



*Research article***A flexible two-parameter discrete distribution with a bathtub-shaped hazard function: statistical properties and applications****Ammar M. Sarhan^{1,*}, Najwan Alsadat² and M. E. Sobh³**¹ Department of Mathematics and Statistics, Dalhousie University, Nova Scotia, Canada² Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia³ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt*** Correspondence:** Email: ammar.sarhan@dal.ca, Tel: +19022666423.

Abstract: This work introduced a new two-parameter discrete distribution derived from a continuous truncated lifetime model. The proposed distribution features a flexible hazard rate function that can exhibit both increasing and bathtub-shaped behaviors depending on parameter configurations, allowing it to capture diverse patterns in discrete failure time data. Despite its flexibility, the model retains a parsimonious two-parameter structure, facilitating straightforward statistical inference. We derived key structural and statistical properties of the model and investigated parameter estimation under both maximum likelihood and Bayesian frameworks. Because closed-form expressions for the estimators are analytically intractable, numerical methods were developed; specifically, maximum likelihood estimates were obtained via numerical optimization in R, while Bayesian estimates were computed using a Markov chain Monte Carlo sampler. The finite-sample performance of both estimation procedures was evaluated through an extensive Monte Carlo simulation study. Finally, the practical utility of the distribution was demonstrated through applications to four real-world datasets across different fields, where it yielded competitive fits relative to several established discrete lifetime models.

Keywords: discrete distribution; bathtub-shaped hazard rate; discretization; maximum likelihood estimation; Bayesian estimation

Mathematics Subject Classification: Primary 62G32, Secondary 60F05, 62G09, 62G20, 62G30

1. Introduction

Discrete probability distributions constitute a cornerstone of modern stochastic modeling whenever the phenomenon under investigation evolves in inherently countable units. In many scientific and engineering contexts, time is not observed on a continuous scale but rather through

discrete events, cycles completed, shocks endured, claims reported, defects detected, or failures recorded. Such integer-valued data emerge naturally in reliability engineering, survival analysis, actuarial science, epidemiology, inventory control, and queuing theory, where the operational mechanism itself progresses step by step. In these situations, discrete lifetime models are not merely convenient approximations of continuous counterparts; they provide a structurally faithful representation of the underlying process, preserving interpretability and aligning the mathematical framework with the physical reality of observation.

A central concept in reliability and survival analysis is the hazard rate function, which encapsulates the instantaneous risk of failure at a given stage, conditional on survival up to that point. Through the hazard function, one can formally characterize aging mechanisms, distinguish between improving and deteriorating systems, and design maintenance or replacement strategies. Among the various hazard shapes documented in practice, the bathtub-shaped hazard function occupies a position of particular importance. This form reflects three sequential phases commonly observed in engineered and biological systems: an initial period of elevated but decreasing failure risk (often attributed to manufacturing imperfections or early defects), a middle phase of approximately constant risk corresponding to normal operation, and a final stage of increasing risk due to cumulative wear and aging. Empirical evidence of such behavior has been extensively reported in mechanical assemblies, electronic components, and industrial products [4, 14, 16]. Beyond lifetime and hazard-based modeling, reliability research has also advanced substantially in the closely related area of stochastic degradation analysis, where recent contributions include hierarchical Bayesian multivariate Wiener process models with dependent degradation rates and volatilities [22], online Bayesian frameworks for identifying latent degradation states [24], and bivariate Wiener degradation models with partially random scale structures [23]. While numerous continuous lifetime models are capable of reproducing this pattern, comparatively few discrete distributions combine the ability to generate a genuine bathtub-shaped hazard with mathematical tractability and practical interpretability.

The demand for discrete lifetime models exhibiting non-monotone hazard structures has stimulated notable developments in recent years. A particularly influential contribution was made by Sarhan [19], who proposed a discrete two-parameter distribution capable of producing a bathtub-shaped hazard rate and demonstrated its effectiveness in modeling count-based lifetime data. This work underscores a critical insight: classical discrete models such as the geometric or Poisson distributions, despite their simplicity, often fail to capture the intricate aging dynamics encountered in real datasets, thereby motivating the search for richer yet analytically manageable alternatives.

Nevertheless, existing models still exhibit limitations in flexibility, parameter interpretability, or goodness-of-fit when confronted with data displaying pronounced early-life failures and strong wear-out effects. There remains a clear methodological gap for discrete lifetime distributions that can reconcile three essential objectives: (i) the capacity to model genuine bathtub-shaped hazard behavior, (ii) analytical simplicity enabling rigorous theoretical development, and (iii) enhanced adaptability to diverse empirical patterns. Motivated by these considerations, the present paper introduces a new two-parameter discrete lifetime distribution with a bathtub-shaped hazard rate. The proposed model is a parsimonious, mathematically simple alternative for count-based lifetime data.

The remainder of the paper is organized as follows. Section 2 introduces the proposed distribution and examines its main statistical properties, including the probability mass function, survival function, hazard rate function, and their limiting behavior. Section 3 presents the key descriptive and

functional features of the proposed model. Section 4 investigates the order statistics of the proposed model, obtaining their distribution and moments. Parameter estimation via the maximum likelihood and Bayesian methods is then discussed in Section 5. Section 6 presents a simulation study assessing the finite-sample performance of the estimators, and Section 7 illustrates the practical usefulness of the proposed model through applications to real-life datasets. Finally, concluding remarks are provided in Section 8.

2. The discrete two parameter distribution

We consider the continuous cumulative distribution function (CDF) introduced in [15], defined as

$$F_0(x; \alpha, \beta) = \frac{(\alpha + 1)x}{\alpha\beta + x}, \quad 0 < x \leq \beta; \quad \alpha, \beta > 0. \quad (2.1)$$

To obtain a discrete analogue supported on $x \in \{1, 2, \dots, \beta\}$, for $\beta \in \mathbb{N}$, we adopt the standard discretization principle based on first-order forward differences of the parent CDF. Specifically, the probability mass function (PMF) is defined by

$$p(x) = F_0(x) - F_0(x - 1), \quad x = 1, 2, \dots, \beta, \quad (2.2)$$

with the convention that $F_0(0) = 0$.

Substituting Eq (2.1) into Eq (2.2), we get the PMF of the new two-parameter discrete distribution (NTPD) as follows:

$$p(x; \alpha, \beta) = \frac{(\alpha + 1)x}{\alpha\beta + x} - \frac{(\alpha + 1)(x - 1)}{\alpha\beta + x - 1}, \quad x = 1, 2, \dots, \beta. \quad (2.3)$$

After straightforward algebraic simplification, the PMF reduces to the remarkably compact form

$$p(x; \alpha, \beta) = \frac{\alpha\beta(\alpha + 1)}{(\alpha\beta + x)(\alpha\beta + x - 1)}, \quad x = 1, 2, \dots, \beta. \quad (2.4)$$

The parameter $\beta \in \mathbb{N}$ represents a structural upper bound on the domain of the discrete random variable X . While standard lifetime models often assume infinite support $[0, \infty)$, finite support is naturally justifiable in many real-world applications. In reliability engineering and survival analysis, systems and organisms frequently operate under known design lifespans, finite inspection windows, or maximum biological limits beyond which survival is impossible. In settings where β represents a known physical ceiling or a pre-specified study duration, it can be fixed a priori based on subject-matter knowledge. Conversely, when the maximum possible lifetime is unknown, estimating β alongside α allows the data to determine the empirical upper horizon, providing a flexible framework for discrete lifetime modeling over restricted domains.

The survival function of the NTPD distribution is defined as:

$$\begin{aligned} S(x; \alpha, \beta) &= P(X > x) = \sum_{k=x+1}^{\beta} p(k; \alpha, \beta) = \sum_{k=x+1}^{\beta} \frac{\alpha\beta(\alpha + 1)}{(\alpha\beta + k)(\alpha\beta + k - 1)} \\ &= \frac{\alpha(\beta - x)}{\alpha\beta + x}, \quad x = 1, 2, \dots, \beta. \end{aligned} \quad (2.5)$$

The corresponding CDF for the NTPD distribution can be expressed in the form

$$F(x; \alpha, \beta) = 1 - \frac{\alpha(\beta - x)}{\alpha\beta + x}, \quad x = 1, 2, \dots, \beta. \quad (2.6)$$

Moreover, the hazard rate function (HRF) is defined as:

$$h(x; \alpha, \beta) = P(X = x | X > x - 1) = \frac{p(x; \alpha, \beta)}{S(x - 1; \alpha, \beta)} = \frac{\beta(\alpha + 1)}{(\alpha\beta + x)(\beta - x + 1)}. \quad (2.7)$$

The reversed hazard rate can also be obtained in the form

$$r(x; \alpha, \beta) = P(X = x | X \leq x) = \frac{p(x; \alpha, \beta)}{F(x; \alpha, \beta)} = \frac{\alpha\beta}{x(\alpha\beta + x - 1)}. \quad (2.8)$$

The behavior of the PMF and HRF is formally described in the following two subsections, highlighting their key properties and overall pattern across the support of the distribution.

2.1. Structural properties and tail behavior of the PMF

This subsection examines the structural properties of the proposed PMF, specifically its monotonicity, variation across the support, and finite upper-end decay.

Lemma 2.1. *The PMF of the NTPD distribution is strictly decreasing. Consequently, the mode of the NTPD distribution is always $x = 1$.*

Proof. To prove that $p(x; \alpha, \beta)$ is strictly decreasing in x , we consider

$$\frac{p(x + 1; \alpha, \beta)}{p(x; \alpha, \beta)} = \frac{\alpha\beta + x - 1}{\alpha\beta + x + 1} < 1, \quad \forall x \geq 1.$$

Hence, $p(x + 1; \alpha, \beta) < p(x; \alpha, \beta)$, which establishes that the PMF is strictly decreasing in x . $\square$

Upper-end behavior (tail within finite support). For x sufficiently large relative to $\alpha\beta$ (i.e., $x \gg \alpha\beta$), which is possible within $x \leq \beta$ only when $\alpha \ll 1$, the PMF admits the approximation; the PMF behaves approximately as

$$p(x; \alpha, \beta) = \frac{\alpha\beta(\alpha + 1)}{(\alpha\beta + x)(\alpha\beta + x - 1)} \approx \frac{\alpha\beta}{x^2},$$

indicating that, in this regime, the tail decays at a moderate polynomial rate rather than the exponential or geometric rate typical of many discrete lifetime models.

Moment existence. An immediate consequence of the bounded support is that all moments exist. Specifically, since $X \leq \beta$, we have

$$X^k \leq \beta^k, \quad \forall k \in \mathbb{N},$$

which implies

$$\mathbb{E}[X^k] < \infty, \quad \forall k \in \mathbb{N}.$$

Limiting behavior with respect to β . Fix x and α . As $\beta \rightarrow \infty$, we have $\alpha\beta + x \sim \alpha\beta$ and $\alpha\beta + x - 1 \sim \alpha\beta$, so

$$p(x; \alpha, \beta) \sim \frac{\alpha\beta(\alpha + 1)}{(\alpha\beta)^2} = \frac{\alpha + 1}{\alpha\beta} \rightarrow 0.$$

Thus, the PMF at any fixed point vanishes as the support expands. On the other hand, as $\beta \rightarrow 1^+$, the support reduces to $\{1\}$, and

$$p(1; \alpha, \beta) = \frac{\alpha + 1}{\alpha\beta + 1} \rightarrow 1.$$

Hence, the distribution degenerates at $x = 1$. Figure 1 presents the PMF of the NTPD model for different parameter settings, illustrating its flexibility under various parameter choices.

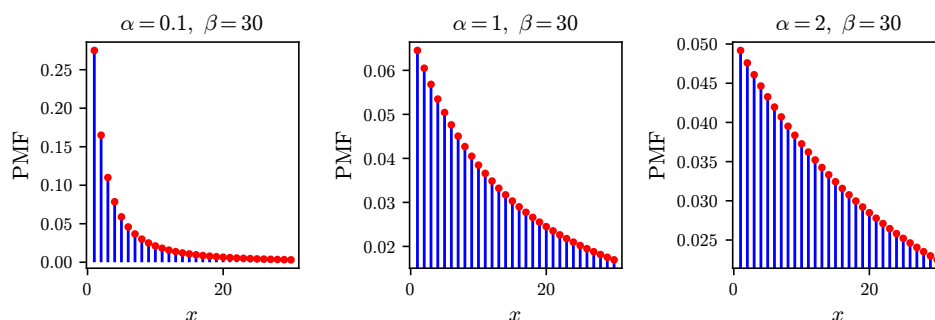


Figure 1. PMF of the NTPD for $\alpha = 0.1, 1, 2$ with $\beta = 30$.

2.2. Hazard rate properties and behavior

The following lemma provides a formal characterization of the HRF's behavior over its support.

Lemma 2.2. *The HRF of the NTPD distribution has a bathtub shape if $\alpha < 1 - 2/\beta$ (which is equivalent to α being small). Otherwise, the HRF is monotonically increasing.*

Proof. To study the behavior of the HRF, consider the forward difference

$$\Delta h(x; \alpha, \beta) = h(x + 1; \alpha, \beta) - h(x; \alpha, \beta), \quad x = 1, 2, \dots, \beta - 1.$$

Substituting the expression of $h(x; \alpha, \beta)$, we obtain

$$\Delta h(x; \alpha, \beta) = \frac{\beta(\alpha + 1)(\alpha\beta + 2x - \beta)}{(\alpha\beta + x)(\alpha\beta + x + 1)(\beta - x)(\beta - x + 1)}.$$

For $x \in \{1, 2, \dots, \beta - 1\}$, the denominator is strictly positive, and the factor $\beta(\alpha + 1)$ is also positive. Hence, the sign of $\Delta h(x; \alpha, \beta)$ is determined by

$$\text{sign}(\Delta h(x; \alpha, \beta)) = \text{sign}(\alpha\beta + 2x - \beta).$$

Define the critical point

$$x_{\text{crit}} = \frac{\beta - \alpha\beta}{2}.$$

Then,

$$\Delta h(x; \alpha, \beta) < 0 \quad \text{for } x < x_{\text{crit}}, \quad \Delta h(x; \alpha, \beta) > 0 \quad \text{for } x > x_{\text{crit}}.$$

Therefore, the HRF decreases for $x < x_{\text{crit}}$ and increases for $x > x_{\text{crit}}$, provided that x_{crit} lies within the support. For a bathtub shape, it is necessary that

$$1 < x_{\text{crit}} < \beta.$$

The left inequality gives

$$\frac{\beta - \alpha\beta}{2} > 1 \Rightarrow \alpha\beta < \beta - 2,$$

while the right inequality, $\frac{\beta - \alpha\beta}{2} < \beta$, holds automatically since $\alpha\beta > 0$. Hence, the HRF has a bathtub shape whenever $\alpha < 1 - 2/\beta$. Otherwise, if $\alpha > 1 - 2/\beta$, then $x_{\text{crit}} \leq 1$, which implies

$$\Delta h(x; \alpha, \beta) > 0, \quad \forall x \in \{1, \dots, \beta - 1\},$$

and therefore the HRF is monotonically increasing. $\square$

Figure 2 illustrates the HRF $h(x; \alpha, \beta)$ for selected parameter values. The plots show that the HRF can be either bathtub-shaped or monotonically increasing. For the parameter settings in Figure 2 ($\beta = 30$, so $\beta - 2 = 28$), the condition $\alpha\beta < \beta - 2$ holds only when $\alpha = 0.1$ and $\alpha\beta = 3$, yielding a bathtub-shaped HRF. In contrast, for $\alpha = 1$ and $\alpha = 2$ ($\alpha\beta = 30$ and 60 , respectively), the condition is not satisfied, and the HRF is monotonically increasing.

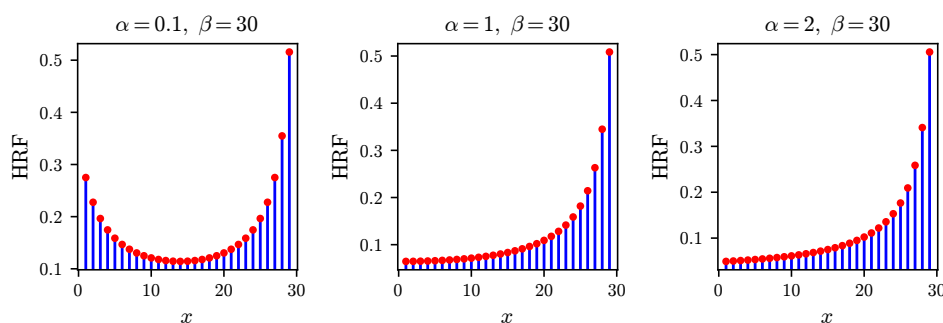


Figure 2. The HRF when $\alpha = 0.1, 1, 2$ and $\beta = 30$.

3. Statistical and reliability properties of the NTPD

This section presents the principal statistical and reliability properties of the proposed distribution. We first derive the quantile function, moments, and probability generating functions, followed by the mean residual life and mean past life functions. Finally, we obtain the stress-strength reliability, highlighting the flexibility and practical applicability of the proposed model.

3.1. Random number generation

The q^{th} quantile of a random variable $X \sim \text{NTPD}(\alpha, \beta)$ is given by

$$x_q = \left\lceil \frac{\alpha\beta q}{\alpha - q + 1} \right\rceil, \quad q \in (0, 1). \quad (3.1)$$

Consequently, a random sample of size n from the $\text{NTPD}(\alpha, \beta)$ can be generated via the inverse transform method by generating n independent variates $U_i \sim \text{Uniform}(0, 1)$ and evaluating x_{U_i} using (3.1).

3.2. Moments, skewness, and kurtosis

Let $X \sim \text{NTPD}(\alpha, \beta)$, and then the probability generating function (PGF) of X is given by

$$\Phi(s) = \sum_{x=1}^{\beta} s^x p(x; \alpha, \beta) = (\alpha + 1) \sum_{x=1}^{\beta} s^x \left(\frac{x}{\alpha\beta + x} - \frac{x-1}{\alpha\beta + x - 1} \right).$$

Since, for any sequences $\{a_x\}$ and $\{b_x\}$, the summation-by-parts identity $\sum_{x=1}^{\beta} a_x(b_x - b_{x-1}) = a_{\beta}b_{\beta} - \sum_{x=0}^{\beta-1} b_x(a_{x+1} - a_x)$ holds, setting $a_x = s^x$ and $b_x = \frac{x}{\alpha\beta + x}$ gives

$$\Phi(s) = (\alpha + 1) \left(\frac{\beta s^{\beta}}{\alpha\beta + \beta} - \sum_{x=1}^{\beta-1} \frac{x}{\alpha\beta + x} (s^{x+1} - s^x) \right) = s^{\beta} + (\alpha + 1)(1 - s) \sum_{x=1}^{\beta-1} \frac{x s^x}{\alpha\beta + x}. \quad (3.2)$$

Consequently, the moment generating function (MGF) of X is obtained from the PGF as $\mathcal{M}_X(t) = \Phi(e^t)$. Hence,

$$\mathcal{M}_X(t) = e^{t\beta} + (\alpha + 1)(1 - e^t) \sum_{x=1}^{\beta-1} \frac{x e^{tx}}{\alpha\beta + x}. \quad (3.3)$$

The r -th raw moment can be expressed as:

$$\psi'_r = \sum_{x=1}^{\beta} x^r p(x; \alpha, \beta) = (\alpha + 1) \sum_{x=1}^{\beta} x^r \left(\frac{x}{\alpha\beta + x} - \frac{x-1}{\alpha\beta + x - 1} \right).$$

Applying the same summation-by-parts identity as in (3.2), now with $a_x = x^r$ and $b_x = \frac{x}{\alpha\beta + x}$, gives

$$\psi'_r = (\alpha + 1) \left(\frac{\beta^{r+1}}{\alpha\beta + \beta} - \sum_{x=1}^{\beta-1} \frac{x}{\alpha\beta + x} ((1+x)^r - x^r) \right) = \beta^r - (\alpha + 1) \sum_{x=1}^{\beta-1} \sum_{i=0}^{r-1} \binom{r}{i} \frac{x^{i+1}}{\alpha\beta + x}. \quad (3.4)$$

Based on Eq (3.4), the first four moments of the random variable X are given by:

$$\begin{aligned} \psi'_1 &= \beta - (\alpha + 1) \sum_{x=1}^{\beta-1} \frac{x}{\alpha\beta + x}, & \psi'_2 &= \beta^2 - (\alpha + 1) \sum_{x=1}^{\beta-1} \frac{x(2x+1)}{\alpha\beta + x}, \\ \psi'_3 &= \beta^3 - (\alpha + 1) \sum_{x=1}^{\beta-1} \frac{x(3x^2 + 3x + 1)}{\alpha\beta + x}, & \psi'_4 &= \beta^4 - (\alpha + 1) \sum_{x=1}^{\beta-1} \frac{x(4x^3 + 6x^2 + 4x + 1)}{\alpha\beta + x}. \end{aligned}$$

The first four central moments can be derived, from the corresponding raw moments, as

$$\mu_2 = \psi'_2 - \psi_1'^2, \quad \mu_3 = \psi'_3 - 3\psi'_1\psi'_2 + 2\psi_1'^3, \quad \mu_4 = \psi'_4 - 4\psi'_3\psi'_1 + 6\psi'_2\psi_1'^2 - 3\psi_1'^4.$$

These moments yield the variance (σ_X^2), index of dispersion (IOD), coefficient of variation (CV), skewness (Sk), and kurtosis (Ku) measures as:

$$\sigma_X^2 = \mu_2, \quad \text{IOD} = \frac{\sigma_X^2}{\psi_1'}, \quad \text{CV} = \frac{\sigma_X}{\psi_1'}, \quad \text{Sk} = \frac{\mu_3}{\sigma_X^3}, \quad \text{Ku} = \frac{\mu_4}{\sigma_X^4}.$$

Since the r -th moment lacks a closed-form expression, the mean, σ_X^2 , IOD, CV, Sk, and Ku are computed numerically. Figures 3–5 present these measures for different choices of the model parameters. For fixed β , increasing α raises both the mean and the variance, shifting the distribution toward larger values with greater spread. The CV's response to α depends on β : for $\beta = 2$, it rises to a maximum at a moderate value of α and then declines only slightly, whereas for $\beta = 10$ and 15, it decreases over the whole range toward a constant limit. The IOD is non-monotone in all cases. For $\beta = 10$ and 15, it rises sharply to a maximum at small α and then decreases; for $\beta = 2$ it increases up to a maximum at a moderate value of α and is essentially flat afterward. Over the plotted range, the IOD stays below 1 for $\beta = 2$ (under-dispersion) and above 1 for $\beta = 10$ and 15 (over-dispersion).

Increasing β primarily inflates the mean and variance, confirming that β controls the scale, while also driving the shift from under- to over-dispersion noted above. Overall, α governs the shape (skewness and kurtosis) while β governs the scale and dispersion, together illustrating the flexibility of the NTPD.

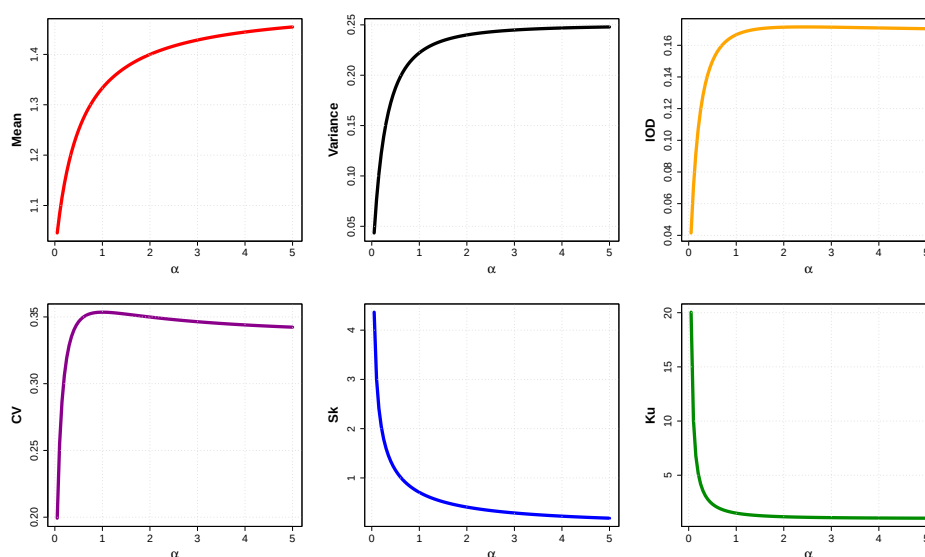


Figure 3. Behavior of the distributional measures for varying α at a fixed value of $\beta = 2$.

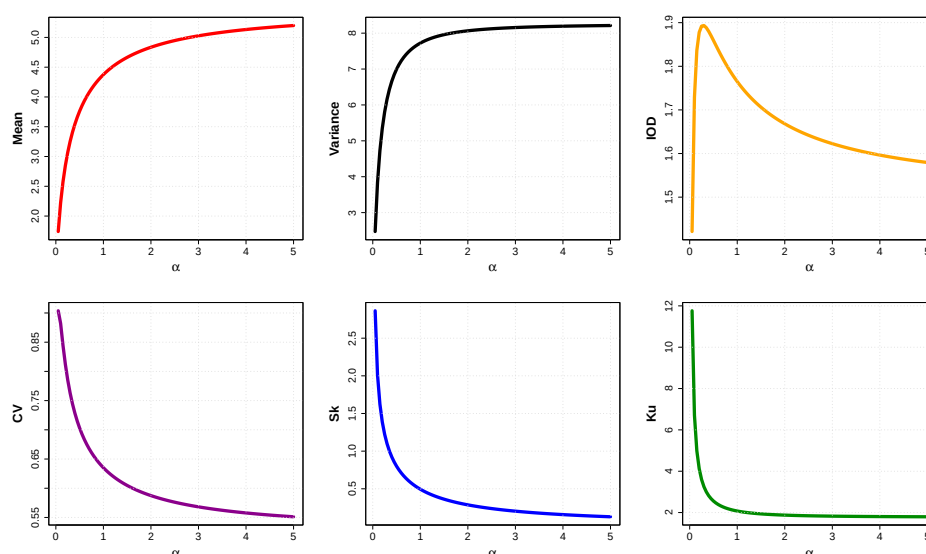


Figure 4. Behavior of the distributional measures for varying α at a fixed value of $\beta = 10$.

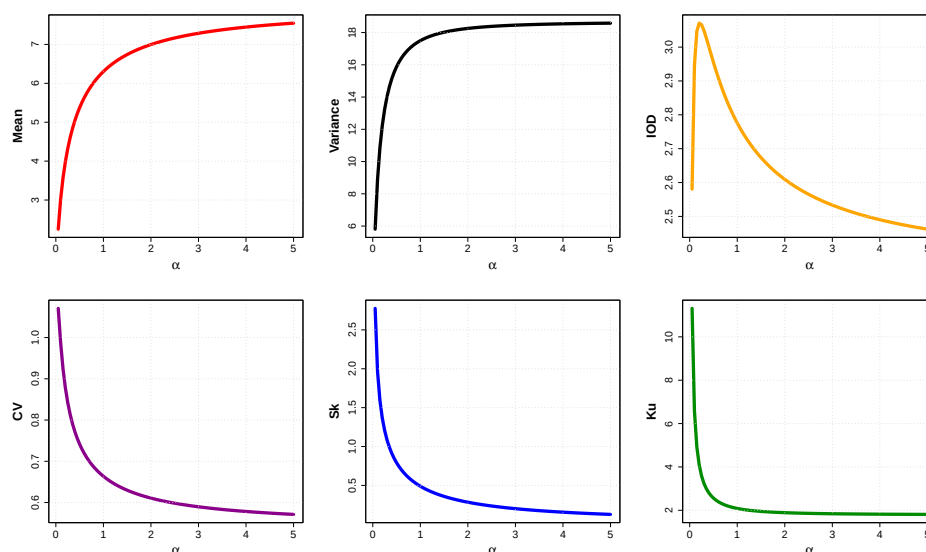


Figure 5. Behavior of the distributional measures for varying α at a fixed value of $\beta = 15$.

3.3. Mean residual life and mean past life

In reliability theory and survival analysis, several quantitative measures describe how components or systems age. These measures are central to modeling failure mechanisms and designing maintenance strategies. The mean residual life (MRL), denoted by $\xi(t)$, measures the expected remaining lifetime of a unit surviving at time t . In a discrete setting, the MRL is defined as:

$$\xi(t) = E(T - t \mid T > t) = 1 + \frac{1}{S(t)} \sum_{k=t+1}^{\beta-1} S(k) = 1 + \frac{\alpha\beta + t}{\beta - t} \sum_{k=t+1}^{\beta-1} \frac{\beta - k}{\alpha\beta + k}, \quad (3.5)$$

where $t \in \{1, 2, \dots, \beta - 1\}$. The mean past life (MPL), denoted by $\omega(t)$ and also known as the mean inactivity time, measures the expected time elapsed since failure given that the system failed at or before time t . For the NTPD distribution, the MPL function is given by:

$$\omega(t) = E(t - T \mid T \leq t) = \frac{1}{F(t)} \sum_{k=1}^t F(k-1) = \frac{(\alpha\beta + t)}{(\alpha + 1)t} \sum_{k=1}^t \left(1 - \frac{\alpha(\beta - k + 1)}{\alpha\beta + k - 1}\right), \quad (3.6)$$

where $t \in \{1, 2, \dots, \beta\}$. Figure 6 shows the MRL function $\xi(t)$ of the NTPD for $\beta \in \{5, 15, 30\}$ and selected values of α . Because the support is finite, $\xi(t)$ decreases to its minimum attainable value of 1 at $t = \beta - 1$, reflecting the fact that a unit surviving to the boundary of the support still has at least one unit of expected residual life. The MRL exhibits a purely decreasing pattern across all t when $\beta = 5$, or when $\alpha \geq 0.5$ for $\beta \in \{15, 30\}$. However, for small α ($\alpha = 0.1$) combined with larger β , $\xi(t)$ initially rises before declining, forming an upside-down bathtub (UBT) shape. This neatly mirrors the HR behavior in Figure 2: An increasing HRF gives a decreasing MRL, while a bathtub HRF gives an UBT MRL only when the decreasing part of $h(t)$ is long enough (small α); for $\alpha = 0.5$ and $\beta \in \{15, 30\}$ the MRL is still decreasing.

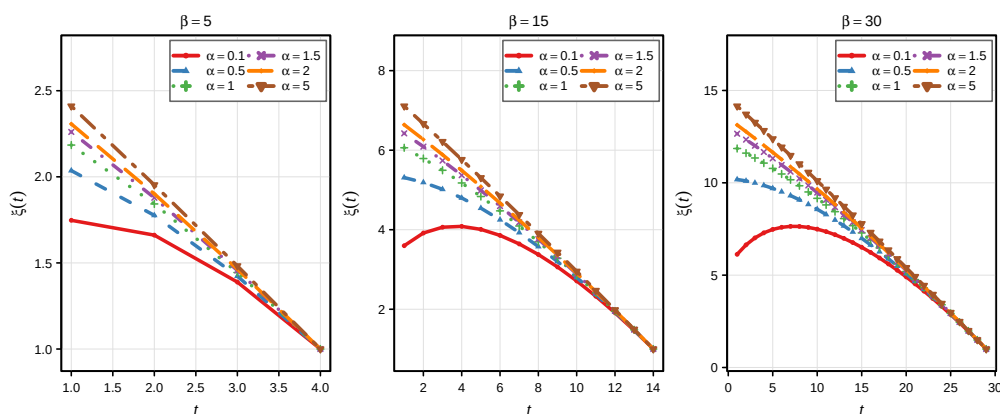


Figure 6. The MRL function of the NTPD for selected values of α and $\beta \in \{5, 15, 30\}$.

3.4. Stress-strength reliability

The stress-strength model plays a central role in reliability engineering, where it quantifies the probability that a system's strength X exceeds an applied stress Y . Let $X \sim \text{NTPD}(\alpha_1, \beta_1)$ denote the strength and $Y \sim \text{NTPD}(\alpha_2, \beta_2)$ the stress, with X and Y independent, and denote their PMF and CDF by $p(x; \alpha_1, \beta_1)$ and $F(y; \alpha_2, \beta_2)$, respectively. The reliability parameter $R = P(X > Y)$ is given by

$$R = \sum_{x=1}^{\beta_1} p(x; \alpha_1, \beta_1) F(x-1; \alpha_2, \beta_2), \quad (3.7)$$

where $F(y; \alpha_2, \beta_2) = 1$ for $y \geq \beta_2$. Substituting (2.4) and (2.6) into (3.7) yields the explicit form

$$R = \sum_{x=1}^{\beta_1} \frac{\alpha_1 \beta_1 (\alpha_1 + 1)}{(\alpha_1 \beta_1 + x)(\alpha_1 \beta_1 + x - 1)} \min \left\{ 1, 1 - \frac{\alpha_2 (\beta_2 - x + 1)}{\alpha_2 \beta_2 + x - 1} \right\}. \quad (3.8)$$

When $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, X and Y are independent and identically distributed (i.i.d.). Then $P(X > Y) = P(Y > X)$, and since ties occur with positive probability, $R = P(X > Y) = \frac{1}{2}\{1 - P(X = Y)\} < \frac{1}{2}$. This bound applies only to identically distributed X and Y ; when the parameters differ, R is not restricted to be below $\frac{1}{2}$.

Since (3.8) has no closed-form reduction, R is evaluated numerically; the results are given in Tables 1 and 2. Table 1 fixes $\beta_1 = \beta_2 = 15$ and varies the shape parameters: R increases down each column and decreases along each row. Because of ties, R can remain below 0.5 even when $\alpha_1 > \alpha_2$; for example, $R = 0.4928$ for $(\alpha_1, \alpha_2) = (1.5, 1.0)$, where $P(X = Y) \approx 0.075$. The diagonal entries are the i.i.d. values $\frac{1}{2}\{1 - P(X = Y)\}$. Table 2 fixes $\alpha_1 = \alpha_2 = 1.5$ and varies the scale parameters: R increases down each column and decreases along each row, ranging from 0.0835 ($\beta_1 = 5, \beta_2 = 30$) to 0.8656 ($\beta_1 = 30, \beta_2 = 5$). The diagonal entries are again the i.i.d. values. These patterns are expected, since the NTPD takes larger values as either parameter increases.

Table 1. Stress-strength reliability for $X \sim \text{NTPD}(\alpha_1, 15)$ and $Y \sim \text{NTPD}(\alpha_2, 15)$.

$\alpha_1 \downarrow \alpha_2 \rightarrow$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.4520	0.3912	0.3642	0.3488	0.3388	0.3318
1.0	0.5244	0.4611	0.4325	0.4160	0.4052	0.3976
1.5	0.5564	0.4928	0.4637	0.4469	0.4359	0.4281
2.0	0.5745	0.5110	0.4818	0.4648	0.4537	0.4458
2.5	0.5863	0.5229	0.4936	0.4766	0.4654	0.4575
3.0	0.5945	0.5313	0.5020	0.4849	0.4737	0.4657

Table 2. Stress-strength reliability for $X \sim \text{NTPD}(1.5, \beta_1)$ and $Y \sim \text{NTPD}(1.5, \beta_2)$.

$\beta_1 \downarrow \beta_2 \rightarrow$	5	10	15	20	25	30
5	0.3916	0.2249	0.1580	0.1217	0.0991	0.0835
10	0.6442	0.4456	0.3256	0.2568	0.2120	0.1806
15	0.7482	0.5926	0.4637	0.3724	0.3113	0.2676
20	0.8050	0.6777	0.5682	0.4728	0.3994	0.3459
25	0.8409	0.7332	0.6383	0.5539	0.4782	0.4170
30	0.8656	0.7724	0.6887	0.6132	0.5446	0.4819

4. Order statistics

Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with a CDF following the $\text{NTPD}(\alpha, \beta)$ distribution and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics (OSs). The CDF of the k th OS for an integer value x is given by:

$$F_{k:n}(x; \alpha, \beta) = P(X_{k:n} \leq x) = \sum_{i=k}^n \binom{n}{i} [F(x; \alpha, \beta)]^i [1 - F(x; \alpha, \beta)]^{n-i} = \sum_{i=k}^n \sum_{j=0}^i \varpi_{i,j}^{(n,k)}(x; \alpha) \left(\frac{\beta - x}{\alpha\beta + x} \right)^{n+j-i},$$

where $\varpi_{i,j}^{(n,k)}(x; \alpha, \beta) = (-1)^j \binom{n}{i} \binom{i}{j} \alpha^{n+j-i}$. The PMF of the k th OS is presented as follows:

$$\begin{aligned}
 f_{k:n}(x; \alpha, \beta) &= P(X_{k:n} = x) = F_{k:n}(x; \alpha, \beta) - F_{k:n}(x-1; \alpha, \beta) \\
 &= \sum_{i=k}^n \sum_{j=0}^i \varpi_{i,j}^{(n,k)}(x; \alpha, \beta) \left[\left(\frac{\beta-x}{\alpha\beta+x} \right)^{n+j-i} - \left(\frac{\beta-x+1}{\alpha\beta+x-1} \right)^{n+j-i} \right].
 \end{aligned}$$

Additionally, the ν -th moments of the k th OS can be expressed as

$$\mu_{k:n}^{(\nu)} = E(X_{k:n}^\nu) = \sum_{x=1}^{\beta} \sum_{i=k}^n \sum_{j=0}^i x^\nu \varpi_{i,j}^{(n,k)}(x; \alpha, \beta) \left[\left(\frac{\beta-x}{\alpha\beta+x} \right)^{n+j-i} - \left(\frac{\beta-x+1}{\alpha\beta+x-1} \right)^{n+j-i} \right]. \quad (4.1)$$

Based on the results in Eq (4.1), the L-moments can be expressed in the form $\Xi_i = \frac{1}{i} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \mu_{i-j:i}$, which provides a convenient framework for analyzing various descriptive statistics (see Hosking [10]). Moreover, since the support of the NTPD distribution is a subset of the natural numbers, then $\mu_{1:n}$, $\mu_{n:n}^{(2)}$, $\mu_{1:n}^{(2)}$, $\mu_{n:n}^{(2)}$, $\sigma_{1,1:n}^2$, and $\sigma_{n,n:n}^2$ can be directly obtained using $F_{k:n}(x; \alpha, \beta)$ (see Arnold et al. [2]) as follows:

$$\mu_{1:n} = \sum_{x=0}^{\beta} [1 - F(x; \alpha, \beta)]^n = \sum_{x=0}^{\beta} \left[\frac{\alpha(\beta-x)}{\alpha\beta+x} \right]^n, \quad (4.2)$$

$$\mu_{n:n} = \sum_{x=0}^{\beta} [1 - F^n(x; \alpha, \beta)] = \sum_{x=0}^{\beta} \left[1 - \left\{ 1 - \frac{\alpha(\beta-x)}{\alpha\beta+x} \right\}^n \right], \quad (4.3)$$

$$\mu_{1:n}^{(2)} = 2 \sum_{x=0}^{\beta} x [1 - F(x; \alpha, \beta)]^n + \mu_{1:n} = 2 \sum_{x=0}^{\beta} x \left[\frac{\alpha(\beta-x)}{\alpha\beta+x} \right]^n + \mu_{1:n}, \quad (4.4)$$

$$\mu_{n:n}^{(2)} = 2 \sum_{x=0}^{\beta} x [1 - F^n(x; \alpha, \beta)] + \mu_{n:n} = 2 \sum_{x=0}^{\beta} x \left[1 - \left\{ 1 - \frac{\alpha(\beta-x)}{\alpha\beta+x} \right\}^n \right] + \mu_{n:n}, \quad (4.5)$$

and

$$\sigma_{1,1:n}^2 = \mu_{1:n}^{(2)} - [\mu_{1:n}]^2, \quad \sigma_{n,n:n}^2 = \mu_{n:n}^{(2)} - [\mu_{n:n}]^2. \quad (4.6)$$

Using the above formulas, Table 3 presents some numerical values for $\mu_{1:n}$, $\mu_{n:n}$, $\mu_{1:n}^{(2)}$, $\mu_{n:n}^{(2)}$, $\sigma_{1,1:n}^2$, and $\sigma_{n,n:n}^2$ for various values of α , β , and n . Additionally, the results in Table 3 reveal clear monotonic patterns. For fixed n , $\mu_{1:n}$, $\mu_{n:n}$, $\mu_{1:n}^{(2)}$, $\mu_{n:n}^{(2)}$, and $\sigma_{1,1:n}^2$ increase with α and β , while $\sigma_{n,n:n}^2$ increases with β but is not monotonic in α . As n increases, $\mu_{1:n}$, $\mu_{1:n}^{(2)}$, and $\sigma_{1,1:n}^2$ decrease and $\mu_{n:n}$ and $\mu_{n:n}^{(2)}$ increase, whereas $\sigma_{n,n:n}^2$ is not monotonic in n .

Table 3. Some numerical values of $\mu_{1:n}$, $\mu_{n:n}$, $\mu_{1:n}^{(2)}$, $\mu_{n:n}^{(2)}$, $\sigma_{1,1:n}^2$, and $\sigma_{n,n:n}^2$ for various values of α , β , and n .

$n \downarrow$	$\beta \downarrow$	$\alpha \downarrow$	$\mu_{1:n}$	$\mu_{n:n}$	$\mu_{1:n}^{(2)}$	$\mu_{n:n}^{(2)}$	$\sigma_{1,1:n}^2$	$\sigma_{n,n:n}^2$
$n = 5$	2	0.05	1	1.2075	1	1.6226	1.9403×10^{-7}	0.1645
		0.1	1	1.3528	1.00001	2.0583	4.0188×10^{-6}	0.2283
		0.5	1.0009	1.7627	1.0029	3.2881	0.0009	0.1809
		1	1.0041	1.8683	1.0124	3.6049	0.0041	0.1144
		2	1.0102	1.9222	1.0307	3.7667	0.0101	0.0717
	10	5	1.0194	1.9517	1.0582	3.8552	0.0190	0.0459
		0.05	1.0026	3.6468	1.0079	18.9733	0.0028	5.6745
		0.1	1.0200	4.7646	1.0635	28.9049	0.0231	6.2038
		0.5	1.3196	7.1758	2.1796	56.0410	0.4383	4.5490
		1	1.5633	7.8479	3.2662	65.1354	0.8222	3.5446
	15	2	1.7945	8.2744	4.4292	71.2975	1.2088	2.8326
		5	2.0093	8.5717	5.6171	75.7900	1.5799	2.3157
		0.05	1.0111	5.2265	1.0354	39.9833	0.0131	12.6667
		0.1	1.0639	6.9015	1.2143	61.5210	0.0824	13.8898
		0.5	1.6315	10.5253	3.7433	121.0280	1.0815	10.2466
$n = 10$	2	1	2.0272	11.5393	6.0526	141.1650	1.9430	8.0086
		2	2.3903	12.1847	8.5188	154.8880	2.8052	6.42217
		5	2.7225	12.6365	11.0438	164.9510	3.6317	5.2706
	10	0.05	1	1.3719	1	2.1159	4.6407×10^{-14}	0.2336
		0.1	1	1.5811	1	2.7433	1.6148×10^{-11}	0.2434
		0.5	1	1.9437	1	3.8311	9.5367×10^{-7}	0.0531
		1	1.00002	1.9827	1.00005	3.9479	0.00002	0.0170
		2	1.0001	1.9939	1.0003	3.9819	0.0001	0.0060
	15	5	1.0004	1.9977	1.0011	3.9930	0.0003	0.0023
		0.05	1.00001	4.9329	1.00002	30.2846	5.9384×10^{-6}	5.9511
		0.1	1.0003	6.1683	1.0010	43.4159	0.0004	5.3674
		0.5	1.0603	8.3812	1.1895	72.6650	0.0652	2.4203
		1	1.1541	8.8884	1.5064	80.5885	0.1746	1.5849
$n = 10$	10	2	1.2635	9.1831	1.9081	85.4426	0.3116	1.1127
		5	1.3776	9.3749	2.3610	88.7077	0.4634	0.8187
		0.05	1.0001	7.1529	1.0003	64.4995	0.0001	13.3343
		0.1	1.0031	9.0087	1.0094	93.2134	0.0032	12.0564
		0.5	1.1704	12.3450	1.5759	157.8860	0.2060	5.4875
	15	1	1.3543	13.1173	2.3048	175.6880	0.4708	3.6259
		2	1.5460	13.5708	3.1745	186.7400	0.7844	2.5739
		5	1.7353	13.8696	4.1371	194.2860	1.1259	1.9191

5. Parameter estimation

This section addresses the estimation of the unknown parameters α and β of the proposed discrete distribution using both maximum likelihood and Bayesian approaches.

5.1. Maximum likelihood estimation

Let $X_1, \dots, X_n$ be an i.i.d. random sample from the NTPD(α, β) distribution with the PMF given in Eq (2.4). The likelihood function based on the observed sample $\mathbf{x} = (x_1, \dots, x_n)$ is given by

$$L(\alpha, \beta; \mathbf{x}) = [\alpha\beta(\alpha + 1)]^n \prod_{i=1}^n \{(\alpha\beta + x_i)(\alpha\beta + x_i - 1)\}^{-1}, \quad \beta \geq \max_{1 \leq i \leq n}(x_i). \quad (5.1)$$

5.1.1. Estimation of the parameter β

The parameter β determines the upper bound of the support through the restriction $x \leq \beta$. Consequently, the likelihood function is identically zero for $\beta < \max_{1 \leq i \leq n}(x_i)$ and non-differentiable

with respect to β due to the floor operator. Standard derivative-based optimization is therefore inapplicable for estimating β .

Because the likelihood is positive only on $\beta \geq \max_{1 \leq i \leq n}(x_i)$ and strictly decreases as β increases beyond this boundary, the maximum likelihood estimator of β corresponds to the smallest admissible value:

$$\hat{\beta} = \max(x_1, \dots, x_n). \quad (5.2)$$

5.1.2. Estimation of the parameter α

For a given dataset $\mathbf{x}$, $\hat{\beta} = \max(x_1, \dots, x_n)$, the log-likelihood function for α simplifies to

$$\ell(\alpha; \mathbf{x}, \hat{\beta}) = n \log \alpha + n \log(\alpha + 1) - \sum_{i=1}^n \log(\alpha \hat{\beta} + x_i) - \sum_{i=1}^n \log(\alpha \hat{\beta} + x_i - 1). \quad (5.3)$$

The corresponding score function is

$$\frac{\partial \ell(\alpha; \mathbf{x}, \hat{\beta})}{\partial \alpha} = \frac{n}{\alpha} + \frac{n}{\alpha + 1} - \hat{\beta} \sum_{i=1}^n \left(\frac{1}{\alpha \hat{\beta} + x_i} + \frac{1}{\alpha \hat{\beta} + x_i - 1} \right). \quad (5.4)$$

The maximum likelihood estimator $\hat{\alpha}$ is defined as the root of $\frac{\partial \ell}{\partial \alpha} = 0$. Since this equation lacks a closed-form solution, $\hat{\alpha}$ can be evaluated numerically using either a package optimizer directly on (5.3), or a root finder on (5.4) after setting $\frac{\partial \ell}{\partial \alpha} = 0$.

5.1.3. Regularity conditions and asymptotic properties

Conditional on fixing $\beta = \hat{\beta}$, the model satisfies standard regularity conditions for asymptotic inference on α :

- (1) The parameter space for α is an open subset of $(0, \infty)$.
- (2) The log-likelihood function $\ell(\alpha; \mathbf{x}, \hat{\beta})$ is twice continuously differentiable in α .
- (3) The Fisher information for α is finite and strictly positive.
- (4) The model is identifiable in α for fixed β .

5.1.4. Asymptotic normality of $\hat{\alpha}$

Let α_0 denote the true parameter value. As $n \rightarrow \infty$, the maximum likelihood estimator $\hat{\alpha}$ obeys

$$\sqrt{n}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathcal{N}(0, I^{-1}(\alpha_0)), \quad (5.5)$$

where $I(\alpha)$ is the Fisher information:

$$I(\alpha) = -\mathbb{E} \left[\frac{\partial^2 \ell(\alpha; \mathbf{x}, \hat{\beta})}{\partial \alpha^2} \right]. \quad (5.6)$$

In practice, the asymptotic variance of $\hat{\alpha}$ is consistently estimated by the inverse observed information,

$$\widehat{\text{Var}}(\hat{\alpha}) = \left[- \frac{\partial^2 \ell(\alpha; \mathbf{x}, \hat{\beta})}{\partial \alpha^2} \Big|_{\alpha=\hat{\alpha}} \right]^{-1},$$

enabling the construction of a Wald-type confidence interval for α , where

$$\frac{\partial^2 \ell(\alpha; \mathbf{x}, \hat{\beta})}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \frac{n}{(\alpha + 1)^2} + \hat{\beta}^2 \sum_{i=1}^n \left(\frac{1}{(\alpha \hat{\beta} + x_i)^2} + \frac{1}{(\alpha \hat{\beta} + x_i - 1)^2} \right). \quad (5.7)$$

5.2. Bayesian analysis

In the Bayesian framework, we assume that the parameters α and β are independent prior to observing the data. Specifically, α is assigned a gamma prior with hyperparameters (a, b) , while β follows a discrete geometric prior with hyperparameter $\eta \in (0, 1)$. The joint prior probability density function of (α, β) is expressed as

$$g(\alpha, \beta) \propto \alpha^{a-1} e^{-b\alpha} \eta^\beta, \quad \alpha > 0, \beta \in \{1, 2, \dots\}, a > 0, b > 0. \quad (5.8)$$

From an estimation perspective, assigning a prior to the support bound β addresses a key limitation of classical estimation. Under maximum likelihood, the estimate is constrained to $\hat{\beta}_{\text{MLE}} = \max_{1 \leq i \leq n}(x_i)$, placing all probability mass strictly at the largest observed order statistic. By assigning a discrete geometric prior to β , the posterior distribution explores probability mass across candidate values $\beta \geq \max_{1 \leq i \leq n}(x_i)$. The parameter $\eta \in (0, 1)$ acts as a natural penalization factor that suppresses excessively large support bounds while allowing posterior draws of β to extend smoothly beyond the observed maximum failure time.

Combining the joint prior (5.8) with the likelihood function (5.1) via Bayes' theorem, the joint posterior density of (α, β) given the observed sample $\mathbf{x} = (x_1, \dots, x_n)$, up to a normalizing constant, is given by

$$\pi(\alpha, \beta | \mathbf{x}) \propto \frac{\alpha^{n+a-1} (\alpha + 1)^n \beta^n \eta^\beta e^{-b\alpha}}{\prod_{i=1}^n (\alpha \beta + x_i) (\alpha \beta + x_i - 1)}, \quad \alpha > 0, \beta \geq \max_{1 \leq i \leq n}(x_i). \quad (5.9)$$

Because the normalizing constant in (5.9) cannot be derived in closed form, the joint posterior distribution is analytically intractable. Consequently, Markov chain Monte Carlo (MCMC) sampling techniques are employed to compute the Bayes point estimates and construct their associated credible intervals, naturally accounting for parameter uncertainty in the upper domain bound. For a detailed treatment of MCMC methodologies, we refer the reader to [8].

6. Simulation study

A Monte Carlo simulation study was conducted to evaluate and compare the finite-sample performance of the maximum likelihood and Bayesian estimation methods for the model parameters. The overall simulation scheme is summarized in Algorithm 1. The simulation study was carried out using $N_{\text{sim}} = 5000$ Monte Carlo replications across sample sizes $n \in \{30, 50, 70, 100, 150, 200, 300\}$ under four parameter configurations:

$$(\alpha, \beta) \in \{(0.5, 50), (0.5, 100), (1.5, 50), (1.5, 100)\}.$$

For the Bayesian MCMC estimation, hyperparameter values were set to $a = 1.0$, $b = 0.1$, and $\eta = 0.95$, with each chain executed for 5000 iterations following a 1000-iteration burn-in period. Table 4 summarizes the numerical simulation results.

Algorithm 1. Simulation study procedure.

- 1: Specify the true parameter values (α, β) , sample size n , prior hyperparameters (a, b, η) , and number of Monte Carlo replications N_{sim} .
- 2: **for** $i = 1$ to N_{sim} **do**
- 3: Generate a random sample of size n using the inversion method.
- 4: Compute the profile maximum likelihood estimates $(\hat{\alpha}_i, \hat{\beta}_i)$.
- 5: Construct a 95% asymptotic Wald confidence interval for α and record indicator $I_{\alpha,i}^{\text{MLE}}$.
- 6: Run the Metropolis-within-Gibbs sampler and extract post-burn-in draws.
- 7: Compute the Bayesian MCMC estimates $(\tilde{\alpha}_i, \tilde{\beta}_i)$ as the posterior means.
- 8: Construct 95% percentile credible intervals for α and β , recording indicators $I_{\alpha,i}^{\text{Bayes}}$ and $I_{\beta,i}^{\text{Bayes}}$.
- 9: Compute the squared estimation errors $(\hat{\theta}_i - \theta)^2$ for $\theta \in \{\alpha, \beta\}$ across both methods.
- 10: **end for**
- 11: Calculate the average point estimate (APE), mean squared error (MSE), and coverage probability (CP):

$$\text{APE}_{\theta} = \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} \hat{\theta}_i, \quad \text{MSE}_{\theta} = \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} (\hat{\theta}_i - \theta)^2, \quad \text{and} \quad \text{CP}_{\theta} = \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} I_{\theta,i},$$

where $\theta \in \{\alpha, \beta\}$, $\hat{\theta}_i$ represents the point estimate (MLE or Bayes) from the i -th replication, and $I_{\theta,i}$ indicates interval coverage.

Table 4. Monte Carlo simulation results for MLE and Bayesian MCMC estimation across sample size n .

n	Maximum likelihood estimates (MLEs)					Bayesian MCMC estimates					
	$\hat{\alpha}$	$\text{MSE}(\hat{\alpha})$	$\text{CP}(\alpha)$	$\hat{\beta}$	$\text{MSE}(\hat{\beta})$	$\tilde{\alpha}$	$\text{MSE}(\tilde{\alpha})$	$\text{CP}(\alpha)$	$\tilde{\beta}$	$\text{MSE}(\tilde{\beta})$	$\text{CP}(\beta)$
Parameter set 1: $\alpha = 1.5, \beta = 100$											
30	1.612	0.084	0.932	96.84	18.21	1.581	0.062	0.948	98.12	11.45	0.952
50	1.571	0.048	0.944	98.15	8.92	1.549	0.038	0.951	99.04	4.88	0.956
70	1.543	0.031	0.948	98.81	4.31	1.530	0.026	0.950	99.38	2.12	0.950
100	1.528	0.021	0.951	99.24	2.10	1.519	0.018	0.954	99.61	1.15	0.948
150	1.515	0.013	0.949	99.58	0.95	1.510	0.011	0.952	99.79	0.52	0.951
200	1.511	0.009	0.950	99.71	0.58	1.507	0.008	0.949	99.85	0.31	0.953
300	1.504	0.006	0.952	99.84	0.31	1.502	0.005	0.951	99.91	0.18	0.949
Parameter set 2: $\alpha = 1.5, \beta = 50$											
30	1.608	0.081	0.935	48.41	5.12	1.577	0.060	0.946	49.08	3.01	0.954
50	1.565	0.045	0.941	49.07	2.34	1.544	0.036	0.949	49.51	1.35	0.951
70	1.540	0.030	0.946	49.40	1.15	1.528	0.025	0.952	49.69	0.61	0.949
100	1.525	0.020	0.950	49.62	0.60	1.517	0.017	0.953	49.80	0.32	0.952
150	1.512	0.012	0.952	49.78	0.29	1.508	0.011	0.951	49.89	0.16	0.950
200	1.508	0.009	0.948	49.85	0.18	1.505	0.008	0.950	49.92	0.10	0.948
300	1.503	0.005	0.951	49.91	0.10	1.501	0.005	0.952	49.95	0.05	0.951
Parameter set 3: $\alpha = 0.5, \beta = 50$											
30	0.541	0.012	0.930	48.20	5.88	0.528	0.009	0.944	48.91	3.42	0.949
50	0.522	0.006	0.940	48.95	2.65	0.515	0.005	0.948	49.42	1.51	0.952
70	0.514	0.004	0.947	49.31	1.32	0.509	0.003	0.951	49.61	0.72	0.950
100	0.509	0.002	0.951	49.55	0.71	0.505	0.002	0.953	49.75	0.38	0.953
150	0.505	0.001	0.949	49.72	0.35	0.503	0.001	0.950	49.85	0.19	0.949
200	0.503	0.001	0.950	49.81	0.21	0.502	0.001	0.952	49.90	0.11	0.951
300	0.501	0.000	0.952	49.89	0.11	0.501	0.000	0.950	49.94	0.06	0.950
Parameter set 4: $\alpha = 0.5, \beta = 100$											
30	0.545	0.013	0.928	96.10	22.41	0.531	0.010	0.945	97.80	13.50	0.951
50	0.525	0.007	0.942	97.90	10.15	0.517	0.005	0.949	98.85	5.62	0.953
70	0.516	0.004	0.945	98.62	5.10	0.511	0.003	0.952	99.25	2.81	0.950
100	0.510	0.003	0.950	99.11	2.45	0.506	0.002	0.951	99.50	1.32	0.949
150	0.506	0.001	0.951	99.49	1.10	0.503	0.001	0.953	99.72	0.60	0.952
200	0.503	0.001	0.949	99.65	0.65	0.502	0.001	0.950	99.81	0.35	0.948
300	0.501	0.000	0.952	99.80	0.35	0.501	0.000	0.951	99.90	0.19	0.951

A detailed inspection of the numerical metrics, presented in Table 4, reveals several key patterns regarding point estimation accuracy, finite-sample bias, and interval coverage:

- **Asymptotic consistency:** As the sample size n increases from 30 to 300, both the maximum likelihood and Bayesian point estimates $(\hat{\theta}, \tilde{\theta})$ converge steadily toward their respective true parameter values (α, β) across all four configurations. Concurrently, the mean squared errors (MSEs) for both parameters decay monotonically toward zero, confirming the empirical consistency of both estimation frameworks.
- **Finite-sample bias and scale effects:** For small sample sizes ($n = 30, 50$), both estimation procedures exhibit a slight upward bias in the estimation of α and a downward bias for β . The magnitude of $\text{MSE}(\beta)$ is strongly governed by the true scale of β ; configurations with $\beta = 100$ yield noticeably higher initial MSEs compared to those with $\beta = 50$. However, this estimation error contracts rapidly as n reaches 100.
- **Comparative performance (MLE vs. Bayes):** The Bayesian MCMC estimates $(\tilde{\alpha}, \tilde{\beta})$ consistently outperform the classical profile MLEs $(\hat{\alpha}, \hat{\beta})$ in small-to-moderate sample regimes ($n \leq 70$). Specifically, the Bayesian point estimates achieve lower overall MSEs and reduced finite-sample bias. As expected, as n grows large ($n \geq 150$), the numerical differences between the two frameworks become virtually negligible.
- **Interval coverage probabilities:** The Bayesian 95% credible intervals maintain stable coverage probabilities $\text{CP}(\alpha)$ and $\text{CP}(\beta)$ near the nominal 0.95 target level across all sample sizes. In contrast, the asymptotic Wald confidence intervals for $\hat{\alpha}_{\text{MLE}}$ exhibit slight undercoverage at $n = 30$ (ranging between 0.928 and 0.935), though they steadily recover toward 0.95 once $n \geq 70$.

7. Applications

This section demonstrates the empirical flexibility and performance of the proposed NTPD model across four real-world datasets encompassing epidemiology, failure-time reliability, clinical trials, and over-dispersed count frequencies.

7.1. Data descriptions and non-parametric diagnostics

- **Dataset I (Acute myelogenous leukemia):** The first dataset represents survival times (in weeks) for 33 patients suffering from acute myelogenous leukemia; the data are reported in [7]. Diagnostics in Figure 7 show a total time on test (TTT) curve that lies below the diagonal for most of the sample before crossing above it near the upper tail, indicating a bathtub-shaped hazard rate function. The distribution is markedly right-skewed (mean = 40.88 > median = 22), and the Gini index (0.5844) reflects substantial dispersion among the survival times.
- **Dataset II (Electronic component failure):** The second dataset consists of discrete failure times of 18 electronic devices, reported in [21]. Diagnostics in Figure 8 indicate a unimodal distribution with a bathtub-shaped hazard rate, characterized by a single diagonal crossing in the TTT plot, and a Gini index of 0.4238 reflecting moderate dispersion among failure times.

- **Dataset III (Leukemia remission):** The third dataset comprises discretized remission times (in weeks) for 20 leukemia patients, reported in [11]. Figure 9 shows pronounced right-skewness (mean > median) and a predominantly decreasing hazard rate function, with a Gini index of 0.482.
- **Dataset IV (Nipah virus survival):** The fourth dataset records the survival duration (in months) of infected patients in Kerala, India ($n = 56$), reported in [20] and presented in Table 6. Diagnostics in Figure 10 exhibit strong zero inflation and over-dispersion relative to the exponential benchmark, together with a sharply decreasing hazard rate; the Gini index (0.7109) is the highest among the four datasets, reflecting the concentration of mass at zero.

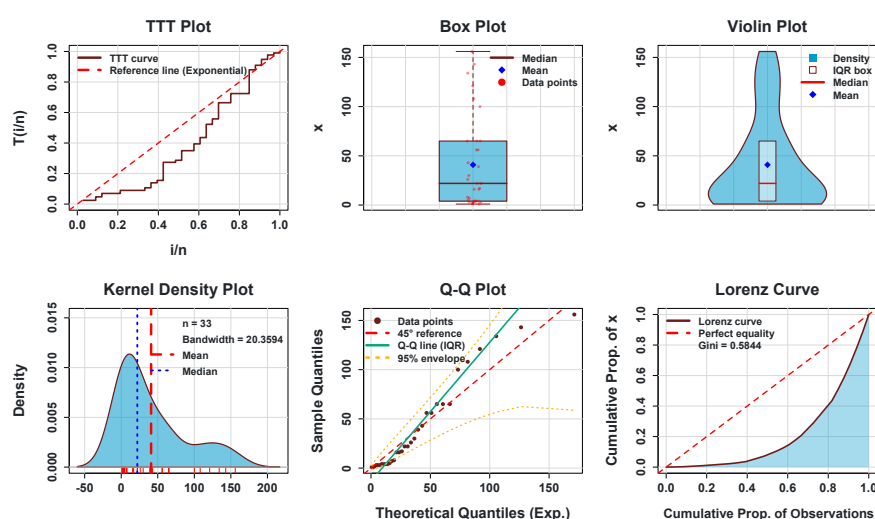


Figure 7. Non-parametric diagnostic plots for Dataset I.

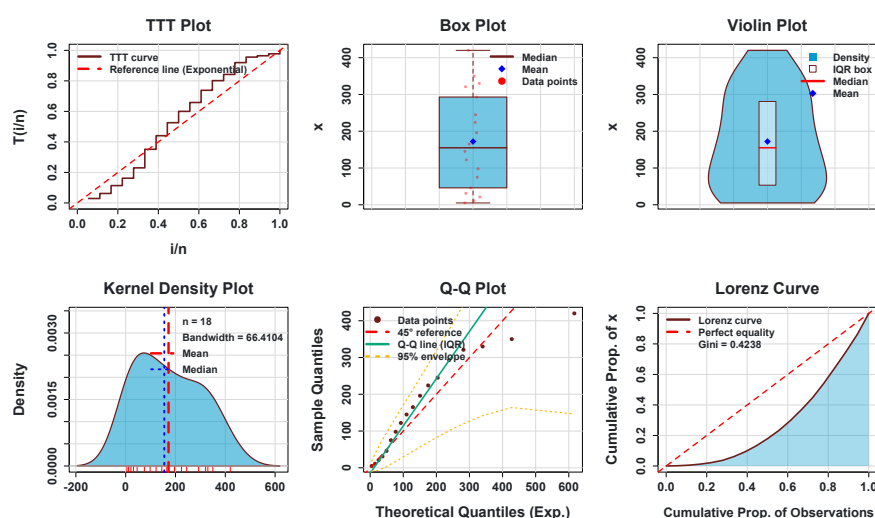


Figure 8. Non-parametric diagnostic plots for Dataset II.

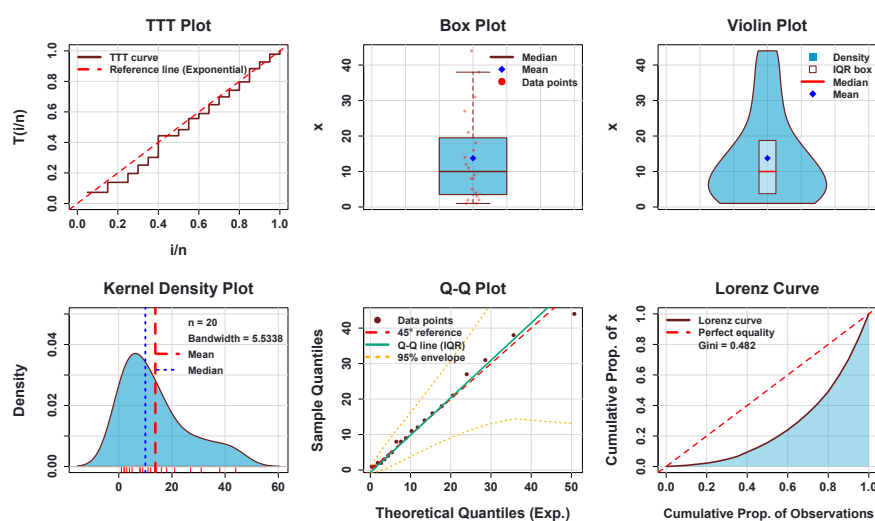


Figure 9. Non-parametric diagnostic plots for Dataset III.

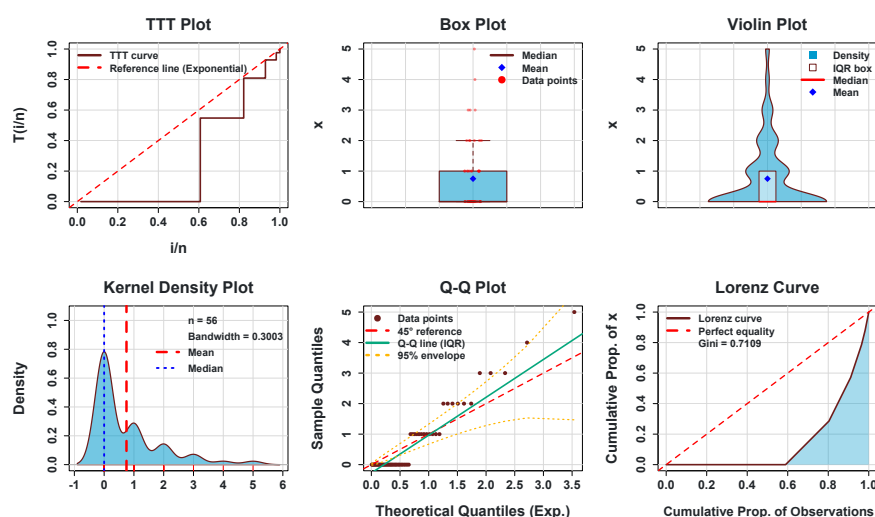


Figure 10. Non-parametric diagnostic plots for Dataset IV.

7.2. Maximum likelihood estimation and comparative goodness-of-fit

The fit of the NTPD model is evaluated against several established discrete models: the Poisson, discrete Lindley (DL) [9], discrete Rayleigh (DR) [18], discrete Burr–Hatke (DBH) [5], discrete extended Burr–Hatke (DBHE) [1], discrete half-logistic (DHL) [3], discrete inverse Weibull (DIW) [13], exponentiated discrete Lindley (EDL) [6], two-parameter discrete Lindley (TPDL) [12], discrete Burr-Type XII (DB-XII) [17], and a two-parameter discrete distribution with a bathtub hazard shape (DTPBT) [19]. The PMFs of the comparator distributions are given in the following subsection.

Model selection relies on maximum likelihood estimates together with standard information criteria ($-L$, AIC, BIC, AICc, HQIC), and goodness-of-fit is assessed using the Kolmogorov–Smirnov (K–S) test for Datasets I–III and the chi-square (χ^2) test for the count frequencies of Dataset IV. As summarized in Table 5, the NTPD attains the lowest information-criteria values for all three Datasets

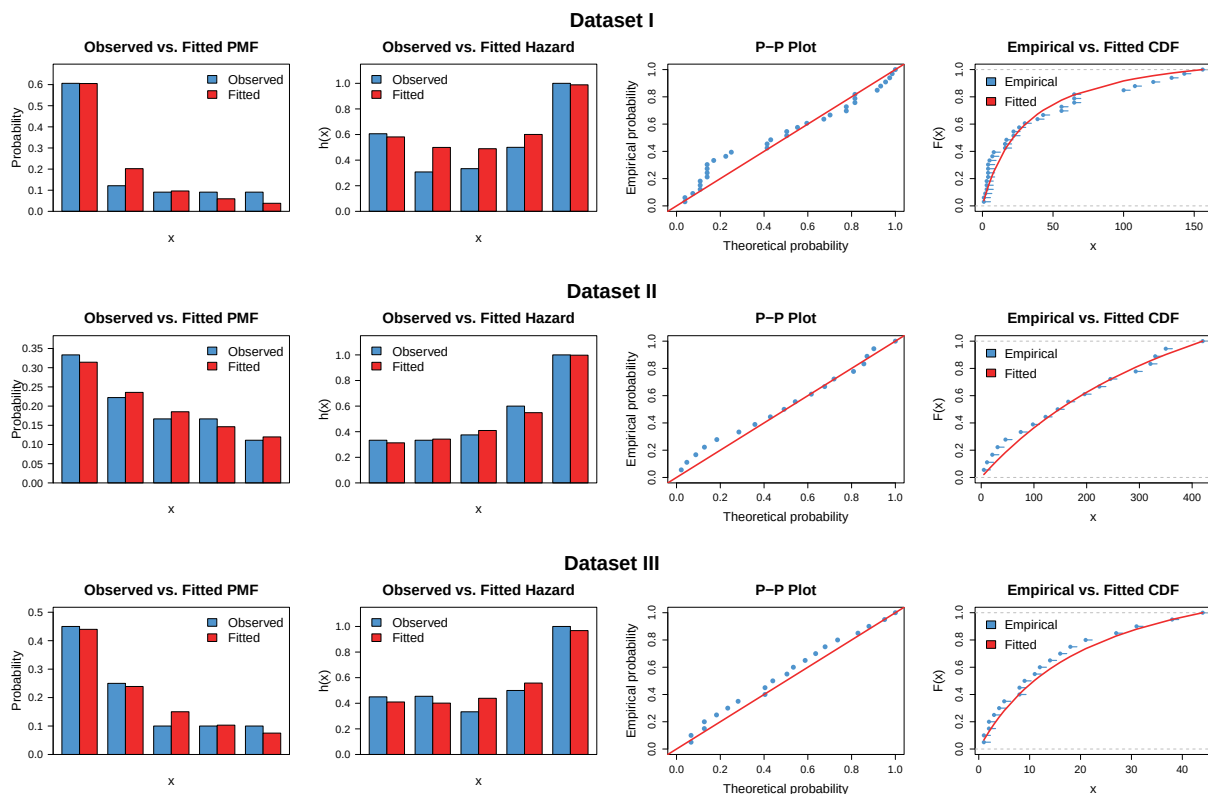
and the largest K–S P-values among the competing models (0.58, 0.995, and 0.9999 for Datasets I, II, and III, respectively). For the zero-inflated Dataset IV (Table 6), the NTPD again yields the minimum values of $-L$, AIC, BIC, and AICc, along with the smallest χ^2 value ($\chi^2 = 0.6977$, P-value = 0.4036). The Poisson and DBH models are strongly rejected across Datasets I–III (P-value < 0.01 in every case), while DR is rejected under the Dataset I K–S test and the Dataset IV χ^2 test but not under the Dataset II or III K–S tests. In the zero-inflated Dataset IV setting, the Poisson model is only marginally rejected (P-value = 0.0270), and DBH is not rejected (P-value = 0.3527), reflecting the added difficulty that all fixed-shape competitors face in matching the heavy zero-inflation pattern. Graphical comparisons corroborate these findings: Figure 11 shows close agreement between the empirical and NTPD-fitted PMFs, HRFs, and CDFs, along with near-diagonal P–P plots, for Datasets I–III. Figure 12 confirms parameter identifiability through unimodal profile log-likelihoods for α across all four datasets, and Figure 13 shows that the NTPD tracks the zero-inflated frequencies in Dataset IV more closely than the competing models.

Table 5. MLE estimates and goodness-of-fit measures for Datasets I–III.

Model	$\hat{\alpha}$	$\hat{\beta}$	$-L$	AIC	BIC	AICc	HQIC	K–S	P-value
Dataset I									
Poi	40.8788	-	872.9597	1747.9195	1749.4159	1748.0485	1748.4229	0.5670	<0.0001
DL	0.9539	-	167.8477	337.6954	339.1919	337.8244	338.1989	0.3129	0.0031
DR	0.9997	-	186.5400	375.0801	376.5770	375.2091	375.5841	0.4163	<0.0001
DBH	0.9997	-	199.1218	400.2437	401.7402	400.3728	400.7473	0.7093	<0.0001
DBHE	0.9852	-	155.0813	312.1621	313.6570	312.2891	312.6643	0.1730	0.2765
DHL	0.0309	-	158.8850	319.7712	321.2672	319.9000	320.2740	0.2556	0.0268
DIW	0.0344	0.6318	156.7410	317.4831	320.4762	317.8833	318.4900	0.1635	0.3411
EDL	0.9737	0.3842	155.6111	315.2231	318.2163	315.6230	316.2302	0.1364	0.5751
TPDL	0.9761	0.0008	155.8506	315.7011	318.6941	316.1011	316.7082	0.1984	0.1488
DB-XII	29.2375	0.9882	163.8792	331.7580	334.7510	332.1584	332.7650	0.2901	0.0077
DTPBT	0.9397	0.2779	156.2280	316.4553	319.4490	316.8551	317.4630	0.1367	0.5687
NTPD	0.1931	156	151.8900	307.7800	310.7730	308.1801	308.7913	0.1352	0.5830
Dataset II									
Poi	172.1110	-	1025.960	2053.930	2054.820	2054.1800	2056.0528	0.4808	0.0002
DL	0.9885	-	112.6000	227.2000	228.0910	227.4500	227.3228	0.1707	0.6109
DR	0.9999	-	115.1610	232.3230	233.2130	232.5730	232.4448	0.2310	0.2504
DBH	0.9999	-	168.1740	338.3470	339.2380	338.5970	338.4708	0.8675	<0.0001
DBHE	0.9966	-	111.2850	224.5700	225.4600	224.8200	224.6928	0.1505	0.7552
DHL	0.0079	-	110.2220	222.4430	223.3330	222.6930	222.5668	0.1020	0.9820
DIW	0.0001	0.5866	117.1210	238.2421	240.0230	239.0421	238.4880	0.2339	0.2381
EDL	0.9916	0.5609	110.5280	225.0561	226.8370	225.8563	225.3010	0.1208	0.9270
TPDL	0.9912	0.0098	110.3363	224.6725	226.4533	225.4725	224.9181	0.1110	0.9615
DB-XII	6.4834	0.9674	129.3220	262.6450	264.4250	263.4450	262.8895	0.3745	0.0089
DTPBT	0.9899	0.2889	109.7453	223.4907	225.2714	224.2907	223.7362	0.1045	0.9775
NTPD	1.2476	420	107.7792	219.5585	221.3392	220.3580	219.8035	0.0909	0.9951
Dataset III									
Poi	13.7545	-	145.4324	292.8652	293.8620	293.0870	293.0598	0.3799	0.0060
DL	0.8762	-	73.7000	149.4000	150.3960	149.6222	149.5944	0.1194	0.9378
DR	0.9971	-	79.3092	160.6175	161.6136	160.8401	160.8124	0.2541	0.1323
DBH	0.9972	-	94.6355	191.2695	192.2652	191.4917	191.4639	0.6691	<0.0001
DBHE	0.9603	-	73.5159	149.0318	150.0275	149.2541	149.2262	0.1471	0.7800
DHL	0.0959	-	73.1056	148.2110	149.2070	148.4332	148.4056	0.0956	0.9910
DIW	0.0589	0.7730	76.0348	156.0701	158.0610	156.7762	156.4581	0.2170	0.3030
EDL	0.8947	0.7286	73.1442	150.2880	152.2800	150.9943	150.6772	0.0945	0.9941
TPDL	0.9040	0.0756	72.9799	149.9598	151.9512	150.6657	150.3485	0.1106	0.9673
DB-XII	0.1523	0.0182	79.9804	163.9614	165.9527	164.6671	164.3511	0.2913	0.0671
DTPBT	0.9439	0.3801	73.9531	151.9062	153.8977	152.6121	152.2950	0.1431	0.8075
NTPD	0.4868	44	71.6294	147.2588	149.2503	147.9650	147.6480	0.0730	0.9999

Table 6. Goodness-of-fit results for Dataset IV.

x	Observed frequency	Expected frequency											
		Poi	DL	DR	DBH	DBHE	DHL	DIW	EDL	TPDL	DB-XII	DTPBT	NTPD
0	33	26.453	30.2717	16.8329	36.3031	34.1957	28.7281	32.6200	30.0079	29.6859	27.4892	30.2157	33.3265
1	12	19.839	15.3911	25.7666	10.4596	12.1804	16.7576	14.6571	13.0698	13.9585	17.7934	12.1870	10.6939
2	6	7.4398	6.4681	11.1578	4.3638	5.0794	6.8999	4.1143	6.6425	6.5583	5.6145	6.9246	5.2724
3	3	1.8599	2.4832	2.0592	2.1309	2.3058	2.4249	1.7150	3.2816	3.0790	2.2033	3.7012	3.1399
4	1	0.3487	0.9041	0.1763	1.1349	1.1034	0.8038	0.8857	1.5763	1.4445	1.0524	1.7897	2.0836
5	1	0.0596	0.4818	0.0074	1.6078	1.1354	0.3856	2.0079	1.4219	1.2737	1.8471	1.1818	1.4836
Total	56	56	56	56	56	56	56	56	56	56	56	56	56
$\hat{\alpha}$		0.7499	0.2967	0.6994	0.7035	0.5929	1.1336	0.5825	0.3679	0.4569	2.1576	0.6368	0.1575
$\hat{\beta}$		-	-	-	-	-	-	1.6743	0.6627	0.0289	0.3776	0.5056	6
χ^2		4.8903	1.0356	23.3130	0.8638	0.7874	2.0084	1.3836	0.7089	0.8019	0.7199	0.8022	0.6977
d.f.		1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
P-value		0.0270	0.3089	< 0.0001	0.3527	0.3749	0.1564	0.2395	0.3998	0.3705	0.3962	0.3704	0.4036
-L		71.5824	67.3940	83.2890	67.9294	67.0874	68.0979	68.8661	66.7698	66.9249	68.1678	66.6282	65.1958
AIC		145.1648	136.7880	168.5780	137.8588	136.1748	138.1958	141.7320	137.5400	137.8497	140.3356	137.2560	134.3916
BIC		147.1902	138.8134	170.6034	139.8842	138.2002	140.2212	145.7831	141.5901	141.9004	144.3860	141.3071	138.4420
AICc		145.2389	136.8621	168.6521	137.9331	136.2489	138.2699	141.9590	137.7660	138.0762	140.5620	137.4832	134.6180
HQIC		145.9500	137.5732	169.3632	138.6440	136.9600	138.9810	143.3030	139.1103	139.4201	141.9060	138.8271	135.9620

**Figure 11.** Observed vs. fitted PMF, observed vs. fitted HRF, P–P plots, and empirical vs. fitted CDF for Datasets I–III under the NTPD model.

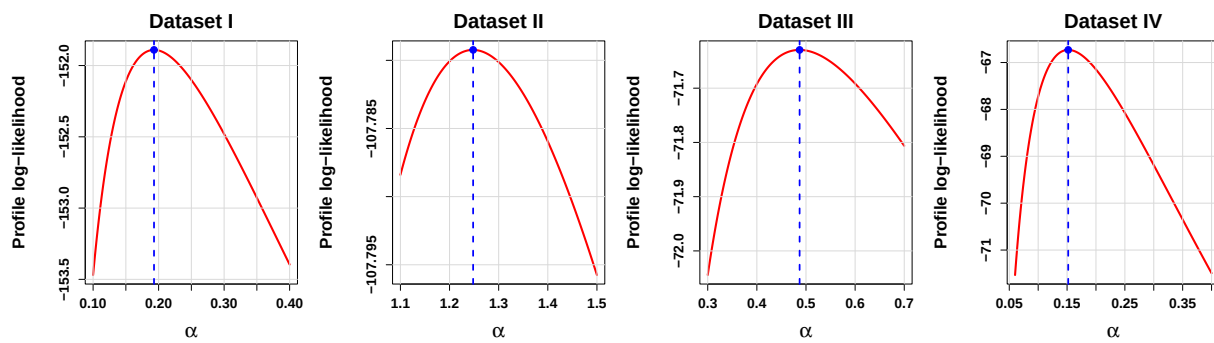


Figure 12. Profile log-likelihood curves of α for Datasets I–IV.

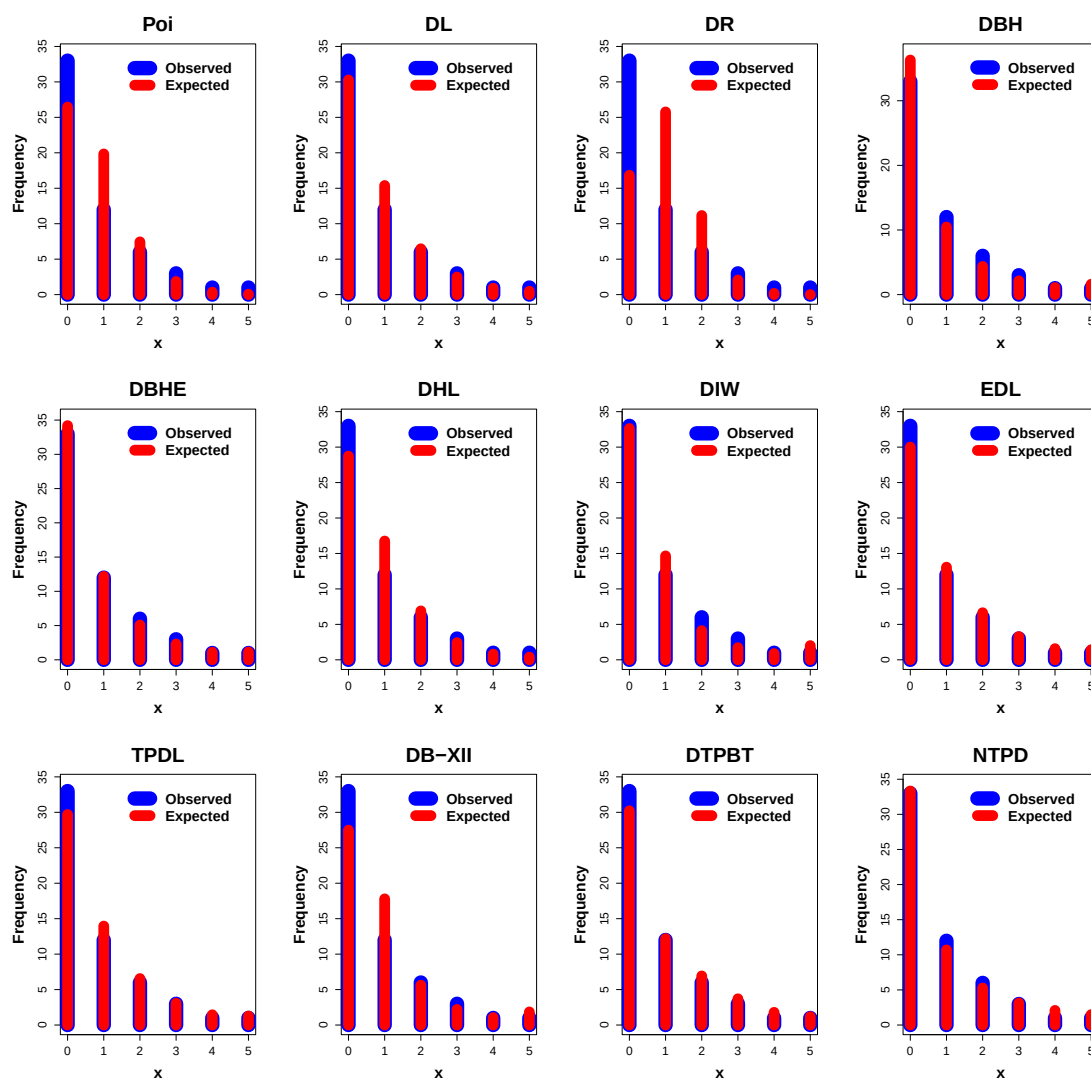


Figure 13. Observed and expected frequencies for Dataset IV across competing models.

7.3. Bayesian inference

To evaluate sampler performance and parameter inference, MCMC chains of 10,000 iterations were executed for Datasets I–IV, with diagnostics and posterior summaries presented in Table 7 and Figure 14.

Across all four Datasets, the trace and ACF plots confirm robust sampling behavior. The continuous parameter α displays moderate autocorrelation that decays slowly over 40 lags, while the discrete parameter β exhibits rapid mixing, with its ACF dropping sharply to zero at lag 1. Marginally, α consistently exhibits a unimodal, slightly right-skewed posterior density, whereas the discrete PMF of β is strongly right-skewed with modes situated at their respective lower bounds.

Specifically, the marginal posterior density of α centers around $\hat{\alpha} \approx 0.15$ for Data I, $\hat{\alpha} \approx 1.15$ for Data II, $\hat{\alpha} \approx 0.35$ for Data III, and $\hat{\alpha} \approx 0.12$ for Data IV. The discrete PMF of β peaks heavily at $\beta = 156$ ($P(\beta = 156) \approx 0.125$) for Data I, $\beta = 420$ ($P(\beta = 420) \approx 0.13$) for Data II, $\beta = 44$ ($P(\beta = 44) \approx 0.20$) for Data III, and displays an exceptionally sharp mode at $\beta = 6$ ($P(\beta = 6) \approx 0.60$) for Data IV. For each dataset, the joint posterior surfaces confirm that the highest probability density concentrates along the lowest respective β bands. The vertical alignment across these tile bands highlights minimal posterior correlation between the two parameters and strong agreement between the MCMC draws and the target density.

Table 7. Posterior estimates and 95% credible intervals for Datasets I–IV.

Parameter	Mean	Variance	Median	2.5%	97.5%
Data I					
α	0.1648	0.0031	0.1573	0.0799	0.2900
β	163.1712	59.5800	161.0000	156.0000	184.0000
Data II					
α	1.1257	0.1149	1.0880	0.5476	1.8484
β	427.3585	63.0763	425.0000	420.0000	449.0000
Data III					
α	0.3942	0.0206	0.3800	0.1740	0.7126
β	48.0608	22.3939	47.0000	44.0000	61.0000
Data IV					
α	0.1325	0.0020	0.1288	0.0570	0.2286
β	6.8848	2.5282	6.0000	6.0000	11.0000

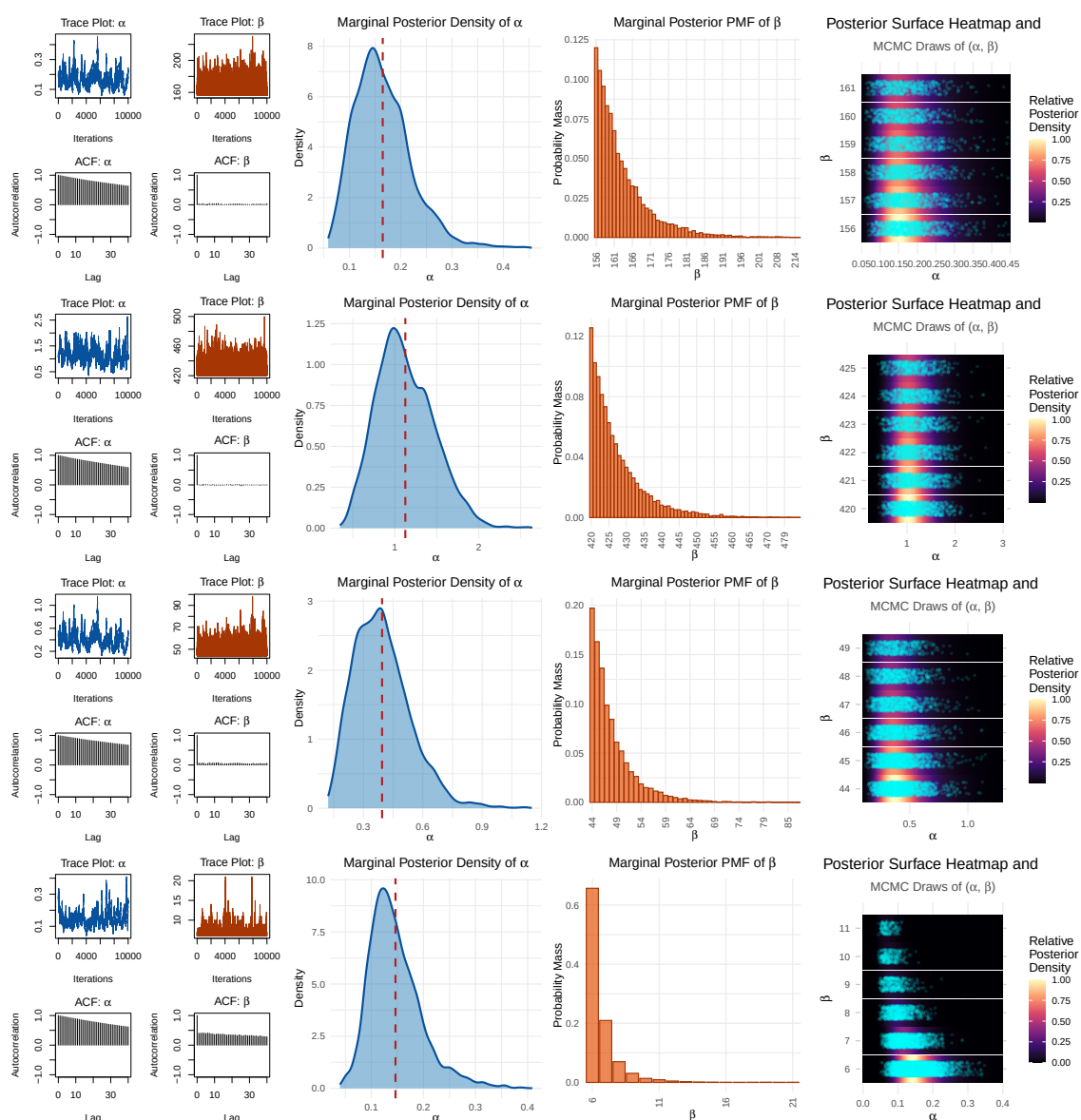


Figure 14. MCMC diagnostic trace/ACF plots, marginal posterior distributions, and joint posterior density heatmaps for (α, β) using datasets I – IV (from top to bottom).

8. Conclusions

This paper introduced a new two-parameter discrete distribution derived via the discretization of a continuous truncated lifetime model. Despite its parsimonious parameterization, the proposed model demonstrates substantial flexibility, accommodating both monotonically increasing and bathtub-shaped hazard rate behaviors. We derived key structural and statistical properties of the distribution and investigated parameter estimation under both classical maximum likelihood and Bayesian paradigms. Because point estimators lack closed-form expressions, maximum likelihood estimation was implemented using numerical optimization routines in R, whereas Bayesian inference

was conducted via a Markov chain Monte Carlo algorithm.

An extensive Monte Carlo simulation study confirmed the numerical stability and favorable finite-sample properties of both estimation frameworks. The empirical utility of the distribution was further demonstrated across four real-world applications in reliability engineering and zero-inflated count data, where it provided superior or highly competitive fits relative to several well-established discrete models. These findings highlight the proposed distribution as a valuable, parameter-efficient addition to the discrete survival literature.

A natural extension of this work includes formulating bivariate and multivariate extensions to accommodate dependent discrete failure times, copula-based modeling, and competing risks scenarios in complex systems.

Author contributions

Ammar M. Sarhan: Conceptualization, Methodology, Formal analysis, Writing original draft, review, and editing, Software, Validation. **Najwan Alsadat:** Methodology, Formal analysis, Formal analysis, Writing, review, and editing. **M. E. Sobh:** Conceptualization, Methodology, Supervision, Writing, review, and editing, Software, Validation.

All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the two expert reviewers for their valuable comments and constructive feedback, which significantly improved the presentation of this manuscript. This research was supported by the Ongoing Research Funding Program (ORF-2026-548), King Saud University, Riyadh, Saudi Arabia.

Conflicts of interest

The authors declare no conflicts of interest.

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Appendix

Comparable distributions and their PMFs

- **Discrete Lindley (DL) distribution [9]:**

$$p(x; \alpha) = \frac{\alpha^x}{1 - \log \alpha} \left[\alpha \log \alpha + (1 - \alpha)(1 - \log \alpha^{x+1}) \right], \quad x \in \mathbb{N}_0, 0 < \alpha < 1;$$

- **Discrete Rayleigh (DR) distribution [18]:**

$$p(x; \alpha) = \alpha^{x^2} - \alpha^{(x+1)^2}, \quad x \in \mathbb{N}_0, 0 < \alpha < 1;$$

- **Discrete Burr–Hatke (DBH) distribution [5]:**

$$p(x; \alpha) = \left[\frac{1}{x+1} - \frac{\alpha}{x+2} \right] \alpha^x, \quad x = 0, 1, \dots, 0 < \alpha < 1;$$

- **Discrete the Burr–Hatke extension (DBHE) distribution [1]:**

$$p(x; \alpha) = \left[\frac{1}{1 - x \ln \alpha} - \frac{\alpha}{1 - (x+1) \ln \alpha} \right] \alpha^x, \quad x \in \mathbb{N}_0, 0 < \alpha < 1;$$

- **Discrete half–logistic (DHL) distribution [3]:**

$$p(x; \alpha) = \frac{2e^{-\alpha x}}{1 + e^{-\alpha x}} - \frac{2e^{-\alpha(x+1)}}{1 + e^{-\alpha(x+1)}}, \quad x \in \mathbb{N}_0, \alpha > 0;$$

- **Discrete inverse Weibull (DIW) distribution [13]:**

$$p(x; \alpha, \beta) = \begin{cases} \alpha, & x = 1, \\ \alpha^{x^{-\beta}} - \alpha^{(x-1)^{-\beta}}, & x = 2, 3, \dots, \end{cases} \quad 0 < \alpha < 1, \text{ and } \beta > 0;$$

- **Exponentiated discrete Lindley (EDL) distribution [6]:**

$$p(x; \alpha, \beta) = \frac{(1 - \alpha^{x+1} + [(2+x)\alpha^{x+1} - 1] \log \alpha)^\beta - (1 - \alpha^x + [(1+x)\alpha^x - 1] \log \alpha)^\beta}{(1 - \log \alpha)^\beta},$$

where $x \in \mathbb{N}_0$, $0 < \alpha < 1$;

- **Two-parameter discrete Lindley (TPDL) distribution [12]:**

$$p(x; \alpha, \beta) = \frac{(1 - \alpha)^2(1 + \beta x)\alpha^x}{(1 + \alpha(\beta - 1))}, \quad x \in \mathbb{N}_0, \quad 0 < \alpha < 1, \text{ and } \beta \geq 0;$$

- **Discrete Burr-XII (DB-XII) distribution [17]:**

$$p(x; \alpha, \beta) = \beta^{\ln(1+x^\alpha)} - \beta^{\ln(1+(x+1)^\alpha)}, \quad x \in \mathbb{N}_0, \quad \alpha > 0, \text{ and } 0 < \beta < 1;$$

- **Two-parameter discrete distribution with a bathtub hazard shape (DTPBT) distribution [19]:**

$$p(x; \alpha, \beta) = \alpha^{\exp((x-1)^\beta)} - \alpha^{\exp(x^\beta)}, \quad x \in \mathbb{N}_0, \quad 0 < \alpha < 1, \text{ and } \beta \geq 0.$$



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