Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 2747–2796.
DOI: 10.3934/math.2026111
Received: 29 September 2025
Revised: 09 January 2026
Accepted: 23 January 2026
Published: 28 January 2026

Research article

Advances in the Shannon capacity of graphs

Nitay Lavi[1] **and Igal Sason**[1,2,*]

[1] The Viterbi Faculty of Electrical and Computer Engineering, Technion-Israel Institute of Technology, Haifa 3200003, Israel

[2] Faculty of Mathematics, Technion–Israel Institute of Technology, Haifa 3200003, Israel

* **Correspondence:** Email: eeigal@technion.ac.il; Tel: +97248294699.

Abstract: We derive exact values and new bounds for the Shannon capacity of two families of graphs: the q-Kneser graphs and the tadpole graphs. We also construct a countably infinite family of connected graphs whose Shannon capacity is not attained by the independence number of any finite strong power. Building on recent work of Schrijver, we establish sufficient conditions under which the Shannon capacity of a polynomial in graphs, formed via disjoint unions and strong products, equals the corresponding polynomial of the individual capacities, thereby reducing the evaluation of such capacities to that of their components. Finally, we prove an inequality relating the Shannon capacities of the strong product of graphs and their disjoint union, which yields alternative proofs of several known bounds as well as new tightness conditions. In addition to contributing to the computation of the Shannon capacity of graphs, this paper is intended to serve as an accessible entry point to those wishing to work in this area.

Keywords: Shannon capacity of graphs; zero-error information theory; strong graph product; graph polynomials; independence number; Kneser graphs; tadpole graphs
Mathematics Subject Classification: 05C35, 05C50, 05C69, 05C76, 94A15

1. Introduction

The Shannon capacity of a graph, introduced by Claude Shannon in 1956 [1], is a central concept bridging zero-error information theory and graph theory. It characterizes the maximum rate of zero-error communication over a noisy channel when the channel is modeled by a graph whose vertices represent the input symbols, and two vertices are adjacent if and only if the corresponding symbols can be confused by the channel with positive probability. This notion establishes a fundamental link between zero-error problems in information theory and graph theory, as surveyed extensively in [2–6].

The calculation of the Shannon capacity of a graph is notoriously difficult (see, e.g., [7–10]), and

only the capacity of a some families of graphs are known (e.g., Kneser graphs [11]). The computational complexity of the Shannon capacity has extended the research towards finding computable bounds on the capacity and exploring its properties [7, 11, 13–21].

While exploring properties of the Shannon capacity, it is worth keeping in mind the more general concept of the asymptotic spectrum of graphs, introduced by Zuiddam [20–23], which delineates a space of graph parameters that remain invariant under graph isomorphisms. This space is characterized by the following unique properties: additivity under disjoint union of graphs, multiplicativity under strong product of graphs, normalization for a simple graph with a single vertex, and monotonicity under graph complement homomorphisms. Building upon Strassen's theory of asymptotic spectra [24], a novel dual characterization of the Shannon capacity of a graph is derived in [22], expressed as the minimum over the elements of its asymptotic spectrum. By confirming that various graph invariants, including the Lovász ϑ-function [11] and the fractional Haemers' bound [14], are elements of the asymptotic spectrum of a graph (spectral points), it can be deduced that these elements indeed serve as upper bounds on the Shannon capacity of a graph. For further exploration, the comprehensive paper by Wigderson and Zuiddam [23] provides a survey on Strassen's theory of the asymptotic spectra and its application areas, including the Shannon capacity of graphs.

Several further properties of the Shannon capacity have been studied. For example, the recent work [25] analyzes the Shannon capacity of the distance-k power of a graph using tools from spectral graph theory and linear optimization methods. Despite this progress, many fundamental questions remain open. Among the most basic is the determination of the Shannon capacity of odd cycle graphs of length greater than 5 [11, 20, 26, 27].

This paper addresses several research directions in the study of the Shannon capacity of graphs, and it is organized as follows.

- Section 2 provides preliminaries that are required for the analysis in this paper. Its focus is on graph invariants, and classes of graphs that are used throughout this paper.
- Section 3 builds on a recent paper by Schrijver [28], and it explores conditions under which, for a family of graphs, the Shannon capacity of any polynomial in these graphs equals the corresponding polynomial of their individual Shannon capacities. This equivalence can substantially simplify the computation of the Shannon capacity for some of structured graphs. Two sufficient conditions are presented, followed by a comparison of their differences and illustrative examples of their use.
- Section 4 explores Tadpole graphs. Exact values and bounds on the capacity of Tadpole graphs are derived, and a direct relation between the capacity of odd-cycles and the capacity of a countably infinite subfamily of the Tadpole graphs is proved, providing an important property of that subfamily that is further discussed in the following section.
- Section 5 determines sufficient conditions for the unattainability of the Shannon capacity by the independence number of any finite strong power of a graph. It first presents in an alternative streamlined way an approach by Guo and Watanabe [10]. It then introduces two other original approaches. One of the novelties in this section is the construction of a countably infinite family of *connected* graphs whose capacity is unattainable by any finite strong power of these graphs.
- Section 6 determines the Shannon capacity of the q-Kneser graphs in a manner analogous to the Lovász calculation of the Shannon capacity of the classical Kneser graphs [11]. The derivation relies on two known results, namely a generalized Erdős–Ko–Rado theorem for finite vector spaces and characterizations of the spectrum of the q-Kneser graphs. This broadens the class of graphs

for which the exact Shannon capacity is explicitly determined.

- Section 7 introduces a new inequality that relates the Shannon capacity of the strong product of graphs to that of their disjoint union, and it identifies several conditions under which equality is attained. As an illustration of the applicability of this inequality, the section also derives an alternative proof of a lower bound on the Shannon capacity of the disjoint union of a graph and its complement, originally due to Alon [7], together with new sufficient conditions for its attainability, as well as analogous bounds for the Lovász ϑ-function.
- Section 8 outlines directions for future research that are largely related to the results of this paper.

2. Preliminaries

This section provides the preliminaries for our analysis and serves as an accessible introduction to the relevant topics.

2.1. Basic definitions and graph families

2.1.1. Terminology

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph, where $\mathsf{V} = \mathsf{V}(\mathsf{G})$ is the *vertex set* of G, and $\mathsf{E} = \mathsf{E}(\mathsf{G})$ is the *edge set* of G.

- An *undirected graph* is a graph whose edges are undirected.
- A *self-loop* is an edge that connects a vertex to itself.
- A *simple graph* is a graph having no self-loops and no multiple edges between any pair of vertices.
- A *finite graph* is a graph with a finite number of vertices.
- The *order* of a finite graph is the number of its vertices, $|\mathsf{V}(\mathsf{G})| = n$.
- The *size* of a finite graph is the number of its edges, $|\mathsf{E}(\mathsf{G})| = m$.
- Vertices $i, j \in \mathsf{V}(\mathsf{G})$ are *adjacent* if they are the endpoints of an edge in G, which is denoted by $\{i, j\} \in \mathsf{E}(\mathsf{G})$ or $i \sim j$.
- An *empty graph* is a graph without edges, so its size is equal to zero.
- The *degree of a vertex v* in G is the number of adjacent vertices to v in G, denoted by $\mathrm{d}_v = \mathrm{d}_v(\mathsf{G})$.
- A graph is *regular* if all its vertices have an identical degree.
- A *d-regular* graph is a regular graph whose all vertices have a fixed degree d.
- A *walk* is a sequence of vertices in a graph G, where every two consecutive vertices in the sequence are adjacent in G.
- A *path* is a walk with no repeated vertices.
- A *cycle* C is obtained from a path P by adding an edge connecting the two endpoints of P (i.e., an edge connecting the two degree-1 vertices of P).
- The *length of a path or a cycle* is equal to its number of edges. A *triangle* is a cycle of length 3.
- A path on n vertices is denoted by P_n, and its size is equal to $n - 1$.
- A cycle on n vertices is called an n-cycle, and it is denoted by C_n with an integer $n \geq 3$. The order and size of C_n are both equal to n.
- A *connected graph* is a graph where every two distinct vertices are connected by a path.
- An *r-partite graph* is a graph whose vertex set is a disjoint union of r subsets such that no two vertices in the same subset are adjacent. If $r = 2$, then G is a *bipartite graph* (e.g., C_n is a bipartite graph if and only if $n \geq 4$ is even).

- A *complete graph* on n vertices, denoted by K_n, is a graph whose all n distinct vertices are pairwise adjacent. Hence, K_n is an $(n-1)$-regular graph of order n.
- A *complete r-partite graph*, denoted by $\mathsf{K}_{n_1,\ldots,n_r}$ with $n_1, \ldots n_r \in \mathbb{N}$, is an r-partite graph whose vertex set is partitioned into r disjoint subsets of cardinalities $n_1, \ldots, n_r$, such that every two vertices in the same subset are not adjacent, and every two vertices in distinct subsets are adjacent.

Throughout this paper, graphs are finite, simple, and undirected. We also use the standard notation $[n] \triangleq \{1, \ldots, n\}$ for all $n \in \mathbb{N}$.

Definition 2.1 (Subgraphs and graph connectivity)**.** A graph F is a *subgraph* of a graph G, and it is denoted by $\mathsf{F} \subseteq \mathsf{G}$, if $\mathsf{V}(\mathsf{F}) \subseteq \mathsf{V}(\mathsf{G})$ and $\mathsf{E}(\mathsf{F}) \subseteq \mathsf{E}(\mathsf{G})$.

- A *spanning subgraph* of G is obtained by edge deletions from G, while its vertex set is left unchanged. A *spanning tree* in G is a spanning subgraph of G that forms a tree.
- An *induced subgraph* is obtained by removing vertices from the original graph, followed by the deletion of their incident edges.

Definition 2.2 (Isomorphic graphs)**.** Graphs G and H are *isomorphic* if there exists a bijection $f\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{H})$ (i.e., a one-to-one and onto mapping) such that $\{i, j\} \in \mathsf{E}(\mathsf{G})$ if and only if $\{f(i), f(j)\} \in \mathsf{E}(\mathsf{H})$. It is denoted by $\mathsf{G} \cong \mathsf{H}$, and f is said to be an *isomorphism* from G to H.

Definition 2.3 (Complement and self-complementary graphs)**.** The *complement* of a graph G, denoted by $\overline{\mathsf{G}}$, is a graph whose vertex set is $\mathsf{V}(\mathsf{G})$, and its edge set is the complement set $\overline{\mathsf{E}(\mathsf{G})}$. Every vertex in $\mathsf{V}(\mathsf{G})$ is nonadjacent to itself in G and $\overline{\mathsf{G}}$, so $\{i, j\} \in \mathsf{E}(\overline{\mathsf{G}})$ if and only if $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ with $i \neq j$. A graph G is *self-complementary* if $\mathsf{G} \cong \overline{\mathsf{G}}$ (i.e., G is isomorphic to $\overline{\mathsf{G}}$).

Example 2.4. It can be verified that P_4 and C_5 are self-complementary graphs.

2.1.2. Graph operations

This subsection presents the basic graph operations used throughout this paper.

Definition 2.5 (Strong product of graphs)**.** Let G and H be simple graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph whose vertices set is $\mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$, and two distinct vertices $(g_1, h_1), (g_2, h_2)$ are adjacent if, in each coordinate, they are either equal or adjacent. This means that one of the following three conditions is needed to be satisfied:

1. $g_1 = g_2$ and $\{h_1, h_2\} \in \mathsf{E}(\mathsf{H})$,
2. $\{g_1, g_2\} \in \mathsf{E}(\mathsf{G})$ and $h_1 = h_2$,
3. $\{g_1, g_2\} \in \mathsf{E}(\mathsf{G})$ and $\{h_1, h_2\} \in \mathsf{E}(\mathsf{H})$.

The interested reader is referred to [31] for an extensive textbook on graph products and their properties.

Define the *k-fold strong power* of G as the strong product of k copies of G:

$$\mathsf{G}^k \triangleq \mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}. \tag{2.1}$$

Throughout this paper, we also use the disjoint union of graphs as the addition operation on graphs.

Definition 2.6 (Disjoint union of graphs)**.** Let G and H be two simple graphs. The disjoint union $\mathsf{G} + \mathsf{H}$ is a graph whose vertices set is $\mathsf{V}(\mathsf{G}) \cup \mathsf{V}(\mathsf{H})$, and the edges set is $\mathsf{E}(\mathsf{G}) \cup \mathsf{E}(\mathsf{H})$. Also, for $m \in \mathbb{N}$, let

$$m\mathsf{G} \triangleq \mathsf{G} + \ldots + \mathsf{G}, \tag{2.2}$$

which is defined as the disjoint union of m copies of G.

2.1.3. Basic graph invariants under isomorphism

Definition 2.7 (Independent sets)**.** Let G be a simple graph. Define

- A set $\mathcal{I} \subseteq \mathsf{V}(\mathsf{G})$ is an independent set in G if every pair of vertices in $\mathcal{I}$ are nonadjacent in G.
- The set $\mathcal{I}(\mathsf{G})$ is the set of independent sets in G.

Now we define the independence number of a graph.

Definition 2.8 (Independence number)**.** The independence number of a graph G, denoted by $\alpha(\mathsf{G})$, is the order of a largest independent set in G, i.e.,

$$\alpha(\mathsf{G}) \triangleq \max\{|\mathcal{I}| : \mathcal{I} \in \mathcal{I}(\mathsf{G})\}. \tag{2.3}$$

Definition 2.9 (Cliques)**.** Let G be a simple graph. Define

- A set $C \subseteq \mathsf{V}(\mathsf{G})$ is a clique in G if every pair of vertices in C are adjacent in G.
- The set $\mathcal{C}(\mathsf{G})$ is the set of cliques in G.

Definition 2.10 (Clique number)**.** The clique number of a graph G, denoted by $\omega(\mathsf{G})$, is the order of a largest clique in G, i.e.,

$$\omega(\mathsf{G}) \triangleq \max\{|C| : C \in \mathcal{C}(\mathsf{G})\}. \tag{2.4}$$

Definition 2.11 (Chromatic number)**.** The chromatic number of a graph G, denoted by $\chi(\mathsf{G})$, is the smallest cardinality of a partition of the vertex set of G into independent sets. Equivalently, it is the minimum number of colors needed to color the vertices of G such that the endpoints of every edge receive distinct colors.

Definition 2.12 (Clique-cover number)**.** The clique-cover number of G, denoted by $\sigma(\mathsf{G})$, is the smallest number of cliques needed to cover all the vertices of G. Hence, $\sigma(\mathsf{G}) = \chi(\overline{\mathsf{G}})$.

Next, we provide required properties of the independence number of a graph.

Theorem 2.13. *Let* G *and* H *be simple graphs. Then,*

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) \geq \alpha(\mathsf{G})\,\alpha(\mathsf{H}), \tag{2.5}$$

$$\alpha(\mathsf{G} + \mathsf{H}) = \alpha(\mathsf{G}) + \alpha(\mathsf{H}). \tag{2.6}$$

Theorem 2.14. *Let* G *be a simple graph, and let* H_1 *and* H_2 *be induced and spanning subgraphs of* G*, respectively. Then,*

$$\alpha(\mathsf{H}_1) \leq \alpha(\mathsf{G}) \leq \alpha(\mathsf{H}_2). \tag{2.7}$$

Furthermore, for every $k \in \mathbb{N}$*,*

$$\alpha(\mathsf{H}_1^k) \leq \alpha(\mathsf{G}^k) \leq \alpha(\mathsf{H}_2^k). \tag{2.8}$$

2.1.4. Fractional invariants of graphs

To properly define four fractional invariants of a graph, the following sets of functions are first introduced.

Definition 2.15. Let G be a simple graph. Define four sets of functions as follows:

- $\mathcal{F}_I(\mathsf{G})$ is the set of non-negative functions, $f\colon\ \mathsf{V}(\mathsf{G}) \to \mathbb{R}$, such that for every $\mathcal{I} \in \mathcal{I}(\mathsf{G})$,
$$\sum_{v\in\mathcal{I}} f(v) \le 1. \tag{2.9}$$
- $\mathcal{F}_C(\mathsf{G})$ is the set of non-negative functions, $f\colon\ \mathsf{V}(\mathsf{G}) \to \mathbb{R}$, such that for every $C \in C(\mathsf{G})$,
$$\sum_{v\in C} f(v) \le 1. \tag{2.10}$$
- $\mathcal{G}_I(\mathsf{G})$ is the set of non-negative functions, $g\colon\ \mathcal{I}(\mathsf{G}) \to \mathbb{R}$, such that for every $v \in \mathsf{V}(\mathsf{G})$,
$$\sum_{\mathcal{I}\in\mathcal{I}(\mathsf{G})\colon v\in\mathcal{I}} g(\mathcal{I}) \ge 1, \tag{2.11}$$
where the summation is over all independent sets in G that include the vertex v.
- $\mathcal{G}_C(\mathsf{G})$ is the set of non-negative functions, $g\colon\ C(\mathsf{G}) \to \mathbb{R}$, such that for every $v \in \mathsf{V}(\mathsf{G})$,
$$\sum_{C\in C(\mathsf{G})\colon v\in C} g(C) \ge 1, \tag{2.12}$$
where the summation is over all cliques in G that include the vertex v.

Next, four basic fractional invariants of graphs are defined by using linear programming.

Definition 2.16 (Fractional invariants of graphs)**.** For a finite, undirected, and simple graph G, fractional invariants of graphs are defined as follows.

- The fractional independence number of G is
$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sup\left\{\sum_{v\in\mathsf{V}(\mathsf{G})} f(v) :\ f \in \mathcal{F}_C(\mathsf{G})\right\}. \tag{2.13}$$
Namely, $\alpha_{\mathrm{f}}(\mathsf{G})$ is the supremum of the total weight assigned to the vertices of G, where nonnegative real weights are assigned so that for every clique of G, the sum of the weights of the vertices in that clique is at most 1.
- The fractional clique-cover number of G is
$$\sigma_{\mathrm{f}}(\mathsf{G}) = \inf\left\{\sum_{C\in C(\mathsf{G})} g(C) :\ g \in \mathcal{G}_C(\mathsf{G})\right\}. \tag{2.14}$$
Namely, $\sigma_{\mathrm{f}}(\mathsf{G})$ is the infimum of the total weight assigned to the cliques of G, where nonnegative real weights are assigned so that for every vertex of G, the sum of the weights of all cliques containing that vertex is at least 1.

- The fractional clique number of G is

$$\omega_{\mathrm{f}}(\mathsf{G}) = \sup\left\{ \sum_{v\in\mathsf{V}(\mathsf{G})} f(v) : \ f \in \mathcal{F}_I(\mathsf{G})\right\}. \tag{2.15}$$

  Namely, $\omega_{\mathrm{f}}(\mathsf{G})$ is the supremum of the total weight assigned to the vertices of G, where nonnegative real weights are assigned so that for every independent set of G, the sum of the weights of the vertices in that independent set is at most 1.
- The fractional chromatic number of G, denoted by $\chi_{\mathrm{f}}(\mathsf{G})$, is given by

$$\chi_{\mathrm{f}}(\mathsf{G}) = \inf\left\{ \sum_{\mathcal{I}\in\mathcal{I}(\mathsf{G})} g(\mathcal{I}) : \ g \in \mathcal{G}_I(\mathsf{G})\right\}. \tag{2.16}$$

  Namely, $\chi_{\mathrm{f}}(\mathsf{G})$ is the infimum of the total weight assigned to the independent sets of G, where nonnegative real weights are assigned so that for every vertex of G, the sum of the weights of all independent sets containing that vertex is at least 1.

The next theorem follows from strong duality in linear programming.

Theorem 2.17. *[37] Let G be a simple graph. Then,*

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sigma_{\mathrm{f}}(\mathsf{G}), \tag{2.17}$$

$$\chi_{\mathrm{f}}(\mathsf{G}) = \omega_{\mathrm{f}}(\mathsf{G}). \tag{2.18}$$

From now on, we will primarily use the fractional independence number and the fractional chromatic number. When computing these quantities, we will use both of their equivalent linear programming formulations.

Another useful property, which is immediate from Definition 2.16, is given next.

Theorem 2.18. *[37] For a simple graph G, the following holds:*

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \sigma_{\mathrm{f}}(\mathsf{G}) = \chi_{\mathrm{f}}(\overline{\mathsf{G}}) = \omega_{\mathrm{f}}(\overline{\mathsf{G}}). \tag{2.19}$$

Some properties of the fractional independence number are presented next.

Theorem 2.19. *[32] Let G and H be simple graphs. Then,*

$$\alpha_{\mathrm{f}}(\mathsf{G} \boxtimes \mathsf{H}) = \alpha_{\mathrm{f}}(\mathsf{G})\,\alpha_{\mathrm{f}}(\mathsf{H}), \tag{2.20}$$

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) \leq \alpha_{\mathrm{f}}(\mathsf{G})\,\alpha(\mathsf{H}). \tag{2.21}$$

2.1.5. Graph spectrum

Definition 2.20 (Adjacency matrix)**.** Let G be a simple undirected graph on n vertices. The *adjacency matrix* of G, denoted by $\mathbf{A} = \mathbf{A}(\mathsf{G})$, is an $n\times n$ symmetric matrix $\mathbf{A} = (\mathrm{A}_{i,j})$ where $\mathrm{A}_{i,j} = 1$ if $\{i, j\} \in \mathsf{E}(\mathsf{G})$, and $\mathrm{A}_{i,j} = 0$ otherwise (so, the entries in the principal diagonal of $\mathbf{A}$ are zeros).

Definition 2.21 (Graph spectrum)**.** Let G be a simple undirected graph on n vertices. The *spectrum* of G is defined as the multiset of eigenvalues of the adjacency matrix of G.

2.1.6. Some structured families of graphs

Vertex- and edge-transitivity, defined as follows, play an important role in characterizing graphs.

Definition 2.22 (Automorphism)**.** An *automorphism* of a graph G is an isomorphism from G to itself.

Definition 2.23 (Vertex-transitivity)**.** A graph G is said to be *vertex-transitive* if for every two vertices $i, j \in \mathsf{V}(\mathsf{G})$, there is an automorphism $f\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{G})$ such that $f(i) = j$.

Definition 2.24 (Edge-transitivity)**.** A graph G is *edge-transitive* if for every two edges $e_1, e_2 \in \mathsf{E}(\mathsf{G})$, there is an automorphism $f\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{G})$ that maps the endpoints of e_1 to the endpoints of e_2.

Definition 2.25 (Kneser graphs)**.** Let $[n]$ be the set with natural numbers from 1 to n, and let $1 \le r \le n$. The Kneser graph $\mathsf{K}(n, r)$ is the graph whose vertex set is composed of the different r-subsets of $[n]$, and every two vertices u, v are adjacent if and only if the respective r-subsets are disjoint.

Kneser graphs are vertex- and edge-transitive.

Definition 2.26 (Perfect graphs)**.** A graph G is perfect if for every induced subgraph H of G,

$$\omega(\mathsf{H}) = \chi(\mathsf{H}). \tag{2.22}$$

Definition 2.27 (Universal graphs)**.** A graph G is universal if for every graph H,

$$\alpha(\mathsf{G} \boxtimes \mathsf{H}) = \alpha(\mathsf{G})\,\alpha(\mathsf{H}). \tag{2.23}$$

Lemma 2.28. *If G is a universal graph, then G^k is universal for all $k \in \mathbb{N}$.*

Proof. This follows easily by Definition 2.27 and mathematical induction on k. □

A corollary by Hales [32] regarding the connection between the graph universality and fractional independence number is presented next.

Theorem 2.29. *[32] A graph G is universal if and only if $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$.*

Definition 2.30 (Strongly regular graphs)**.** A graph G is strongly regular with parameters $\mathsf{srg}(n, d, \lambda, \mu)$ if it satisfies the following:

- The order of G is n.
- G is d-regular.
- Every pair of adjacent vertices has exactly λ common neighbors.
- Every pair of distinct, nonadjacent vertices has exactly μ common neighbors.

Definition 2.31 (Paley graphs)**.** Let $q = p^n$ be a prime power with p prime, $n \in \mathbb{N}$, and $q \equiv 1 \bmod 4$. The Paley graph of order q, denoted by $P(q)$, is defined as follows:

- The vertex set of $P(q)$ is $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$.
- Two distinct vertices $a, b \in \mathbb{F}_q$ are adjacent if $a - b$ is a square in $\mathbb{F}_q$.

Theorem 2.32. *[33] Let $q \equiv 1$ mod 4 be a prime power, and let $P(q)$ be a Paley graph. Then, the following hold:*

- *$P(q)$ is a self-complementary graph.*
- *$P(q)$ is strongly regular with parameters $\mathsf{srg}(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.*
- *$P(q)$ is vertex-transitive.*
- *$P(q)$ is edge-transitive.*

2.2. The Shannon capacity of graphs

The Shannon capacity of a graph G was introduced in [1] as a measure of the largest effective alphabet size permitting zero-error communication over a given noisy channel. A discrete memoryless channel (DMC) with finite input and output alphabets is specified by a finite input set $\mathcal{X}$, a finite output set $\mathcal{Y}$, and a non-empty fan-out set $\mathcal{S}_x \triangleq \{y \in \mathcal{Y} : \ p_{Y|X}(y|x) > 0\}$ for each $x \in \mathcal{X}$. In each channel use, the sender transmits an input $x \in \mathcal{X}$, and the receiver observes an arbitrary output in the corresponding fan-out set $\mathcal{S}_x$. The channel can be represented by a *confusion graph* G that is defined as follows:

- $\mathsf{V}(\mathsf{G}) = \mathcal{X}$ represents the set of symbols of the channel's input alphabet.
- $\mathsf{E}(\mathsf{G})$ is the edge set of G, where two distinct vertices are adjacent if the corresponding two input symbols from $\mathcal{X}$ are not distinguishable by the channel, i.e., they can produce an identical output symbol with positive probability. In other words, two vertices in G are adjacent if the fan-out sets of the corresponding input symbols intersect, so

$$\mathsf{E}(\mathsf{G}) = \Big\{\{x, x'\} : \ x, x' \in \mathcal{X}, \ x \neq x', \ \mathcal{S}_x \cap \mathcal{S}_{x'} \neq \varnothing\Big\}. \tag{2.24}$$

The largest cardinality of a set of input symbols that can be communicated without error in a single use of the channel equals the independence number $\alpha(\mathsf{G})$. Indeed, in this single-use setting, the sender and the receiver agree in advance on an independent set $\mathcal{I}$ of a maximum size $\alpha(\mathsf{G})$, the sender transmits only inputs in $\mathcal{I}$, every received output is in the fan-out set of exactly one input in $\mathcal{I}$, and the receiver can correctly determine the transmitted input.

Consider the transmission of k-length strings over a channel, where the channel is used $k \geq 1$ times. The sender transmits a sequence $x_1, \ldots, x_k$, and the receiver observes an output sequence $y_1, \ldots, y_k$, where $y_i \in \mathcal{S}_{x_i}$ for all $i \in [k]$. In this setup, k uses of the channel are viewed as a single use of a super-channel whose input alphabet is $\mathcal{X}^k$, its output alphabet is $\mathcal{Y}^k$, and the fan-out set of $(x_1, \ldots, x_k) \in \mathcal{X}^k$ is given by the Cartesian product $\mathcal{S}_{x_1} \times \ldots \times \mathcal{S}_{x_k}$. Note that two distinct sequences $(x_1, \ldots, x_k), (x'_1, \ldots, x'_k) \in \mathcal{X}^k$ are distinguishable by the channel if and only if

$$S_{x_i} \cap S_{x'_i} = \varnothing \tag{2.25}$$

for *some* index $i \in [k]$. Thus, it is possible to represent the super-channel by the kth confusion graph that is defined as the k-fold strong power of G. Using the kth confusion graph, the largest amount of information that can be sent by k uses of the channel with error-free communication is given by the independence number of the kth confusion graph, i.e., it is equal to $\alpha(\mathsf{G}^k)$. The maximum information rate per symbol that is achieved by k uses of the channel, and with zero-error communication, is equal to

$$\frac{1}{k} \log \alpha(\mathsf{G}^k) = \log \sqrt[k]{\alpha(\mathsf{G}^k)}, \quad k \in \mathbb{N}. \tag{2.26}$$

Finally, by omitting the logarithm (as a monotonically increasing function) and supremizing over $k \in \mathbb{N}$, the Shannon capacity of the graph G is defined as follows.

Definition 2.33 (Shannon capacity)**.** Let G be a simple graph. The Shannon capacity of G is defined as

$$\Theta(\mathsf{G}) \triangleq \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^k)} \tag{2.27}$$

$$= \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^k)}, \tag{2.28}$$

where equality (2.28) holds by inequality (2.5) and Fekete's lemma.

Next, we provide several properties of the Shannon capacity, which are used throughout this paper.

Theorem 2.34. *Let* G *be a simple graph and let* $m \in \mathbb{N}$*. Then,*

$$\Theta(\mathsf{G}^m) = \Theta(\mathsf{G})^m. \tag{2.29}$$

Proof. Equality (2.29) holds by (2.28). □

Theorem 2.35. *Let* G_1 *and* G_2 *be simple graphs. Then,*

$$\Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) \geq \Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2). \tag{2.30}$$

Theorem 2.36 (Shannon's inequality)**.** *[1] Let* G_1 *and* G_2 *be simple graphs. Then,*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) \geq \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2). \tag{2.31}$$

Proof. An elegant proof of Shannon's inequality (2.31) is presented in [28]. □

In light of Definition 2.6, if G and H are the confusion graphs of two channels, then their disjoint union represents the sum of the channels corresponding to the situation where either one of the two channels may be used, a new choice being made for each transmitted symbol. The Shannon capacity of the disjoint union of graphs may be strictly larger than the sum of their capacities [7], disproving Shannon's conjecture in the following strong sense.

Theorem 2.37 (The Shannon capacity of disjoint union of graphs)**.** *[7, Theorem 1.1] For every* $k \in \mathbb{N}$*, there is a graph* G *so that the Shannon capacity of the graph and that of its complement* $\overline{\mathsf{G}}$ *satisfy* $\Theta(\mathsf{G}) \leq k$ *and* $\Theta(\overline{\mathsf{G}}) \leq k$*, whereas*

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq k^{(1+o(1))\frac{\ln k}{8 \ln\ln k}}, \tag{2.32}$$

and the $o(1)$*-term tends to zero as we let* k *tend to infinity.*

The following known result is central in this paper, so its short and nice proof from [28] is included here.

Theorem 2.38 (Duality theorem)**.** *[23, 28] Let* G *and* H *be simple graphs. Then,*

$$\Theta(\mathsf{G} + \mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H}) \iff \Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H}). \tag{2.33}$$

Proof. See Appendix A. □

Theorem 2.39. *Let* G *be a simple graph and let* $m \in \mathbb{N}$*. Then,*

$$\Theta(m\mathsf{G}) = m\,\Theta(\mathsf{G}). \tag{2.34}$$

Proof. Equality (2.34) follows from Theorem 2.38 by relying on equality (2.29). □

Theorem 2.40. *Let* G *be a simple graph, and let* H_1 *and* H_2 *be induced and spanning subgraphs of* G*, respectively. Then,*

$$\Theta(\mathsf{H}_1) \leq \Theta(\mathsf{G}) \leq \Theta(\mathsf{H}_2). \tag{2.35}$$

Proof. This holds by the definitions of induced and spanning subgraphs, together with (2.27). □

2.3. The Lovász ϑ-function of graphs

Before we present the Lovász ϑ-function of graphs, we define an orthogonal representation of a graph [11] (see also [12, Chapter 11]).

Definition 2.41 (Orthogonal representations)**.** Let G be a simple graph. An *orthogonal representation* of G in $\mathbb{R}^d$ assigns each vertex $i \in \mathsf{V}(\mathsf{G})$ to a nonzero vector $u_i \in \mathbb{R}^d$ such that, for every distinct nonadjacent vertices $i, j \in \mathsf{V}(\mathsf{G})$, the vectors u_i, u_j are orthogonal. An orthogonal representation is called an *orthonormal representation* if all the representing vectors of G have a unit length.

In an orthogonal representation of a graph G, distinct nonadjacent vertices are mapped to orthogonal vectors, but adjacent vertices may not be necessarily mapped to non-orthogonal vectors. If the latter condition also holds, then it is called a *faithful* orthogonal representation.

Definition 2.42 (Lovász ϑ-function)**.** Let G be a simple graph of order n. The Lovász ϑ-function of G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{u,\mathbf{c}} \max_{1\le i\le n} \frac{1}{(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2}, \tag{2.36}$$

where the minimum is taken over all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G and all unit vectors $\mathbf{c}$. The unit vector $\mathbf{c}$ attaining the minimum is called the handle of the orthonormal representation.

An orthonormal representation of the pentagon C_5, along with its handle $\mathbf{c}$, is shown in Figure 1. This figure is reproduced, with permission of the author [34], by combining Figures 1.3 and 1.4 in [12].

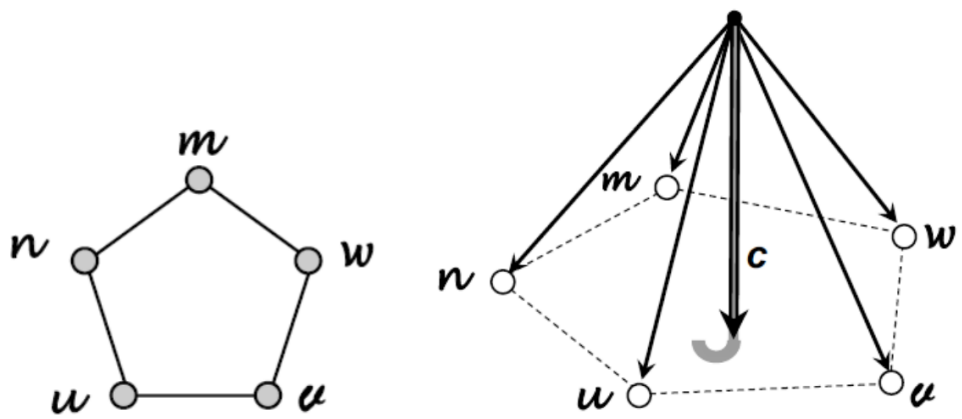

Figure 1. A 5-cycle graph and its orthonormal representation (Lovász umbrella).

The Lovász ϑ-function can be expressed as a solution of a semidefinite programming (SDP) problem. To that end, let $\mathbf{A} = (A_{i,j})$ be the $n \times n$ adjacency matrix of G with $n \triangleq |\mathsf{V}(\mathsf{G})|$. The Lovász ϑ-function $\vartheta(\mathsf{G})$ can be expressed by the following convex optimization problem:

$$\boxed{\begin{array}{l} \text{maximize} \quad \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}} \tag{2.37}$$

The SDP formulation in (2.37) yields the existence of an algorithm that computes $\vartheta(\mathsf{G})$, for every graph G, with a precision of r decimal digits, and a computational complexity that is polynomial in n and r. Thus, the Lovász ϑ-function can be computed in polynomial time in n, and by [11], it is an upper bound on the Shannon capacity, whose computation requires the computation of an infinite series of independence number, which is a known NP-hard problem.

Adding the inequality constraints $B_{i,j} \geq 0$ for all $i, j \in [n]$ to (2.37) yields the Schrijver ϑ-function of G, denoted by $\vartheta'(\mathsf{G})$, which therefore yields

$$\vartheta'(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.38}$$

In light of (2.38), one may ask whether the Schrijver ϑ-function can serve as a tighter upper bound on the Shannon capacity of graphs. However, a recent paper by the second author [35] shows that the Schrijver ϑ-function does not, in general, provide an upper bound on the Shannon capacity: An explicit graph is constructed whose Shannon capacity is strictly larger than its Schrijver ϑ-value. Nevertheless, the Schrijver ϑ-function is an upper bound on the independence number, and can therefore be used to derive an improved bound on the independence number.

Next, we provide an alternative representation of the Lovász ϑ-function of a graph [11].

Theorem 2.43. *[11] Let* $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ *range over all orthonormal representations of* $\overline{\mathsf{G}}$*, and let* $\mathbf{d}$ *range over all unit vectors. Then,*

$$\vartheta(\mathsf{G}) = \max \sum_{i=1}^{n} (\mathbf{d}^{\mathrm{T}} \mathbf{v}_i)^2. \tag{2.39}$$

Next, we provide several properties of the Lovász ϑ-function, regarding graph operations and subgraphs.

Theorem 2.44. *[11] Let* G_1 *and* G_2 *be simple graphs. Then,*

$$\vartheta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) = \vartheta(\mathsf{G}_1)\, \vartheta(\mathsf{G}_2). \tag{2.40}$$

The following result was first stated and proved by Knuth (Section 18 of [17]). We suggest an alternative elementary proof in Appendix B.

Theorem 2.45. *[17] Let* G *and* H *be simple graphs. Then,*

$$\vartheta(\mathsf{G} + \mathsf{H}) = \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \tag{2.41}$$

The following result is an easy consequence of Definition 2.42, and it is an analog of Theorem 2.40.

Theorem 2.46. *Let* G *be a simple graph, and let* H_1 *and* H_2 *be induced and spanning subgraphs of* G*, respectively. Then,*

$$\vartheta(\mathsf{H}_1) \leq \vartheta(\mathsf{G}) \leq \vartheta(\mathsf{H}_2). \tag{2.42}$$

Next, we present a few known formulas and bounds on the Lovász function.

Theorem 2.47. *[11, 36] Let* G *be a simple graph on n vertices. Then,*

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq n, \tag{2.43}$$

with an equality in (2.43) *if* G *is a vertex-transitive or strongly regular graph.*

Theorem 2.48. *[11] Let* G *be a d-regular graph of order n, and let* λ_n *be its smallest eigenvalue. Then,*

$$\vartheta(\mathsf{G}) \leq -\frac{n\lambda_n}{d-\lambda_n}, \tag{2.44}$$

with an equality in (2.44) *if* G *is an edge-transitive graph.*

2.4. Concluding preliminaries

We provide a useful lower bound for the fractional independence number and the fractional chromatic number (see [37, Proposition 3.1.1]).

Theorem 2.49. *Let* G *be a simple graph of order n. Then,*

$$\alpha_{\mathrm{f}}(\mathsf{G}) \geq \frac{n}{\omega(\mathsf{G})}, \tag{2.45}$$

$$\chi_{\mathrm{f}}(\mathsf{G}) \geq \frac{n}{\alpha(\mathsf{G})}. \tag{2.46}$$

Both inequalities hold with equality for vertex-transitive graphs.

The following sandwich theorem for the Lovász ϑ-function is remarkable in that it provides a computable bound on graph invariants that are NP-hard to compute.

Theorem 2.50. *Let* G *be a simple graph. Then,*

$$\alpha(\mathsf{G}) \leq \Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \alpha_{\mathrm{f}}(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}) = \sigma(\mathsf{G}). \tag{2.47}$$

In continuation to Definition 2.25, we provide some known invariants of Kneser graphs.

Theorem 2.51. *Let* $\mathsf{G} = \mathsf{K}(n, r)$ *be a Kneser graph with* $n \geq 2r$*. The invariants of* G *are*

$$\alpha(\mathsf{G}) = \binom{n-1}{r-1}, \tag{2.48}$$

$$\omega(\mathsf{G}) = \left\lfloor \frac{n}{r} \right\rfloor, \tag{2.49}$$

$$\Theta(\mathsf{G}) = \binom{n-1}{r-1}, \tag{2.50}$$

$$\vartheta(\mathsf{G}) = \binom{n-1}{r-1}, \tag{2.51}$$

$$\vartheta(\overline{\mathsf{G}}) = \frac{n}{r}, \tag{2.52}$$

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor}, \tag{2.53}$$

$$\chi_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r}, \tag{2.54}$$

$$\chi(\mathsf{G}) = n - 2r + 2, \tag{2.55}$$

$$\sigma(\mathsf{G}) = \left\lceil \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor} \right\rceil. \tag{2.56}$$

Proof. See [11, Theorem 13], [38], and Theorem 2.49. □

Theorem 2.52. *The path P_ℓ, for $\ell \in \mathbb{N}$, is a universal graph with parameters*

$$\alpha(\mathsf{P}_\ell) = \Theta(\mathsf{P}_\ell) = \vartheta(\mathsf{P}_\ell) = \alpha_{\mathrm{f}}(\mathsf{P}_\ell) = \sigma(\mathsf{P}_\ell) = \left\lceil \frac{\ell}{2} \right\rceil. \tag{2.57}$$

Theorem 2.53. *Let $k \geq 4$. Then, the cycle graph C_k has*

- *If k is an even number, then C_k is a universal graph with*

$$\alpha(\mathsf{C}_k) = \Theta(\mathsf{C}_k) = \vartheta(\mathsf{C}_k) = \frac{k}{2}. \tag{2.58}$$

- *If $k \geq 5$ is an odd number, then*

$$\alpha(\mathsf{C}_k) = \left\lfloor \frac{k}{2} \right\rfloor, \quad \vartheta(\mathsf{C}_k) = \frac{k}{1 + \sec \frac{\pi}{k}}. \tag{2.59}$$

Lemma 2.54. *If G is a universal graph, then $\Theta(\mathsf{G}) = \alpha(\mathsf{G})$.*

Proof. If G is a universal graph, then by Theorems 2.29 and 2.50,

$$\Theta(\mathsf{G}) = \alpha(\mathsf{G}). \tag{2.60}$$

□

Remark 2.55. The converse of Lemma 2.54 is in general false, see Example 2.57.

Corollary 2.56. *Let $\mathsf{G} = \mathsf{K}(n, r)$ be a Kneser graph with $n \geq 2r$. Then, G is universal if and only if $r \mid n$.*

Proof. By Theorem 2.51,

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor}. \tag{2.61}$$

For G to be a universal graph, the equality $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$ should hold (see Theorem 2.29). Thus, G is universal if and only if

$$\binom{n-1}{r-1} = \frac{\binom{n}{r}}{\left\lfloor \frac{n}{r} \right\rfloor}, \tag{2.62}$$

which holds if and only if $r \mid n$, as required. □

Example 2.57. The Petersen graph $\mathsf{G} = \mathsf{K}(5, 2)$ satisfies $\alpha(\mathsf{G}) = 4 = \Theta(\mathsf{G})$, yet it is not universal by Corollary 2.56.

3. The Shannon capacity of polynomials in graphs

The Shannon capacity of the disjoint union of graphs is at least the sum of their individual capacities [1]; however, it may in fact be significantly larger [7]. This phenomenon is strikingly demonstrated by a result of Alon [7, Theorem 1.1], which we restate here as Theorem 2.37. Moreover, Alon showed in [7, Theorem 1.2] that if we let G be the Schläfli graph, then

$$\Theta(\mathsf{G}) + \Theta(\overline{\mathsf{G}}) \leq 10 < 2\sqrt{27} \leq \Theta(\mathsf{G} + \overline{\mathsf{G}}). \tag{3.1}$$

We note that the simple counterexample constructed by Haemers [15] plays a key role in this argument and was later used by Alon [7] to establish (3.1).

We provide sufficient conditions on a sequence of graphs under which the Shannon capacity of their disjoint union equals the sum of their individual capacities. Our results rely on a recent result by Schrijver [28], which we include with proof due to its central role (see Theorem 2.38). It states that for any two simple graphs G and H,

$$\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H}) \iff \Theta(\mathsf{G} + \mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H}). \tag{3.2}$$

This result was independently proved by Wigderson and Zuiddam [23], with credit to Holzman for personal communication.

Let $\mathbb{N}[x_1, \ldots, x_\ell]$ be the set of nonzero polynomials in the variables $x_1, \ldots, x_\ell$ with nonnegative integer coefficients. A polynomial in graphs, $p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)$, is defined by interpreting graph addition as the disjoint union and graph multiplication as the strong product.

In this section, we derive sufficient conditions for a sequence of graphs $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ to satisfy the property that, for every polynomial $p \in \mathbb{N}[x_1, \ldots, x_\ell]$, the following equality holds:

$$\Theta(p(\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell)) = p(\Theta(\mathsf{G}_1), \Theta(\mathsf{G}_2), \ldots, \Theta(\mathsf{G}_\ell)). \tag{3.3}$$

By a corollary of the duality theorem (Theorem 2.38), which was proved by Schrijver [28], the following surprising result holds.

Lemma 3.1. *Let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ be a sequence of simple graphs. Then, equality* (3.3) *holds for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$ if and only if*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2) + \ldots + \Theta(\mathsf{G}_\ell). \tag{3.4}$$

By Lemma 3.1 (see [28, Theorem 3]), we focus on finding sufficient conditions for the satisfiability of (3.4). In this section, we present two main results that provide such sufficient conditions, discuss the differences between them, and illustrate their application. We start by providing the following result.

Theorem 3.2. *Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be simple graphs, with $\ell \in \mathbb{N}$. Then, for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$,*

$$p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell)) \leq \Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)) \leq p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_\ell)). \tag{3.5}$$

Proof. By Theorems 2.35 and 2.36,

$$p(\Theta(\mathsf{G}_1), \ldots, \Theta(\mathsf{G}_\ell)) \leq \Theta(p(\mathsf{G}_1, \ldots, \mathsf{G}_\ell)), \tag{3.6}$$

and by Theorems 2.44 and 2.45,

$$\begin{aligned}\Theta(p(\mathsf{G}_1,\ldots,\mathsf{G}_\ell)) &\leq \vartheta(p(\mathsf{G}_1,\ldots,\mathsf{G}_\ell))\\ &= p(\vartheta(\mathsf{G}_1),\ldots,\vartheta(\mathsf{G}_\ell)).\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) gives both inequalities in (3.5). □

Corollary 3.3. *Let* $\mathsf{G}_1,\ldots,\mathsf{G}_\ell$ *be simple graphs, and let* $p \in \mathbb{N}[x_1,\ldots,x_\ell]$ *with* $\ell \in \mathbb{N}$. *If* $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ *for every* $i \in [\ell]$, *then*

$$\Theta(p(\mathsf{G}_1,\ldots,\mathsf{G}_\ell)) = p(\vartheta(\mathsf{G}_1),\ldots,\vartheta(\mathsf{G}_\ell)). \tag{3.8}$$

Proof. By assumption, it follows that

$$p(\Theta(\mathsf{G}_1),\ldots,\Theta(\mathsf{G}_\ell)) = p(\vartheta(\mathsf{G}_1),\ldots,\vartheta(\mathsf{G}_\ell)). \tag{3.9}$$

Thus, equality (3.8) holds by Theorem 3.2. □

The next result readily follows.

Theorem 3.4. *Let* $\mathsf{G}_1,\ldots,\mathsf{G}_\ell$ *be simple graphs, with* $\ell \in \mathbb{N}$. *If* $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ *for every* $i \in [\ell]$, *then equality* (3.3) *holds for every* $p \in \mathbb{N}[x_1,\ldots,x_\ell]$.

Proof. This holds by Corollary 3.3 upon replacing $\Theta(\mathsf{G}_i)$ with $\vartheta(\mathsf{G}_i)$ for all $i \in [\ell]$. □

Example 3.5. If $\mathsf{G}_1,\ldots,\mathsf{G}_\ell$ are all Kneser graphs or self-complementary vertex-transitive graphs, then by Theorem 3.4, equality (3.3) holds for every polynomial $p \in \mathbb{N}[x_1,\ldots,x_\ell]$. In particular, if $\mathsf{G}_i = \mathsf{K}(n_i, r_i)$ (with $n_i \geq 2r_i$) for every $i \in [\ell]$, then by Theorem 2.51,

$$\Theta(\mathsf{K}(n_1,r_1) + \ldots + \mathsf{K}(n_\ell,r_\ell)) = \binom{n_1-1}{r_1-1} + \ldots + \binom{n_\ell-1}{r_\ell-1}. \tag{3.10}$$

Similarly, if G_i is a self-complementary and vertex-transitive graph for all $i \in [\ell]$, then by [39, Theorem 3.26],

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \sqrt{n_1} + \ldots + \sqrt{n_\ell}, \tag{3.11}$$

where n_i is the order of G_i.

Next, we give a second sufficient condition for equality (3.3) to hold.

Lemma 3.6. *Let* G_1 *and* G_2 *be simple graphs. If* $\Theta(\mathsf{G}_1) = \alpha_{\mathrm{f}}(\mathsf{G}_1)$, *then*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2). \tag{3.12}$$

Proof. By Theorem 2.35,

$$\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) \leq \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2), \tag{3.13}$$

and by Theorem 2.19,

$$\begin{aligned}\Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) &= \lim_{k\to\infty} \sqrt[k]{\alpha((\mathsf{G}_1 \boxtimes \mathsf{G}_2)^k)} \\ &\leq \lim_{k\to\infty} \sqrt[k]{\alpha_{\mathrm{f}}(\mathsf{G}_1^k)\,\alpha(\mathsf{G}_2^k)} \\ &= \alpha_{\mathrm{f}}(\mathsf{G}_1)\,\Theta(\mathsf{G}_2), \end{aligned} \tag{3.14}$$

where (3.14) holds since the equality $\alpha_{\mathrm{f}}(\mathsf{G}_1) = \sqrt[k]{\alpha_{\mathrm{f}}(\mathsf{G}_1^k)}$ is satisfied for all $k \in \mathbb{N}$, and by definition $\Theta(\mathsf{G}_2) = \lim_{k\to\infty} \sqrt[k]{\alpha(\mathsf{G}_2^k)}$. Combining inequalities (3.13) and (3.14) gives

$$\begin{aligned}\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) &\leq \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2) \\ &\leq \alpha_{\mathrm{f}}(\mathsf{G}_1)\,\Theta(\mathsf{G}_2). \end{aligned} \tag{3.15}$$

Hence, by the assumption,

$$\Theta(\mathsf{G}_1)\,\Theta(\mathsf{G}_2) = \Theta(\mathsf{G}_1 \boxtimes \mathsf{G}_2), \tag{3.16}$$

and equality (3.12) follows from Theorem 2.38. □

Theorem 3.7. *Let* $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ *be simple graphs. If* $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ *for (at least)* $\ell - 1$ *of these graphs, then equality* (3.3) *holds for every* $p \in \mathbb{N}[x_1, \ldots, x_\ell]$.

Proof. Without loss of generality, assume $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ for every $i \in [\ell - 1]$. By a recursive application of Lemma 3.6,

$$\Theta\left(\sum_{i=1}^{\ell} \mathsf{G}_i\right) = \Theta(\mathsf{G}_1) + \Theta\left(\sum_{i=2}^{\ell} \mathsf{G}_i\right) = \ldots = \sum_{i=1}^{\ell} \Theta(\mathsf{G}_i). \tag{3.17}$$

Thus, by Lemma 3.1, equality (3.3) holds for every $p \in \mathbb{N}[x_1, \ldots, x_\ell]$. □

It is worth noting that our sufficient conditions for equality (3.3) to hold are less restrictive than those hinted at in Shannon's paper [1], which are presented next. The following theorem is derived directly from Lemma 3.6 and the sandwich theorem. We provide the original proof in Appendix C.

Theorem 3.8. *Let* G_1 *and* G_2 *be simple graphs with* $\alpha(\mathsf{G}_1) = \chi(\overline{\mathsf{G}}_1)$. *Then,*

$$\Theta(\mathsf{G}_1 + \mathsf{G}_2) = \Theta(\mathsf{G}_1) + \Theta(\mathsf{G}_2). \tag{3.18}$$

Remark 3.9. The sufficient condition in Theorem 3.8 is more restrictive than the one in Lemma 3.6. This holds since by Theorem 2.50,

$$\alpha(\mathsf{G}) \leq \Theta(\mathsf{G}) \leq \alpha_{\mathrm{f}}(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}). \tag{3.19}$$

The next example suggests a family of graphs for which $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) < \chi(\overline{\mathsf{G}})$ holds for every graph G in that family, thus showing the possible applicability of Lemma 3.6 in cases where the conditions in Theorem 3.8 are not satisfied.

Example 3.10. Let $\mathsf{G} = \overline{\mathsf{K}(n,r)}$ be the complement of a Kneser graph where $n > 2r$, $r > 1$, and $r \mid n$. It is claimed that the graph G satisfies

$$\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) < \chi(\overline{\mathsf{G}}). \tag{3.20}$$

Indeed, the invariants of the Kneser graph and its complement are known, and by Theorem 2.51 (since $\alpha(\mathsf{G}) = \omega(\overline{\mathsf{G}})$, where $\overline{\mathsf{G}} = \mathsf{K}(n,r)$ with r and n as above),

$$\alpha(\mathsf{G}) = \frac{n}{r}, \tag{3.21}$$

$$\alpha_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r}, \tag{3.22}$$

$$\chi(\overline{\mathsf{G}}) = n - 2r + 2. \tag{3.23}$$

Since the independence number and the fractional independence number coincide and are both equal to $\frac{n}{r}$, then $\Theta(\mathsf{G}) = \frac{n}{r}$. Hence, the assumption $n > 2r$ implies that

$$\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G}) = \frac{n}{r} < n - 2r + 2 = \chi(\overline{\mathsf{G}}). \tag{3.24}$$

Remark 3.11. The sufficient conditions provided by Theorems 3.4 and 3.7 do not supersede each other (see Examples 3.12 and 3.13). More explicitly, for every graph G,

$$\Theta(\mathsf{G}) \le \vartheta(\mathsf{G}) \le \alpha_{\mathrm{f}}(\mathsf{G}), \tag{3.25}$$

so the condition $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$ implies that the equality $\Theta(\mathsf{G}) = \vartheta(\mathsf{G})$ holds. Hence, Theorem 3.7 imposes a stronger condition than the one in Theorem 3.4 on $\ell - 1$ of the graphs, while no condition is imposed on the ℓth graph.

The next two examples (Examples 3.12 and 3.13) show that neither Theorem 3.4 nor Theorem 3.7 implies the other.

Example 3.12. Let $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ be Kneser graphs, where

$$\mathsf{G}_i = \mathsf{K}(n_i, r_i), \quad n_i \ge 2r_i. \tag{3.26}$$

For these non-empty graphs, by Theorem 2.51,

$$\alpha(\mathsf{G}_i) = \Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i) = \binom{n_i - 1}{r_i - 1}, \quad \forall\, i \in [\ell], \tag{3.27}$$

and

$$\alpha_{\mathrm{f}}(\mathsf{G}_i) = \frac{\binom{n_i}{r_i}}{\left\lfloor \frac{n_i}{r_i} \right\rfloor}, \quad \forall\, i \in [\ell]. \tag{3.28}$$

If $r_i \nmid n_i$, then it follows from (3.27) and (3.28) that

$$\alpha_{\mathrm{f}}(\mathsf{G}_i) > \frac{\binom{n_i}{r_i}}{\frac{n_i}{r_i}} = \binom{n_i - 1}{r_i - 1} = \Theta(\mathsf{G}_i), \tag{3.29}$$

so, if $r_i \nmid n_i$, then $\alpha_{\mathrm{f}}(\mathsf{G}_i) > \Theta(\mathsf{G}_i)$.

Let the parameters $n_1, \ldots, n_\ell$ and $r_1, \ldots, r_\ell$ be chosen such that $r_i \nmid n_i$ for at least two of the Kneser graphs $\{\mathsf{G}_i\}_{i=1}^{\ell}$. For each such index $i \in [\ell]$, we have $\alpha_{\mathrm{f}}(\mathsf{G}_i) > \Theta(\mathsf{G}_i)$, which violates the sufficient conditions of Theorem 3.7. By (3.27), it follows from Theorem 3.4 that

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \sum_{i=1}^{\ell} \binom{n_i - 1}{r_i - 1}, \tag{3.30}$$

whereas equality (3.30) does not follow from Theorem 3.7.

Example 3.13. Let G_i be a perfect graph for every $i \in [\ell - 1]$, and let G_ℓ be the complement of the Schläfli graph. Then,

- $\Theta(\mathsf{G}_i) = \alpha_{\mathrm{f}}(\mathsf{G}_i)$ for every $i \in [\ell - 1]$.
- $\Theta(\mathsf{G}_\ell) < \vartheta(\mathsf{G}_\ell)$ (by [15]).

Hence, the sufficient conditions of Theorem 3.7 hold, in contrast to those of Theorem 3.4.

4. The Shannon capacity of Tadpole graphs

The present section explores Tadpole graphs, where exact values and bounds on their Shannon capacity are derived.

Definition 4.1 (Tadpole graphs)**.** Let $k, \ell \in \mathbb{N}$ with $k \geq 3$ and $\ell \geq 1$. The graph $\mathrm{T}(k, \ell)$, called the Tadpole graph of order (k, ℓ), is obtained by taking a cycle C_k of order k and a path P_ℓ of order ℓ, and then joining one pendant vertex of P_ℓ (i.e., one of its two vertices of degree 1) to a vertex of C_k by an edge. For completeness, if $\ell = 0$, the Tadpole graph is defined trivially as a cycle graph of order k, $\mathrm{T}(k, 0) = \mathsf{C}_k$.

Every Tadpole graph is a connected graph with

$$|\mathsf{V}(\mathrm{T}(k, \ell))| = |\mathsf{E}(\mathrm{T}(k, \ell))| = k + \ell. \tag{4.1}$$

The Tadpole graph $\mathrm{T}(k, \ell)$ with $\ell \geq 1$ is irregular since it has one vertex of degree 3, $k + \ell - 2$ vertices of degree 2, and one vertex of degree 1 (see Figure 2).

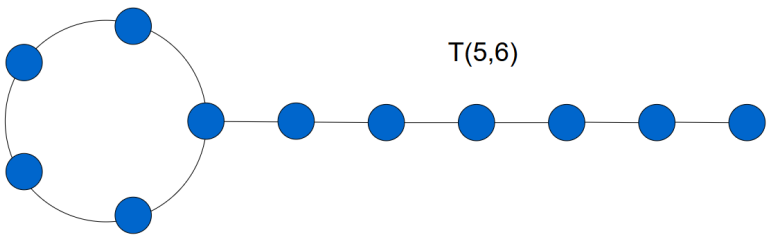

Figure 2. The Tadpole graph $\mathrm{T}(5, 6)$.

The motivation for studying Tadpole graphs stems from their similarity to cycle graphs, both as graphs and in terms of their equivalent DMCs, as illustrated in Figure 3.

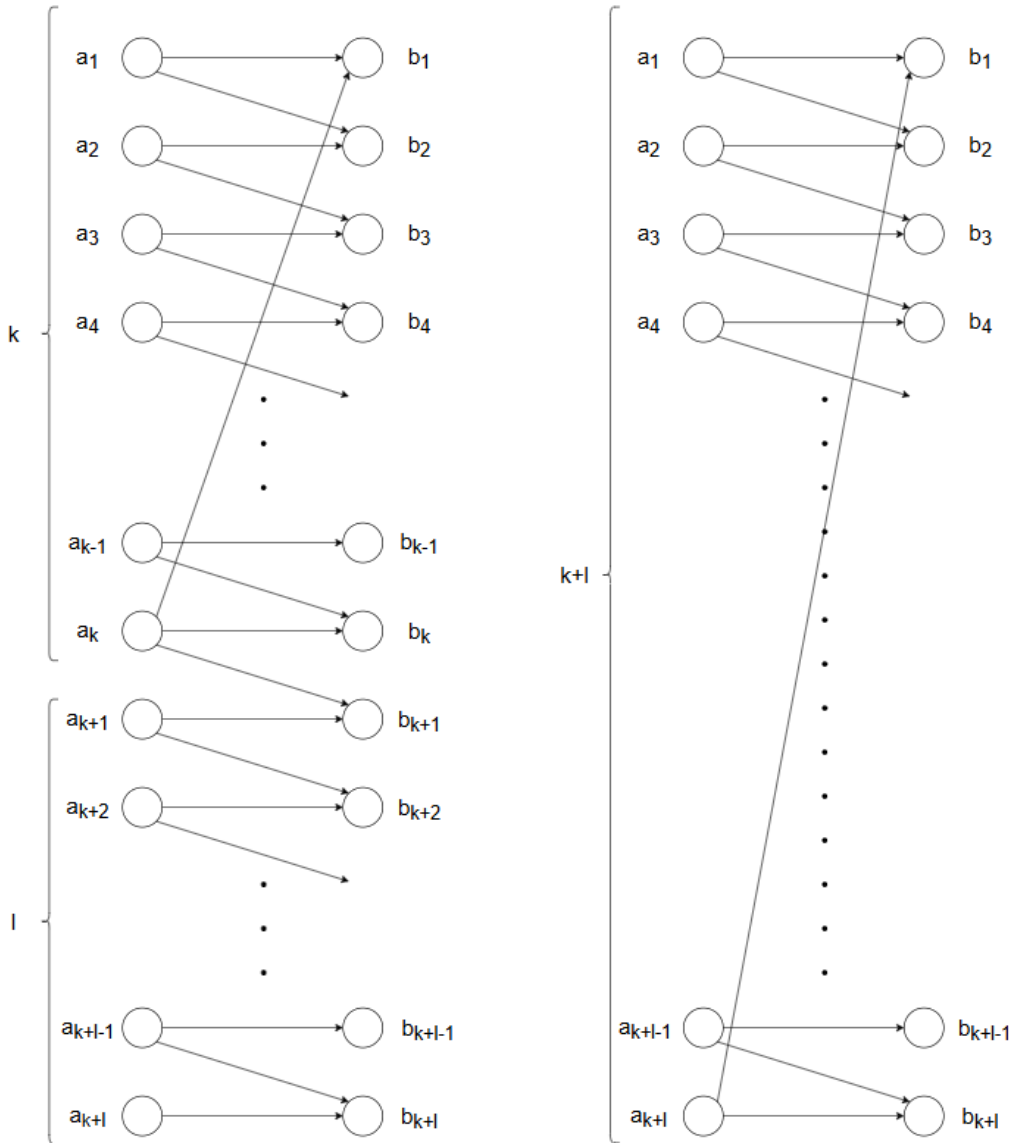

Figure 3. The DMCs of $\mathrm{T}(k,\ell)$ (left plot) and $\mathsf{C}_{k+\ell}$ (right plot).

Lemma 4.2. *Let $k \geq 3$ and $\ell \geq 0$. The independence number of the Tadpole graph $\mathrm{T}(k,\ell)$ is given by*

$$\alpha(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.2}$$

Proof. The independence number of C_k is $\left\lfloor \frac{k}{2} \right\rfloor$, and the independence number of P_ℓ is $\left\lceil \frac{\ell}{2} \right\rceil$. One can select a maximal independent set by excluding the vertex of C_k that is adjacent to the vertex of P_ℓ, which gives (4.2). □

Lemma 4.3. *Let $k \geq 3$ and $\ell \geq 0$.*

1. *If one of the following two conditions holds:*
   - *$k = 3$ or $k \geq 4$ is even,*

- *$k \geq 5$ is odd and $\ell \geq 1$ is odd,*

then

$$\vartheta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.3}$$

2. *If $k \geq 5$ is odd and $\ell \geq 0$ is even, then*

$$\vartheta(\mathrm{T}(k,\ell)) = \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.4}$$

Proof. First, if $k = 3$, then C_3 is a clique of order 3. Since the clique-cover number of P_ℓ is $\left\lceil \frac{\ell}{2} \right\rceil$, it follows that

$$\sigma(\mathrm{T}(3,\ell)) = 1 + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.5}$$

By Lemma 4.2 and Theorem 2.50, the lower and upper bounds on $\vartheta(\mathrm{T}(k,\ell))$ coincide, which gives

$$\vartheta(\mathrm{T}(3,\ell)) = 1 + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.6}$$

Next, if $k > 3$, then the clique number of $\mathrm{T}(k,\ell)$ is 2 (since there are no triangles), thus every clique in $\mathrm{T}(k,\ell)$ is either an edge or a single vertex. In this case, if k is even, it is possible to cover the cycle C_k by $\frac{k}{2}$ edges (cliques), the path P_ℓ can be covered by $\left\lceil \frac{\ell}{2} \right\rceil$ edges, and thus $\sigma(\mathrm{T}(k,\ell)) = \frac{k}{2} + \left\lceil \frac{\ell}{2} \right\rceil$. Moreover, if k and ℓ are odd, then one can cover $\mathrm{T}(k,\ell)$ by $\frac{k+\ell}{2}$ edges (by covering C_k by $\frac{k-1}{2}$ edges, excluding the vertex of C_k that is adjacent to a pendant vertex of P_ℓ, and then covering the remaining vertices of $\mathrm{T}(k,\ell)$ by $\frac{\ell+1}{2}$ edges). Overall, in both instances we obtain

$$\sigma(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil, \tag{4.7}$$

which gives, by again using the sandwich theorem in (2.47) and Lemma 4.2,

$$\vartheta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.8}$$

Finally, assume that $k \geq 5$ is odd and $\ell \geq 2$ is even (if $\ell = 0$, then $\mathrm{T}(k,0) = \mathsf{C}_k$, and the result follows from Theorem 2.53). By deleting the edge that connects the cycle C_k with the path P_ℓ, it follows that $\mathsf{C}_k + \mathsf{P}_\ell$ is a spanning subgraph of $\mathrm{T}(k,\ell)$. On the other hand, by deleting the leftmost vertex of P_ℓ (i.e., the vertex that is of distance 1 from the cycle C_k), and the two edges that are incident to it in $\mathrm{T}(k,\ell)$, it follows that $\mathsf{C}_k + \mathsf{P}_{\ell-1}$ is an induced subgraph of $\mathrm{T}(k,\ell)$. Hence, by Theorem 2.46,

$$\vartheta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) \leq \vartheta(\mathrm{T}(k,\ell)) \leq \vartheta(\mathsf{C}_k + \mathsf{P}_\ell). \tag{4.9}$$

Furthermore, by Theorems 2.52 and 2.53,

$$\vartheta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) = \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2} = \vartheta(\mathsf{C}_k + \mathsf{P}_\ell). \tag{4.10}$$

Thus,

$$\vartheta(\mathrm{T}(k,\ell)) = \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.11}$$

□

Theorem 4.4. *Let $k \geq 3$ and $\ell \geq 0$ be integers.*

1. *If one of the following two conditions holds:*
   - *$k = 3$ or $k \geq 4$ is even,*
   - *$k \geq 5$ is odd and $\ell \geq 1$ is odd,*

   then

$$\Theta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.12}$$

2. *If $k \geq 5$ is odd and $\ell \geq 0$ is even, then*

$$\Theta(\mathrm{T}(k,\ell)) = \Theta(\mathsf{C}_k) + \frac{\ell}{2}. \tag{4.13}$$

Proof. First, if $k = 3$ or $k \geq 4$ is even, or $k \geq 5$ is odd and $\ell \geq 1$ is odd, then by Lemmas 4.2 and 4.3,

$$\Theta(\mathrm{T}(k,\ell)) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{\ell}{2} \right\rceil. \tag{4.14}$$

Second, if $k \geq 5$ is odd and $\ell \geq 0$ is even, then by using the same subgraphs of $\mathrm{T}(k,\ell)$ from the proof of Lemma 4.3, and by Theorem 2.40,

$$\Theta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) \leq \Theta(\mathrm{T}(k,\ell)) \leq \Theta(\mathsf{C}_k + \mathsf{P}_\ell). \tag{4.15}$$

Furthermore, as path graphs are universal (see Theorem 2.52), it follows from Theorem 3.7 that

$$\begin{aligned} \Theta(\mathsf{C}_k + \mathsf{P}_{\ell-1}) &= \Theta(\mathsf{C}_k) + \Theta(\mathsf{P}_{\ell-1}) \\ &= \Theta(\mathsf{C}_k) + \frac{\ell}{2} \\ &= \Theta(\mathsf{C}_k + \mathsf{P}_\ell). \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) gives equality (4.13). □

The next result follows from equality (4.13).

Corollary 4.5. *Let $\ell \geq 0$ be an even number and let $k \geq 5$ be an odd number.*

1. *If $k = 5$, then*

$$\Theta(\mathrm{T}(5,\ell)) = \sqrt{5} + \frac{\ell}{2}. \tag{4.17}$$

2. *If $k \geq 7$, then*

$$\frac{k+\ell-1}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.18}$$

Proof. If $k = 5$, then $\Theta(\mathsf{C}_5) = \sqrt{5}$ [11], and by Theorem 4.4,

$$\begin{aligned}\Theta(\mathrm{T}(5,\ell)) &= \Theta(\mathsf{C}_5) + \frac{\ell}{2} \\ &= \sqrt{5} + \frac{\ell}{2}.\end{aligned} \tag{4.19}$$

If $k \geq 7$, then by Theorem 2.53,

$$\begin{aligned}\frac{k-1}{2} &= \alpha(\mathsf{C}_k) \\ &\leq \Theta(\mathsf{C}_k) \\ &\leq \vartheta(\mathsf{C}_k) = \frac{k}{1+\sec\frac{\pi}{k}}.\end{aligned} \tag{4.20}$$

Thus, by Theorem 4.4,

$$\begin{aligned}\frac{k+\ell-1}{2} &\leq \Theta(\mathrm{T}(k,\ell)) \\ &\leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}.\end{aligned} \tag{4.21}$$

□

Example 4.6. Let $\mathsf{G}_1 = \mathrm{T}(5,6)$. Then, by Corollary 4.5,

$$\Theta(\mathrm{T}(5,6)) = 3 + \sqrt{5} = 5.23607\ldots \tag{4.22}$$

For comparison, using the SageMath software [40] gives the values

$$\sqrt{\alpha(\mathsf{G}_1 \boxtimes \mathsf{G}_1)} = \sqrt{26} = 5.09902, \tag{4.23}$$

$$\sqrt[3]{\alpha(\mathsf{G}_1 \boxtimes \mathsf{G}_1 \boxtimes \mathsf{G}_1)} = \sqrt[3]{136} = 5.14256. \tag{4.24}$$

Let $\mathsf{G}_2 = \mathrm{T}(7,6)$. By the lower bound in [26] on the Shannon capacity of the cycle C_7,

$$\Theta(\mathsf{C}_7) \geq \sqrt[5]{\alpha(\mathsf{C}_7^5)} \geq \sqrt[5]{367}, \tag{4.25}$$

which implies by Corollary 4.5 and (4.25) that

$$\begin{aligned}6.2578659\ldots = \sqrt[5]{367} + 3 &\leq \Theta(\mathsf{C}_7) + 3 \\ &= \Theta(\mathrm{T}(7,6)) \\ &\leq \frac{7}{1+\sec\frac{\pi}{7}} + 3 = 6.3176672.\end{aligned} \tag{4.26}$$

The last lower bound on $\Theta(\mathrm{T}(7,6))$ improves the previous lower bound in the leftmost term of (4.18), thus closing the gap between the upper and lower bounds from 0.3176 to 0.0598.

The last example shows that the lower bound in Corollary 4.5 is not tight. The following result provides an improved lower bound.

Theorem 4.7. *If $k \geq 5$ is odd and $\ell \geq 0$ is even, then*

$$\sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2} \leq \Theta(\mathrm{T}(k,\ell)) \leq \frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}. \tag{4.27}$$

Proof. The upper bound was proved in Corollary 4.5. Next, we prove the improved lower bound. By Hales' result [32, Theorem 7.1], if $k \geq 3$ is odd, then

$$\alpha(\mathsf{C}_k^2) = \left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor, \tag{4.28}$$

which implies that

$$\Theta(\mathsf{C}_k) \geq \sqrt{\alpha(\mathsf{C}_k^2)} = \sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor}. \tag{4.29}$$

Thus, by combining (4.29) with Theorem 4.4, it follows that

$$\begin{aligned} \Theta(\mathrm{T}(k,\ell)) &= \Theta(\mathsf{C}_k) + \frac{\ell}{2} \\ &\geq \sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2}. \end{aligned} \tag{4.30}$$

□

The gap between the upper and lower bounds on $\Theta(\mathrm{T}(k,\ell))$ in (4.27), for an arbitrary odd $k \geq 7$ and even $\ell \geq 0$, satisfies

$$\begin{aligned} &\left(\frac{k}{1+\sec\frac{\pi}{k}} + \frac{\ell}{2}\right) - \left(\sqrt{\left(\frac{k-1}{2}\right)^2 + \left\lfloor \frac{k-1}{4} \right\rfloor} + \frac{\ell}{2}\right) \\ &< \frac{k+\ell}{2} - \frac{k+\ell-1}{2} \\ &= \frac{1}{2}, \end{aligned} \tag{4.31}$$

where inequality (4.31) follows by dropping the term $\left\lfloor \frac{k-1}{4} \right\rfloor$ inside the square root of the first line, and since $\sec\frac{\pi}{k} > 1$.

Remark 4.8. Theorem 4.7 gives an improved lower bound that relies on a lower bound on the Shannon capacity of odd cycles that was constructed by Hales (see [32]). Since Hales' result, better lower bounds for odd cycles were constructed, and can be used to improve the lower bound in special cases. For better lower bounds on higher powers of odd cycles see [27, 41]. The next theorem shows an improvement to the bound in (4.27) for special cases using a result from [27].

Theorem 4.9. *Let $n, d \in \mathbb{N}$, let $\ell \geq 0$ be an even number, and define*

$$k = n2^d + 2^{d-1} + 1. \tag{4.32}$$

Then,

$$\sqrt[d]{nk^{d-1} + \left(\frac{k-1}{2}\right)k^{d-2}} + \frac{\ell}{2} \leq \Theta(\mathrm{T}(k, \ell)). \tag{4.33}$$

Proof. By Theorem 4.4,

$$\Theta(\mathrm{T}(k, \ell)) = \frac{\ell}{2} + \Theta(\mathsf{C}_k), \tag{4.34}$$

and, by [27, Theorem 1.4],

$$\begin{aligned} \Theta(\mathsf{C}_k) &\geq \sqrt[d]{\alpha(\mathsf{C}_k^d)} \\ &\geq \sqrt[d]{nk^{d-1} + \left(\frac{k-1}{2}\right)k^{d-2}}. \end{aligned} \tag{4.35}$$

Finally, combining (4.34) and (4.35) gives (4.33). □

Remark 4.10. The proof of Theorem 4.9 relies on the lower bound on $\alpha(\mathsf{C}_k^d)$ given in (4.35), as presented in [27, Theorem 1.4]. It is worth noting that in [27], Bohman conjectured that the lower bound is the exact value of the independence number (see [27, Conjecture 1.5]). In a subsequent paper by Bohman, Holzman, and Natarajan [41], the conjecture was confirmed for a countably infinite subset of these values. In particular, by substituting $d = 3$, which gives $k = 8n + 5$, it was proved in [41] that if $8n + 5$ is a prime number, then the lower bound coincides with the exact value of the independence number. Specifically, by [41, Theorem 1], the independence number of the strong cubed power of the cycle graph C_{8n+5} is given by

$$\alpha(\mathsf{C}_{8n+5}^3) = \tfrac{1}{2}(8n + 5)\left[(2n + 1)(8n + 5) - 1\right]. \tag{4.36}$$

5. When the graph capacity is not attained by the independence number of any finite power?

In Section 4, we proved that if $k = 3$, or if $k \geq 4$ is even, or if $k \geq 5$ and $\ell \geq 1$ are odd, then the capacity of the Tadpole graph $\mathrm{T}(k, \ell)$ coincides with its independence number. Additional families of graphs also share this property; e.g., by [11], the capacities of all Kneser graphs are equal to their independence numbers. It is, however, well known that not all graphs posses this property; e.g., $\Theta(\mathsf{C}_5) = \sqrt{\alpha(\mathsf{C}_5^2)} = \sqrt{5}$ and $\alpha(\mathsf{C}_5) = 2$. The property that the Shannon capacity coincides with the square root of the independence number of the second (strong) power of the graph was proved, more generally for all self-complementary vertex transitive graphs [11] and for all self-complementary strongly regular graphs [39], provided that the order of the graph is not a square of an integer. Families of graphs whose capacity is attained at a finite strong power that is strictly larger than 2 are yet unknown. In

this section, we explore sufficient conditions on a graph G such that its Shannon capacity is unattainable by the independence number of any finite strong power of G, i.e.,

$$\Theta(\mathsf{G}) > \sqrt[k]{\alpha(\mathsf{G}^k)}, \quad \forall\, k \in \mathbb{N}. \tag{5.1}$$

This problem was explored by Guo and Watanabe [10], and a family of disconnected graphs that satisfies (5.1) was constructed. In this section, we start by presenting the method from [10], and then we provide two original approaches. The first original approach relies on Dedekind's lemma in number theory (see Lemma 5.5), and it uses a similar concept of proof to the one in Example 5.1. The second original approach uses the result from [10] to construct a countably infinite family of *connected* graphs whose capacity is strictly larger than the independence number of any finite (strong) power of the graph.

The following example exhibits a disconnected graph whose capacity is not attained by the independence number of any strong finite power of the graph. The proof of this example relies on an important concept that is later employed in our first original approach. This example, appearing explicitly in [23], was essentially already contained in [10] and was later explicitly pointed out in the 1998 paper on zero-error information theory by Körner and Orlitsky [5] (at the end of Section IV), where it is attributed to Arikan.

Example 5.1. Let $\mathsf{G} = \mathsf{C}_5 + \mathsf{K}_1$. We have $\mathsf{C}_5 \boxtimes \mathsf{K}_1 \cong \mathsf{C}_5$, so

$$\Theta(\mathsf{C}_5 \boxtimes \mathsf{K}_1) = \Theta(\mathsf{C}_5) = \Theta(\mathsf{C}_5)\,\Theta(\mathsf{K}_1). \tag{5.2}$$

Consequently, by Theorem 2.38,

$$\Theta(\mathsf{G}) = \Theta(\mathsf{C}_5) + \Theta(\mathsf{K}_1) = \sqrt{5} + 1, \tag{5.3}$$

and, for every $k \in \mathbb{N}$,

$$\begin{aligned}
\Theta(\mathsf{G})^k &= (\sqrt{5} + 1)^k \\
&= \sum_{i=0}^{k} \binom{k}{i} 5^{\frac{i}{2}} \\
&= \sum_{0 \le \ell \le \lfloor \frac{k}{2} \rfloor} \binom{k}{2\ell} 5^{\ell} + \sum_{0 \le \ell \le \lfloor \frac{k-1}{2} \rfloor} \binom{k}{2\ell+1} 5^{\ell + \frac{1}{2}} \\
&= c_k + d_k \sqrt{5} \notin \mathbb{N},
\end{aligned} \tag{5.4}$$

where

$$c_k = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2\ell} 5^{\ell} \in \mathbb{N}, \tag{5.5}$$

and

$$d_k = \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2\ell+1} 5^{\ell} \in \mathbb{N}. \tag{5.6}$$

Therefore, it follows that for all $k \in \mathbb{N}$, $\Theta(\mathsf{G})^k \neq \alpha(\mathsf{G}^k)$ since the independence number is an integer. Consequently, (5.1) holds.

The interested reader is referred to [42, Section 5], where Example 5.1 is studied in the context of zero-error variable-length coding.

Next, we consider three approaches for the construction of families of graphs whose Shannon capacities satisfy the condition in (5.1). The first approach is due to Guo and Watanabe [10] with a simplified proof (the original proof from [10] is provided in Appendix D), and the second and third approaches of constructing such graph families are original.

5.1. First approach

Theorem 5.2. *[10] Let* G *be a universal graph (see Definition 2.27), and let* H *satisfy the inequality* $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$*. Then, the Shannon capacity of* $\mathsf{K} \triangleq \mathsf{G} + \mathsf{H}$ *is not attained at any finite power of* K*.*

We present here a new and shortened proof, while the proof from [10] is presented in Section D.

Proof. Let $k \in \mathbb{N}$. By the universality of G,

$$\begin{aligned}\alpha((\mathsf{G}+\mathsf{H})^k) &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G}^i \boxtimes \mathsf{H}^{k-i}) \\ &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G})^i \, \alpha(\mathsf{H}^{k-i}) \\ &= \sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \, \alpha(\mathsf{H}^{k-i}),\end{aligned} \tag{5.7}$$

where the last equality holds by Lemma 2.54. Next, by the assumption that $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$ and since $\Theta(\mathsf{H}^m) = \Theta(\mathsf{H})^m \geq \alpha(\mathsf{H})^m$ for all $m \in \mathbb{N}$ with strict inequality if $m = 1$ (by assumption), it follows that for all $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \, \alpha(\mathsf{H}^{k-i}) &< \sum_{i=0}^{k} \binom{k}{i} \Theta(\mathsf{G})^i \, \Theta(\mathsf{H})^{k-i} \\ &= \big(\Theta(\mathsf{G}) + \Theta(\mathsf{H})\big)^k.\end{aligned} \tag{5.8}$$

Finally, by Shannon's inequality (see Theorem 2.36),

$$\Theta(\mathsf{G}) + \Theta(\mathsf{H}) \leq \Theta(\mathsf{G}+\mathsf{H}). \tag{5.9}$$

Combining (5.7)–(5.9) and raising both sides of the resulting inequality to the power of $\frac{1}{k}$ gives

$$\sqrt[k]{\alpha((\mathsf{G}+\mathsf{H})^k)} < \Theta(\mathsf{G}+\mathsf{H}), \tag{5.10}$$

thus confirming the satisfiability of the condition in (5.1). □

Corollary 5.3. *[10] Let* G *be a universal graph, and let* $\mathsf{K} = \mathsf{G} + \mathsf{C}_{2k+1}$ *with* $k \geq 2$ *be the disjoint union of a universal graph and an odd cycle of length at least* 5*. Then,* $\Theta(\mathsf{K})$ *is not attained at any finite strong power of* K*.*

Proof. This follows from Theorem 5.2 by showing that $\Theta(\mathsf{C}_{2k+1}) > \alpha(\mathsf{C}_{2k+1})$. For an odd cycle graph of length $2k+1$, $\alpha(\mathsf{C}_{2k+1}) = k$. By [32, Theorem 7.1], for all integers j and k such that $2 \leq j \leq k$,

$$\alpha(\mathsf{C}_{2j+1} \boxtimes \mathsf{C}_{2k+1}) = jk + \left\lfloor \frac{j}{2} \right\rfloor, \tag{5.11}$$

which yields

$$\begin{aligned} \Theta(\mathsf{C}_{2k+1}) &\geq \sqrt{\alpha(\mathsf{C}_{2k+1}^2)} \\ &= \sqrt{k^2 + \left\lfloor \frac{k}{2} \right\rfloor} \\ &> k \\ &= \alpha(\mathsf{C}_{2k+1}). \end{aligned} \tag{5.12}$$

□

Remark 5.4. Corollary 5.3 applies in particular to $\mathsf{K}_1 + \mathsf{C}_5$ since K_1 is universal.

5.2. Second approach

The next approach is based on a classical number-theoretic lemma established by Dedekind in 1858. The lemma is stated below, and for completeness we present the simple and elegant proof given in [43, p. 309].

Lemma 5.5. *If the square-root of a natural number is rational, then it must be an integer; equivalently, the square-root of a natural number is either an integer or an irrational number.*

Proof. Let $m \in \mathbb{N}$ be such that $\sqrt{m} \in \mathbb{Q}$. Let $n_0 \in \mathbb{N}$ be the smallest natural number such that $n_0\sqrt{m} \in \mathbb{N}$. Suppose that $\sqrt{m} \notin \mathbb{N}$. Then, there exists $\ell \in \mathbb{N}$ such that $0 < \sqrt{m} - \ell < 1$. Define $n_1 = n_0(\sqrt{m} - \ell)$. Since $n_0\sqrt{m} \in \mathbb{N}$ and $n_0\ell \in \mathbb{N}$, it follows that $n_1 \in \mathbb{N}$, and also $0 < n_1 < n_0$. In addition, $n_1\sqrt{m} = n_0 m - \ell(n_0\sqrt{m}) \in \mathbb{N}$, which contradicts the minimality of n_0. Hence, $\sqrt{n} \in \mathbb{N}$. □

Next, we show the main result of this subsection.

Theorem 5.6. *Let $r \geq 2$, and let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_r$ be graphs such that for every $\ell \in [r]$:*

1. *The graph G_ℓ is either a Kneser graph, a self-complementary strongly regular graph, or a self-complementary vertex-transitive graph.*
2. *There exists a single $\ell_0 \in [r]$ such that G_{ℓ_0} is either self-complementary vertex-transitive or self-complementary strongly regular on n_{ℓ_0} vertices, where n_{ℓ_0} is not the square of an integer.*

Let $p \in \mathbb{N}[x_1, \ldots, x_r]$ be a polynomial with nonzero coefficients in $\mathbb{N}$ such that the following hold:

1. *There exists a monomial in $p(x_1, \ldots, x_r)$ whose degree in x_{ℓ_0} is odd.*
2. *The variable x_{ℓ_0} does not occur in at least one monomial of $p(x_1, \ldots, x_r)$.*

Let $\mathsf{G} = p(\mathsf{G}_1, \ldots, \mathsf{G}_r)$. Then, the Shannon capacity $\Theta(\mathsf{G})$ is not attained at any finite strong power of G (i.e., condition (5.1) *holds).*

Proof. For all $\ell \in [r]$, $\Theta(\mathsf{G}_\ell) = \vartheta(\mathsf{G}_\ell)$ (this follows from the first assumption on G_ℓ). Thus, by Corollary 3.3,

$$\Theta(\mathsf{G}) = p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_r)). \tag{5.13}$$

Furthermore, for $\ell \in [r]$, if G_ℓ is a Kneser graph, then $\vartheta(\mathsf{G}_\ell) = \binom{n_\ell - 1}{r_\ell - 1} \in \mathbb{N}$ (by [11, Theorem 13]), and if G_ℓ is self-complementary and vertex-transitive or self-complementary and strongly regular, then $\vartheta(\mathsf{G}_\ell) = \sqrt{n_\ell}$ (by [39, Theorem 3.26]). By the second assumption on the graphs, it follows that if $\ell \neq \ell_0$, then $\vartheta(\mathsf{G}_\ell) \in \mathbb{N}$, and $\vartheta(\mathsf{G}_{\ell_0}) \notin \mathbb{N}$. Next, by the assumptions of the polynomial $p \in \mathbb{N}[x_1, \ldots, x_r]$, there exists a monomial in p whose degree in x_{ℓ_0} is odd, and there exists a monomial in p where the variable x_{ℓ_0} does not occur. Thus, there exist $a, b \in \mathbb{N}$ such that

$$p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_r)) = a\sqrt{n_{\ell_0}} + b. \tag{5.14}$$

Similarly to the concept of proof in Example 5.1, for every $k \in \mathbb{N}$, it follows from (5.14) that there exist $c_k, d_k \in \mathbb{N}$ such that

$$p(\vartheta(\mathsf{G}_1), \ldots, \vartheta(\mathsf{G}_r))^k = c_k\sqrt{n_{\ell_0}} + d_k. \tag{5.15}$$

Since $\sqrt{n_{\ell_0}} \notin \mathbb{N}$ (by assumption), we have $\sqrt{n_{\ell_0}} \notin \mathbb{Q}$ by Lemma 5.5. By (5.13) and (5.15), it implies that

$$\Theta(\mathsf{G})^k \notin \mathbb{N}, \tag{5.16}$$

so, for all $k \in \mathbb{N}$,

$$\Theta(\mathsf{G}) > \sqrt[k]{\alpha(\mathsf{G}^k)}. \tag{5.17}$$

This shows that the Shannon capacity of G is not attained by the independence number of any finite strong power of G. □

Corollary 5.7. *Let $r \geq 2$, and let $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_r$ be graphs such that, for every $\ell \in [r]$:*

1. *The graph G_ℓ is self-complementary on n_ℓ vertices, and it is either strongly regular or vertex-transitive.*
2. *There exists a single $\ell_0 \in [r]$ such that n_{ℓ_0} is not a square of an integer.*

Then, the Shannon capacity of the disjoint union of these graphs is not attained at any finite strong power.

Proof. The result is achieved by applying Theorem 5.6 to the linear polynomial

$$p(x_1, \ldots, x_r) = \sum_{j=1}^{r} x_j. \tag{5.18}$$

□

Example 5.8. The special case of $\mathsf{G} = \mathsf{C}_5 + \mathsf{K}_1$ (presented earlier) is obtained by selecting in the previous corollary

- $\mathsf{G}_1 = \mathsf{C}_5$, which is a self-complementary and vertex-transitive graph.
- $\mathsf{G}_2 = \mathsf{K}_1$, which is a Kneser graph.
- $r = 2$.

Hence, the Shannon capacity of G is not attained at any finite power of G.

Next, we present an additional example based on Paley graphs.

Example 5.9. Let $q_1, \ldots, q_\ell$ be integer powers of prime numbers, $p_i \equiv 1 \bmod 4$ for every $i \in [\ell]$, where only one of the q_i's is an odd power of the prime p_i. Define $\mathsf{G}_i = P(q_i)$ (where $P(q)$ is the Paley graph of order q) for every $i \in [\ell]$. Then, the Shannon capacity of the disjoint union of these graphs,

$$\mathsf{G} = \mathsf{G}_1 + \mathsf{G}_2 + \ldots + \mathsf{G}_\ell \tag{5.19}$$

is not attained by the independence number of any finite strong power of G. This follows from Corollary 5.7, as the Paley graphs are self-complementary and strongly regular, and the restrictions on $q_1, \ldots, q_\ell$ guarantee that the conditions of Corollary 5.7 hold.

5.3. Third approach

In (4.17), one already observes a novel example of a countably infinite family of *connected* graphs, where the Shannon capacity of every such graph is not achieved by the independence number of any finite strong power. Indeed, this holds since raising the right-hand side of (4.17) to any power $k \in \mathbb{N}$ does not result in an integer (similarly to (5.4)). In the present last approach, we build a generalized family of *connected* graphs, whose capacity is not attained at any of their finite strong powers. In particular, we prove that an extended infinite family of Tadpole graphs is not attained at a finite strong power.

Theorem 5.10. *Let H be a graph with $\alpha(\mathsf{H}) < \Theta(\mathsf{H})$, let $\ell \geq 2$ be an even number, and let $v \in \mathsf{V}(\mathsf{H})$ be an arbitrary vertex in H. Define the graph G as the disjoint union of H and P_ℓ with an extra edge between v and one of the two endpoints of P_ℓ. Then, the capacity of G is unattainable by any of its finite strong powers.*

Proof. Since $\mathsf{H} + \mathsf{P}_{\ell-1}$ is an induced subgraph of G, and $\mathsf{H} + \mathsf{P}_\ell$ is a spanning subgraph of G, it follows that

$$\Theta(\mathsf{H} + \mathsf{P}_{\ell-1}) \leq \Theta(\mathsf{G}) \leq \Theta(\mathsf{H} + \mathsf{P}_\ell). \tag{5.20}$$

Path graphs are universal, and for an even $\ell \geq 2$, the capacities of $\mathsf{P}_{\ell-1}$ and P_ℓ coincide (see Theorem 2.52). The latter holds because path graphs are bipartite and therefore perfect, so their Shannon capacities coincide with their independence numbers, and also $\alpha(\mathsf{P}_{\ell-1}) = \frac{\ell}{2} = \alpha(\mathsf{P}_\ell)$ if $\ell \geq 2$ is even. By the universality of path graphs, and since the independence numbers of P_ℓ and $\mathsf{P}_{\ell-1}$ coincide for an even $\ell \geq 2$ (see Theorem 2.52), for every $k \in \mathbb{N}$,

$$\begin{aligned}\alpha((\mathsf{H} + \mathsf{P}_{\ell-1})^k) &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i \boxtimes \mathsf{P}_{\ell-1}^{k-i}) \\ &= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i)\, \alpha(\mathsf{P}_{\ell-1})^{k-i}\end{aligned}$$

$$= \sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{H}^i)\, \alpha(\mathsf{P}_\ell)^{k-i}$$
$$= \alpha((\mathsf{H} + \mathsf{P}_\ell)^k). \tag{5.21}$$

Consequently, by raising both sides of (5.21) to the power $\frac{1}{k}$ and letting $k \to \infty$, it follows that $\Theta(\mathsf{H} + \mathsf{P}_{\ell-1}) = \Theta(\mathsf{H} + \mathsf{P}_\ell)$. Combining the last equality with (5.20), hence gives

$$\Theta(\mathsf{G}) = \Theta(\mathsf{H} + \mathsf{P}_\ell). \tag{5.22}$$

By the same argument that yields (5.20), for all $k \in \mathbb{N}$, $(\mathsf{H} + \mathsf{P}_{\ell-1})^k$ is an induced subgraph of G^k, and $(\mathsf{H} + \mathsf{P}_\ell)^k$ is a spanning subgraph of G^k, so

$$\alpha((\mathsf{H} + \mathsf{P}_{\ell-1})^k) \leq \alpha(\mathsf{G}^k) \leq \alpha((\mathsf{H} + \mathsf{P}_\ell)^k). \tag{5.23}$$

Thus, by combining (5.21) and (5.23), it follows that for every $k \in \mathbb{N}$,

$$\alpha(\mathsf{G}^k) = \alpha((\mathsf{H} + \mathsf{P}_\ell)^k). \tag{5.24}$$

Finally, by Theorem 5.2 and equalities (5.22) and (5.24), for every $k \in \mathbb{N}$,

$$\Theta(\mathsf{G})^k = (\Theta(\mathsf{H} + \mathsf{P}_\ell))^k > \alpha((\mathsf{H} + \mathsf{P}_\ell)^k) = \alpha(\mathsf{G}^k), \tag{5.25}$$

so G is not attained at any of its finite strong powers. □

Corollary 5.11. *Let $k \geq 5$ be an odd number, and let $\ell \geq 2$ be an even number. Then, the Shannon capacity of the Tadpole graph $\mathrm{T}(k, \ell)$ is unattainable by any of its strong powers.*

Proof. This follows directly from Theorem 5.10 and Definition 4.1, by selecting $\mathsf{H} = \mathsf{C}_k$ for an odd $k \geq 5$. In the latter case, $\alpha(\mathsf{C}_k) < \Theta(\mathsf{C}_k)$, as required by Theorem 5.10. □

Remark 5.12. Corollary 5.11 provides a countably infinite set of *connected* graphs whose Shannon capacities are unattainable by any of its strong powers. This is the first infinite family of connected graphs with that property. All previous constructions with that property were disconnected graphs.

6. The Shannon capacity of q-Kneser graphs

In this section, we determine the exact Shannon capacity of the family of q-Kneser graphs. For the classical Kneser graphs, the Shannon capacity was determined by Lovász [11, Theorem 13] (see (2.50)).

Definition 6.1. Let $n, k \in \mathbb{N}$ with $k \leq n$, let p be a prime, let $q = p^m$ be a prime power with $m \in \mathbb{N}$, and let $\mathbb{F}_q$ denote the Galois field of order q. The Gaussian coefficient, denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$, is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}. \tag{6.1}$$

The Gaussian coefficient admits the following combinatorial interpretation. Let V be an n-dimensional vector space over $\mathbb{F}_q$. Then, the number of distinct k-dimensional subspaces of V is equal to $\left[\begin{array}{c} n \\ k \end{array}\right]_q$. Consequently, we define $\left[\begin{array}{c} 0 \\ 0 \end{array}\right]_q \triangleq 1$.

Letting q vary continuously and taking the limit $q \to 1$ gives

$$\begin{aligned} \lim_{q\to 1}\left[\begin{array}{c} n \\ k \end{array}\right]_q &= \frac{n}{k}\cdot\frac{n-1}{k-1}\cdots\frac{n-(k-1)}{k-(k-1)} \\ &= \binom{n}{k}, \end{aligned} \tag{6.2}$$

thus converging to the binomial coefficient.

Definition 6.2 (q-Kneser graphs)**.** Let $V(n,q)$ denote the n-dimensional vector space over the finite field $\mathbb{F}_q$, where q is a prime power. The q-Kneser graph $\mathsf{K}_q(n,k)$ has as its vertices the k-dimensional subspaces of $V(n,q)$, and any two vertices are adjacent if and only if their intersection is the zero vector.

Some of the properties of the q-Kneser graphs are presented next (see [52, Proposition 3.4]).

Theorem 6.3. *Let $\mathsf{G} = \mathsf{K}_q(n,k)$ be a q-Kneser graph, where $n \geq 2k$ and q is a prime power. Then, the following hold:*

1. *The order of $\mathsf{K}_q(n,k)$ is given by*

$$|\mathsf{V}(\mathsf{K}_q(n,k))| = \left[\begin{array}{c} n \\ k \end{array}\right]_q. \tag{6.3}$$

2. *The size of $\mathsf{K}_q(n,k)$ is given by*

$$|\mathsf{E}(\mathsf{K}_q(n,k))| = \tfrac{1}{2}\, q^{k^2}\left[\begin{array}{c} n-k \\ k \end{array}\right]_q\left[\begin{array}{c} n \\ k \end{array}\right]_q. \tag{6.4}$$

3. *$\mathsf{K}_q(n,k)$ is d-regular with $d = q^{k^2}\left[\begin{array}{c} n-k \\ k \end{array}\right]_q$.*

4. *$\mathsf{K}_q(n,k)$ is vertex-transitive and edge-transitive.*

Next, we determine the Shannon capacity of $\mathsf{K}_q(n,k)$ in a manner analogous to the Lovász calculation of the Shannon capacity of the Kneser graph $\mathsf{K}(n,k)$ (see [11, Theorem 13]). To that end, we derive the independence number and the Lovász ϑ-function of the q-Kneser graph and show that they coincide, thereby yielding the Shannon capacity of the graph.

Lemma 6.4. *[29] Let $\mathsf{G} = \mathsf{K}_q(n,k)$ be a q-Kneser graph, where $n \geq 2k$ and q is a prime power. Then,*

$$\alpha(\mathsf{K}_q(n,k)) = \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q. \tag{6.5}$$

Proof. By the version of the Erdős-Ko-Rado theorem for finite vector spaces (see [29, Theorem 9.8.1]), it follows that

$$\alpha(\mathsf{K}_q(n,k)) \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.6}$$

Moreover, we can construct a family of k-subspaces of $V(n,q)$ containing a fixed 1-dimensional subspace of $V(n,q)$. This family of subspaces has $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ subspaces (see Definition 6.1 and [29, Theorem 9.8.1]), and it is an independent set in $\mathsf{K}_q(n,k)$, thus

$$\alpha(\mathsf{K}_q(n,k)) \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{6.7}$$

which proves equality (6.5). □

Lemma 6.5. *Let $\mathsf{G} = \mathsf{K}_q(n,k)$ be a q-Kneser graph, where $n \geq 2k$ and q is a prime power. Then,*

$$\vartheta(\mathsf{K}_q(n,k)) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.8}$$

Proof. By Theorem 2.48, and since the q-Kneser graphs are edge-transitive (see Theorem 6.3), it follows that

$$\vartheta(\mathsf{K}_q(n,k)) = -\frac{|\mathsf{V}(\mathsf{K}_q(n,k))|\,\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}, \tag{6.9}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are, respectively, the largest and smallest eigenvalues of the adjacency matrix of $\mathsf{K}_q(n,k)$. By [30, Theorem 2], these eigenvalues are given by

$$\lambda_{\max} = q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \tag{6.10}$$

$$\lambda_{\min} = -q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q. \tag{6.11}$$

Substituting (6.3), (6.10), and (6.11) into (6.9) gives

$$\begin{aligned} \vartheta(\mathsf{K}_q(n,k)) &= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}{q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q + q^{k^2-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q} \\ &= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}{q^{k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q}. \end{aligned} \tag{6.12}$$

This expression can be simplified, based on the following identity:

$$\left[\begin{array}{c} m \\ r \end{array}\right]_q = \frac{q^m - 1}{q^r - 1}\left[\begin{array}{c} m-1 \\ r-1 \end{array}\right]_q, \tag{6.13}$$

where using (6.13) with $m = n - k$ and $r = k$ simplifies the denominator on the right-hand side of (6.12) to

$$\begin{aligned} & q^k \left[\begin{array}{c} n-k \\ k \end{array}\right]_q + \left[\begin{array}{c} n-k-1 \\ k-1 \end{array}\right]_q \\ &= q^k \cdot \frac{q^{n-k} - 1}{q^k - 1} \cdot \left[\begin{array}{c} n-k-1 \\ k-1 \end{array}\right]_q + \left[\begin{array}{c} n-k-1 \\ k-1 \end{array}\right]_q \\ &= \frac{q^n - 1}{q^k - 1} \cdot \left[\begin{array}{c} n-k-1 \\ k-1 \end{array}\right]_q. \end{aligned} \tag{6.14}$$

Combining (6.12) and (6.14) finally gives the simplified form

$$\begin{aligned} \vartheta(\mathsf{K}_q(n,k)) &= \left[\begin{array}{c} n \\ k \end{array}\right]_q \cdot \frac{q^k - 1}{q^n - 1} \\ &= \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q, \end{aligned} \tag{6.15}$$

where (6.15) holds by (6.13). □

Due to the coincidence of the independence number and the Lovász ϑ-function in Lemmas 6.4 and 6.5, their common value is equal to the Shannon capacity of the graph. This gives the following closed-form expression for the Shannon capacity of q-Kneser graphs.

Theorem 6.6. *The Shannon capacity of the q-Kneser graph $\mathsf{K}_q(n,k)$, where $n \geq 2k$ and q is a prime power, is given by*

$$\Theta(\mathsf{K}_q(n,k)) = \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_q. \tag{6.16}$$

7. A new inequality for the capacity of graphs

The following result provides a relation between the Shannon capacity of any strong product of graphs, and the capacity of the disjoint union of the component graphs. If these component graphs are connected, then their strong product is a connected graph on a number of vertices that is equal to the product of the number of vertices in each component graph, whereas the disjoint union of these component graphs is a disconnected graph on a number of vertices that is equal to the sum of the orders of the component graphs (the latter order is typically much smaller than the former). The motivation for our inequality comes from the following result.

Theorem 7.1 (Unique Prime Factorization for Connected Graphs)**.** *Every connected graph has a unique prime factor decomposition with respect to the strong product.*

The proof of Theorem 7.1 was introduced by Dörfler and Imrich [44], and Mckenzie [45]. See Section 7.3 in the comprehensive book on graph products [31] (Theorem 7.14). In a paper by Feigenbaum and Schäffer [46], a polynomial-time algorithm was introduced for finding that unique prime factorization (with respect to strong products).

Next, we provide and prove the main result of this section.

Theorem 7.2. *Let* $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_\ell$ *be simple graphs. Then,*

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq \left(\frac{\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)}{\ell}\right)^\ell. \tag{7.1}$$

Furthermore, if $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ *for every* $i \in [\ell]$*, then inequality* (7.1) *holds with equality if and only if*

$$\Theta(\mathsf{G}_1) = \Theta(\mathsf{G}_2) = \ldots = \Theta(\mathsf{G}_\ell). \tag{7.2}$$

In particular, if for every $i \in [\ell]$*, one of the following statements holds:*

- G_i *is a perfect graph,*
- $\mathsf{G}_i = \mathsf{K}(n, r)$ *for some* $n, r \in \mathbb{N}$ *with* $n \geq 2r$*,*
- $\mathsf{G}_i = \mathsf{K}_q(n, r)$ *for a prime factor* q *and some* $n, r \in \mathbb{N}$ *with* $n \geq 2r$*,*
- G_i *is vertex-transitive and self-complementary,*
- G_i *is strongly regular and self-complementary,*

then inequality (7.1) *holds with equality if and only if the condition in* (7.2) *is satisfied.*

Proof. Let $k \in \mathbb{N}$. By (2.29),

$$\begin{aligned}
&\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k} \\
&= \Theta((\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k}) \\
&= \Theta\left(\sum_{k_1,\ldots,k_\ell:\, k_1+\ldots+k_\ell=\ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}\right).
\end{aligned} \tag{7.3}$$

By (2.29) and (2.30),

$$\begin{aligned}
&\Theta\left(\sum_{k_1,\ldots,k_\ell:\, k_1+\ldots+k_\ell=\ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}\right) \\
&\geq \sum_{k_1,\ldots,k_\ell:\, k_1+\ldots+k_\ell=\ell k} \binom{\ell k}{k_1, \ldots, k_\ell} \Theta(\mathsf{G}_1^{k_1} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k_\ell}) \\
&\geq \binom{\ell k}{k, \ldots, k} \Theta(\mathsf{G}_1^{k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{k}) \\
&= \binom{\ell k}{k, \ldots, k} \Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell)^{k},
\end{aligned} \tag{7.4}$$

thus, by combining (7.3) and (7.4), it follows that

$$\binom{\ell k}{k, \ldots, k} \Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell)^{k} \leq \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^{\ell k}.$$

Raising both sides of the inequality to the power of $\frac{1}{k}$, and letting k tend to infinity gives

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq \left(\frac{\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)}{\ell}\right)^\ell, \tag{7.5}$$

which holds by the equality

$$\lim_{k\to\infty} \sqrt[k]{\binom{\ell k}{k, \ldots, k}} = \ell^\ell, \quad \forall\, \ell \in \mathbb{N}. \tag{7.6}$$

If $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for every $i = 1, \ldots, \ell$, then by Theorem 3.4,

$$\Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell) = \Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell). \tag{7.7}$$

By (7.7), inequality (7.1) is equivalent to

$$\sqrt[\ell]{\Theta(\mathsf{G}_1) \ldots \Theta(\mathsf{G}_\ell)} \leq \frac{\Theta(\mathsf{G}_1) + \ldots + \Theta(\mathsf{G}_\ell)}{\ell}, \tag{7.8}$$

and, by the conditions for equality in the AM-GM inequality, equality holds in (7.8) if and only if (7.2) holds. Finally, all the graphs that are listed in this theorem (Theorem 7.2) satisfy the equality $\Theta(\mathsf{G}_i) = \vartheta(\mathsf{G}_i)$ for $i \in [\ell]$. Thus, if G_i is one of these graphs for each $i \in [\ell]$, then inequality (7.1) holds with equality if and only if the condition in (7.2) is satisfied. □

Remark 7.3. An anonymous reviewer brought to our attention that inequality (7.1) can be related to Eq. (1) in the preprint [47] for the complementary graph entropy, via the results of [48, Lemma 1] and [6, Equation (11.2)].

The next result strengthens [7, Theorem 2.1] by identifying several sufficient conditions under which the inequality holds with equality.

Corollary 7.4. *Let* G *be a graph on n vertices. Then,*

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n}, \tag{7.9}$$

with equality in (7.9) *if the graph* G *is either self-complementary and vertex-transitive, self-complementary and strongly regular, a conference graph, a Latin square graph, or the complement of any of these graphs.*

Proof. By Theorem 7.2,

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{\Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}})}. \tag{7.10}$$

Since $\{(1,1), \ldots, (n,n)\}$ is a trivial independent set of $\mathsf{G} \boxtimes \overline{\mathsf{G}}$, it follows that

$$\Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) \geq \alpha(\mathsf{G} \boxtimes \overline{\mathsf{G}}) \geq n. \tag{7.11}$$

Combining inequalities (7.10) and (7.11) yields (7.9). Furthermore, suppose that the graph G is one of the following: (1) self-complementary and vertex-transitive, (2) self-complementary and strongly regular, (3) a conference graph, (4) a Latin square graph, or (5) the complement of any of the graphs in (1)–(4). Then, by Theorem 7.2 and [39, Theorems 3.23, 3.26, and 3.28], inequalities (7.10) and (7.11) hold with equality, and consequently inequality (7.9) also holds with equality. □

Remark 7.5. The original proof of inequality (7.9) in [7, Theorem 2.1] is presented in Appendix E. Our proof of Corollary 7.4 follows a different approach, which enables us to identify sufficient conditions for equality in (7.9).

By (7.1), some additional inequalities are derived in the next two corollaries.

Corollary 7.6. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, let* $m_1, \ldots, m_\ell \in \mathbb{N}$*, and let* $m = \sqrt[\ell]{m_1 \ldots m_\ell}$*. Then,*

$$\Theta(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_\ell) \leq (m\ell)^{-\ell}\, \Theta(m_1\mathsf{G}_1 + \ldots + m_\ell\mathsf{G}_\ell)^\ell. \tag{7.12}$$

Proof. By (2.34), inequality (7.12) is equivalent to

$$\Theta(m_1\mathsf{G}_1 \boxtimes \ldots \boxtimes m_\ell\mathsf{G}_\ell) \leq \ell^{-\ell}\, \Theta(m_1\mathsf{G}_1 + \ldots + m_\ell\mathsf{G}_\ell)^\ell. \tag{7.13}$$

Thus, by Theorem 7.2, inequality (7.12) holds. □

Theorem 7.7. *Let* $\mathsf{G}_1, \ldots, \mathsf{G}_\ell$ *be simple graphs, with some* $\ell \in \mathbb{N}$*, and let* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ *be a probability vector with* $\alpha_j \in \mathbb{Q}$ *for all* $j \in [\ell]$*. Let*

$$K(\underline{\alpha}) \triangleq \Big\{k \in \mathbb{N} :\ \alpha_j k \in \mathbb{N},\ \forall\, j \in [\ell]\Big\}. \tag{7.14}$$

Then, for all $k \in K(\underline{\alpha})$*,*

$$\Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) \leq \exp(-k\, H(\underline{\alpha}))\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k, \tag{7.15}$$

where the entropy function H *is given by*

$$H(\underline{\alpha}) \triangleq -\sum_{j=1}^{\ell} \alpha_j \log \alpha_j, \quad \forall\, \underline{\alpha} :\ \alpha_1, \ldots, \alpha_\ell \geq 0,\ \ \sum_{j=1}^{\ell} \alpha_j = 1. \tag{7.16}$$

Proof. Let

$$A \triangleq \operatorname{lcm}(\alpha_1 k, \ldots, \alpha_\ell k),$$

and, for every $j \in [\ell]$, define

$$n_j \triangleq \frac{A}{\alpha_j k}, \tag{7.17}$$

$$a_j \triangleq \sum_{i=1}^{j} \alpha_i k \in \mathbb{N}, \tag{7.18}$$

with $a_0 \triangleq 0$. For each $i \in [k]$, let $j \in [\ell]$ be the index that satisfies

$$a_{j-1} + 1 \leq i \leq a_j, \tag{7.19}$$

and define

$$m_i \triangleq n_j, \quad \forall\, i \in [k]. \tag{7.20}$$

By Corollary 7.6 (note that the strong product in the next line involves k graphs),

$$\begin{aligned}\Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) &\leq (mk)^{-k}\, \Theta(m_1\mathsf{G}_1 + \ldots + m_{a_1}\mathsf{G}_1 + \ldots + m_{a_{\ell-1}+1}\mathsf{G}_\ell + \ldots + m_k\mathsf{G}_\ell)^k \\ &= (mk)^{-k}\, \Theta((m_1 + \ldots + m_{a_1})\,\mathsf{G}_1 + \ldots + (m_{a_{\ell-1}+1} + \ldots + m_k)\,\mathsf{G}_\ell)^k \\ &= (mk)^{-k}\, \Theta(\alpha_1\, n_1\, k\mathsf{G}_1 + \ldots + \alpha_\ell\, n_\ell\, k\mathsf{G}_\ell)^k, \end{aligned} \tag{7.21}$$

where the last equality holds since, by (7.17) and (7.18),

$$m_1 = \ldots = m_{a_1} = n_1, \tag{7.22}$$

$$m_{a_1+1} = \ldots = m_{a_2} = n_2, \tag{7.23}$$

$$\vdots$$

$$m_{a_{\ell-1}+1} = \ldots = m_k = n_\ell, \quad a_\ell = k. \tag{7.24}$$

So, for every $j \in [\ell]$,

$$\begin{aligned} m_{a_{j-1}+1} + \ldots + m_{a_j} &= (a_j - a_{j-1})\, n_j \\ &= \alpha_j\, n_j\, k. \end{aligned} \tag{7.25}$$

The calculation of m in the right-hand side of (7.21) gives, by (7.17) and (7.22)–(7.25),

$$\begin{aligned} m &= \sqrt[k]{m_1 \ldots m_k} \\ &= \sqrt[k]{n_1^{a_1-a_0} \ldots n_\ell^{a_\ell-a_{\ell-1}}} \\ &= \sqrt[k]{n_1^{\alpha_1 k} \ldots n_\ell^{\alpha_\ell k}} \\ &= \sqrt[k]{\left(\frac{A}{\alpha_1 k}\right)^{\alpha_1 k} \ldots \left(\frac{A}{\alpha_\ell k}\right)^{\alpha_\ell k}} \\ &= \frac{A}{k}\left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right)^{-\frac{1}{k}}, \end{aligned} \tag{7.26}$$

and also, by (2.34) and (7.17),

$$\begin{aligned} \Theta(n_1\alpha_1 k\mathsf{G}_1 + \ldots + n_\ell\alpha_\ell k\mathsf{G}_\ell)^k &= \Theta(A\mathsf{G}_1 + \ldots + A\mathsf{G}_\ell)^k \\ &= A^k\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k. \end{aligned} \tag{7.27}$$

Finally, combining (7.21), (7.26), and (7.27), we get

$$\begin{aligned} \Theta(\mathsf{G}_1^{\alpha_1 k} \boxtimes \ldots \boxtimes \mathsf{G}_\ell^{\alpha_\ell k}) &\leq (mk)^{-k} A^k\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\ &= \left(A\left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right)^{-\frac{1}{k}}\right)^{-k} A^k\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\ &= \left(\prod_{i=1}^{\ell} \alpha_i^{\alpha_i k}\right) \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k \\ &= \exp(-k\, H(\underline{\alpha}))\, \Theta(\mathsf{G}_1 + \ldots + \mathsf{G}_\ell)^k. \end{aligned} \tag{7.28}$$

□

Remark 7.8. Let $\alpha_j = \frac{p_j}{q_j}$ with $(p_j, q_j) = 1$ for all $j \in [\ell]$. Then,

$$K(\underline{\alpha}) = \mathrm{lcm}(q_1, \ldots, q_\ell)\,\mathbb{N}.$$

This is true because if we choose k that is not a multiple of $\mathrm{lcm}(q_1, \ldots, q_\ell)$, then there exists $i \in [\ell]$ such that $\alpha_i \notin \mathbb{N}$.

In analogy to Corollary 7.4, we next derive upper and lower bounds on $\vartheta(\mathsf{G} + \overline{\mathsf{G}})$.

Theorem 7.9. *For every simple graph* G *on n vertices*

$$\vartheta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n} + \frac{(\vartheta(\mathsf{G}) - \sqrt{n})^2}{\vartheta(\mathsf{G})}, \tag{7.29}$$

and for every d-regular graph G*, whose eigenvalues are ordered in non-increasing order as*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \tag{7.30}$$

the following holds:

$$2 + \frac{n-d-1}{1+\lambda_2} - \frac{d}{\lambda_n} \leq \vartheta(\mathsf{G} + \overline{\mathsf{G}}) \leq \frac{n(1+\lambda_2)}{n-d+\lambda_2} - \frac{n\lambda_n}{d-\lambda_n}. \tag{7.31}$$

Furthermore,

1. *If* G *is either vertex-transitive or strongly regular, then inequality* (7.29) *holds with equality.*
2. *If both* G *and* $\overline{\mathsf{G}}$ *are edge-transitive, or if* G *is strongly regular, then the right-hand inequality in* (7.31) *is attained with equality.*
3. *If* G *is strongly regular, then the left-hand inequality in* (7.31) *is attained with equality.*

Proof. Using Theorems 2.45 and 2.47, we get

$$\begin{aligned}\vartheta(\mathsf{G} + \overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\geq \vartheta(\mathsf{G}) + \frac{n}{\vartheta(\mathsf{G})} \\ &= 2\sqrt{n} + \frac{(\vartheta(\mathsf{G}) - \sqrt{n})^2}{\vartheta(\mathsf{G})}.\end{aligned} \tag{7.32}$$

In addition, by Theorem 2.47, inequality (7.32) holds with equality if G is vertex-transitive or strongly regular. Furthermore, if G is a d-regular graph, then by [36, Proposiotion 1],

$$\begin{aligned}\vartheta(\mathsf{G} + \overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\leq \frac{n(1+\lambda_2)}{n-d+\lambda_2} - \frac{n\lambda_n}{d-\lambda_n},\end{aligned} \tag{7.33}$$

and

$$\begin{aligned}\vartheta(\mathsf{G} + \overline{\mathsf{G}}) &= \vartheta(\mathsf{G}) + \vartheta(\overline{\mathsf{G}}) \\ &\geq 2 + \frac{n-d-1}{1+\lambda_2} - \frac{d}{\lambda_n},\end{aligned} \tag{7.34}$$

which proves (7.31). Finally, by [36, Proposition 1], if both G and $\overline{\mathsf{G}}$ are edge-transitive, or if G is strongly regular, then inequality (7.33) holds with equality. Moreover, if G is strongly regular, then inequality (7.34) also holds with equality. This completes the proof of the stated sufficient conditions for the attainability with equality of the two inequalities in (7.31). □

8. Outlook

This section suggests some potential directions for further research that are related to the findings in this paper.

1. In Theorem 5.6, a construction of graphs was provided whose Shannon capacity is not attained by the independence number of any of their finite strong powers. The proof relied on Dedekind's lemma from number theory (see Lemma 5.5), showing that the capacity of this construction equals a value whose finite powers are not natural numbers, and hence cannot correspond to the finite root of an independence number. This method of proof also applies to rational numbers that are not integers, since none of their finite powers are natural numbers either. This raises an interesting question: *Does there exist a graph whose Shannon capacity is a rational number that is not an integer?*. At present, no such graphs are known. However, if the answer is positive, it would immediately follow that these graphs also possess the property of having a Shannon capacity that is not attained by the independence number of any of their finite strong powers (thus potentially leading to a fourth approach in Section 5).

2. It was proved in [18] that if the Shannon capacity of a graph is attained at some finite power, then the Shannon capacity of its Mycielskian is strictly larger than that of the original graph. In view of the constructions presented in Section 5, which yield graph families whose Shannon capacity is not attained at any finite power, it would be interesting to determine whether this property also holds in such cases. If it does not, then the graph constructions from Section 5 could provide potential candidates for a counterexample.

3. In light of earlier studies on the Shannon capacity of graphs, as well as the present work, it remains unknown whether there exists a finite, undirected, and simple graph whose Shannon capacity is attained by the independence number of some finite strong power, but not by that of the first or second strong powers. Further study of this problem is therefore of interest.

4. By combining Theorems 2.19 and 2.29, the equality $\alpha(\mathsf{G} \boxtimes \mathsf{H}) = \alpha(\mathsf{G})\,\alpha(\mathsf{H})$ holds for every simple graph H if $\alpha(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$. Moreover, by Theorem 2.38 and Lemma 3.6, if $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$, then $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H})$ holds for all H. From these results, a natural question arises: *Is it true that the equality* $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H})$ *holds for all* H *if and only if* $\Theta(\mathsf{G}) = \alpha_{\mathrm{f}}(\mathsf{G})$*?*. This question was already discussed to some extent in [53], showing that

$$\sup_{\mathsf{H}} \frac{\Theta(\mathsf{G} \boxtimes \mathsf{H})}{\Theta(\mathsf{H})} \leq \alpha_{\mathrm{f}}(\mathsf{G}), \tag{8.1}$$

   while raising a question about a possible gap between the two sides of (8.1).

5. The new inequality in Theorem 7.2 can be viewed as an analogue of the arithmetic–geometric mean inequality for the Shannon capacity of graphs, relating the Shannon capacity of the strong product of graphs to that of their disjoint union. Part of the interest in this inequality arises from the fact that every finite, undirected, simple, and connected graph admits a unique prime factorization with respect to the strong product (see Theorem 7.1). Several applications of the inequality in Theorem 7.2 are presented in Section 7, and additional applications are likely to follow.

6. This last open issue, while not directly related to the results of this paper, is mentioned here because of its fundamental importance. The random graph $G(n, \frac{1}{2})$ is a graph on n vertices in which each pair of distinct vertices is joined by an edge independently with probability $\frac{1}{2}$. For sufficiently large n, with high probability, the independence number of such a graph is equal to $2\log_2 n + O(\log\log n)$ [54, Chapter 11], and its Lovász ϑ-function ranges between $(\frac{1}{2} + o(1))\sqrt{n}$ and $(2 + o(1))\sqrt{n}$ [55]. In contrast, a comparable probabilistic result for the Shannon capacity of $G(n, \frac{1}{2})$ is yet unknown. It has been conjectured by Alon that there exists a positive constant b, independent of n, such that the Shannon capacity of $G(n, \frac{1}{2})$ is at most $b\log_2 n$ almost surely (i.e., with probability tending to 1 as $n \to \infty$) [3, Conjecture 2.4].

This sample of open problems illustrates that, despite the existence of classical results on the Shannon capacity of graphs and a growing body of recent work, many fundamental avenues of research remain open.

Appendix

A. Proof of the duality result in Theorem 2.38

By Theorems 2.35 and 2.36, the claim is equivalent to

$$\Theta(\mathsf{G} + \mathsf{H}) > \Theta(\mathsf{G}) + \Theta(\mathsf{H}) \iff \Theta(\mathsf{G} \boxtimes \mathsf{H}) > \Theta(\mathsf{G})\,\Theta(\mathsf{H}). \tag{A.1}$$

Next, we prove both directions of the equivalence in (A.1).

1. Assume that $\Theta(\mathsf{G} \boxtimes \mathsf{H}) > \Theta(\mathsf{G})\,\Theta(\mathsf{H})$. By equality (2.29),

$$\begin{aligned}\Theta(\mathsf{G} + \mathsf{H})^2 &= \Theta((\mathsf{G} + \mathsf{H})^2)\\ &= \Theta(\mathsf{G}^2 + 2\mathsf{G} \boxtimes \mathsf{H} + \mathsf{H}^2),\end{aligned} \tag{A.2}$$

   where (A.2) holds by the commutative semiring argument in [28]. By Theorems 2.35 and 2.36,

$$\Theta(\mathsf{G}^2 + 2\mathsf{G} \boxtimes \mathsf{H} + \mathsf{H}^2) \geq \Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G} \boxtimes \mathsf{H}) + \Theta(\mathsf{H})^2. \tag{A.3}$$

   Using the above assumption, we get

$$\begin{aligned}\Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G} \boxtimes \mathsf{H}) + \Theta(\mathsf{H})^2 &> \Theta(\mathsf{G})^2 + 2\Theta(\mathsf{G})\,\Theta(\mathsf{H}) + \Theta(\mathsf{H})^2\\ &= \big(\Theta(\mathsf{G}) + \Theta(\mathsf{H})\big)^2,\end{aligned} \tag{A.4}$$

   which gives $\Theta(\mathsf{G} + \mathsf{H}) > \Theta(\mathsf{G}) + \Theta(\mathsf{H})$.

2. Next, assume $\Theta(\mathsf{G} \boxtimes \mathsf{H}) = \Theta(\mathsf{G})\,\Theta(\mathsf{H})$. Then, for all $i, j \in \mathbb{N}$, we get

$$\begin{aligned}\Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j)\,\Theta(\mathsf{G})^j\,\Theta(\mathsf{H})^i &= \Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j)\,\Theta(\mathsf{G}^j)\,\Theta(\mathsf{H}^i)\\ &\leq \Theta(\mathsf{G}^{i+j} \boxtimes \mathsf{H}^{i+j})\\ &= \Theta(\mathsf{G} \boxtimes \mathsf{H})^{i+j}\\ &= \Theta(\mathsf{G})^{i+j}\,\Theta(\mathsf{H})^{i+j}.\end{aligned} \tag{A.5}$$

Thus, $\Theta(\mathsf{G}^i \boxtimes \mathsf{H}^j) \leq \Theta(\mathsf{G})^i\,\Theta(\mathsf{H})^j$, which implies that for all $k \in \mathbb{N}$,

$$\begin{aligned}
\alpha((\mathsf{G}+\mathsf{H})^k) &= \alpha\left(\sum_{\ell=0}^{k}\binom{k}{\ell}\mathsf{G}^\ell \boxtimes \mathsf{H}^{k-\ell}\right)\\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\alpha(\mathsf{G}^\ell \boxtimes \mathsf{H}^{k-\ell})\\
&\leq \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta(\mathsf{G}^\ell \boxtimes \mathsf{H}^{k-\ell})\\
&\leq \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta(\mathsf{G}^\ell)\,\Theta(\mathsf{H}^{k-\ell})\\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}\Theta(\mathsf{G})^\ell\,\Theta(\mathsf{H})^{k-\ell}\\
&= (\Theta(\mathsf{G})+\Theta(\mathsf{H}))^k. \qquad \text{(A.6)}
\end{aligned}$$

Finally, letting $k \to \infty$ gives $\Theta(\mathsf{G}+\mathsf{H}) \leq \Theta(\mathsf{G}) + \Theta(\mathsf{H})$, and by Theorem 2.36,

$$\Theta(\mathsf{G}+\mathsf{H}) = \Theta(\mathsf{G}) + \Theta(\mathsf{H}). \qquad \text{(A.7)}$$

B. A new proof of Theorem 2.45

Theorem B.1. *[17] Let* G *and* H *be simple graphs. Then,*

$$\vartheta(\mathsf{G}+\mathsf{H}) = \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \qquad \text{(B.1)}$$

Proof. By Theorem 2.43, let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an orthonormal representation of $\overline{\mathsf{G}}$, and let $\mathbf{c}$ be a unit vector such that

$$\vartheta(\mathsf{G}) = \sum_{i=1}^{n}(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2. \qquad \text{(B.2)}$$

Likewise, let $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ be an orthonormal representation of $\overline{\mathsf{H}}$, and let $\mathbf{d}$ be a unit vector such that

$$\vartheta(\mathsf{H}) = \sum_{i=1}^{r}(\mathbf{d}^{\mathrm{T}}\mathbf{v}_i)^2. \qquad \text{(B.3)}$$

Assume without loss of generality that the dimensions of $\mathbf{u}_i, \mathbf{c}, \mathbf{v}_j$, and $\mathbf{d}$ are identical and equal to m (if the dimensions are distinct, the vectors of the lower dimension can be padded by zeros). Next, let $\mathbf{A}$ be an orthogonal matrix of order $m \times m$ such that $\mathbf{Ad} = \mathbf{c}$. Such a matrix $\mathbf{A}$, satisfying $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_m$, exists (e.g., the householder matrix defined as $\mathbf{A} = \mathbf{I}_m - \frac{2(\mathbf{c}-\mathbf{d})(\mathbf{c}-\mathbf{d})^{\mathrm{T}}}{\|\mathbf{c}-\mathbf{d}\|^2}$ provided that $\mathbf{c} \neq \mathbf{d}$, and $\mathbf{A} = \mathbf{I}_m$ if $\mathbf{c} = \mathbf{d}$). Let $w = (\mathbf{w}_1, \ldots, \mathbf{w}_r)$ be defined as $\mathbf{w}_i = \mathbf{Av}_i$ for every $i \in [r]$. Since the pairwise inner products

are preserved under an orthogonal (orthonormal) transformation and $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{H}}$, so is $(\mathbf{w}_1, \ldots, \mathbf{w}_r)$. Likewise, we get

$$\sum_{i=1}^{r}(\mathbf{c}^{\mathrm{T}}\mathbf{w}_i)^2 = \sum_{i=1}^{r}((\mathbf{A}\mathbf{d})^{\mathrm{T}}(\mathbf{A}\mathbf{v}_i))^2 = \sum_{i=1}^{r}(\mathbf{d}^{\mathrm{T}}\mathbf{v}_i)^2 = \vartheta(\mathsf{H}). \tag{B.4}$$

Next, the representation $(\mathbf{x}_1, \ldots, \mathbf{x}_{n+r}) = (\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_r)$ is an orthonormal representation of $\overline{\mathsf{G}+\mathsf{H}}$ (since there are no additional non-adjacencies in the graph $\overline{\mathsf{G}+\mathsf{H}}$ in comparison to the disjoint union of the pairs of nonadjacent vertices in $\overline{\mathsf{G}}$ and $\overline{\mathsf{H}}$). Hence, by Theorem 2.43, the equality

$$\vartheta(\mathsf{G}) + \vartheta(\mathsf{H}) = \sum_{i=1}^{n}(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2 + \sum_{i=1}^{r}(\mathbf{c}^{\mathrm{T}}\mathbf{w}_i)^2 = \sum_{i=1}^{n+r}(\mathbf{c}^{\mathrm{T}}\mathbf{x}_i)^2, \tag{B.5}$$

yields the inequality

$$\vartheta(\mathsf{G}) + \vartheta(\mathsf{H}) \leq \vartheta(\mathsf{G}+\mathsf{H}). \tag{B.6}$$

Next, by Theorem 2.43, let $(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_r)$ be an orthonormal representation of $\overline{\mathsf{G}+\mathsf{H}}$, where the vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ correspond to the vertices of G and the vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ correspond to the vertices of H, and let $\mathbf{c}$ be a unit vector such that

$$\vartheta(\mathsf{G}+\mathsf{H}) = \sum_{i=1}^{n}(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2 + \sum_{i=1}^{r}(\mathbf{c}^{\mathrm{T}}\mathbf{v}_i)^2. \tag{B.7}$$

By definition, since $(\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{G}+\mathsf{H}}$, it follows that if i and j are nonadjacent vertices in $\overline{\mathsf{G}}$, then they are nonadjacent in $\overline{\mathsf{G}+\mathsf{H}}$ and thus, $\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j = 0$. Hence, $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is an orthonormal representation of $\overline{\mathsf{G}}$. Similarly, it follows that $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is an orthonormal representation of $\overline{\mathsf{H}}$. Thus, by Theorem 2.43,

$$\vartheta(\mathsf{G}+\mathsf{H}) = \sum_{i=1}^{n}(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2 + \sum_{i=1}^{r}(\mathbf{c}^{\mathrm{T}}\mathbf{v}_i)^2 \leq \vartheta(\mathsf{G}) + \vartheta(\mathsf{H}). \tag{B.8}$$

Combining inequalities (B.6) and (B.8) gives the equality in (2.41). □

C. Proof of Theorem 3.8

Theorem 3.8 is a direct corollary of [1, Theorem 4]. To show this, we define an adjacency-reducing mapping.

Definition C.1. Let G be a simple graph, and let $f\colon \mathsf{V}(\mathsf{G}) \to \mathcal{A}$ be a mapping from the vertices of G to a subset $\mathcal{A}$ of $\mathsf{V}(\mathsf{G})$. The mapping f is called *adjacency-reducing* if for every pair of nonadjacent vertices $u, v \in \mathsf{V}(\mathsf{G})$, the vertices $f(u)$ and $f(v)$ are also nonadjacent.

Proof. Let $\mathcal{U} = \{u_1, u_2, \ldots, u_k\}$ be a maximal independent set of vertices in G_1. By assumption, $\mathsf{V}(\mathsf{G}_1)$ can be partitioned into k cliques. Let $C = \{C_1, \ldots, C_k\}$ be such cliques. Obviously, every $u_i, u_j \in \mathcal{U}$ cannot be in the same clique because they are nonadjacent. So, every clique in C contains exactly one

vertex from $\mathcal{U}$. For simplicity, assume that $\mathcal{U}$ and C are ordered in a way that $u_j \in C_j$ for every $j \in [k]$. Next, define the mapping $f : \mathsf{V}(\mathsf{G}_1) \to \mathcal{U}$ as follows. Let $v \in \mathsf{V}(\mathsf{G}_1)$ be a vertex, and let C_j be the clique that has $v \in C_j$. Then, $f(v) = u_j$. If v_1 and v_2 are nonadjacent, then they belong to different cliques in C. Thus, $f(v_1)$ and $f(v_2)$ are mapped to different vertices in $\mathcal{U}$. Since $\mathcal{U}$ is an independent set, $f(v_1)$ and $f(v_2)$ are nonadjacent. Thus, f is an adjacency-reducing mapping of G_1 into $\mathcal{U}$. Then, by [1, Theorem 4], equality (3.12) holds. □

D. The original proof from [10] of Theorem 5.2

Theorem D.1. *[10] Let* G *be a universal graph, and let* H *satisfy* $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$. *The Shannon capacity of* $\mathsf{K} \triangleq \mathsf{G} + \mathsf{H}$ *is not attained at any finite power of* K.

Proof. Let G be a universal graph, and let $k \in \mathbb{N}$. Then, since G is universal,

$$\begin{aligned}
\alpha(\mathsf{K}^{2k}) &= \sum_{\ell=0}^{2k} \binom{2k}{\ell} \alpha(\mathsf{G}^{2k-\ell})\, \alpha(\mathsf{H}^{\ell}) \\
&= \sum_{\ell=0}^{2k} \sum_{0\le i,j\le k; i+j=\ell} \binom{k}{i}\binom{k}{j} \alpha(\mathsf{G}^{2k-(i+j)})\, \alpha(\mathsf{H}^{i+j}) \\
&= \sum_{\ell=0}^{2k} \sum_{0\le i,j\le k; i+j=\ell} \binom{k}{i}\binom{k}{j} \alpha(\mathsf{G})^{2k-(i+j)}\, \alpha(\mathsf{H}^{i+j}) \\
&= \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i}\binom{k}{j} \alpha(\mathsf{G})^{2k-i-j}\, \alpha(\mathsf{H}^{i+j}). \qquad \text{(D.1)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha(\mathsf{K}^{k})^2 &= \left(\sum_{i=0}^{k} \binom{k}{i} \alpha(\mathsf{G})^{k-i}\, \alpha(\mathsf{H}^{i}) \right)^2 \\
&= \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i}\binom{k}{j} \alpha(\mathsf{G})^{2k-i-j}\, \alpha(\mathsf{H}^{i})\, \alpha(\mathsf{H}^{j}). \qquad \text{(D.2)}
\end{aligned}$$

Subtracting (D.2) from (D.1) gives

$$\alpha(\mathsf{K}^{2k}) - \alpha(\mathsf{K}^{k})^2 = \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i}\binom{k}{j} \alpha(\mathsf{G})^{2k-i-j} \left(\alpha(\mathsf{H}^{i+j}) - \alpha(\mathsf{H}^{i})\, \alpha(\mathsf{H}^{j}) \right). \qquad \text{(D.3)}$$

Since by assumption $\Theta(\mathsf{H}) > \alpha(\mathsf{H})$, there exists $i_0, j_0 \in \mathbb{N}$ such that

$$\alpha(\mathsf{H}^{i_0+j_0}) - \alpha(\mathsf{H}^{i_0})\, \alpha(\mathsf{H}^{j_0}) > 0. \qquad \text{(D.4)}$$

Otherwise, by (2.5) and (D.4), for all $m \in \mathbb{N}$, $\alpha(\mathsf{H}^m) = \alpha(\mathsf{H})^m$, so $\Theta(\mathsf{H}) = \alpha(\mathsf{H})$, contradicting our assumption. Hence, $\alpha(\mathsf{K}^{2k}) > \alpha(\mathsf{K}^k)^2$ for all $k \ge \max\{i_0, j_0\}$. Set $k_0 \triangleq \max\{i_0, j_0\}$.

Suppose by contradiction that $\Theta(\mathsf{K}) = \sqrt[k]{\alpha(\mathsf{K}^k)}$ for some $k \geq k_0$. Then, by (2.5), (D.3), and (D.4), we get $\alpha(\mathsf{K}^{2k}) > \alpha(\mathsf{K}^k)^2$, which gives

$$\Theta(\mathsf{K}) \geq \sqrt[2k]{\alpha(\mathsf{K}^{2k})} > \sqrt[k]{\alpha(\mathsf{K}^k)} = \Theta(\mathsf{K}), \tag{D.5}$$

thus leading to a contradiction. Hence, the equality $\Theta(\mathsf{K}) = \sqrt[k]{\alpha(\mathsf{K}^k)}$ cannot hold for any $k \geq k_0$.

Note that if we assume that the equality $\Theta(\mathsf{K}) = \sqrt[k]{\alpha(\mathsf{K}^k)}$ holds for some $k < k_0$, then for every $n \in \mathbb{N}$,

$$\Theta(\mathsf{G}) \geq \sqrt[nk]{\alpha(\mathsf{K}^{nk})} \geq \sqrt[k]{\alpha(\mathsf{K}^k)} = \Theta(\mathsf{G}), \tag{D.6}$$

which, using the same argument for $nk \geq k_0$, leads to a contradiction. Hence, the Shannon capacity of the graph K is not attained by the independence number of any finite strong power of this graph. □

E. The original proof from [7] of inequality (7.9)

Theorem E.1. *[7] Let G be a simple graph on n vertices. Then,*

$$\Theta(\mathsf{G} + \overline{\mathsf{G}}) \geq 2\sqrt{n}. \tag{E.1}$$

Proof. Let $\mathcal{A} \cup \mathcal{B} = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be the vertex set of $\mathsf{G} + \overline{\mathsf{G}}$, where $\mathcal{A}$ and $\mathcal{B}$ are the vertex sets of a copy of G and of its complement $\overline{\mathsf{G}}$, respectively. The labeling is chosen so that, for $i \neq j$, $\{a_i, a_j\} \in \mathsf{E}(\mathsf{G})$ if and only if $\{b_i, b_j\} \notin \mathsf{E}(\overline{\mathsf{G}})$. Moreover, since $\mathsf{G} + \overline{\mathsf{G}}$ is a disjoint union, there are no edges of the form $\{a_i, b_j\}$. For each $k \in \mathbb{N}$, we construct an independent set in the strong power $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$, which is then used to bound the Shannon capacity of $\mathsf{G} + \overline{\mathsf{G}}$. For every k, define $\mathcal{S}_k$ as the set of all vectors $\mathbf{v} = (v_1, v_2, \ldots, v_{2k})$ with coordinates in $\mathcal{A} \cup \mathcal{B}$ that satisfy the following two conditions:

1. $\left|\{i : v_i \in \mathcal{A}\}\right| = \left|\{j : v_j \in \mathcal{B}\}\right| = k$,
2. For every $i \in [k]$, if a_r is the ith coordinate of $\mathbf{v}$ (from left to right) belonging to $\mathcal{A}$, and b_s is the ith coordinate of $\mathbf{v}$ belonging to $\mathcal{B}$, then $r = s$.

Next, we prove that $\mathcal{S}_k$ is an independent set in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vectors in $\mathcal{S}_k$. We consider the following two cases:

- Case 1: If there exists an index t such that $u_t \in \mathcal{A}$ and $v_t \in \mathcal{B}$ (or vice versa), then the vertices $\mathbf{u}$ and $\mathbf{v}$ are clearly nonadjacent in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$.
- Case 2: If there is no such an index, then there exist two indices $1 \leq i, j \leq k$, and two positions $1 \leq r, s \leq 2k$ such that $u_r = a_i$, $u_s = b_i$, $v_r = a_j$, and $v_s = b_j$. Since $\{a_i, a_j\} \in \mathsf{E}(\mathsf{G})$ if and only if $\{b_i, b_j\} \notin \mathsf{E}(\overline{\mathsf{G}})$, the vertices $\mathbf{u}$ and $\mathbf{v}$ are again nonadjacent in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$.

Hence, $\mathcal{S}_k$ is an independent set in $(\mathsf{G} + \overline{\mathsf{G}})^{2k}$. The cardinality of $\mathcal{S}_k$ is $\binom{2k}{k} n^k$. Indeed, there are $\binom{2k}{k}$ ways to choose the positions of the coordinates from $\mathcal{A}$, which uniquely determines the positions of the coordinates from $\mathcal{B}$. For each such choice, there are n possible values for each of the k coordinates, yielding $|\mathcal{S}_k| = \binom{2k}{k} n^k$. Consequently,

$$\alpha((\mathsf{G} + \overline{\mathsf{G}})^{2k}) \geq |\mathcal{S}_k| = \binom{2k}{k} n^k, \tag{E.2}$$

which yields

$$\begin{aligned}\Theta(\mathsf{G}+\overline{\mathsf{G}}) &\geq \lim_{k\to\infty} \sqrt[2k]{\alpha((\mathsf{G}+\overline{\mathsf{G}})^{2k})} \\ &\geq \lim_{k\to\infty} \sqrt[2k]{\binom{2k}{k} n^k} \\ &= 2\sqrt{n}. \end{aligned} \tag{E.3}$$

□

Remark E.2. It is noted in [7] that the inequality $\Theta(\mathsf{G}+\overline{\mathsf{G}}) \geq \sqrt{2n}$, which is a loosened version of inequality (E.1), easily follows for every simple graph G on n vertices. This holds since

$$\{(a_i, b_i)\}_{i\in[n]} \bigcup \{(b_i, a_i)\}_{i\in[n]}$$

is an independent set of the square $(\mathsf{G}+\overline{\mathsf{G}})^2$, consisting of $2n$ elements. Furthermore, it is also noted in [7] that inequality (E.1) is easily obtained for every self-complementary and vertex-transitive graph G. Indeed, by [11, Theorem 12], $\Theta(\mathsf{G}) = \sqrt{n} = \Theta(\overline{\mathsf{G}})$ for every such graph on n vertices, so it follows from Shannon's inequality (2.31) that $\Theta(\mathsf{G}+\overline{\mathsf{G}}) \geq 2\sqrt{n}$. Our strengthened result in Corollary 7.4 asserts that inequality (E.1) holds with equality for all self-complementary vertex-transitive graphs, as well as for some additional graph families.

Author contributions

Nitay Lavi and Igal Sason: Analysis, writing original draft, revision, and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were utilized in creating this article.

Acknowledgments

The authors gratefully acknowledge the timely and constructive reports of the three anonymous referees, which contributed to improving the presentation.

Conflict of interest

Igal Sason served as the Guest Editor of the special issue on "Mathematical Foundations of Information Theory" for *AIMS Mathematics* and was not involved in the editorial review or the decision to publish this article.

The authors declare no conflicts of interest.

References

1. C. E. Shannon, The zero error capacity of a noisy channel, *IEEE T. Inform. Theory*, **2** (1956), 8–19. https://doi.org/10.1109/TIT.1956.1056798
2. N. Alon, *Graph powers*, In: Contemporary Combinatorics (B. Bollobás, Ed.), Bolyai Soc. Math. Stud., Springer, Budapest, Hungary, **10** (2002), 11–28. Available from: `https://www.tau.ac.il/~nogaa/PDFS/cap2.pdf`.
3. N. Alon, *Lovász, vectors, graphs and codes*, In: Building Bridges II—Mathematics of László Lovász (I. Bárány, G. O. H. Katona and A. Sali, Eds.), Bolyai Soc. Math. Stud., Springer, Budapest, Hungary, **28** (2019), 1–16. https://doi.org/10.1007/978-3-662-59204-5_1
4. M. Jurkiewicz, *A survey on known values and bounds on the Shannon capacity*, In: Selected Topics in Modern Mathematics - Edition 2014, eds. G. Gancarzewicz, M. Skrzyński, Publishing House AKAPIT, Kraków, Poland, 2014, 115–128. Available from: `https://repozytorium.biblos.pk.edu.pl/resources/25729`.
5. J. Körner, A. Orlitsky, Zero-error information theory, *IEEE T. Inform. Theory*, **44** (1998), 2207–2229. https://doi.org/10.1109/18.720537
6. I. Csiszár, J. Körner, *Information theory, coding theorems for discrete memoryless systems*, 2 Eds., Cambridge University Press, 2011.
7. N. Alon, The Shannon capacity of a union, *Combinatorica*, **18** (1998), 301–310. https://doi.org/10.1007/PL00009824
8. N. Alon, E. Lubetzky, The Shannon capacity of a graph and the independence numbers of its powers, *IEEE T. Inform. Theory*, **52** (2006), 2172–2176. https://doi.org/10.1109/TIT.2006.872856
9. H. Boche, C. Deppe, Computability of the zero-error capacity of noisy channels, *Information*, **16** (2025), 571. https://doi.org/10.1109/ITW48936.2021.9611383
10. F. Guo, Y. Watanabe, On graphs in which the Shannon capacity is unachievable by finite product, *IEEE T. Inform. Theory*, **36** (1990), 622–623. https://doi.org/10.1109/18.54907
11. L. Lovász, On the Shannon capacity of a graph, *IEEE T. Inform. Theory*, **25** (1979), 1–7. https://doi.org/10.1109/TIT.1979.1055985
12. L. Lovász, *Graphs and geometry*, American Mathematical Society, **65** (2019). https://doi.org/10.1090/coll/065
13. Y. Bi, A. Tang, On upper bounding Shannon capacity of graph through generalized conic programming, *Optim. Lett.*, **13** (2019), 1313–1323. https://doi.org/10.1007/s11590-019-01436-7
14. B. Bukh, C. Cox, On a fractional version of Haemers' bound, *IEEE T. Inform. Theory*, **65** (2019), 3340–3348. https://doi.org/10.1109/TIT.2018.2889108
15. W. H. Haemers, On some problems of Lovász concerning the Shannon capacity of a graph, *IEEE T. Inform. Theory*, **25** (1979), 231–232. https://doi.org/10.1109/TIT.1979.1056027
16. S. Hu, I. Tamo, O. Shayevitz, A bound on the Shannon capacity via a linear programming variation, *SIAM J. Discrete Math.*, **32** (2018), 2229–2241. https://doi.org/10.1137/17M115565X

17. D. E. Knuth, The sandwich theorem, *Electron. J. Combin.*, **1** (1994), 1–48. https://doi.org/10.37236/1193
18. B. Csonka, G. Simonyi, Shannon capacity, Lovász ϑ number, and the Mycielski construction, *IEEE T. Inform. Theory*, **70** (2024), 7632–7646. https://doi.org/10.1109/TIT.2024.3394775
19. G. Simonyi, Shannon capacity and the categorical product, *Electron. J. Combin.*, **28** (2021), 1–23. https://doi.org/10.37236/9113
20. D. de Boer, P. Buys, J. Zuiddam, The asymptotic spectrum distance, graph limits, and the Shannon capacity, *preprint*, 2024. https://doi.org/10.48550/arXiv.2404.16763
21. P. Buys, S. Polak, J. Zuiddam, A group-theoretic approach to Shannon capacity of graphs and a limit theorem from lattice packings, *preprint*, 2025. https://doi.org/10.48550/arXiv.2506.14654
22. J. Zuiddam, The asymptotic spectrum of graphs and the Shannon capacity, *Combinatorica*, **39** (2019), 1173–1184. https://doi.org/10.1007/s00493-019-3992-5
23. A. Wigderson, J. Zuiddam, Asymptotic spectra: Theory, applications and extensions, *preprint*, 2025. Available from: `https://staff.fnwi.uva.nl/j.zuiddam//papers/convexity.pdf`.
24. V. Strassen, The asymptotic spectrum of tensors, *J. Reine Angew. Math.*, **384** (1988), 102–152. https://doi.org/10.1515/crll.1988.384.102
25. A. Abiad, C. Dalfó, M. A. Fiol, The Shannon capacity of graph powers, *IEEE T. Inform. Theory*, **72** (2026), 100–111. https://doi.org/10.1109/TIT.2025.3628011
26. S. C. Polak, A. Schrijver, New lower bound on the Shannon capacity of C_7 from circular graphs, *Inform. Process. Lett.*, **143** (2019), 37–40. https://doi.org/10.1016/j.ipl.2018.11.006
27. T. Bohman, A limit theorem for the Shannon capacities of odd cycles. II, *Proc. Amer. Math. Soc.*, **133** (2005), 537–543. Available from: `https://www.jstor.org/stable/4097960`.
28. A. Schrijver, On the Shannon capacity of sums and products of graphs, *Indag. Math.*, **34** (2023), 37–41. https://doi.org/10.1016/j.indag.2022.08.009
29. C. Godsil, K. Meagher, *Erdős-Ko-Rado theorems*, Cambridge University Press, 2016. https://doi.org/10.1017/CBO9781316414958
30. B. Lv, K. Wang, The eigenvalues of q-Kneser graphs, *Discrete Math.*, **312** (2012), 1144–1147. https://doi.org/10.1016/j.disc.2011.11.042
31. R. Hammack, W. Imrich, S. Klavžar, *Handbook of product graphs*, 2 Eds., 2011. https://doi.org/10.1201/b10959
32. R. S. Hales, Numerical invariants and the strong product of graphs, *J. Combin. Theory Ser. B*, **15** (1973), 146–155. https://doi.org/10.1016/0095-8956(73)90014-2
33. C. Godsil, G. Royle, *Algebraic graph theory*, Springer, 2001. https://doi.org/10.1007/978-1-4613-0163-9
34. L. Lovász, *personal communication*, January 2026.
35. I. Sason, An example showing that Schrijver's ϑ-function need not upper bound the Shannon capacity of a graph, *AIMS Math.*, **10** (2025), 15294–15301. https://doi.org/10.3934/math.2025685
36. I. Sason, Observations on Lovász ϑ-function, graph capacity, eigenvalues, and strong products, *Entropy*, **25** (2023), 104. https://doi.org/10.3390/e25010104

37. E. R. Scheinerman, D. H. Ullman, *Fractional graph theory: A rational approach to the theory of graphs*, Dover Publications, 2013. Available from: `https://www.ams.jhu.edu/ers/wp-content/uploads/2015/12/fgt.pdf`.

38. A. E. Brouwer, A. Schrijver, *Uniform hypergraphs*, in: A. Schrijer (ed.), Packing and Covering in Combinatorics, Mathematical Centre Tracts 106, Amsterdam, 1979, 39–73.

39. I. Sason, Observations on graph invariants with the Lovász ϑ-function, *AIMS Math.*, **9** (2024), 15385–15468. https://doi.org/10.3934/math.2024747

40. *The Sage Developers, SageMath, the Sage Mathematics Software System*, Version 9.3, August 2021. Available from: `https://www.sagemath.org/`.

41. T. Bohman, R. Holzman, V. Natarajan, On the independence numbers of the cubes of odd cycles, *Electron. J. Combin.*, **20** (2013), P10. https://doi.org/10.37236/2598

42. N. Charpenay, M. Le Treust, *Zero-error coding with a generator set of variable-length words*, In: Proceedings of the 2020 IEEE International Symposium on Inform. Theory, Los Angeles, CA, USA, 2020, 78–83. https://doi.org/10.1109/ISIT44484.2020.9174059

43. M. Aigner, G. M. Ziegler, *Proofs from the book*, 6 Eds., Springer, Berlin, Germany, 2018. https://doi.org/10.1007/978-3-662-57265-8

44. W. Dörfler, W. Imrich, Über das starke produkt von graphen, *Monatsh. Math.*, **74** (1970), 97–102.

45. R. McKenzie, Cardinal multiplication of structures with a reflexive relation, *Fund. Math.*, **70** (1971), 59–101. Available from: `https://scispace.com/pdf/cardinal-multiplication-of-structures-with-a-reflexive-1bw5w0tzrt.pdf`.

46. J. Feigenbaum, A. A. Schäffer, Finding the prime factors of strong direct product graphs in polynomial time, *Discrete Math.*, **109** (1992), 77–102. https://doi.org/10.1016/0012-365X(92)90280-S

47. N. Charpenay, M. Le Treust, A. Roumy, On the additivity of optimal rates for independent zero-error source and channel problems, *IEEE T. Inform. Theory*, 2025. https://doi.org/10.1109/TIT.2025.3639292

48. K. Marton, On the Shannon capacity of probabilistic graphs, *J. Comb. Theory B*, **57** (1993), 183–195. https://doi.org/10.1006/jctb.1993.1015

49. C. Greene, D. J. Kleitman, *Proof techniques in the theory of finite sets*, in MAA Studies in Math. Assoc. of Amer., Washington, D.C., 1978. Available from: `https://www.semanticscholar.org/paper/Proof-techniques-in-the-theory-of-finite-sets-Greene-Kleitman/d6d94511006c1f9a7e9212cbb05f636060cebd78`.

50. S. Jukna, *Extremal combinatorics with applications in computer science*, 2 Eds., Springer, 2011. https://doi.org/10.1007/978-3-642-17364-6

51. L. Lovász, On the ratio of optimal integral and fractional covers, *Discrete Math.*, **13** (1975), 383–390. https://doi.org/10.1016/0012-365X(75)90058-8

52. T. J. J. Mussche, *Extremal combinatorics in generalized Kneser graphs*, Discrete Algebra and Geometry, Technische Universiteit Eindhoven, 2009. https://doi.org/10.6100/IR642440

53. A. Acín, R. Duanc, D. E. Roberson, A. B. Sainz, A. Winter, A new property of the Lovász number and duality relations between graph parameters, *Discrete Appl. Math.*, **216** (2017), 489–501. https://doi.org/10.1016/j.dam.2016.04.028

54. B. Bollobás, *Random graphs*, 2 Eds., Cambridge University Press, 2001. https://doi.org/10.1017/CBO9780511814068

55. F. Juhász, On the asymptotic behavior of the Lovász theta function for random graphs, *Combinatorica*, **2** (1982), 153–155. https://doi.org/10.1007/BF02579314

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)