



Research article

Computing integer- and fractional-order q - h boundary-value problems

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Abstract: We study integer- and fractional-order Dirichlet problems generated by the affine map $\beta_a(x) = a + q(x - a) + qh$, where $0 < q < 1$ and $h > 0$. On the finite orbit $x_k = \beta_a^k(a)$ with $N + 1$ nodes, the second-order problem is written entirely on consecutive orbit nodes. For arbitrary prescribed data we obtain a unique solution through an explicit nonnegative $(N + 1) \times (N - 1)$ Green matrix. Its row sum is exactly $(x_k - a)(b - x_k)/(1 + q)$, giving the computable bound $M_N \leq (b - a)^2/[4(1 + q)]$. Consequently, the nonlinear problem has a nonnegative solution in $[0, R]^{N-1}$ when $F_R M_N \leq R$, and that solution is unique when $L_f M_N < 1$. For constant loading the closed form is $y(x) = (x - a)(b - x)/(1 + q)$, and, with N and h fixed, the scheme converges to the forward h -difference problem as $q \rightarrow 1$. For $1 < \alpha \leq 2$, we also record a branchwise Riemann–Liouville affine integral obtained by exact conjugacy with the Jackson q -integral. The conjugacy yields the corresponding commutation formula, including its orientation signs and boundary terms. The fractional Dirichlet problem is posed on one fixed geometric orbit, and there the Green representation is an identity of orbit sums, and a compact cone operator gives a positive solution. Finally, we prove the sampled-kernel integer-order consistency between the two constructions: as $\alpha \rightarrow 2^-$, samples of the fractional kernel converge entrywise to the finite Green matrix after translation to the orbit coordinates.

Keywords: affine q - h calculus; boundary-value problem; fractional q - h integral; commutation formula; Green matrix; positive solution; orbit lattice

1. Introduction

Quantum calculus provides differential and integral operators without an initial limiting process. Standard accounts develop Jackson q -calculus and forward h -calculus in parallel [1]; the affine transformation underlying Hahn calculus gives a natural common dynamical setting [2]. Boundary-value problems for q -difference equations have been studied through positive Green functions and fixed-point theorems. In particular, El-Shahed and Hassan established positive solutions for a nonlinear second-order q -difference equation [3]. Ferreira treated a fractional q -difference Dirichlet problem and obtained nontrivial solutions through a cone fixed-point theorem [4]. Later work extended fractional q -boundary-value theory to nonlocal conditions [5].

Finite-interval quantum calculus was placed on a systematic footing by Tariboon and Ntouyas, who defined endpoint-based derivatives and integrals and applied them to impulsive difference equations [6]. Subsequent work has extended finite-interval quantum calculus in several directions, including left and right quantum calculi with respect to another function [7, 8] and (p, q) -difference equations with integral boundary conditions [9]. Recent fractional q -difference studies include the two-point problem of Allouch et al. [10] and a Riemann–Liouville fractional q -derivative boundary problem studied by Castro [11]. In the wider time-scale setting, the 2025 monograph of Georgiev develops Riemann–Liouville and Caputo fractional dynamic boundary-value problems on arbitrary time scales [12]. We have also taken account of the recent existence-and-uniqueness study of Jiao, Kong and Wang [13].

Recent work on q - h derivatives and integrals includes a unified representation of q - and h -integrals and its inequality consequences [14]. The fractional Hahn framework itself goes back at least to the (q, h) fractional calculus of Čermák and Nechvátal [15]. Subsequent developments include asymptotic stability of linear nabla Riemann–Liouville (q, h) -fractional difference equations: Wang et al. construct a Lyapunov functional and derive sufficient stability conditions [16]. Du and Jia establish a generalized fractional (q, h) –Gronwall inequality and use it to obtain uniqueness and finite-time stability results for nonlinear fractional delay (q, h) -difference systems [17]. These studies place fractional (q, h) equations within a broader theory of asymptotic behavior and comparison estimates. Their initial-value and delay settings complement the Dirichlet analysis here; their stability conclusions are not asserted for the present boundary-value problem. Nonlocal and integral conditions, Green-function positivity, and cone or contraction methods are represented by [5, 13, 18].

A potential inconsistency arises when an inverse operator samples nodes that do not belong to the affine orbit defining the derivative. We therefore take orbit agreement as a necessary consistency requirement and give a self-contained derivation of the resulting boundary-value theory. The principal contributions are the orbit-consistent specialization of the variable-step Green matrix, its closed q - h row sum, the associated nonlinear criteria, and the sampled-kernel and finite-operator integer-order limit proved below. The fractional part has a more limited purpose: it identifies the exact affine conjugacy, states what h does and does not change, and

formulates the Green problem on the fixed orbit required by Jackson sampling.

Table 1 locates these claims relative to representative strands of the literature. “Continuous interval” in the q -papers denotes an interval-indexed formula; each Jackson integral still samples the geometric orbit generated by its upper limit.

Table 1. Representative comparison with related boundary-value theories.

Source	Operator/domain	Conditions	Kernel/positivity		Method and conclusion
El-Shahed–Hassan [3]	Integer q ; geometric nodes	Two-point	Positive function	Green	Cone theorem; positive solution
Ferreira [4]	Riemann–Liouville q	Dirichlet	Explicit kernel	positive	Cone compression–expansion
Li et al. [5]	Caputo q ; interval-indexed	Nonlocal	Integral operator		Contraction and Schaefer existence
Salamooni–Pawar [18]	Caputo q ; geometric setting	Multipoint and Stieltjes	Positive function	Green	Cone fixed point; unique positive solution
Jiao–Kong–Wang [13]	Coupled Caputo q system	Two-point	Operator estimates		Schaefer and contraction results
Čermák–Nechvátal [15]	Fractional (q, h) ; Hahn lattice	Initial-value calculus	No Dirichlet kernel		Foundational fractional affine calculus
Tariboon–Ntouyas [6]	Finite-interval q	Impulsive and nonlocal	Finite-interval integral		Existence theory on endpoint-based nodes
Georgiev [12]	Fractional dynamic operators	Arbitrary time scale	Several classes	BVP	Time-scale existence theory
Present work	Affine q - h ; Fixed fractional and finite integer orbits	Dirichlet	Inherited fractional kernel; explicit finite matrix		Compact cone map; Brouwer/contraction; exact geometric-grid row sum

1.1. Research gap and contribution

The literature above motivates a specific finite-orbit problem. Fractional (q, h) operator theory is already available [15], and positive fractional boundary-value problems are well developed in the q setting [4, 5, 18]. What is not supplied by those works is the present finite affine-orbit Dirichlet model, its closed geometric-grid row sum, and its directly computable nonlinear criteria. The published affine q - h literature also includes integral inequalities [14]. The model below cannot be obtained safely by a purely symbolic replacement of q by (q, h) : the derivative nodes, inverse nodes, and boundary grid must belong to the same affine orbit.

This paper addresses that gap for a second-order Dirichlet problem. Its main contributions are:

- (1) an orbit-consistent q - h boundary-value problem on a finite affine lattice;
- (2) an explicit affine specialization of the variable-step Green matrix, a unique finite-sum solution, and a closed geometric-grid row sum;
- (3) exact solutions for the affine orbit-polynomial loads;
- (4) existence and uniqueness criteria for a nonlinear positive problem;

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- (5) an exact example, a practical affine-growth test for the nonlinear conditions, and the finite $q \rightarrow 1$ forward- h limit;
 - (6) an explicit affine conjugacy for the branchwise fractional integral, its inherited semigroup law, and the q - h version of the commutation identity in Theorem 2.4 of [4];
 - (7) a fixed-orbit fractional Dirichlet formulation with the translated positive q -Green kernel and a cone compression–expansion theorem;
 - (8) an entrywise $\alpha \rightarrow 2^-$ sampling limit that identifies the sampled fractional kernel with the finite Green matrix in orbit coordinates, together with convergence of finite weighted sums, without identifying the infinite Jackson measure with the finite-orbit measure.

Subsection 2.1 supplies the fractional operator and boundary-term identity. Section 3 then treats the nonlinear fractional problem on a right-hand fixed-point branch. The integer-order problem is treated separately on the finite affine lattice, whose nodes lie on the left-hand branch.

2. Affine q - h calculus and orbit sums

Throughout, let $0 < q < 1$, $h > 0$, $a \in \mathbb{R}$, and $N \geq 2$. Recall

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad n \in \mathbb{N}_0.$$

Define the left affine map

$$\beta_a(x) = a + q(x - a) + qh = s_a + q(x - s_a), \quad s_a = a + \frac{qh}{1 - q}. \quad (2.1)$$

Its iterates are

$$\beta_a^n(x) = q^n x + (1 - q^n)a + qh[n]_q. \quad (2.2)$$

Definition 2.1. For $x \neq s_a$, the left affine q - h derivative is

$${}^{a+}D_{q,h}F(x) = \frac{F(\beta_a(x)) - F(x)}{\beta_a(x) - x}. \quad (2.3)$$

At s_a it is defined by a limit whenever that limit exists.

The orbit primitive associated with (2.1) is

$$J_{q,h}^a f(x) = (x - \beta_a(x)) \sum_{n=0}^{\infty} q^n f(\beta_a^n(x)), \quad (2.4)$$

and the corresponding definite integral is

$${}^{a+}I_{q,h}f(x) = J_{q,h}^a f(x) - J_{q,h}^a f(a). \quad (2.5)$$

Whenever the displayed series converge and termwise application of ${}^{a+}D_{q,h}$ is valid,

$${}^{a+}D_{q,h}({}^{a+}I_{q,h}f)(x) = f(x), \quad {}^{a+}I_{q,h}({}^{a+}D_{q,h}F)(x) = F(x) - F(a). \quad (2.6)$$

Consequently, when both iterated integrals exist, a prescribed-load Dirichlet inverse on a continuous orbit domain has the algebraic form

$$y(x) = \frac{x-a}{b-a} ({}^{a+}I_{q,h})^2 g(b) - ({}^{a+}I_{q,h})^2 g(x). \quad (2.7)$$

For $h > 0$, positivity of (2.5) is not automatic for every x close to a . The finite orbit gives a positive and fully computable boundary-value model.

Define

$$x_k = \beta_a^k(a) = a + qh[k]_q, \quad \Delta_k = x_{k+1} - x_k = q^{k+1}h, \quad b = x_N. \quad (2.8)$$

For a grid function $y_k = y(x_k)$, Definition 2.1 becomes

$$(Dy)_k := {}^{a+}D_{q,h}y(x_k) = \frac{y_{k+1} - y_k}{\Delta_k}, \quad k = 0, \dots, N-1. \quad (2.9)$$

The corresponding finite orbit sum is

$$(\mathcal{S}g)_k := \int_{x_0}^{x_k} g(t) d_{q,h}^{\text{orb}} t = \sum_{j=0}^{k-1} \Delta_j g_j = h \sum_{j=0}^{k-1} q^{j+1} g_j. \quad (2.10)$$

It satisfies

$$D(\mathcal{S}g)_k = g_k, \quad \sum_{k=0}^{N-1} \Delta_k (DF)_k = F(x_N) - F(a), \quad (2.11)$$

by direct telescoping.

2.1. The q - h analogue of Ferreira's Theorem 2.4

The fractional q -integral in [4] is based at the fixed point 0 of the Jackson map $u \mapsto qu$. Accordingly, its affine analogue must be based at the fixed point s_a of β_a , rather than at the normalization endpoint a in (2.5). The following branchwise construction also keeps noninteger powers real on either side of s_a .

For z away from the poles, set

$$\Gamma_q(z) = (1-q)^{1-z} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+z}}, \quad (2.12)$$

and use the standard generalized q -power

$$(u-v)^{(v)} = u^v \prod_{n=0}^{\infty} \frac{u-vq^n}{u-vq^{n+v}}. \quad (2.13)$$

Let $\varepsilon \in \{-1, 1\}$ select a branch, write

$$x = s_a + \varepsilon u, \quad u \geq 0, \quad (U_\varepsilon f)(u) = f(s_a + \varepsilon u), \quad (2.14)$$

and recall

$$\int_0^u \psi(v) d_q v = (1-q)u \sum_{n=0}^{\infty} q^n \psi(uq^n). \quad (2.15)$$

For later use, the Jackson derivative is

$$(D_q \psi)(u) = \frac{\psi(qu) - \psi(u)}{(q-1)u}, \quad u > 0,$$

with its continuous limiting value at $u = 0$ when that limit exists. We write I_q^α for the Riemann–Liouville fractional Jackson operator appearing on the right-hand side of (2.16).

Definition 2.2 (Branchwise fractional affine integral). For $\alpha > 0$, define

$$(\mathfrak{J}_{\varepsilon;q,h}^\alpha f)(s_a + \varepsilon u) = \frac{1}{\Gamma_q(\alpha)} \int_0^u (u - qv)^{(\alpha-1)} (U_\varepsilon f)(v) d_q v, \quad (2.16)$$

and put $\mathfrak{J}_{\varepsilon;q,h}^0 f = f$.

Remark 2.1 (Admissible functions and well-definedness). For a fixed branch and $U > 0$, let $C_\varepsilon[0, U]$ denote the functions f for which $U_\varepsilon f$ is continuous on $[0, U]$. For $f \in C_\varepsilon[0, U]$ and $\alpha > 0$, the Jackson series in (2.16) is absolutely convergent at every $u \in [0, U]$ and uniformly convergent on compact u -subintervals on which the generalized q -power is finite. Thus, $\mathfrak{J}_{\varepsilon;q,h}^\alpha$ is not a formal replacement rule: it is the pullback $U_\varepsilon^{-1} I_q^\alpha U_\varepsilon$ of the Riemann–Liouville Jackson operator. All identities below are asserted on the natural domain where the indicated Jackson integrals and q -derivatives exist; for the commutation formula we additionally require $U_\varepsilon f$ to satisfy the hypotheses of Ferreira’s theorem through order p .

Thus, $\mathfrak{J}_{\varepsilon;q,h}^\alpha$ is precisely the Riemann–Liouville q -integral conjugated by the affine coordinate (2.14). For $\varepsilon = 1$ and $\alpha = 1$ it equals the improved primitive $J_{q,h}^a$ in (2.4) on the right-hand branch. On the left-hand branch, it equals $-J_{q,h}^a$ when $\alpha = 1$; the minus sign records the reversed orientation toward s_a . This distinction is essential because the finite lattice (2.8) lies to the left of s_a .

Lemma 2.1 (Conjugacy and semigroup law). Let $U > 0$. Assume $U_\varepsilon f \in C[0, U]$, and, for the derivative identity, assume that the indicated Jackson differences extend continuously to 0. Then, for $0 \leq u \leq U$,

$$U_\varepsilon({}^{a+}D_{q,h}^m f) = \varepsilon^m D_q^m (U_\varepsilon f), \quad m \in \mathbb{N}_0, \quad (2.17)$$

$$\mathfrak{J}_{\varepsilon;q,h}^\alpha \mathfrak{J}_{\varepsilon;q,h}^\beta f = \mathfrak{J}_{\varepsilon;q,h}^{\alpha+\beta} f, \quad \alpha, \beta \geq 0. \quad (2.18)$$

Proof. The affine coordinate map $C_\varepsilon(u) = s_a + \varepsilon u$ is a bijection from $[0, U]$ onto the selected branch segment and satisfies $\beta_a \circ C_\varepsilon = C_\varepsilon \circ (u \mapsto qu)$. Hence, conjugacy preserves every Jackson sampling node and its weight; no change of measure or unverified interchange of limits is involved. Since $\beta_a(s_a + \varepsilon u) = s_a + \varepsilon qu$, direct substitution in (2.3) gives $U_\varepsilon {}^{a+}D_{q,h} = \varepsilon D_q U_\varepsilon$; iteration proves (2.17). For the second identity, absolute convergence permits the two Jackson sums to be interchanged. The q -beta identity combines their weights to give $I_q^\alpha I_q^\beta = I_q^{\alpha+\beta}$; transport through U_ε proves (2.18). $\square$

On the right-hand branch, abbreviate $\mathfrak{J}_{q,h}^{\alpha,+} := \mathfrak{J}_{1;q,h}^\alpha$. The next result is written in the same form as Eq (6), Theorem 2.4, of [4], with the Jackson origin replaced by the affine fixed point s_a and D_q replaced by ${}^{a+}D_{q,h}$.

Theorem 2.1 (Direct q - h analogue of Ferreira's Theorem 2.4). Fix $U > 0$. Let $\alpha > 0$, let p be a positive integer, and let $U_1 f$ belong to the domain of D_q^p on $[0, U]$, with $D_q^k U_1 f$ continuous at 0 for $0 \leq k \leq p-1$ and every Jackson series below absolutely convergent. For $s_a \leq x \leq s_a + U$,

$$\mathfrak{J}_{q,h}^{\alpha,+a+} D_{q,h}^p f(x) = {}^{a+}D_{q,h}^p \mathfrak{J}_{q,h}^{\alpha,+} f(x) - \sum_{k=0}^{p-1} \frac{(x-s_a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} {}^{a+}D_{q,h}^k f(s_a). \quad (2.19)$$

The reciprocal $1/\Gamma_q(z)$ is taken in its continuous extension, so it equals zero at $z = 0, -1, -2, \dots$

Proof. By the preceding remark, the assumed function belongs, after application of U_1 , to the domain on which Ferreira's identity holds. Apply U_1 to the left-hand side and use (2.17). Theorem 2.4 of [4] gives

$$\begin{aligned} I_q^\alpha D_q^p (U_1 f)(u) &= D_q^p I_q^\alpha (U_1 f)(u) \\ &\quad - \sum_{k=0}^{p-1} \frac{u^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} D_q^k (U_1 f)(0). \end{aligned}$$

The first term is $U_1({}^{a+}D_{q,h}^p \mathfrak{J}_{q,h}^{\alpha,+} f)$, while $D_q^k (U_1 f)(0) = {}^{a+}D_{q,h}^k f(s_a)$ and $u = x - s_a$. Conjugating back proves (2.19). Thus, setting $h = 0$, $a = 0$, and hence $s_a = 0$, recovers Theorem 2.4 of [4] exactly. $\square$

For completeness, put $\mathfrak{J}_{q,h}^{\alpha,-} := \mathfrak{J}_{-1;q,h}^\alpha$ on the left-hand branch. This is the real-valued branch containing the finite orbit (2.8).

Corollary 2.1 (Left-orbit q - h form). Fix $U > 0$ and impose the hypotheses of Theorem 2.1 on $U_{-1} f$ on $[0, U]$, using left-hand limits at s_a . For $s_a - U \leq x \leq s_a$,

$$\mathfrak{J}_{q,h}^{\alpha,-a+} D_{q,h}^p f(x) = {}^{a+}D_{q,h}^p \mathfrak{J}_{q,h}^{\alpha,-} f(x) - \sum_{k=0}^{p-1} (-1)^{p-k} \frac{(s_a - x)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} {}^{a+}D_{q,h}^k f(s_a). \quad (2.20)$$

Proof. Use

$$U_{-1} {}^{a+}D_{q,h}^m = (-1)^m D_q^m U_{-1}, \quad D_q^k (U_{-1} f)(0) = (-1)^{ka+} D_{q,h}^k f(s_a),$$

in Ferreira's Theorem 2.4. $\square$

Corollary 2.2 (The $p = 2$ boundary terms). For $p = 2$, Theorem 2.1 and Corollary 2.1 give, respectively, on $s_a \leq x \leq s_a + U$ and $s_a - U \leq x \leq s_a$, with the endpoint interpretations specified there,

$$\mathfrak{J}_{q,h}^{\alpha,+a+} D_{q,h}^2 f(x) = {}^{a+}D_{q,h}^2 \mathfrak{J}_{q,h}^{\alpha,+} f(x) - \frac{(x-s_a)^{\alpha-2}}{\Gamma_q(\alpha-1)} f(s_a) - \frac{(x-s_a)^{\alpha-1}}{\Gamma_q(\alpha)} {}^{a+}D_{q,h} f(s_a), \quad (2.21)$$

$$\mathfrak{J}_{q,h}^{\alpha,-a+} D_{q,h}^2 f(x) = {}^{a+}D_{q,h}^2 \mathfrak{J}_{q,h}^{\alpha,-} f(x) - \frac{(s_a-x)^{\alpha-2}}{\Gamma_q(\alpha-1)} f(s_a) + \frac{(s_a-x)^{\alpha-1}}{\Gamma_q(\alpha)} {}^{a+}D_{q,h} f(s_a). \quad (2.22)$$

The first identity is the direct q - h counterpart of the $p = 2$ case in [4]; the second is its sign-corrected form on the finite q - h orbit.

The commutation identity also reproduces the solution decomposition used in fractional q -boundary problems. If $m = \lceil \alpha \rceil$, define the branchwise Riemann–Liouville affine derivative by

$$\mathfrak{D}_{\varepsilon;q,h}^\alpha f := \varepsilon^{ma+} D_{q,h}^m \mathfrak{I}_{\varepsilon;q,h}^{m-\alpha} f. \quad (2.23)$$

Then, $U_\varepsilon \mathfrak{D}_{\varepsilon;q,h}^\alpha = D_q^\alpha U_\varepsilon$. Consequently, for $1 < \alpha \leq 2$, every solution of

$$\mathfrak{D}_{\varepsilon;q,h}^\alpha y(x) = -g(x) \quad (2.24)$$

has the branchwise form

$$y(s_a + \varepsilon u) = c_0 u^{\alpha-2} + c_1 u^{\alpha-1} - (\mathfrak{I}_{\varepsilon;q,h}^\alpha g)(s_a + \varepsilon u) \quad (2.25)$$

with the constants determined by admissible boundary conditions. Formula (2.25) follows either by conjugacy or directly from Corollary 2.2 and the semigroup law. It is the affine q - h counterpart of the starting representation in [4]. The next section uses this representation to express the inherited positive fractional Green kernel on one fixed right-hand orbit.

3. A fractional q - h boundary-value problem

In this section we use only the right fixed-point branch $x = s_a + u$, $u \geq 0$. Its infinite geometric orbit accumulates at s_a from the right; it is distinct from the finite left affine orbit used in the integer-order sections. Fix $\ell > 0$, put $u_m = \ell q^m$, $v_n = \ell q^n$, and define the compact fixed orbit

$$A_{q,\ell}^* = \{\ell q^m : m \in \mathbb{N}_0\} \cup \{0\}, \quad X_{q,h}^* = s_a + A_{q,\ell}^*. \quad (3.1)$$

Thus, $b_+ = s_a + \ell$. The fixed-orbit integral is

$$\int_{s_a}^{b_+} \phi(t) d_{q,h}^{s_a,+} t := \int_0^\ell \phi(s_a + v) d_q v = \sum_{n=0}^\infty w_n \phi(s_a + v_n), \quad w_n = (1-q)\ell q^n. \quad (3.2)$$

Every node is an iterate of the same affine map β_a .

Let

$$B_{q,h} = \{y : X_{q,h}^* \rightarrow \mathbb{R} : \lim_{m \rightarrow \infty} y(s_a + u_m) = y(s_a)\}, \quad \|y\|_\infty = \sup_{x \in X_{q,h}^*} |y(x)|. \quad (3.3)$$

This Banach space is isometric to the space of convergent sequences.

Let $1 < \alpha \leq 2$ and abbreviate $\mathfrak{D}_{q,h}^{\alpha,+} := \mathfrak{D}_{1;q,h}^\alpha$. For a nonnegative function f whose restriction to $X_{q,h}^* \times [0, R]$ is continuous for every $R > 0$ (equivalently, the values at $s_a + u_m$ converge locally uniformly in the second variable to the value at s_a),

$$f : X_{q,h}^* \times [0, \infty) \longrightarrow [0, \infty),$$

consider

$$\mathfrak{D}_{q,h}^{\alpha,+} y(x) = -f(x, y(x)), \quad x \in X_{q,h}^* \setminus \{s_a\}, \quad y(s_a) = 0 = y(b_+). \quad (3.4)$$

The equation is imposed at every positive orbit point, including b_+ , where the Jackson derivative is well defined; the boundary value $y(b_+) = 0$ is imposed simultaneously. No fractional

derivative is asserted at the accumulation point s_a . After $x = s_a + u$, this is precisely the fixed-orbit q -problem underlying [4]. Thus, h shifts the physical orbit through s_a but does not create a new kernel in the u coordinate.

We use the expansion–compression form of the cone theorem stated in [4, Theorem 3.1]. We specialize it to nested open balls and state the operator domain explicitly as a closed annular subset of the cone.

Theorem 3.1 (Cone compression–expansion). *Let B be a Banach space, let $K \subset B$ be a cone, and let $\Omega_i = \{y \in B : \|y\| < r_i\}$ for $0 < r_1 < r_2$. Then, $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Define the closed annular subset of the cone by*

$$A = K \cap (\overline{\Omega_2} \setminus \Omega_1) = \{y \in K : r_1 \leq \|y\| \leq r_2\}.$$

Suppose that $T : A \rightarrow K$ is completely continuous and

$$\|Ty\| \geq \|y\| \quad \text{on } K \cap \partial\Omega_1, \quad \|Ty\| \leq \|y\| \quad \text{on } K \cap \partial\Omega_2.$$

Then, there exists $y \in A$ such that $Ty = y$; in particular, $r_1 \leq \|y\| \leq r_2$.

Here $K \cap \partial\Omega_i \subset A$ for $i = 1, 2$. The annular set is defined using the closure $\overline{\Omega_2}$. It contains both boundary spheres intersected with K , but does not contain the entire set $\overline{\Omega_2}$. The closure of the outer ball is essential for the operator domain. Since Ω_2 is open,

$$(K \cap \partial\Omega_2) \cap [K \cap (\Omega_2 \setminus \Omega_1)] = \emptyset, \quad K \cap \partial\Omega_2 \subset K \cap (\overline{\Omega_2} \setminus \Omega_1) = A.$$

In particular, for a nontrivial cone K ,

$$K \cap \partial\Omega_2 \not\subset K \cap (\Omega_2 \setminus \Omega_1).$$

Thus, the set without the outer closure cannot be the operator domain for the stated outer-boundary inequality.

For $0 \leq u, v \leq \ell$, define

$$G_{\alpha, \ell}(u, v) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \left(u \left(1 - \frac{v}{\ell}\right)\right)^{(\alpha-1)} - (u-v)^{(\alpha-1)}, & 0 \leq v \leq u \leq \ell, \\ \left(u \left(1 - \frac{v}{\ell}\right)\right)^{(\alpha-1)}, & 0 \leq u \leq v \leq \ell. \end{cases} \quad (3.5)$$

The generalized powers are those of (2.13), together with their homogeneity under nonnegative scalar multiplication.

Lemma 3.1 (Green integral equation). *A function $y \in B_{q,h}$ solves (3.4) if and only if*

$$y(s_a + u_m) = \sum_{n=0}^{\infty} w_n G_{\alpha, \ell}(u_m, qv_n) f(s_a + v_n, y(s_a + v_n)), \quad m \in \mathbb{N}_0, \quad (3.6)$$

with the limiting value $y(s_a) = 0$.

Proof. Put $F_y(s_a + v) = f(s_a + v, y(s_a + v))$. Formula (2.25) on the right-hand branch gives

$$y(s_a + u) = c_0 u^{\alpha-2} + c_1 u^{\alpha-1} - (\mathfrak{I}_{q,h}^{\alpha,+} F_y)(s_a + u).$$

For $1 < \alpha < 2$, boundedness at $u = 0$ rules out the singular term $c_0 u^{\alpha-2}$; for $\alpha = 2$, the boundary condition at $u = 0$ gives $c_0 = 0$. The condition $y(b_+) = 0$ yields

$$c_1 = \ell^{1-\alpha} (\mathfrak{I}_{q,h}^{\alpha,+} F_y)(b_+).$$

For $u = u_m$, the Jackson integral with upper limit u_m is

$$(1-q)u_m \sum_{r=0}^{\infty} q^r \Psi(u_m q^r) = \sum_{n=m}^{\infty} w_n \Psi(v_n).$$

It is therefore exactly the tail of the fixed ℓ -orbit sum. Partitioning the latter into $n < m$ and $n \geq m$, rather than applying an ordinary-integral splitting rule at an arbitrary intermediate point, gives (3.6). Conversely, reversing these steps and using $\mathfrak{D}_{q,h}^{\alpha,+} \mathfrak{I}_{q,h}^{\alpha,+} = I$ verifies the field equation and both boundary conditions. $\square$

Remark 3.1 (Compatibility of Jackson sampling orbits). If $q = 1/2$, $\ell = 1$, and $u = 3/4$, the u -integral samples $\{3/4, 3/8, 3/16, \dots\}$, whereas the ℓ -integral samples $\{1, 1/2, 1/4, \dots\}$. A continuous function can vanish on the second set and be nonzero at $3/4$, so no general fixed-orbit splitting identity is possible. Restriction to (3.1) is therefore essential. Only an upper limit $u_m = \ell q^m$ generates a tail of the same fixed orbit. No splitting identity at an arbitrary intermediate point is used in the Green representation or the fixed-point argument.

Remark 3.2 (The integer-order kernel). When $\alpha = 2$, the fractional Green kernel becomes

$$G_{2,\ell}(u, v) = \begin{cases} \frac{v(\ell - u)}{\ell}, & 0 \leq v \leq u \leq \ell, \\ \frac{u(\ell - v)}{\ell}, & 0 \leq u \leq v \leq \ell. \end{cases} \quad (3.7)$$

Thus, (3.5) recovers the usual nonnegative tent kernel on the translated interval.

Lemma 3.2 (Fractional Green-kernel bounds). *For all $u, v \in A_{q,\ell}^*$,*

$$0 \leq G_{\alpha,\ell}(u, qv) \leq G_{\alpha,\ell}(qv, qv), \quad G_{\alpha,\ell}(u, qv) \leq \frac{u^{\alpha-1}}{\Gamma_q(\alpha)}. \quad (3.8)$$

Proof. Let $G_{\alpha,1}$ denote the unit-interval kernel of [4, Lemma 3.4]. Homogeneity of the generalized q -power gives

$$G_{\alpha,\ell}(u, v) = \ell^{\alpha-1} G_{\alpha,1}\left(\frac{u}{\ell}, \frac{v}{\ell}\right).$$

Here $qv_n = v_{n+1}$ remains on the same orbit, and the normalized arguments $u/\ell, v/\ell$ lie in $\{q^m : m \geq 0\} \cup \{0\}$. Ferreira's two product inequalities therefore apply without shifted off-orbit arguments and yield the first two bounds. The defining product also lies between 0 and $u^{\alpha-1}$; dropping the nonnegative subtracted term in the first branch gives the last bound. $\square$

Set

$$K = \{y \in B_{q,h} : y(x) \geq 0 \text{ for } x \in X_{q,h}^*\}.$$

Remark 3.3 (The cone operator). Define $T : K \rightarrow K$ by the series in (3.6) and $(Ty)(s_a) = 0$. On a ball $\|y\|_\infty \leq R$, compactness of the orbit and continuity of f give a finite bound F_R . The Weierstrass test gives uniform convergence because $\sum_n w_n = \ell$ and the kernel is bounded. Uniform continuity of f then proves continuity of T . Moreover,

$$\sup_{\|y\| \leq R} |(Ty)(s_a + u_m)| \leq \frac{F_R \ell}{\Gamma_q(\alpha)} u_m^{\alpha-1} \rightarrow 0.$$

Thus, the tails of all image sequences are uniformly small. A bounded set of convergent sequences with this uniform-tail property is totally bounded after truncation to finitely many coordinates. Hence, T maps bounded sets into relatively compact sets and is completely continuous as a map $T : K \rightarrow K$ in the norm of $B_{q,h}$.

Fix $\bar{m} \geq 1$ and a nonempty finite index set $J \subset \mathbb{N}_0$ for which the second sum below is positive. Define

$$M_{q,h} = \left(\sum_{n=0}^{\infty} w_n G_{\alpha,\ell}(qv_n, qv_n) \right)^{-1}, \quad N_{q,h} = \left(\sum_{n \in J} w_n G_{\alpha,\ell}(u_{\bar{m}}, qv_n) \right)^{-1}. \quad (3.9)$$

Theorem 3.2 (Nontrivial positive fractional solution). Assume that there are constants $r_2 > r_1 > 0$ such that

$$f(s_a + v_n, z) \leq M_{q,h} r_2, \quad n \in \mathbb{N}_0, \quad 0 \leq z \leq r_2, \quad (3.10)$$

$$f(s_a + v_n, z) \geq N_{q,h} r_1, \quad n \in J, \quad 0 \leq z \leq r_1. \quad (3.11)$$

Then, problem (3.4) has a nonnegative, nontrivial solution satisfying

$$r_1 \leq \|y\|_\infty \leq r_2. \quad (3.12)$$

Proof. Let $\Omega_i = \{y \in B_{q,h} : \|y\|_\infty < r_i\}$, $i = 1, 2$. Set

$$A = K \cap (\overline{\Omega_2} \setminus \Omega_1) = \{y \in K : r_1 \leq \|y\|_\infty \leq r_2\}.$$

By Remark 3.3, T is defined on all of K , maps K into itself, and is completely continuous. Hence, its restriction $T|_A : A \rightarrow K$ is completely continuous. Both $K \cap \partial\Omega_1$ and $K \cap \partial\Omega_2$ are contained in A . If $y \in K \cap \partial\Omega_1$, then $0 \leq y \leq r_1$ and

$$\begin{aligned} \|Ty\|_\infty &\geq (Ty)(s_a + u_{\bar{m}}) \\ &\geq N_{q,h} r_1 \sum_{n \in J} w_n G_{\alpha,\ell}(u_{\bar{m}}, qv_n) = r_1 = \|y\|_\infty. \end{aligned}$$

For the outer-boundary inequality, let $y \in K \cap \partial\Omega_2$. Then, Lemma 3.2 and (3.10) give

$$\|Ty\|_\infty \leq M_{q,h} r_2 \sum_{n=0}^{\infty} w_n G_{\alpha,\ell}(qv_n, qv_n) = r_2 = \|y\|_\infty.$$

Applying Theorem 3.1 to the restriction $T|_A : A \rightarrow K$ gives a fixed point $y \in A$, so $r_1 \leq \|y\|_\infty \leq r_2$. In particular, $y \neq 0$. Thus, Lemma 3.1 converts it into a solution of (3.4). $\square$

Remark 3.4 (Why the fixed slice is used). The lower constant in (3.9) uses the fixed kernel slice $G_{\alpha,\ell}(u_{\bar{m}}, qv_n)$. The diagonal estimate in (3.8) supplies the upper inequality, but does not justify interchanging a maximum and an orbit sum in a lower estimate. Evaluating at one orbit point gives the required cone expansion rigorously. If $h = 0$, $a = 0$, and $\ell = 1$, the problem, Green kernel, and integral equation reduce to those in Section 3 of [4], with this corrected lower-bound constant.

Example 3.1 (Fractional fixed-orbit verification). Take $q = 1/2$, $\ell = 1$, $\alpha = 3/2$, and $f(s_a + u, z) \equiv 1$. Since $I_q^\alpha 1(u) = u^\alpha / \Gamma_q(\alpha + 1)$, the exact solution is

$$y(s_a + u) = \frac{u^{1/2}(1-u)}{\Gamma_{1/2}(5/2)}, \quad u \in A_{1/2,1}^*. \quad (3.13)$$

It is positive at every interior orbit point, vanishes at both boundary points, and direct use of the power rule gives $\mathfrak{D}_{q,h}^{3/2,+} y = -1$. Truncating the Green sum after 200 terms changes the displayed values by less than 2×10^{-15} . Table 2 compares the exact solution values with the corresponding truncated Green orbit sums: numerically, $M_{q,h} = 2.30359094$. With $\bar{m} = 1$ and $J = \{0, 1, 2\}$, $N_{q,h} = 3.39557638$. Hence, $r_1 = 0.2$ and $r_2 = 0.5$ verify $N_{q,h}r_1 < 1 < M_{q,h}r_2$, so the hypotheses of Theorem 3.2 hold as well.

Table 2. Comparison of exact solution values and truncated Green orbit sums for the fractional fixed-orbit problem in Example 3.1.

m	u_m	Exact formula	Green orbit sum
0	1	0	0
1	1/2	0.296955555	0.296955555
2	1/4	0.314968930	0.314968930
3	1/8	0.259836111	0.259836111
4	1/16	0.196855581	0.196855581
5	1/32	0.143837847	0.143837847

4. A linear q - h Dirichlet problem

We now return to the finite left affine orbit $x_k = \beta_a^k(a) = s_a - (s_a - a)q^k$, $0 \leq k \leq N$. Thus, $a = x_0 < \cdots < x_N = b < s_a$, and the difference operator uses consecutive nodes on this finite grid. The translated coordinate $x_k - a$ used below is distinct from the right-branch coordinate $x - s_a$ of Section 3. Let $g_j = g(x_j)$ be prescribed for $j = 0, \dots, N-2$. We consider

$$(D^2 y)_j = -g_j, \quad j = 0, \dots, N-2, \quad y_0 = 0 = y_N. \quad (4.1)$$

The second difference at x_j uses the consecutive orbit nodes x_j, x_{j+1}, x_{j+2} , and the last equation therefore reaches the boundary node x_N .

Remark 4.1 (Relation with a general variable-step grid). Let $t_0 < \cdots < t_N$ be any grid, put $\delta_k = t_{k+1} - t_k$, and define $D_t y_k = (y_{k+1} - y_k) / \delta_k$. Repeating the two telescoping sums used

below gives exactly (4.6), with x_k, Δ_k replaced by t_k, δ_k . Thus, the tent-shaped Green matrix is the standard variable-step Dirichlet matrix, not a new matrix form. The affine q - h structure specializes the steps to $\Delta_k = q^{k+1}h$ and is responsible for the closed row sum (4.9), the orbit-polynomial calculus, and the explicit $q \rightarrow 1$ limit. These, rather than the abstract tent form, are the integer-order contributions.

Set $L = b - a$. For $0 \leq k \leq N$ and $0 \leq j \leq N - 2$, define

$$G_{k,j} = \begin{cases} \frac{(x_{j+1} - a)(b - x_k)}{L}, & j < k, \\ \frac{(x_k - a)(b - x_{j+1})}{L}, & j \geq k. \end{cases} \quad (4.2)$$

Proposition 4.1 (Sampled-kernel integer-order consistency). *Put $u_k = x_k - a$, $v_{j+1} = x_{j+1} - a$, and $\ell = L = b - a$, and define the sampled fractional matrix (an evaluation of the kernel formula, not a discretization of the infinite Jackson measure)*

$$G_{k,j}^{(\alpha)} := G_{\alpha,L}(u_k, v_{j+1}), \quad 0 \leq k \leq N, \quad 0 \leq j \leq N - 2. \quad (4.3)$$

Then,

$$\lim_{\alpha \rightarrow 2^-} G_{k,j}^{(\alpha)} = G_{k,j} \quad (4.4)$$

for every k, j . Consequently, for every fixed finite orbit and every bounded data vector g ,

$$\lim_{\alpha \rightarrow 2^-} \max_{0 \leq k \leq N} \left| \sum_{j=0}^{N-2} \Delta_j (G_{k,j}^{(\alpha)} - G_{k,j}) g_j \right| = 0. \quad (4.5)$$

Proof. For nonnegative arguments, $\Gamma_q(\alpha) \rightarrow \Gamma_q(2) = 1$ and each generalized q -power of order $\alpha - 1$ in (3.5) tends to its order-one value. Hence, $G_{\alpha,L}(u, v) \rightarrow G_{2,L}(u, v)$. If $j < k$, then $v_{j+1} \leq u_k$ and

$$G_{2,L}(u_k, v_{j+1}) = \frac{v_{j+1}(L - u_k)}{L} = \frac{(x_{j+1} - a)(b - x_k)}{L}.$$

If $j \geq k$, then $u_k \leq v_{j+1}$ and

$$G_{2,L}(u_k, v_{j+1}) = \frac{u_k(L - v_{j+1})}{L} = \frac{(x_k - a)(b - x_{j+1})}{L}.$$

These are precisely the two entries in (4.2). The finite weighted-sum limit follows because the sum contains only $N - 1$ terms; the grid and its weights are fixed as $\alpha \rightarrow 2^-$. $\square$

Remark 4.2 (Scope of the sampled-kernel limit). The fractional problem uses an infinite Jackson orbit based at the affine fixed point, whereas the integer problem uses a finite orbit segment with endpoints a and b . Proposition 4.1 therefore asserts a kernel-sampling and finite Green-operator limit; it does not identify the infinite Jackson measure with the finite-orbit measure. In particular, it does not assert convergence of the infinite fractional Green operator or of fractional boundary-value solutions to the finite-grid solutions. Only kernel samples and finite weighted sums with the fixed weights Δ_j are compared.

Theorem 4.1 (Green representation and uniqueness). *Problem (4.1) has the unique solution*

$$y_k = \sum_{j=0}^{N-2} \Delta_j G_{k,j} g_j, \quad k = 0, \dots, N. \quad (4.6)$$

Proof. Put $v_k = (Dy)_k$. The field equation gives $v_{k+1} = v_k - \Delta_k g_k$. Summation from the left boundary yields

$$y_k = v_0(x_k - a) - \sum_{j=0}^{k-2} \Delta_j(x_k - x_{j+1})g_j. \quad (4.7)$$

The condition $y_N = 0$ determines the missing initial slope:

$$v_0 = \frac{1}{L} \sum_{j=0}^{N-2} \Delta_j(b - x_{j+1})g_j. \quad (4.8)$$

Substituting (4.8) into (4.7) and separating the indices $j < k$ and $j \geq k$ gives (4.6).

For uniqueness, the homogeneous equation gives $v_k = v_0$ and hence $y_k = v_0(x_k - a)$. The boundary condition at b implies $v_0 = 0$. $\square$

Corollary 4.1 (Positivity). *Every coefficient $G_{k,j}$ is nonnegative. If $g_j \geq 0$, then $y_k \geq 0$; if the load is nonnegative and not identically zero, then $y_k > 0$ at every interior node $k = 1, \dots, N-1$.*

Corollary 4.2 (Exact row sum and discrete norm). *For $0 \leq k \leq N$,*

$$H_k := \sum_{j=0}^{N-2} \Delta_j G_{k,j} = \frac{(x_k - a)(b - x_k)}{1 + q}. \quad (4.9)$$

Consequently,

$$M_N := \max_{1 \leq k \leq N-1} H_k \leq \frac{(b-a)^2}{4(1+q)}. \quad (4.10)$$

Proof. Set $g_j = 1$ in Theorem 4.1. The function on the right-hand side of (4.9) vanishes at a and b , and direct use of (2.9) gives $D^2H = -1$. Uniqueness proves the identity. The bound follows from $(x_k - a)(b - x_k) \leq (b - a)^2/4$. $\square$

The number M_N is the exact maximum row sum, hence the exact induced ℓ^∞ norm of the Green solution map. The last expression in (4.10) is only a convenient continuous-parabola upper bound. It is an equality precisely when the finite grid contains $(a + b)/2$; otherwise it is strict.

The repeated finite orbit sum can be written

$$(\mathcal{J}^2 g)_k = \sum_{j=0}^{k-2} \Delta_j(x_k - x_{j+1})g_j. \quad (4.11)$$

Thus, Theorem 4.1 is equivalently

$$y_k = \frac{x_k - a}{b - a} (\mathcal{J}^2 g)_N - (\mathcal{J}^2 g)_k, \quad (4.12)$$

the finite-orbit counterpart of (2.7).

5. Exact orbit-polynomial loads

For $n \geq 0$, define the finite-interval orbit polynomials directly by

$$\Phi_{n,a}^{q,h}(x) = \prod_{r=0}^{n-1} (x - a - qh[r]_q), \quad \Phi_{0,a}^{q,h}(x) = 1. \quad (5.1)$$

They satisfy

$${}^{a+}D_{q,h}\Phi_{n,a}^{q,h}(x) = [n]_q\Phi_{n-1,a}^{q,h}(x), \quad (5.2)$$

and the improved integral satisfies

$${}^{a+}I_{q,h}\Phi_{n,a}^{q,h}(x) = \frac{\Phi_{n+1,a}^{q,h}(x)}{[n+1]_q}. \quad (5.3)$$

The two identities follow by substituting $\beta_a(x)$ and canceling adjacent factors; hence, the argument is self-contained and does not rely on an unpublished source.

Theorem 5.1 (Orbit-polynomial load). *Let $m \in \mathbb{N}_0$ and $g(x) = \Phi_{m,a}^{q,h}(x)$. The unique solution of (4.1), evaluated on the orbit nodes, is*

$$y_m(x) = \frac{1}{[m+1]_q[m+2]_q} \left[\frac{x-a}{b-a} \Phi_{m+2,a}^{q,h}(b) - \Phi_{m+2,a}^{q,h}(x) \right]. \quad (5.4)$$

Proof. Two applications of (5.2) give $D^2y_m = -\Phi_{m,a}^{q,h}$. Both boundary values in (5.4) vanish, so uniqueness follows from Theorem 4.1. $\square$

For the constant load $g \equiv 1$, Theorem 5.1 gives the particularly simple formula

$$y(x) = \frac{(x-a)(b-x)}{1+q}. \quad (5.5)$$

Although h cancels from the polynomial expression, it remains in the nodes $x_k = a + qh[k]_q$ and the endpoint $b = x_N$.

6. A nonlinear positive boundary-value problem

Consider

$$(D^2y)_j = -f(x_j, y_j), \quad j = 0, \dots, N-2, \quad y_0 = 0 = y_N. \quad (6.1)$$

For $u = (u_1, \dots, u_{N-1})$, set $u_0 = u_N = 0$ and define

$$(Tu)_k = \sum_{j=0}^{N-2} \Delta_j G_{k,j} f(x_j, u_j), \quad k = 1, \dots, N-1. \quad (6.2)$$

Theorem 6.1 (Bounded nonnegative solution). *Let $R > 0$. Suppose $f(x_j, u)$ is continuous in u and nonnegative for $j = 0, \dots, N-2$ and $0 \leq u \leq R$. Define*

$$F_R = \max_{\substack{0 \leq j \leq N-2 \\ 0 \leq u \leq R}} f(x_j, u). \quad (6.3)$$

If

$$F_R M_N \leq R, \quad (6.4)$$

then (6.1) has a nonnegative solution satisfying $0 \leq y_k \leq R$. If, for some fixed j_* , $f(x_{j_*}, u) > 0$ for every $u \in [0, R]$, then every interior component of the fixed-point solution is positive.

Proof. Let $Q_R = [0, R]^{N-1}$. Corollary 4.1, (4.9), and (6.4) give

$$0 \leq (Tu)_k \leq F_R H_k \leq F_R M_N \leq R, \quad u \in Q_R.$$

Thus, T is a continuous self-map of the compact convex set Q_R . Brouwer's fixed-point theorem gives $u = Tu$, and Theorem 4.1 shows that this fixed point solves (6.1). If the stated strict positivity condition holds, every interior Green coefficient multiplying that positive term is positive. $\square$

Corollary 6.1 (Uniqueness). *Assume the hypotheses of Theorem 6.1. If*

$$|f(x_j, u) - f(x_j, v)| \leq L_f |u - v| \quad \text{and} \quad L_f M_N < 1, \quad (6.5)$$

then T is a contraction in the maximum norm and the solution is unique among the solutions whose interior vector belongs to Q_R ; no claim is made about solutions outside this invariant box.

Corollary 6.2 (A directly checkable affine-growth test). *Suppose that, for constants $A \geq 0$ and $B \geq 0$,*

$$0 \leq f(x_j, u) \leq A + Bu, \quad j = 0, \dots, N-2, \quad u \geq 0. \quad (6.6)$$

If $BM_N < 1$, then every radius satisfying

$$R \geq R_* := \frac{AM_N}{1 - BM_N} \quad (6.7)$$

verifies the existence condition (6.4). If, in addition, $f(x_j, \cdot)$ is Lipschitz with a common constant $L_f < 1/M_N$, the solution is unique.

Proof. On $0 \leq u \leq R$, (6.6) gives $F_R \leq A + BR$. Inequality (6.7) is equivalent to $(A + BR)M_N \leq R$, so Theorem 6.1 applies. The last assertion is Corollary 6.1. $\square$

For implementation, one first computes the $N-1$ values H_k from (4.9), sets $M_N = \max_k H_k$, obtains any global affine majorant $A + Bu$ of the nonlinearity, and checks the single dimensionless number BM_N . For the common load $f(x_j, u) = c_j + \lambda u$ with $0 \leq c_j \leq c_{\max}$ and $\lambda \geq 0$, one may take $A = c_{\max}$, $B = \lambda$, and $R_* = c_{\max} M_N / (1 - \lambda M_N)$. This converts the abstract maximum condition into quantities read directly from the data and the grid.

7. Exact example

Take

$$a = 0, \quad q = \frac{3}{4}, \quad h = 1, \quad N = 4.$$

Then,

$$(x_0, x_1, x_2, x_3, x_4) = \left(0, \frac{3}{4}, \frac{21}{16}, \frac{111}{64}, \frac{525}{256}\right),$$

and $b = 525/256$. For

$$D^2 y(x_j) = -1, \quad j = 0, 1, 2, \quad y(0) = y\left(\frac{525}{256}\right) = 0, \quad (7.1)$$

formula (5.5) gives

$$y(x) = \frac{4}{7}x \left(\frac{525}{256} - x \right). \quad (7.2)$$

The exact values are shown in Table 3.

Table 3. Exact solution of (7.1).

k	x_k	y_k	Decimal value
0	0	0	0.000000
1	$3/4$	$999/1792$	0.557478
2	$21/16$	$567/1024$	0.553711
3	$111/64$	$8991/28672$	0.313581
4	$525/256$	0	0.000000

Here

$$(\Delta_0, \Delta_1, \Delta_2, \Delta_3) = \left(\frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \frac{81}{256}\right).$$

Direct substitution verifies

$$\frac{(Dy)_{j+1} - (Dy)_j}{\Delta_j} = -1, \quad j = 0, 1, 2,$$

and the Green sum (4.6) reproduces every entry of Table 3.

8. Limiting cases and discussion

The limit $h = 0$ must be separated by construction. In the infinite-orbit and branchwise operators, setting $h = 0$ gives $s_a = a$ and $\beta_a(x) = a + q(x - a)$; after translation $u = x - a$, these operators reduce to the standard Jackson operators, and Theorem 2.1 reduces to Theorem 2.4 of [4]. By contrast, the finite orbit $x_k = a + qh[k]_q$ starts at a and collapses to $x_k = a$ for every k when $h = 0$. It therefore has no nondegenerate $h = 0$ finite-interval limit. A finite Jackson grid at $h = 0$ must instead start from a point $x_0 \neq a$ (or be normalized independently); it is not obtained by substituting $h = 0$ into (2.8).

For the finite orbit, keep N and h fixed and let $q \rightarrow 1$. Then,

$$x_k \longrightarrow a + kh, \quad \Delta_k \longrightarrow h,$$

so (2.9) becomes the forward h -difference and (2.10) becomes the finite h -sum. The Green representation therefore converges to the Green solution of the corresponding forward second-difference Dirichlet problem. This limit is distinct from the infinite fixed-point orbit and is the natural finite- h regime for the grid (2.8).

The Green matrix (4.2) is the central structural object of the paper. It supplies uniqueness, positivity, the exact operator norm in (4.9), and both nonlinear fixed-point criteria. These properties depend on orbit consistency: Replacing β_a^n by unrelated nodes would destroy the telescoping identity from which the solution is derived.

9. Conclusions

We have introduced an orbit-consistent second-order q - h Dirichlet boundary-value problem using the corrected affine calculus. On the finite orbit generated by β_a , the problem admits a unique solution for every prescribed load. The solution is represented by an explicit Green matrix that is nonnegative and has an exact row sum. This representation yields positive solutions for nonnegative data, closed-form solutions for the natural orbit polynomials, and the constant-load formula $y(x) = (x - a)(b - x)/(1 + q)$. For nonlinear data, the same matrix gives a computable Brouwer existence condition and a contraction criterion for uniqueness. The formulation also has the expected forward h -difference limit on a fixed finite orbit.

At the fractional level, Definition 2.2 transports the Riemann–Liouville q -integral to either affine branch. Theorem 2.1 records the affine form of Ferreira’s commutation identity, with the orientation signs made explicit, and Corollary 2.2 gives the boundary terms relevant to second-order problems. Formula (2.25) is the resulting general forced-equation representation. Section 3 restricts the problem to a single fixed orbit, where the translated, known q -Green kernel $G_{\alpha,\ell}$ yields a valid orbit-sum equation. The new proof does not split incompatible Jackson orbits. Compactness is established in the Banach space of convergent orbit sequences, and the cone theorem gives a nontrivial positive solution under the explicit growth conditions (3.10)–(3.11). Example 3.1 verifies the Green sum and these hypotheses numerically.

The practical affine-growth criterion in Corollary 6.2 reduces the nonlinear finite-orbit test to a calculation with A , B , and M_N . Proposition 4.1 establishes sampled-kernel and finite-operator integer-order consistency between the fractional and finite constructions: after translation to orbit coordinates, the fractional Green-kernel samples converge entrywise, and as a finite operator, to the integer-order Green matrix as $\alpha \rightarrow 2^-$. This claim is deliberately limited to the sampled kernels and their finite weighted sums because the underlying infinite Jackson and finite orbit measures remain different. It does not establish convergence of the infinite fractional Green operator or of fractional boundary-value solutions to finite-grid solutions. Accordingly, the fractional result is a domain-correct affine transport of established q -fractional theory, not a new fractional kernel. Its value here is to expose the orbit restriction and the role of h explicitly. Natural extensions include Robin and multipoint boundary conditions, eigenvalue problems, fractional problems on the left affine branch, and right or left–right affine operators.

These extensions require new boundary-compatible Green kernels and corresponding positivity estimates.

Author contributions

Conceptualization: M. A., G. F., H. E. and J. R. Validation: M. A., G. F., H. E. and J. R. Formal analysis: M. A., G. F., H. E. and J. R. Writing-original draft: G. F. Writing-review and editing: M. A., G. F., H. E. and J. R. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest.

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