. <https://doi.org/10.3934/math.2020080>
26. R. Kaur, S. Singh, D. Kumar, New analytical solutions of the (3+1)-dimensional generalised B-type Kadomtsev-Petviashvili equation, *Pramana*, **99** (2025), 101. <https://doi.org/10.1007/s12043-025-02942-3>
27. Y. H. Liang, K. J. Wang, Chaotic pattern, phase portrait, bifurcation analysis, variational principle, Hamiltonian and diverse wave solutions of the generalised (3+1)-dimensional B-type Kadomtsev-Petviashvili equation, *Pramana*, **100** (2026), 39. <https://doi.org/10.1007/s12043-025-03057-5>
28. M. G. Asaad, W. X. Ma, Pfaffian solutions to a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation and its modified counterpart, *Appl. Math. Comput.*, **218** (2012), 5524–5542. <https://doi.org/10.1016/j.amc.2011.11.042>
29. H. C. Ma, N. Su, A. P. Deng, Exact solutions of the time-fractional extended (3+1)-dimensional Kadomtsev-Petviashvili equation, *Nonlinear Dyn.*, **112** (2024), 5575–590. <https://doi.org/10.1007/s11071-024-09318-z>
30. U. K. Mandal, S. Malik, S. Kumar, Y. Zhang, A. Das, Integrability aspects, rational type solutions and invariant solutions of an extended (3+1)-dimensional B-type Kadomtsev-Petviashvili equation, *Chaos Solitons Fract.*, **181** (2024), 114689. <https://doi.org/10.1016/j.chaos.2024.114689>
31. H. H. Lu, Y. F. Zhang, Dynamics of abundant solutions to the generalized (3+1)-dimensional B-type Kadomtsev-Petviashvili equation, *Modern Phys. Lett. B*, **35** (2021), 2150110. <https://doi.org/10.1142/S0217984921501104>

32. X. Y. Wu, B. Tian, H. P. Chai, Y. Sun, Rogue waves and lump solutions for a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation in fluid mechanics, *Modern Phys. Lett. B*, **31** (2017), 1750122. <https://doi.org/10.1142/S0217984917501226>
33. Y. Sun, B. Tian, L. Liu, H. P. Chai, Y. Q. Yuan, Rogue waves and lump solitons of the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation for water waves, *Commun. Theor. Phys.*, **68** (2017), 693. <https://doi.org/10.1088/0253-6102/68/6/693>
34. X. F. Cao, Lump solutions to the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation, *Adv. Math. Phys.*, **2018** (2018), 7843498. <https://doi.org/10.1155/2018/7843498>
35. X. W. Yan, S. F. Tian, X. B. Wang, T. T. Zhang, Solitons to rogue waves transition, lump solutions and interaction solutions for the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation in fluid dynamics, *Int. J. Comput. Math.*, **96** (2019), 1839–1848. <https://doi.org/10.1080/00207160.2018.1535708>
36. I. Siddique, R. M. Zulqarnain, M. A. Akbar, S. Ali, Lump collision dynamics in the generalized (3+1)-dimensional variable coefficient B-type Kadomtsev-Petviashvili equation, *Phys. Scr.*, **100** (2025), 015231. <https://doi.org/10.1088/1402-4896/ad986c>
37. S. Liu, Y. Ding, J. G. Liu, A class of exact solutions of (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation, *Int. J. Nonlinear Sci. Numer. Simul.*, **18** (2017), 137–143. <https://doi.org/10.1515/ijnsns-2016-0086>
38. N. M. Elsonbaty, H. M. Ahmed, N. M. Badra, W. B. Rabie, Dynamic soliton solutions and stability analysis of the (2+1)-dimensional Wazwaz Kaur Boussinesq equation using an efficient method, *Sci. Rep.*, **16** (2026), 5950. <https://doi.org/10.1038/s41598-025-28602-5>
39. X. R. Xie, R. F. Zhang, Neural network-based symbolic calculation approach for solving the Korteweg-de Vries equation, *Chaos Solitons Fract.*, **194** (2025), 116232. <https://doi.org/10.1016/j.chaos.2025.116232>
40. R. F. Zhang, S. Bilige, Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation, *Nonlinear Dyn.*, **95** (2019), 3041–3048. <https://doi.org/10.1007/s11071-018-04739-z>
41. J. Muhammad, G. H. Tipu, Y. Alrashedi, M. Alhazmi, U. Younas, Analytical study of the nonlinear dynamical systems: application of the neural networks method, *AIMS Math.*, **10** (2025), 14596–14616. <https://doi.org/10.3934/math.2025657>
42. Y. Zhu, K. H. Li, C. Y. Huang, Y. Z. Xu, J. J. Zhong, J. J. Li, Abundant exact solutions of the fractional (3+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation using the Bell polynomial-based neural network method, *Chaos Solitons Fract.*, **196** (2025), 116333. <https://doi.org/10.1016/j.chaos.2025.116333>
43. Z. L. Zhao, R. F. Zhang, Artificial intelligence-enhanced mathematical derivation method: exact solutions of the Benjamin-Bona-Mahony equation, *Eng. Appl. Artif. Intell.*, **173** (2026), 114464. <https://doi.org/10.1016/j.engappai.2026.114464>
44. H. W. Zhang, R. F. Zhang, Q. R. Liu, A novel multi-modal neurosymbolic reasoning intelligent algorithm for BLMP equation, *Chinese Phys. Lett.*, **42** (2025), 100002. <https://doi.org/10.1088/0256-307X/42/10/100002>

45. Z. Li, Solving the exactly explicit solutions of the conformable space-time fractional Phi-4 equation via neural networks method, *Phys. Lett. A*, **581** (2026), 131535. <https://doi.org/10.1016/j.physleta.2026.131535>
46. J. J. Li, C. L. Chen, Y. Zhu, C. Y. Huang, X. Zhen, R. F. Zhang, Analytical solutions of generalized (3+1)-dimensional KP equation via neural network-augmented bilinear framework, *Eur. Phys. J. Spec. Top.*, **234** (2025), 5583–5600. <https://doi.org/10.1140/epjs/s11734-025-01859-7>
47. M. Raissi, P. Perdikaris, G. E. Karniadakis, Physics-informed neural networks: a deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations, *J. Comput. Phys.*, **378** (2019), 686–707. <https://doi.org/10.1016/j.jcp.2018.10.045>
48. P. F. Wang, Y. Z. Sun, H. Triki, Q. Zhou, Informed reconstruction-oriented numerical networks: a more efficient method for solving multiple unknown parameters of Manakov equations, *Chinese Phys. Lett.*, **43** (2026), 040004. <https://doi.org/10.1088/0256-307X/43/4/040004>
49. A. C. Yang, Y. Z. Sun, J. H. Wang, P. F. Wang, N. Li, Predicting bright and dark solitons in (2+1)-dimensional Bose-Einstein condensates by two types of neural operator networks, *Chinese J. Phys.*, **99** (2026), 12–24. <https://doi.org/10.1016/j.cjph.2025.11.016>

Appendix

Supplementary 1: PINN architecture comparison

Table A.1 summarizes the PINN architecture comparison experiments mentioned in Section 4.1.

Table A.1. Comparison of PINN architectures for the u_{22} numerical reproduction.

Configuration	Hidden layers	Activation	Relative L_2 error	Training time (s)
Original	4-512-512-512-512-1	tanh	2.66%	~5700
Shallower	4-256-256-256-1	tanh	23.12%	~3200
Narrower	4-256-256-256-256-1	tanh	13.58%	~3500
Swish	4-512-512-512-512-1	swish	4.21%	~6000

Supplementary 2: Maple verification of wavefield degeneracy

The following steps outline the verification procedure for the wavefield degeneracy discussed in Section 5.1 of the main text (Type I, Case 1):

- (1) Define the τ -function F for Type I, Case 1.
- (2) Substitute parameter set P (Eq (3.4)) and compute $u_P = 2(\ln F_P)_x$.
- (3) Substitute parameter set Q (Eq (3.5)) and compute $u_Q = 2(\ln F_Q)_x$.
- (4) Verify $u_P \equiv u_Q$ using `simplify(u_P - u_Q)` in Maple.
- (5) Generalize to other parameter variations and check for explicit dependencies.



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