Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 2722–2746.
DOI: 10.3934/math.2026110
Received: 05 January 2026
Revised: 22 January 2026
Accepted: 26 January 2026
Published: 28 January 2026

Research article

Coupled system of (k, ψ)-Caputo fractional boundary value problems involving multi-point fractional closed boundary conditions

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Abstract: This work is devoted to the analysis of a coupled system of (k, ψ)-Caputo fractional differential equations equipped with fractional closed boundary conditions. We establish the uniqueness of solutions by applying Banach's contraction principle, and existence is obtained through the Leray–Schauder alternative. Illustrative examples are included to demonstrate the effectiveness and practical relevance of the theoretical results.

Keywords: coupled systems; (k, ψ)-Caputo fractional derivative; existence and uniqueness; fixed point theory; Leray–Schauder alternative
Mathematics Subject Classification: 34A08, 34B10, 34B15

1. Introduction

Fractional calculus is a rapidly developing area of mathematical analysis that extends classical differentiation and integration to noninteger orders. This broader framework is well suited for modeling nonlocal phenomena and memory-dependent processes, which arise in numerous applications such as diffusion process, viscoelasticity, control theory, biological systems, immune systems, financial modeling, etc. A detailed treatment of the theory and applications of fractional calculus can be found in [1–3].

Fractional integrals and derivatives play a central role in modeling memory and nonlocal effects across diverse fields such as physics, biology, chemistry, and finance. This wide applicability has led to the development of numerous fractional operators tailored to different dynamical behaviors. Hilfer [4] introduced a unifying operator interpolating between the Riemann–Liouville and Caputo derivatives systematically presented in [1] and elaborated its applications in physics.

Further generalizations have incorporated differentiation with respect to auxiliary functions and extensions within k-calculus. Notable contributions include the Caputo-type derivative with respect to another function, ψ-Hilfer derivative and (k, ψ)-fractional operators [5–7]. These operators have recently been employed in the study of boundary value problems, coupled systems, Langevin-type equations, variational models, and the controllability of fractional dynamical systems [8–10].

Closed boundary conditions are fundamental in the modeling of fluid flows, as they describe systems with zero mass transfer across the boundary. They commonly represent impermeable or thermally insulated boundaries, including free-slip configurations that allow tangential motion while restricting normal flow. Such boundary conditions arise in a wide range of applications, including gravitational and radiative processes, elastic wave propagation, and heat transfer, and they are extensively used in computational fluid dynamics, image processing, and transport phenomena in structured media [11–13]. Although the settings in these studies differ from those examined here, they demonstrate the widespread applicability and practical importance of closed boundary formulations in both theoretical and applied contexts.

The study of boundary value problems associated with fractional differential equations and differential inclusions subject to closed boundary conditions was first developed in [14]. In that work, the authors focused on the analysis of the following family of problems:

$$\begin{cases} {}^{C}D^{\mu}y(t) = f(t, y(t)), \quad t \in J := [0, T], \\ {}^{C}D^{\mu}y(t) \in F(t, y(t)), \quad t \in J := [0, T], \\ y(T) = p_1 y(0) + p_2 T y'(0), \\ T y'(T) = q_1 y(0) + q_2 T y'(0), \end{cases}$$

where ${}^{C}D^{\mu}$ denotes the Caputo fractional derivative of order μ, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map (here we denote the family of all nonempty subsets of $\mathbb{R}$ by $\mathcal{P}(\mathbb{R})$), and $p_1, p_2, q_1, q_2 \in \mathbb{R}$.

In [15], the authors studied a system of nonlinear Caputo fractional differential equations supplemented with coupled closed boundary conditions,

$$\begin{cases} {}^{C}D^{q_1}\varphi(t) = \rho_1(t, \varphi(t), \psi(t)), \qquad t \in \mathcal{J} = [0, T], \\ {}^{C}D^{q_2}\psi(t) = \rho_2(t, \varphi(t), \psi(t)), \qquad t \in \mathcal{J} = [0, T], \\ \varphi(T) = \alpha_1 \psi(0) + \beta_1 T \psi'(0), \quad T\varphi'(T) = \gamma_1 \psi(0) + \delta_1 T \psi'(0), \\ \psi(T) = \alpha_2 \varphi(0) + \beta_2 T \varphi'(0), \quad T\psi'(T) = \gamma_2 \varphi(0) + \delta_2 T \varphi'(0), \end{cases}$$

where ${}^{C}D^{q_1}$, ${}^{C}D^{q_2}$ denote the Caputo fractional derivatives of order q_1, q_2, $1 < q_1, q_2 < 2$, respectively, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$, $T > 0$, and $\rho_1, \rho_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

A comprehensive overview of fractional boundary value problems with closed-type boundary conditions is available in the recent survey reported in [16].

Moreover, the authors of [16] analyzed sufficient criteria ensuring the existence and uniqueness of solutions to a (k,ψ)-Caputo fractional differential equation equipped with multipoint fractional closed boundary conditions of the form

$$\begin{cases} {}^{k,C}D_{a^+}^{\vartheta;\psi}\varphi(\varkappa) = \nu(\varkappa, \varphi(\varkappa)), \quad a \le \varkappa \le b, \\ \varphi(b) = \sum_{i=1}^{m} \left(\eta_i\, \varphi(\xi_i) + b\,\mu_i\, {}^{k,C}D_{a^+}^{\sigma_i;\psi}\varphi(\zeta_i)\right), \\ b\, {}^{k,C}D_{a^+}^{\gamma;\psi}\varphi(b) = \sum_{i=1}^{m} \left(p_i\, \varphi(\xi_i) + b\, q_i\, {}^{k,C}D_{a^+}^{\sigma_i;\psi}\varphi(\zeta_i)\right), \end{cases} \tag{1.1}$$

where ${}^{k,C}D_{a^+}^{\vartheta;\psi}$ and ${}^{k,C}D_{a^+}^{\gamma;\psi}$ represent the (k,ψ)-Caputo fractional derivatives of orders satisfying $1 < \vartheta/k < 2$ and $0 < \gamma/k < 1$, respectively. For a fixed $k > 0$, the operator ${}^{k,C}D_{a^+}^{\sigma_i;\psi}$ denotes the (k,ψ)-Caputo fractional derivative of order $0 < \sigma_i/k < 1$, for each $i = 1, 2, \ldots, m$. The interior points are chosen such that $a < \xi_1 < \xi_2 < \cdots < \xi_m < b$ and $a < \zeta_1 < \zeta_2 < \cdots < \zeta_m < b$. All coefficients η_i, μ_i, p_i, q_i are real constants, and the nonlinear term $\nu : [a,b] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous.

The aim of this study is to further develop the current body of research by analyzing the coupled framework associated with problem (1.1). In precise terms, we investigate a system of coupled (k,ψ)-Caputo fractional differential equations subject to multipoint closed fractional boundary conditions of the following type:

$$\begin{cases} {}^{k_1,C}D_{a^+}^{\vartheta_1;\psi_1}x_1(t) = \nu_1(t, x_1(t), x_2(t)), \quad a \le t \le b, \\ {}^{k_2,C}D_{a^+}^{\vartheta_2;\psi_2}x_2(t) = \nu_2(t, x_1(t), x_2(t)), \quad a \le t \le b, \\ x_1(b) = \sum_{i=1}^{m} \left(\eta_i\, x_2(\xi_i) + b\,\mu_i\, {}^{k_2,C}D_{a^+}^{\sigma_i;\psi_2}x_2(\zeta_i)\right), \\ x_2(b) = \sum_{i=1}^{m} \left(\beta_i\, x_1(\xi_i) + b\,\alpha_i\, {}^{k_1,C}D_{a^+}^{\varphi_i;\psi_1}x_1(\zeta_i)\right), \\ b\, {}^{k_1,C}D_{a^+}^{\gamma_1;\psi_1}x_1(b) = \sum_{i=1}^{m} \left(p_i\, x_2(\xi_i) + b\, q_i\, {}^{k_2,C}D_{a^+}^{\sigma_i;\psi_2}x_2(\zeta_i)\right), \\ b\, {}^{k_2,C}D_{a^+}^{\gamma_2;\psi_2}x_2(b) = \sum_{i=1}^{m} \left(r_i\, x_1(\xi_i) + b\, s_i\, {}^{k_1,C}D_{a^+}^{\varphi_i;\psi_1}x_1(\zeta_i)\right), \end{cases} \tag{1.2}$$

where ${}^{k_j,C}D_{a^+}^{\vartheta_j;\psi_j}$, ${}^{k_j,C}D_{a^+}^{\gamma_j;\psi_j}$ denote the (k_j,ψ_j)-Caputo fractional derivative operators of orders $1 < \frac{\vartheta_j}{k_j} < 2$ and $0 < \frac{\gamma_j}{k_j} < 1$ for $j = 1, 2$, respectively, and ${}^{k_j,C}D_{a^+}^{\sigma_i;\psi_j}$, ${}^{k_j,C}D_{a^+}^{\varphi_i;\psi_j}$ represent the (k_j,ψ_j)-Caputo fractional derivative operators of orders $0 < \frac{\sigma_i}{k_1} < 1$ and $0 < \frac{\varphi_i}{k_2} < 1$ for all $i = 1, 2, \cdots, m$ and $j = 1, 2$, respectively, for each $k > 0$. Moreover, $a < \xi_1 < \xi_2 < \cdots < \xi_m < b$, $a < \zeta_1 < \zeta_2 < \cdots < \zeta_m < b$, $\eta_i, \mu_i, \beta_i, \alpha_i, p_i, q_i, r_i, s_i \in \mathbb{R}$ for all $i = 1, 2, \cdots, m$, and $x_1, x_2 : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Systems of coupled fractional differential equations are of considerable importance in modeling complex processes encountered in financial mathematics, bioengineering and fractional dynamics [2, 17, 18]. Several relevant analytical results for such coupled frameworks have been reported in the literature; see, for example, [19, 20].

It is worth noting that, for specific choices of ψ and k, the (k,ψ)-Caputo fractional derivative reduces to several well-known operators. For instance, setting $\psi(t) = t$ and $k = 1$ recovers the classical Caputo derivative, whereas choosing $\psi(t) = \log t$ with $k = 1$ yields the Caputo–Hadamard derivative. The present study is motivated by the aim of unifying two key features:

(i) a coupled system of fractional differential equations formulated via the (k,ψ)-Caputo operator, and

(ii) closed fractional boundary conditions applied at multiple points— a setting that, to the best of our knowledge, has not yet been explored in the literature.

The findings obtained in this work are original and constitute a meaningful addition to the existing literature on coupled fractional systems, especially those characterized by fractional closed boundary conditions.

The paper is structured as follows. In Section 2, we present the essential background from fractional calculus, along with the notation, definitions, and core concepts required for the subsequent analysis. Section 3 focuses on the solvability of the proposed system, where uniqueness of solutions is established through Banach's contraction principle, and the existence result is derived by means of the Leray–Schauder alternative. Finally, Section 4 contains examples that illustrate the relevance and practical applicability of the obtained theoretical results.

2. Preliminaries

In this section, we recall some related definitions of fractional calculus required to obtain our main results.

Definition 2.1. *[21]. For $\vartheta \in \mathbb{C}$ with a positive real part and $k \in \mathbb{R}^+$, the k-gamma function is defined by*

$$\Gamma_k(\vartheta) = \int_0^\infty \tau^{\vartheta-1} e^{-\frac{\tau^k}{k}}\, d\tau.$$

Moreover, the following relations hold:

$$\Gamma(\vartheta) = \lim_{k\to 1}\Gamma_k(\vartheta), \quad \Gamma_k(\vartheta) = k^{\frac{\vartheta}{k}-1}\Gamma\left(\frac{\vartheta}{k}\right) \quad \textit{and} \quad \vartheta\,\Gamma_k(\vartheta) = \Gamma_k(\vartheta + k).$$

Definition 2.2. *[22]. Let $x : [a,b] \to \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on $(a,b]$, having a continuous derivative ψ' on (a,b). Then, the (k,ψ)-Riemann–Liouville fractional integral of a function x with respect to another function ψ on $[a,b]$ of order ϑ and $k > 0$ is defined by*

$${}^{k}I_{a^+}^{\vartheta;\psi} x(t) = \frac{1}{k\Gamma_k(\vartheta)} \int_a^t \psi'(\tau)\,(\psi(t) - \psi(\tau))^{\frac{\vartheta}{k}-1}\, x(\tau)\, d\tau, \quad t > a.$$

Definition 2.3. *[7]. Let $\vartheta, k \in \mathbb{R}^+$, $\psi \in C^n([a,b],\mathbb{R})$ be such that ψ is increasing, and $\psi'(t) > 0$ for all $t \in [a,b]$ and $x \in C^n([a,b],\mathbb{R})$. Then, the (k,ψ)-Caputo fractional derivative of order ϑ for a function x is defined by*

$${}^{k,C}D_{a^+}^{\vartheta;\psi} x(t) = \frac{1}{k\Gamma_k(nk - \vartheta)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{n-\frac{\vartheta}{k}-1} \left(\frac{k}{\psi'(\tau)}\frac{d}{d\tau}\right)^n x(\tau)\, d\tau,$$

where $n = \left\lceil \frac{\vartheta}{k} \right\rceil$ *is the ceiling function of* $\frac{\vartheta}{k}$.

Lemma 2.4. *[23]. Let* $\vartheta, k \in \mathbb{R}^+$ *and* $n = \left\lceil \frac{\vartheta}{k} \right\rceil$. *Suppose that* $x \in C^n([a,b],\mathbb{R})$, *Then,*

$$\left({}^{k}I_{a^+}^{\vartheta;\psi}\,{}^{k,C}D_{a^+}^{\vartheta;\psi}\,x\right)(t) = x(t) - \sum_{j=0}^{n-1} \frac{(\psi(t)-\psi(a))^j}{\Gamma_k(jk+k)} \left[\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^j x(t)\right]_{t=a}.$$

Lemma 2.5. *[23]. Let* $\vartheta_1, \vartheta_2, k \in \mathbb{R}^+$ *with* $\vartheta_2 > \vartheta_1$. *Then*

$${}^{k,C}D_{a^+}^{\vartheta_1;\psi}\,{}^{k}I_{a^+}^{\vartheta_2;\psi} x(t) = {}^{k}I_{a^+}^{\vartheta_2-\vartheta_1;\psi} x(t).$$

Lemma 2.6. *[23]. Let* $\vartheta, k \in \mathbb{R}^+$ *and* $\mu \in \mathbb{R}$ *such that* $\frac{\mu}{k} > -1$; *then,*

$${}^{k,C}D_{a^+}^{\vartheta;\psi}\,(\psi(t)-\psi(a))^{\frac{\mu}{k}} = \frac{\Gamma_k(\mu+k)}{\Gamma_k(\mu+k-\vartheta)}\,(\psi(t)-\psi(a))^{\frac{\mu-\vartheta}{k}}.$$

The following lemma, dealing with a linear variant of the coupled system (1.2), plays an important role in the forthcoming analysis.

Lemma 2.7. *Assume that* $\Delta \neq 0$; $k_j > 0$; $1 < \frac{\vartheta_j}{k_j} < 2$; $0 < \frac{\gamma_j}{k_j} < 1$; *for* $j = 1,2$; $a < \xi_1 < \xi_2 < \cdots < \xi_m < b$; $a < \zeta_1 < \zeta_2 < \cdots < \zeta_m < b$; $\eta_i, \mu_i, \beta_i, \alpha_i, p_i, q_i, r_i, s_i \in \mathbb{R}$; $0 < \frac{\sigma_i}{k_1} < 1$; *and* $0 < \frac{\varphi_i}{k_2} < 1$ *for all* $i = 1,2,\cdots,m$, *and* $z_j \in C([a,b],\mathbb{R})$ *for* $j = 1,2$. *Then, the solution of the linear system*

$$\begin{cases} {}^{k_1,C}D_{a^+}^{\vartheta_1;\psi_1} x_1(t) = z_1(t), \quad a \le t \le b, \\ {}^{k_2,C}D_{a^+}^{\vartheta_2;\psi_2} x_2(t) = z_2(t) \quad a \le t \le b, \\ x_1(b) = \displaystyle\sum_{i=1}^{m} \left(\eta_i\, x_2(\xi_i) + b\,\mu_i\,{}^{k_2,C}D_{a^+}^{\sigma_i;\psi_2} x_2(\zeta_i)\right), \\ x_2(b) = \displaystyle\sum_{i=1}^{m} \left(\beta_i\, x_1(\xi_i) + b\,\alpha_i\,{}^{k_1,C}D_{a^+}^{\varphi_i;\psi_1} x_1(\zeta_i)\right), \\ b\,{}^{k_1,C}D_{a^+}^{\gamma_1;\psi_1} x_1(b) = \displaystyle\sum_{i=1}^{m} \left(p_i\, x_2(\xi_i) + b\,q_i\,{}^{k_2,C}D_{a^+}^{\sigma_i;\psi_2} x_2(\zeta_i)\right), \\ b\,{}^{k_2,C}D_{a^+}^{\gamma_2;\psi_2} x_2(b) = \displaystyle\sum_{i=1}^{m} \left(r_i\, x_1(\xi_i) + b\,s_i\,{}^{k_1,C}D_{a^+}^{\varphi_i;\psi_1} x_1(\zeta_i)\right), \end{cases} \tag{2.1}$$

is given by a pair of the integral equations,

$$\begin{aligned} x_1(t) = {}&{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(t) + \frac{1}{\Delta}\left[\Upsilon_1 + \Upsilon_2 \frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\Bigg\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(b) \\ &- \sum_{i=1}^{m}\left(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(\xi_i) + b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} z_2(\zeta_i)\right)\Bigg\} \\ &+ \frac{1}{\Delta}\left[\Upsilon_3 + \Upsilon_4 \frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\Bigg\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(b) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} \Big(\beta_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(\xi_i) + b\, \alpha_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} z_1(\zeta_i) \Big) \Bigg\} \\
&+ \frac{1}{\Delta}\Bigg[\Upsilon_5 + \Upsilon_6 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)} \Bigg] \Bigg\{ b \, {}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1} z_1(b) \\
&- \sum_{i=1}^{m} \Big(p_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(\xi_i) + b\, q_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} z_2(\zeta_i) \Big) \Bigg\} \\
&+ \frac{1}{\Delta}\Bigg[\Upsilon_7 + \Upsilon_8 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)} \Bigg] \Bigg\{ b \, {}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2} z_2(b) \\
&- \sum_{i=1}^{m} \Big(r_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(\xi_i) + b\, s_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} z_1(\zeta_i) \Big) \Bigg\},
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
x_2(t) = {}& {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(t) + \frac{1}{\Delta}\Bigg[\Upsilon_9 + \Upsilon_{10} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} \Bigg] \Bigg\{ {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(b) \\
&- \sum_{i=1}^{m} \Big(\eta_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(\xi_i) + b\, \mu_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} z_2(\zeta_i) \Big) \Bigg\} \\
&+ \frac{1}{\Delta}\Bigg[\Upsilon_{11} + \Upsilon_{12} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} \Bigg] \Bigg\{ {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(b) \\
&- \sum_{i=1}^{m} \Big(\beta_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(\xi_i) + b\, \alpha_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} z_1(\zeta_i) \Big) \Bigg\} \\
&+ \frac{1}{\Delta}\Bigg[\Upsilon_{13} + \Upsilon_{14} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} \Bigg] \Bigg\{ b \, {}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1} z_1(b) \\
&- \sum_{i=1}^{m} \Big(p_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(\xi_i) + b\, q_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} z_2(\zeta_i) \Big) \Bigg\} \\
&+ \frac{1}{\Delta}\Bigg[\Upsilon_{15} + \Upsilon_{16} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} \Bigg] \Bigg\{ b \, {}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2} z_2(b) \\
&- \sum_{i=1}^{m} \Big(r_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(\xi_i) + b\, s_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} z_1(\zeta_i) \Big) \Bigg\},
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
\Upsilon_1 &= \Big[-Q_5Q_8Q_{12} + Q_7Q_{12} - Q_9Q_{11} + Q_6Q_8Q_{11} \Big], \\
\Upsilon_2 &= \Big[-Q_1Q_8Q_{12} + Q_2Q_7Q_{12} - Q_2Q_9Q_{11} + Q_3Q_8Q_{11} \Big], \\
\Upsilon_3 &= \Big[-Q_1Q_{12} + Q_2Q_5Q_{12} - Q_2Q_6Q_{11} + Q_3Q_{11} \Big], \\
\Upsilon_4 &= \Big[-Q_1Q_9 + Q_1Q_6Q_8 + Q_2Q_5Q_9 - Q_2Q_6Q_7 - Q_3Q_5Q_8 + Q_3Q_7 \Big], \\
\Upsilon_5 &= \Big[Q_4Q_8Q_{12} + Q_9Q_{10} - Q_6Q_8Q_{10} \Big], \\
\Upsilon_6 &= \Big[Q_8Q_{12} + Q_2Q_9Q_{10} - Q_3Q_8Q_{10} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_7 &= \Big[Q_{12} - Q_2Q_4Q_{12} + Q_2Q_6Q_{10} - Q_3Q_{10}\Big],\\
\Upsilon_8 &= \Big[Q_9 - Q_6Q_8 - Q_2Q_4Q_9 + Q_3Q_4Q_8\Big],\\
\Upsilon_9 &= \Big[Q_4Q_7Q_{12} - Q_4Q_9Q_{11} + Q_5Q_9Q_{10} - Q_6Q_7Q_{10}\Big],\\
\Upsilon_{10} &= \Big[Q_7Q_{12} - Q_9Q_{11} + Q_1Q_9Q_{10} - Q_3Q_7Q_{10}\Big],\\
\Upsilon_{11} &= \Big[Q_5Q_{12} - Q_6Q_{11} - Q_1Q_4Q_{12} + Q_1Q_6Q_{10} + Q_3Q_4Q_{11} - Q_3Q_5Q_{10}\Big],\\
\Upsilon_{12} &= \Big[Q_5Q_9 - Q_6Q_7 - Q_1Q_4Q_9 + Q_3Q_4Q_7\Big],\\
\Upsilon_{13} &= \Big[Q_4Q_8Q_{11} - Q_5Q_8Q_{10} + Q_7Q_{10}\Big],\\
\Upsilon_{14} &= \Big[Q_8Q_{11} - Q_1Q_8Q_{10} + Q_2Q_7Q_{10}\Big],\\
\Upsilon_{15} &= \Big[Q_{11} - Q_1Q_{10} - Q_2Q_4Q_{11} + Q_2Q_5Q_{10}\Big],\\
\Upsilon_{16} &= \Big[-Q_5Q_8 + Q_7 + Q_1Q_4Q_8 - Q_2Q_4Q_7\Big],
\end{aligned}
\tag{2.4}
$$

with

$$
\begin{aligned}
Q_1 &= \frac{(\psi_1(b) - \psi_1(a))}{\Gamma_{k_1}(2k_1)},\\
Q_2 &= \sum_{i=1}^{m}\eta_i,\\
Q_3 &= \sum_{i=1}^{m}\left(\eta_i\frac{(\psi_2(\xi_i) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} + b\,\mu_i\frac{(\psi_2(\zeta_i) - \psi_2(a))^{1-\frac{\sigma_i}{k_2}}}{\Gamma_{k_2}(2k_2 - \sigma_i)}\right),\\
Q_4 &= \sum_{i=1}^{m}\beta_i,\\
Q_5 &= \sum_{i=1}^{m}\left(\beta_i\frac{(\psi_1(\xi_i) - \psi_1(a))}{\Gamma_{k_1}(2k_1)} + b\,\alpha_i\frac{(\psi_1(\zeta_i) - \psi_1(a))^{1-\frac{\varphi_i}{k_1}}}{\Gamma_{k_1}(2k_1 - \varphi_i)}\right),\\
Q_6 &= \frac{(\psi_2(b) - \psi_2(a))}{\Gamma_{k_2}(2k_2)},\\
Q_7 &= b\,\frac{(\psi_1(b) - \psi_1(a))^{1-\frac{\gamma_1}{k_1}}}{\Gamma_{k_1}(2k_1 - \gamma_1)},\\
Q_8 &= \sum_{i=1}^{m}p_i,\\
Q_9 &= \sum_{i=1}^{m}\left(p_i\frac{(\psi_2(\xi_i) - \psi_2(a))}{\Gamma_{k_2}(2k_2)} + b\,q_i\frac{(\psi_2(\zeta_i) - \psi_2(a))^{1-\frac{\sigma_i}{k_2}}}{\Gamma_{k_2}(2k_2 - \sigma_i)}\right),\\
Q_{10} &= \sum_{i=1}^{m}r_i,\\
Q_{11} &= \sum_{i=1}^{m}\left(r_i\frac{(\psi_1(\xi_i) - \psi_1(a))}{\Gamma_{k_1}(2k_1)} + b\,s_i\frac{(\psi_1(\zeta_i) - \psi_1(a))^{1-\frac{\varphi_i}{k_1}}}{\Gamma_{k_1}(2k_1 - \varphi_i)}\right),
\end{aligned}
$$

$$Q_{12} = b\,\frac{(\psi_2(b)-\psi_2(a))^{1-\frac{\gamma_2}{k_2}}}{\Gamma_{k_2}(2k_2-\gamma_2)}, \tag{2.5}$$

and

$$\begin{aligned}\Delta = {} & Q_5Q_8Q_{12} - Q_7Q_{12} + Q_9Q_{11} - Q_6Q_8Q_{11} - Q_1Q_4Q_8Q_{12} - Q_1Q_9Q_{10} + Q_1Q_6Q_8Q_{10} \\ & + Q_2Q_4Q_7Q_{12} - Q_2Q_4Q_9Q_{11} + Q_2Q_5Q_9Q_{10} - Q_2Q_6Q_7Q_{10} + Q_3Q_4Q_8Q_{11} \\ & - Q_3Q_5Q_8Q_{10} + Q_3Q_7Q_{10}.\end{aligned}$$

Proof. Applying the (k_j,ψ_j)-Riemann fractional integral for $j=1,2$ to both sides of Eq (1.2) and using Lemma 2.4, we obtain

$$x_1(t) = {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} z_1(t) + c_0 + \frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}c_1 \tag{2.6}$$

and

$$x_2(t) = {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} z_2(t) + d_0 + \frac{(\psi_2(t)-\psi_2(a))}{\Gamma_{k_2}(2k_2)}d_1, \tag{2.7}$$

where

$$\begin{aligned} c_0 &= [z_1(t)]_{t=a}, \quad c_1 = \left[\left(\frac{k_1}{\psi_1'(t)}\frac{d}{dt}\right)z_1(t)\right]_{t=a}, \\ d_0 &= [z_2(t)]_{t=a}, \quad d_1 = \left[\left(\frac{k_2}{\psi_2'(t)}\frac{d}{dt}\right)z_2(t)\right]_{t=a}. \end{aligned}$$

Now, using (2.6) and (2.7) in the multipoint closed boundary conditions in (2.1) together with the notations in (2.5) and Lemma 2.5, we obtain the following system:

$$\begin{cases} -c_0 - Q_1\,c_1 + Q_2\,d_0 + Q_3\,d_1 = {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(b) - \sum\limits_{i=1}^{m}\left(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i) + b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\right), \\ Q_4\,c_0 + Q_5\,c_1 - d_0 - Q_6\,d_1 = {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(b) - \sum\limits_{i=1}^{m}\left(\beta_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i) + b\,\alpha_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\right), \\ -Q_7\,c_1 + Q_8\,d_0 + Q_9\,d_1 = b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}z_1(b) - \sum\limits_{i=1}^{m}\left(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i) + b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\right), \\ Q_{10}\,c_0 + Q_{11}\,c_1 - Q_{12}\,d_1 = b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}z_2(b) - \sum\limits_{i=1}^{m}\left(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i) + b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\right). \end{cases} \tag{2.8}$$

Solving system (2.8) for c_0, c_1, d_0, and d_1 and using the notations in (2.4), we find that

$$\begin{aligned} c_0 = \frac{1}{\Delta}\Bigg[& \Upsilon_1\Bigg\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(b) - \sum_{i=1}^{m}\left(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i) + b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\right) \\ & + \Upsilon_3\Bigg\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(b) - \sum_{i=1}^{m}\left(\beta_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i) + b\,\alpha_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\right)\Bigg\} \\ & + \Upsilon_5\Bigg\{b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}z_1(b) - \sum_{i=1}^{m}\left(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i) + b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\right)\Bigg\} \end{aligned}$$

$$+\Upsilon_7\Big\{b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}\Big],$$

$$c_1=\frac{1}{\Delta}\Big[\Upsilon_2\Big\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)$$
$$+\Upsilon_4\Big\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(\beta_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,\alpha_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_6\Big\{b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_8\Big\{b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}\Big],$$

$$d_0=\frac{1}{\Delta}\Big[\Upsilon_9\Big\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)$$
$$+\Upsilon_{11}\Big\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(\beta_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,\alpha_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_{13}\Big\{b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_{15}\Big\{b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}\Big],$$

and

$$d_1=\frac{1}{\Delta}\Big[\Upsilon_{10}\Big\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(\eta_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,\mu_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)$$
$$+\Upsilon_{12}\Big\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(\beta_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,\alpha_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_{14}\Big\{b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}z_1(b)-\sum_{i=1}^{m}\Big(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}z_2(\xi_i)+b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}z_2(\zeta_i)\Big)\Big\}$$
$$+\Upsilon_{16}\Big\{b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}z_2(b)-\sum_{i=1}^{m}\Big(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}z_1(\xi_i)+b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}z_1(\zeta_i)\Big)\Big\}\Big].$$

Replacing c_0, c_1, d_0, and d_1 in (2.6) and (2.7) with the above values, we obtain the solution of (2.2) and (2.3). The converse of the lemma can be verified by direct computation. Thus, the proof is completed. □

3. Main results

Let us introduce the space $X_1=\{x_1(t)\mid x_1(t)\in C([a,b],\mathbb{R})\}$ endowed with the norm $\|x_1\|_{X_1}=\sup\{|x_1(t)|:t\in[a,b]\}$. Obviously, $(X_1,\|\cdot\|_{X_1})$ is a Banach space, and consequently, the product space

$(X_1 \times X_2, \|\cdot\|_{X_1\times X_2})$ is also a Banach space with the norm $\|(x_1, x_2)\|_{X_1\times X_2} = \|x_1\|_{X_1} + \|x_2\|_{X_2}$ for $(x_1, x_2) \in X_1 \times X_2$, where $X_2 = \{x_2(t) \mid x_2(t) \in C([a, b], \mathbb{R})\}$, and $\|x_2\|_{X_2} = \sup\{|x_2(t)| : t \in [a, b]\}$.

According to Lemma 2.7, we define an operator $T : X_1 \times X_2 \to X_1 \times X_2$ associated with the coupled system (1.2) by

$$T(x_1, x_2)(t) = \begin{pmatrix} T_1(x_1, x_2)(t) \\ T_2(x_1, x_2)(t) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} T_1(x_1, x_2)(t) = {}& {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(t, x_1(t), x_2(t)) + \frac{1}{\Delta}\left[\Upsilon_1 + \Upsilon_2 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\ & \times \left\{ {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(\eta_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,\mu_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} \nu_2(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\} + \frac{1}{\Delta}\left[\Upsilon_3 + \Upsilon_4 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\ & \times \left\{ {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(\beta_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,\alpha_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} \nu_1(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\} + \frac{1}{\Delta}\left[\Upsilon_5 + \Upsilon_6 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\ & \times \left\{ b\, {}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1} \nu_1(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(p_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,q_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} \nu_2(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\} + \frac{1}{\Delta}\left[\Upsilon_7 + \Upsilon_8 \frac{(\psi_1(t) - \psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\ & \times \left\{ b\, {}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2} \nu_2(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(r_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,s_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} \nu_1(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\}, \quad t \in [a, b], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} T_2(x_1, x_2)(t) = {}& {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(t, x_1(t), x_2(t)) + \frac{1}{\Delta}\left[\Upsilon_9 + \Upsilon_{10} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)}\right] \\ & \times \left\{ {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(\eta_i \, {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,\mu_i \, {}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2} \nu_2(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\} + \frac{1}{\Delta}\left[\Upsilon_{11} + \Upsilon_{12} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)}\right] \\ & \times \left\{ {}^{k_2}I_{a^+}^{\vartheta_2;\psi_2} \nu_2(b, x_1(b), x_2(b)) - \sum_{i=1}^{m} \left(\beta_i \, {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1} \nu_1(\xi_i, x_1(\xi_i), x_2(\xi_i)) \right.\right. \\ & \left.\left. + b\,\alpha_i \, {}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1} \nu_1(\zeta_i, x_1(\zeta_i), x_2(\zeta_i)) \right) \right\} + \frac{1}{\Delta}\left[\Upsilon_{13} + \Upsilon_{14} \frac{(\psi_2(t) - \psi_2(a))}{\Gamma_{k_2}(2k_2)}\right] \end{aligned}$$

$$\times\left\{b\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}\nu_1(b,x_1(b),x_2(b))-\sum_{i=1}^{m}\Big(p_i\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}\nu_2(\xi_i,x_1(\xi_i),x_2(\xi_i))\right.$$
$$\left.+\,b\,q_i\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}\nu_2(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))\Big)\right\}+\frac{1}{\Delta}\left[\Upsilon_{15}+\Upsilon_{16}\frac{(\psi_2(t)-\psi_2(a))}{\Gamma_{k_2}(2k_2)}\right]$$
$$\times\left\{b\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}\nu_2(b,x_1(b),x_2(b))-\sum_{i=1}^{m}\Big(r_i\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}\nu_1(\xi_i,x_1(\xi_i),x_2(\xi_i))\right.$$
$$\left.+\,b\,s_i\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}\nu_1(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))\Big)\right\},\quad t\in[a,b], \tag{3.3}$$

where Υ_i $(i=1,2,\ldots,16)$ are given in (2.4).

For the sake of computational convenience, we set

$$\Omega_1=\sum_{j=1}^{6}\Phi_j,\ \ \Omega_2=\sum_{j=7}^{12}\Phi_j, \tag{3.4}$$

where

$$\begin{aligned}
\Phi_1 &= \frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\Big[1+\Theta_1\Big]+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}\Theta_3,\\
\Phi_2 &= \frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\Theta_5+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}\Theta_7,\\
\Phi_3 &= \sum_{i=1}^{m}[\Theta_2\,|\beta_i|+\Theta_4|r_i|]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)},\\
\Phi_4 &= \sum_{i=1}^{m}[\Theta_6\,|\beta_i|+\Theta_8|r_i|]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)},\\
\Phi_5 &= |b|\sum_{i=1}^{m}[\Theta_2\,|\alpha_i|+\Theta_4|s_i|]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)},\\
\Phi_6 &= |b|\sum_{i=1}^{m}[\Theta_6\,|\alpha_i|+\Theta_8|s_i|]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)},\\
\Phi_7 &= \sum_{i=1}^{m}[\Theta_1\,|\eta_i|+\Theta_3|p_i|]\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)},\\
\Phi_8 &= \sum_{i=1}^{m}[\Theta_5\,|\eta_i|+\Theta_7|p_i|]\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)},\\
\Phi_9 &= |b|\sum_{i=1}^{m}[\Theta_1\,|\mu_i|+\Theta_3\,|q_i|]\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)},\\
\Phi_{10} &= |b|\sum_{i=1}^{m}[\Theta_5\,|\mu_i|+\Theta_7|q_i|]\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)},
\end{aligned} \tag{3.5}$$

$$\Phi_{11} = \frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\Theta_2 + |b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)}\Theta_4,$$
$$\Phi_{12} = \frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}[1+\Theta_6] + |b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)}\Theta_8$$

with

$$\begin{aligned}\Theta_i &= \frac{1}{|\Delta|}\left[|\Upsilon_{2i-1}| + |\Upsilon_{2i}|\frac{(\psi_1(b)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right], \quad i=1,2,3,4,\\ \Theta_j &= \frac{1}{|\Delta|}\left[|\Upsilon_{2j-1}| + |\Upsilon_{2j}|\frac{(\psi_2(b)-\psi_2(a))}{\Gamma_{k_2}(2k_2)}\right], \quad j=5,6,7,8.\end{aligned} \tag{3.6}$$

3.1. Uniqueness result via Banach's contraction mapping principle

In this subsection, we prove the existence of a unique solution to the coupled system (1.2) by employing Banach's contraction mapping principle (see [24]).

Theorem 3.1. *Suppose that* $\Delta \neq 0$, *and let* $\nu_1, \nu_2 : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *be continuous mappings satisfying the following condition:*

(A_1) *There exist constants* $L_1, L_2 > 0$ *such that*

$$|\nu_1(t,x_1,x_2) - \nu_1(t,\overline{x}_1,\overline{x}_2)| \le L_1\left(|x_1-\overline{x}_1| + |x_2-\overline{x}_2|\right),$$
$$|\nu_2(t,x_1,x_2) - \nu_2(t,\overline{x}_1,\overline{x}_2)| \le L_2\left(|x_1-\overline{x}_1| + |x_2-\overline{x}_2|\right),$$

for all $t \in [a,b]$ *and* $x_i, \overline{x}_i \in \mathbb{R}$, $i=1,2$.

Then the coupled system (1.2) *admits a unique solution on the interval* $[a,b]$, *provided that*

$$L_1\,\Omega_1 + L_2\,\Omega_2 < 1, \tag{3.7}$$

where the constants Ω_1 *and* Ω_2 *are defined in* (3.4).

Proof. Let us define a closed ball $B_{r_1} = \{(x_1,x_2) \in X_1\times X_2 : \|(x_1,x_2)\|_{X_1\times X_2} \le r_1\}$ with

$$r_1 \ge \frac{M_1\,\Omega_1 + M_2\,\Omega_2}{1-(L_1\,\Omega_1 + L_2\,\Omega_2)}, \tag{3.8}$$

where $M_1 = \sup_{t\in[a,b]} |\nu_1(t,0,0)| < \infty$, and $M_2 = \sup_{t\in[a,b]} |\nu_2(t,0,0)| < \infty$. In the first step, we show that $T(B_{r_1}) \subseteq B_{r_1}$, where $T : B_{r_1} \to X_1\times X_2$ is defined by (3.1). By using the assumption (A_1), for each $(x_1,x_2) \in B_{r_1}$, we have

$$\begin{aligned}|\nu_1(t,x_1(t),x_2(t))| &\le |\nu_1(t,x_1(t),x_2(t)) - \nu_1(t,0,0)| + |\nu_1(t,0,0)|\\ &\le L_1(\|\nu_1\|_{X_1} + \|\nu_2\|_{X_2}) + M_1 = L_1\,\|(\nu_1,\nu_2)\|_{X_1\times X_2} + M_1 \le L_1\,r_1 + M_1.\end{aligned}$$

Similarly, we have $|\nu_2(t,x_1(t),x_2(t))| \le L_2\,r_1 + M_2$. By virtue of the above inequalities and the notations in (3.6), we obtain

$$\begin{aligned}
|T_1(x_1,x_2)(t)| \leq\ & {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(t,x_1(t),x_2(t))| + \frac{1}{|\Delta|}\left[|\Upsilon_1| + |\Upsilon_2|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(b,x_1(b),x_2(b))| + \sum_{i=1}^{m}\left(|\eta_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i,x_1(\xi_i),x_2(\xi_i))|\right.\right. \\
& \left.\left. + |b|\,|\mu_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))|\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_3| + |\Upsilon_4|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(b,x_1(b),x_2(b))| + \sum_{i=1}^{m}\left(|\beta_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i,x_1(\xi_i),x_2(\xi_i))|\right.\right. \\
& \left.\left. + |b|\,|\alpha_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))|\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_5| + |\Upsilon_6|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{|b|\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}|\nu_1(b,x_1(b),x_2(b))| + \sum_{i=1}^{m}\left(|p_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i,x_1(\xi_i),x_2(\xi_i))|\right.\right. \\
& \left.\left. + |b|\,|q_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))|\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_7| + |\Upsilon_8|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{|b|\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}|\nu_2(b,x_1(b),x_2(b))| + \sum_{i=1}^{m}\left(|r_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i,x_1(\xi_i),x_2(\xi_i))|\right.\right. \\
& \left.\left. + |b|\,|s_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))|\right)\right\} \\
\leq\ & (L_1 r_1 + M_1)\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)} + \frac{1}{|\Delta|}\left[|\Upsilon_1| + |\Upsilon_2|\frac{(\psi_1(b)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{(L_1 r_1 + M_1)\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)} + (L_2 r_1 + M_2)\sum_{i=1}^{m}\left(|\eta_i|\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\right.\right. \\
& \left.\left. + |b|\,|\mu_i|\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)}\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_3| + |\Upsilon_4|\frac{(\psi_1(b)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{(L_2 r_1 + M_2)\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)} + (L_1 r_1 + M_1)\sum_{i=1}^{m}\left(|\beta_i|\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\right.\right. \\
& \left.\left. + |b|\,|\alpha_i|\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_5| + |\Upsilon_6|\frac{(\psi_1(b)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right] \\
& \times\left\{(L_1 r_1 + M_1)\,|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)} + (L_2 r_1 + M_2)\sum_{i=1}^{m}\left(|p_i|\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\right.\right. \\
& \left.\left. + |b|\,|q_i|\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)}\right)\right\} + \frac{1}{|\Delta|}\left[|\Upsilon_7| + |\Upsilon_8|\frac{(\psi_1(b)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg\{(L_2\, r_1 + M_2)\,|b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)} + (L_1\, r_1 + M_1)\sum_{i=1}^{m}\Bigg(|r_i|\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\\
&+|b|\,|s_i|\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\Bigg)\Bigg\}\\
= &\ (L_1\, r_1 + M_1)\Bigg[\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)} + \frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\Theta_1\\
&+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}\Theta_3 + \sum_{i=1}^{m}[\Theta_2\,|\beta_i| + \Theta_4|r_i|]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\\
&+|b|\sum_{i=1}^{m}[\Theta_2\,|\alpha_i| + \Theta_4|s_i|]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\Bigg]\\
&+(L_2\, r_1 + M_2)\Bigg[\sum_{i=1}^{m}[\Theta_1\,|\eta_i| + \Theta_3|p_i|]\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\\
&+|b|\sum_{i=1}^{m}[\Theta_1\,|\mu_i| + \Theta_3\,|q_i|]\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)} + \frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\Theta_2\\
&+|b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)}\Theta_4\Bigg].
\end{aligned}
$$

In view of the notations in (3.5), the above inequality takes the form

$$\|T_1\,(x_1,x_2)\|_{X_1} \le (\Phi_1+\Phi_3+\Phi_5)(L_1\, r_1 + M_1) + (\Phi_7+\Phi_9+\Phi_{11})(L_2\, r_1 + M_2). \tag{3.9}$$

In the same way, we find that

$$\|T_2\,(x_1,x_2)\|_{X_2} \le (\Phi_2+\Phi_4+\Phi_6)(L_1\, r_1 + M_1) + (\Phi_8+\Phi_{10}+\Phi_{12})(L_2\, r_1 + M_2). \tag{3.10}$$

Combining the inequalities (3.9) and (3.10) and using (3.8), we get

$$
\begin{aligned}
\|T\,(x_1,x_2)\|_{X_1\times X_2} &= \|T_1\,(x_1,x_2)\|_{X_1} + \|T_2\,(x_1,x_2)\|_{X_2}\\
&\le r_1\big(L_1(\Phi_1+\Phi_2+\Phi_3+\Phi_4+\Phi_5+\Phi_6) + L_2(\Phi_7+\Phi_8+\Phi_9+\Phi_{10}+\Phi_{11}+\Phi_{12})\big)\\
&\quad + M_1(\Phi_1+\Phi_2+\Phi_3+\Phi_4+\Phi_5+\Phi_6) + M_2(\Phi_7+\Phi_8+\Phi_9+\Phi_{10}+\Phi_{11}+\Phi_{12})\\
&= r_1(L_1\,\Omega_1 + L_2\,\Omega_7) + M_1\,\Omega_1 + M_2\,\Omega_2\\
&\le r_1,
\end{aligned}
$$

which implies that $T(B_{r_1}) \subseteq B_{r_1}$.

Next, we will show that the operator T is a contraction. For each pair of elements $(x_1,x_2),(\overline{x}_1,\overline{x}_2) \in B_{r_1}$ and for any $t \in [a,b]$, we have

$$
\begin{aligned}
&|T_1(x_1,x_2)(t)-T_1(\overline{x}_1,\overline{x}_2)(t)|\\
&\le {}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(t,x_1(t),x_2(t))-\nu_1(t,\overline{x}_1(t),\overline{x}_2(t))|\\
&\quad+\frac{1}{|\Delta|}\left[|\Upsilon_1|+|\Upsilon_2|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\left\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(b,x_1(b),x_2(b))-\nu_1(b,\overline{x}_1(b),\overline{x}_2(b))|\right.\\
&\quad+\sum_{i=1}^{m}\Big(|\eta_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i,x_1(\xi_i),x_2(\xi_i))-\nu_2(\xi_i,\overline{x}_1(\xi_i),\overline{x}_2(\xi_i))|\\
&\quad\left.+|b|\,|\mu_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))-\nu_2(\zeta_i,\overline{x}_1(\zeta_i),\overline{x}_2(\zeta_i))|\Big)\right\}\\
&\quad+\frac{1}{|\Delta|}\left[|\Upsilon_3|+|\Upsilon_4|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\left\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(b,x_1(b),x_2(b))-\nu_2(b,\overline{x}_1(b),\overline{x}_2(b))|\right.\\
&\quad+\sum_{i=1}^{m}\Big(|\beta_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i,x_1(\xi_i),x_2(\xi_i))-\nu_1(\xi_i,\overline{x}_1(\xi_i),\overline{x}_2(\xi_i))|\\
&\quad\left.+|b|\,|\alpha_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))-\nu_1(\zeta_i,\overline{x}_1(\zeta_i),\overline{x}_2(\zeta_i))|\Big)\right\}\\
&\quad+\frac{1}{|\Delta|}\left[|\Upsilon_5|+|\Upsilon_6|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\left\{|b|\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}|\nu_1(b,x_1(b),x_2(b))-\nu_1(b,\overline{x}_1(b),\overline{x}_2(b))|\right.\\
&\quad+\sum_{i=1}^{m}\Big(|p_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i,x_1(\xi_i),x_2(\xi_i))-\nu_2(\xi_i,\overline{x}_1(\xi_i),\overline{x}_2(\xi_i))|\\
&\quad\left.+|b|\,|q_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))-\nu_2(\zeta_i,\overline{x}_1(\zeta_i),\overline{x}_2(\zeta_i))|\Big)\right\}\\
&\quad+\frac{1}{|\Delta|}\left[|\Upsilon_7|+|\Upsilon_8|\frac{(\psi_1(t)-\psi_1(a))}{\Gamma_{k_1}(2k_1)}\right]\left\{|b|\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}|\nu_2(b,x_1(b),x_2(b))-\nu_2(b,\overline{x}_1(b),\overline{x}_2(b))|\right.\\
&\quad+\sum_{i=1}^{m}\Big(|r_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i,x_1(\xi_i),x_2(\xi_i))-\nu_1(\xi_i,\overline{x}_1(\xi_i),\overline{x}_2(\xi_i))|\\
&\quad\left.+|b|\,|s_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i,x_1(\zeta_i),x_2(\zeta_i))-\nu_1(\zeta_i,\overline{x}_1(\zeta_i),\overline{x}_2(\zeta_i))|\Big)\right\}\\
&\le L_1\Big(\|x_1-\overline{x}_1\|_{X_1}+\|x_2-\overline{x}_2\|_{X_2}\Big)\left[\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}+\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\Theta_1\right.\\
&\quad+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}\Theta_3+\sum_{i=1}^{m}\big[\Theta_2\,|\beta_i|+\Theta_4|r_i|\big]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\\
&\quad\left.+|b|\sum_{i=1}^{m}\big[\Theta_2\,|\alpha_i|+\Theta_4|s_i|\big]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ L_2\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big)\Bigg[\sum_{i=1}^{m} [\Theta_1 \, |\eta_i| + \Theta_3 |p_i|] \, \frac{(\psi_2(\xi_i) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} \\
&+ |b| \sum_{i=1}^{m} [\Theta_1 \, |\mu_i| + \Theta_3 \, |q_i|] \, \frac{(\psi_2(\zeta_i) - \psi_2(a))^{\frac{\vartheta_2 - \sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \sigma_i + k_2)} + \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} \Theta_2 \\
&+ |b| \, \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2 - \gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \gamma_2 + k_2)} \Theta_4 \Bigg],
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|T_1(x_1, x_2) - T_1(\overline{x}_1, \overline{x}_2)\|_{X_1} \leq\ & L_1(\Phi_1 + \Phi_3 + \Phi_5)\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big) \\
& + L_2(\Phi_7 + \Phi_9 + \Phi_{11})\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big).
\end{aligned}
\tag{3.11}
$$

In a similar manner, one can find that

$$
\begin{aligned}
\|T_2(x_1, x_2) - T_2(\overline{x}_1, \overline{x}_2)\|_{X_2} \leq\ & L_1(\Phi_2 + \Phi_4 + \Phi_6)\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big) \\
& + L_2(\Phi_8 + \Phi_{10} + \Phi_{12})\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big).
\end{aligned}
\tag{3.12}
$$

From (3.11) and (3.12), we deduce that

$$
\begin{aligned}
&|T(x_1, x_1)(t) - T(\overline{x}_1, \overline{x}_2)(t)| \\
&\leq L_1(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6)\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big) \\
&\quad + L_2(\Phi_7 + \Phi_8 + \Phi_9 + \Phi_{10} + \Phi_{11} + \Phi_{12})\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big).
\end{aligned}
$$

Consequently, it follows by using the notations in (3.4) that

$$
\|T(x_1, x_2) - T(\overline{x}_1, \overline{x}_2)\|_{X_1 \times X_2} \leq \Big(L_1 \, \Omega_1 + L_2 \, \Omega_2\Big)\Big(\|x_1 - \overline{x}_1\|_{X_1} + \|x_2 - \overline{x}_2\|_{X_2} \Big).
$$

By the assumption (3.7), it follows from the above inequality that the operator T is a contraction. An application of Banach's contraction principle guarantees that the operator T possesses a single fixed point in B_{r_1}. Consequently, this fixed point represents the unique solution to the coupled system (1.2) on the interval $[a, b]$, thereby concluding the proof. □

3.2. Existence result via Leray–Schauder alternative

In this subsection, we address the existence of solutions to the coupled system (1.2) through the use of the Leray–Schauder alternative (see [25]).

Theorem 3.2. *Let* $\Delta \neq 0$, *and suppose that the functions* $\nu_1, \nu_2 : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous and satisfy the following condition:*

(A_2) *There exist constants* $f_i, g_i \geq 0$, $i = 1, 2$ *and* $f_0 > 0$, $g_0 > 0$ *such that*

$$
\begin{aligned}
|\nu_1(t, x_1, x_2)| &\leq f_0 + f_1 \, |x_1| + f_2 \, |x_2| \, , \\
|\nu_2(t, x_1, x_2)| &\leq g_0 + g_1 \, |x_1| + g_2 \, |x_2| \, .
\end{aligned}
$$

Then, the coupled system (1.2) *has at least one solution on* $[a, b]$, *provided that*

$$\Omega_1 f_1 + \Omega_2 g_1 < 1 \ \text{ and } \ \Omega_1 f_2 + \Omega_2 g_2 < 1, \tag{3.13}$$

where Ω_1 *and* Ω_2 *are given in* (3.4).

Proof. Let us first show that the operator $T : X_1 \times X_2 \to X_1 \times X_2$ is completely continuous. Because the functions v_1 and v_2 are continuous, it follows that the operators T_1 and T_2 are continuous. Consequently, the operator T is continuous. Let $B_{r_2} \subseteq X_1 \times X_2$ be a bounded set. By (A_2), we have

$$\begin{aligned} |v_1(t, x_1, x_2)| &\leq f_0 + f_1\|x_1\|_{X_1} + f_2\|x_2\|_{X_2} \\ &\leq f_0 + (f_1 + f_2)(\|x_1\|_{X_1} + \|x_2\|_{X_2}) \\ &= f_0 + (f_1 + f_2)\|(x_1, x_2)\| \\ &\leq f_0 + (f_1 + f_2)r_2 := N_1. \end{aligned}$$

Similarly, we have $|v_2(t, x_1, x_2)| \leq g_0 + (g_1 + g_2)\|(x_1, x_2)\| \leq g_0 + (g_1 + g_2)r_2 := N_2$. Then, for any $(x_1, x_2) \in B_{r_2}$, we obtain

$$\begin{aligned} |T_1\,(x_1, x_2)\,(t)| \leq N_1 \Bigg[& \frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} + \frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)}\Theta_1 + |b|\,\frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \gamma_1 + k_1)}\Theta_3 \\ & + \sum_{i=1}^{m}[\Theta_2\,|\beta_i| + \Theta_4|r_i|]\,\frac{(\psi_1(\xi_i) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} \\ & + |b|\sum_{i=1}^{m}[\Theta_2\,|\alpha_i| + \Theta_4|s_i|]\,\frac{(\psi_1(\zeta_i) - \psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \varphi_i + k_1)} \Bigg] \\ & + N_2 \Bigg[\sum_{i=1}^{m}[\Theta_1\,|\eta_i| + \Theta_3|p_i|]\,\frac{(\psi_2(\xi_i) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} \\ & + |b|\sum_{i=1}^{m}[\Theta_1\,|\mu_i| + \Theta_3\,|q_i|]\,\frac{(\psi_2(\zeta_i) - \psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \sigma_i + k_2)} + \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)}\Theta_2 \\ & + |b|\,\frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \gamma_2 + k_2)}\Theta_4 \Bigg], \end{aligned}$$

which leads to

$$\|T_1(x_1, x_2)\|_{X_1} \leq (\Phi_1 + \Phi_3 + \Phi_5)N_1 + (\Phi_7 + \Phi_9 + \Phi_{11})N_2.$$

In the same way, we have

$$\|T_2(x_1, x_2)\|_{X_2} \leq (\Phi_2 + \Phi_4 + \Phi_6)N_1 + (\Phi_8 + \Phi_{10} + \Phi_{12})N_2.$$

Thus,

$$
\begin{aligned}
\|T(x_1, x_2)\|_{X_1\times X_2} &= \|T_1(x_1, x_2)\|_{X_1} + \|T_2(x_1, x_2)\|_{X_2} \\
&\le (\Phi_1 + \Phi_3 + \Phi_5 + \Phi_2 + \Phi_4 + \Phi_6)N_1 \\
&\quad + (\Phi_7 + \Phi_9 + \Phi_{11} + \Phi_8 + \Phi_{10} + \Phi_{12})N_2,
\end{aligned}
$$

which means that the operator $T(B_{r_2})$ is uniformly bounded.

Next, we show that the operator T is equicontinuous. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|T_1(x_1, x_2)(t_2) - T_1(x_1, x_2)(t_1)| \\
&\le \frac{1}{k_1\,\Gamma_{k_1}(\vartheta_1)}\Bigg|\int_a^{t_1} \psi_1'(z)\left((\psi_1(t_2) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1} - (\psi_1(t_1) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1}\right)\nu_1(z, x_1(z), x_2(z))dz \\
&\quad + \int_{t_1}^{t_2} \psi_1'(z)\,(\psi_1(t_2) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1}\,\nu_1(z, x_1(z), x_2(z))dz\Bigg| + \frac{(\psi_1(t_2) - \psi_1(t_1))}{|\Delta|\,\Gamma_{k_1}(2k_1)} \\
&\quad \times\Bigg[|\Upsilon_2|\Bigg\{{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(b, x_1(b), x_2(b))| + \sum_{i=1}^{m}\Big(|\eta_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i, x_1(\xi_i), x_2(\xi_i))| \\
&\quad + |b|\,|\mu_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i, x_1(\zeta_i), x_2(\zeta_i))|\Big)\Bigg\} + |\Upsilon_4|\Bigg\{{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(b, x_1(b), x_2(b))| \\
&\quad + \sum_{i=1}^{m}\Big(|\beta_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i, x_1(\xi_i), x_2(\xi_i))| + |b|\,|\alpha_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i, x_1(\zeta_i), x_2(\zeta_i))|\Big)\Bigg\} \\
&\quad + |\Upsilon_6|\Bigg\{|b|\,{}^{k_1}I_{a^+}^{\vartheta_1-\gamma_1;\psi_1}|\nu_1(b, x_1(b), x_2(b))| + \sum_{i=1}^{m}\Big(|p_i|\,{}^{k_2}I_{a^+}^{\vartheta_2;\psi_2}|\nu_2(\xi_i, x_1(\xi_i), x_2(\xi_i))| \\
&\quad + |b|\,|q_i|\,{}^{k_2}I_{a^+}^{\vartheta_2-\sigma_i;\psi_2}|\nu_2(\zeta_i, x_1(\zeta_i), x_2(\zeta_i))|\Big)\Bigg\} + |\Upsilon_8|\Bigg\{|b|\,{}^{k_2}I_{a^+}^{\vartheta_2-\gamma_2;\psi_2}|\nu_2(b, x_1(b), x_2(b))| \\
&\quad + \sum_{i=1}^{m}\Big(|r_i|\,{}^{k_1}I_{a^+}^{\vartheta_1;\psi_1}|\nu_1(\xi_i, x_1(\xi_i), x_2(\xi_i))| + |b|\,|s_i|\,{}^{k_1}I_{a^+}^{\vartheta_1-\varphi_i;\psi_1}|\nu_1(\zeta_i, x_1(\zeta_i), x_2(\zeta_i))|\Big)\Bigg\}\Bigg] \\
&\le \frac{N_1}{k_1\,\Gamma_{k_1}(\vartheta_1)}\Bigg|\int_a^{t_1} \psi_1'(z)\left((\psi_1(t_2) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1} - (\psi_1(t_1) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1}\right)dz \\
&\quad + \int_{t_1}^{t_2} \psi_1'(z)\,(\psi_1(t_2) - \psi_1(z))^{\frac{\vartheta_1}{k_1}-1}\,dz\Bigg| + \frac{(\psi_1(t_2) - \psi_1(t_1))}{|\Delta|\,\Gamma_{k_1}(2k_1)}\Bigg\{N_1\Bigg[\frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)}|\Upsilon_2| \\
&\quad + |b|\,\frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \gamma_1 + k_1)}|\Upsilon_6| + \sum_{i=1}^{m}[|\Upsilon_4|\,|\beta_i| + |\Upsilon_8|\,|r_i|]\,\frac{(\psi_1(\xi_i) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} \\
&\quad + |b|\sum_{i=1}^{m}[|\Upsilon_4|\,|\alpha_i| + |\Upsilon_8|\,|s_i|]\,\frac{(\psi_1(\zeta_i) - \psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \varphi_i + k_1)}\Bigg] \\
&\quad + N_2\Bigg[\sum_{i=1}^{m}[|\Upsilon_2|\,|\eta_i| + |\Upsilon_6||p_i|]\,\frac{(\psi_2(\xi_i) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)}
\end{aligned}
$$

$$
\begin{aligned}
&+|b|\sum_{i=1}^{m}\big[|\Upsilon_2|\,|\mu_i|+|\Upsilon_6|\,|q_i|\big]\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)}+\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}|\Upsilon_4|\\
&+|b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)}|\Upsilon_8|\Bigg]\Bigg\}\\
\le{}& \frac{N_1}{k_1\,\Gamma_{k_1}(\vartheta_1+k_1)}\Bigg\{2\,(\psi_1(t_2)-\psi_1(t_1))^{\frac{\vartheta_1}{k_1}}+\Big|\,(\psi_1(t_2)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}-(\psi_1(t_1)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}\,\Big|\Bigg\}\\
&+\frac{(\psi_1(t_2)-\psi_1(t_1))}{|\Delta|\,\Gamma_{k_1}(2k_1)}\Bigg\{N_1\Bigg[\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}|\Upsilon_2|+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}|\Upsilon_6|\\
&+\sum_{i=1}^{m}\big[|\Upsilon_4|\,|\beta_i|+|\Upsilon_8|\,|r_i|\big]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\\
&+|b|\sum_{i=1}^{m}\big[|\Upsilon_4|\,|\alpha_i|+|\Upsilon_8|\,|s_i|\big]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\Bigg]\\
&+N_2\Bigg[\sum_{i=1}^{m}\big[|\Upsilon_2|\,|\eta_i|+|\Upsilon_6||p_i|\big]\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}\\
&+|b|\sum_{i=1}^{m}\big[|\Upsilon_2|\,|\mu_i|+|\Upsilon_6|\,|q_i|\big]\,\frac{(\psi_2(\zeta_i)-\psi_2(a))^{\frac{\vartheta_2-\sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\sigma_i+k_2)}+\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}|\Upsilon_4|\\
&+|b|\,\frac{(\psi_2(b)-\psi_2(a))^{\frac{\vartheta_2-\gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2-\gamma_2+k_2)}|\Upsilon_8|\Bigg]\Bigg\},
\end{aligned}
$$

which implies that

$$
|T_1(x_1,x_2)(t_2)-T_1(x_1,x_2)(t_1)|\to 0,\quad as\quad t_1\to t_2,
$$

independently of $(x_1,x_2)\in B_{r_2}$. Analogously, one can obtain

$$
\begin{aligned}
&|T_2(x_1,x_2)(t_2)-T_2(x_1,x_2)(t_1)|\\
\le{}& \frac{N_2}{k_2\,\Gamma_{k_2}(\vartheta_2+k_2)}\Bigg\{2\,(\psi_2(t_2)-\psi_2(t_1))^{\frac{\vartheta_2}{k_2}}+\Big|\,(\psi_2(t_2)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}-(\psi_2(t_1)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}\,\Big|\Bigg\}\\
&+\frac{(\psi_1(t_2)-\psi_1(t_1))}{|\Delta|\,\Gamma_{k_1}(2k_1)}\Bigg\{N_1\Bigg[\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}|\Upsilon_{10}|\\
&+|b|\,\frac{(\psi_1(b)-\psi_1(a))^{\frac{\vartheta_1-\gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\gamma_1+k_1)}|\Upsilon_{14}|+\sum_{i=1}^{m}\big[|\Upsilon_{12}|\,|\beta_i|+|\Upsilon_{16}|\,|r_i|\big]\,\frac{(\psi_1(\xi_i)-\psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1+k_1)}\\
&+|b|\sum_{i=1}^{m}\big[|\Upsilon_{12}|\,|\alpha_i|+|\Upsilon_{16}|\,|s_i|\big]\,\frac{(\psi_1(\zeta_i)-\psi_1(a))^{\frac{\vartheta_1-\varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1-\varphi_i+k_1)}\Bigg]\\
&+N_2\Bigg[\sum_{i=1}^{m}\big[|\Upsilon_{10}|\,|\eta_i|+|\Upsilon_{14}||p_i|\big]\,\frac{(\psi_2(\xi_i)-\psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2+k_2)}
\end{aligned}
$$

$$+ |b| \sum_{i=1}^{m} [|\Upsilon_{10}| \, |\mu_i| + |\Upsilon_{14}| \, |q_i|] \, \frac{(\psi_2(\zeta_i) - \psi_2(a))^{\frac{\vartheta_2 - \sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \sigma_i + k_2)} + \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} |\Upsilon_{12}|$$
$$+ |b| \, \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2 - \gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \gamma_2 + k_2)} |\Upsilon_{16}| \Bigg] \Bigg\},$$

which implies that

$$|T_2(x_1, x_2)(t_2) - T_2(x_1, x_2)(t_1)| \to 0, \quad \text{as} \quad t_1 \to t_2;$$

independently of $(x_1, x_2) \in B_{r_2}$. Hence, the set $T(B_{r_2})$ is equicontinuous. By applying the Arzelà–Ascoli theorem, $T(B_{r_2})$ is relatively compact, which implies that the operator T is completely continuous.

It remains to show that the set

$$U = \{(x_1, x_2) \in X_1 \times X_2 : (x_1, x_2) = \lambda \, T(x_1, x_2), \; 0 < \lambda < 1\}$$

is bounded. Let $(x_1, x_2) \in U$; then, $(x_1, x_2) = \lambda \, T(x_1, x_2)$. For any $t \in [a, b]$, we have

$$x_1(t) = \lambda \, T_1(x_1, x_2)(t), \quad x_2(t) = \lambda \, T_2(x_1, x_2)(t).$$

Then, we get

$$\begin{aligned}
\|x_1\|_{X_1} \leq & \Big(f_0 + f_1 \|x_1\|_{X_1} + f_2 \|x_2\|_{X_2}\Big) \Bigg[\frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} + \frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} \Theta_1 \\
& + |b| \, \frac{(\psi_1(b) - \psi_1(a))^{\frac{\vartheta_1 - \gamma_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \gamma_1 + k_1)} \Theta_3 + \sum_{i=1}^{m} [\Theta_2 \, |\beta_i| + \Theta_4 |r_i|] \, \frac{(\psi_1(\xi_i) - \psi_1(a))^{\frac{\vartheta_1}{k_1}}}{\Gamma_{k_1}(\vartheta_1 + k_1)} \\
& + |b| \sum_{i=1}^{m} [\Theta_2 \, |\alpha_i| + \Theta_4 |s_i|] \, \frac{(\psi_1(\zeta_i) - \psi_1(a))^{\frac{\vartheta_1 - \varphi_i}{k_1}}}{\Gamma_{k_1}(\vartheta_1 - \varphi_i + k_1)} \Bigg] \\
& + \Big(g_0 + g_1 \|x_1\|_{X_1} + g_2 \|x_2\|_{X_2}\Big) \Bigg[\sum_{i=1}^{m} [\Theta_1 \, |\eta_i| + \Theta_3 |p_i|] \, \frac{(\psi_2(\xi_i) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} \\
& + |b| \sum_{i=1}^{m} [\Theta_1 \, |\mu_i| + \Theta_3 \, |q_i|] \, \frac{(\psi_2(\zeta_i) - \psi_2(a))^{\frac{\vartheta_2 - \sigma_i}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \sigma_i + k_2)} + \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 + k_2)} \Theta_2 \\
& + |b| \, \frac{(\psi_2(b) - \psi_2(a))^{\frac{\vartheta_2 - \gamma_2}{k_2}}}{\Gamma_{k_2}(\vartheta_2 - \gamma_2 + k_2)} \Theta_4 \Bigg] \\
\leq & (\Phi_1 + \Phi_3 + \Phi_5) f_0 + (\Phi_7 + \Phi_9 + \Phi_{11}) g_0 \\
& + \Big((\Phi_1 + \Phi_3 + \Phi_5) f_1 + (\Phi_7 + \Phi_9 + \Phi_{11}) g_1\Big) \|x_1\|_{X_1} \\
& + \Big((\Phi_1 + \Phi_3 + \Phi_5) f_2 + (\Phi_7 + \Phi_9 + \Phi_{11}) g_2\Big) \|x_2\|_{X_2},
\end{aligned}$$

and similarly,

$$
\begin{aligned}
\|x_2\|_{X_2} \leq {} & (\Phi_2+\Phi_4+\Phi_6) f_0 + (\Phi_8+\Phi_{10}+\Phi_{12}) g_0 \\
& + \Big((\Phi_2+\Phi_4+\Phi_6) f_1 + (\Phi_8+\Phi_{10}+\Phi_{12}) g_1\Big) \|x_1\|_{X_1} \\
& + \Big((\Phi_2+\Phi_4+\Phi_6) f_2 + (\Phi_8+\Phi_{10}+\Phi_{12}) g_2\Big) \|x_2\|_{X_2},
\end{aligned}
$$

which leads to

$$
\|x_1\|_{X_1} + \|x_2\|_{X_2} \leq \Omega_1 f_0 + \Omega_2 g_0 + (\Omega_1 f_1 + \Omega_2 g_1) \|x_1\|_{X_1} + (\Omega_1 f_2 + \Omega_2 g_2) \|x_2\|_{X_2}.
$$

Consequently,

$$
\|(x_1, x_2)\|_{X_1 \times X_2} \leq \frac{\Omega_1 f_0 + \Omega_2 g_0}{\min\{1 - [\Omega_1 f_1 + \Omega_2 g_1], 1 - [\Omega_1 f_2 + \Omega_2 g_2]\}},
$$

which proves that U is bounded. In consequence, we deduce by the Leray–Schauder alternative that there exists at least one fixed point for the operator T. Therefore, the coupled system (1.2) has at least one solution on $[a, b]$. This completes the proof. □

4. Illustrative examples

In this section, we present examples that illustrate the practical application of our theoretical results. Consider a coupled system of the form:

$$
\begin{cases}
{}^{\frac{4}{3},C}D_{0^+}^{\frac{8}{5};t^2+t+1} x_1(t) = \nu_1(t, x_1(t), x_2(t)), \quad 0 \leq t \leq \frac{1}{2}, \\
{}^{\frac{5}{4},C}D_{0^+}^{\frac{7}{4};e^t} x_2(t) = \nu_2(t, x_1(t), x_2(t)), \quad 0 \leq t \leq \frac{1}{2}, \\
x_1\left(\frac{1}{2}\right) = x_2\left(\frac{1}{4}\right) + {}^{\frac{5}{4},C}D_{0^+}^{\frac{2}{3};e^t} x_2\left(\frac{1}{6}\right), \\
x_2\left(\frac{1}{2}\right) = 3\, x_1\left(\frac{1}{4}\right) + 2\, {}^{\frac{4}{3},C}D_{0^+}^{\frac{2}{5};t^2} x_1\left(\frac{1}{6}\right), \\
\frac{1}{2}\, {}^{\frac{4}{3},C}D_{0^+}^{\frac{1}{3};t^2} x_1\left(\frac{1}{2}\right) = \frac{1}{4} x_2\left(\frac{1}{4}\right) + \frac{1}{6}\, {}^{\frac{5}{4},C}D_{0^+}^{\frac{2}{3};e^t} x_2\left(\frac{1}{6}\right), \\
\frac{1}{2}\, {}^{\frac{5}{4},C}D_{0^+}^{\frac{1}{5};e^t} x_2\left(\frac{1}{2}\right) = \frac{1}{2} x_1\left(\frac{1}{4}\right) + \frac{1}{2}\, {}^{\frac{4}{3},C}D_{0^+}^{\frac{2}{5};t^2} x_1\left(\frac{1}{6}\right),
\end{cases}
\tag{4.1}
$$

where $a = 0$, $b = \frac{1}{2}$, $k_1 = \frac{4}{3}$, $k_2 = \frac{5}{4}$, $\psi_1(t) = t^2 + t + 1$, $\psi_2(t) = e^t$, $\vartheta_1 = \frac{8}{5}$, $\vartheta_2 = \frac{7}{4}$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{1}{5}$, $m = 1$, $\xi_1 = \frac{1}{4}$, $\zeta_1 = \frac{1}{6}$, $\sigma_1 = \frac{2}{3}$, $\varphi_1 = \frac{2}{5}$, $\eta_1 = 1$, $\mu_1 = 2$, $\beta_1 = 3$, $\alpha_1 = 4$, $p_1 = \frac{1}{4}$, $q_1 = \frac{1}{3}$, $r_1 = \frac{1}{2}$, and $s_1 = 1$.

Using the given data, we find that $Q_1 = 0.5625$, $Q_2 = 1$, $Q_3 \approx 0.685908$, $Q_4 = 3$, $Q_5 \approx 1.275049$, $Q_6 \approx 0.518977$, $Q_7 \approx 0.353359$, $Q_8 = 0.25$, $Q_9 \approx 0.133253$, $Q_{10} = 0.5$, $Q_{11} \approx 0.260168$, $Q_{12} \approx 0.305746$, $\Delta \approx 0.219434$, $\Upsilon_1 \approx 0.009664$, $\Upsilon_2 \approx 0.074987$, $\Upsilon_3 \approx 0.261289$, $\Upsilon_4 \approx 0.008275$, $\Upsilon_5 \approx 0.231063$, $\Upsilon_6 \approx 0.0573245$, $\Upsilon_7 \approx -0.694957$, $\Upsilon_8 \approx 0.11818$, $\Upsilon_9 \approx 0.213369$, $\Upsilon_{10} \approx -0.010338$, $\Upsilon_{11} \approx$

-0.017093, $\Upsilon_{12} \approx 0.488769$, $\Upsilon_{13} \approx 0.212424$, $\Upsilon_{14} \approx 0.171408$, $\Upsilon_{15} \approx -0.164061$, $\Upsilon_{16} \approx -0.603605$, $\Theta_1 \approx 0.236263$, $\Theta_2 \approx 1.211952$, $\Theta_3 \approx 1.199941$, $\Theta_4 \approx 3.469987$, $\Theta_5 \approx 0.99681$, $\Theta_6 \approx 1.233869$, $\Theta_7 \approx 1.373446$, $\Theta_8 \approx 2.175224$, $\Phi_1 \approx 0.917005$, $\Phi_2 \approx 0.8593$, $\Phi_3 \approx 0.854737$, $\Phi_4 \approx 0.762174$, $\Phi_5 \approx 0.764504$, $\Phi_6 \approx 0.653558$, $\Phi_7 \approx 0.0542264$, $\Phi_8 \approx 0.13552$, $\Phi_9 \approx 0.086113$, $\Phi_{10} \approx 0.241949$, $\Phi_{11} \approx 1.072353$, $\Phi_{12} \approx 1.145989$, $\Omega_1 \approx 4.811278$, and $\Omega_2 \approx 2.736151$.

Case I. Let $\nu_1, \nu_2 : \left[0, \frac{1}{2}\right] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given nonlinear functions defined by

$$\nu_1(t, x_1, x_2) = \frac{e^{-(1-2t)}}{\sqrt[3]{729 + t^3}} \left(\frac{|x_1|}{1 + |x_1|} + \sin|x_2| + 2 \right) \tag{4.2}$$

and

$$\nu_2(t, x_1, x_2) = \frac{\cos^2(\pi t)}{3(t + 2)} \left(\tan^{-1}|x_1| + \frac{|x_2|}{1 + |x_2|} + 1 \right). \tag{4.3}$$

It is easy to verify that ν_1 and ν_2 satisfy the Lipschitz condition. Indeed,

$$|\nu_1(t, x_1, x_2) - \nu_1(t, \overline{x}_1, \overline{x}_2)| \le \frac{1}{9}\Big(|x_1 - \overline{x}_1| + |x_2 - \overline{x}_2| \Big), \tag{4.4}$$

and

$$|\nu_2(t, x_1, x_2) - \nu_2(t, x_1, x_2)| \le \frac{1}{6}\Big(|x_1 - \overline{x}_1| + |x_2 - \overline{x}_2| \Big). \tag{4.5}$$

Accordingly, the corresponding Lipschitz constants are $L_1 = \frac{1}{9}$ and $L_2 = \frac{1}{6}$. Hence, the functions ν_1 and ν_2 fulfill assumption (A_1) of Theorem 3.1. Moreover, a direct computation yields

$$L_1\,\Omega_1 + L_2\,\Omega_2 \approx 0.990611 < 1,$$

showing that condition (3.7) is met. Because all hypotheses of Theorem 3.1 are satisfied, its conclusion applies to the coupled system (4.1) with ν_1 and ν_2 defined by (4.4) and (4.5), respectively.

Case II. Let the functions $\nu_1, \nu_2 : \left[0, \frac{1}{2}\right] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\nu_1(t, x_1, x_2) = \sin(\pi t) + \frac{\cos^4 x_2}{2(t + 5)} x_1 + \frac{e^{-x_1^2}}{\sqrt{t^2 + 121}} \left(\frac{x_2^2}{1 + |x_2|} \right) \tag{4.6}$$

and

$$\nu_2(t, x_1, x_2) = \frac{1}{6} + \frac{1}{7(t + 1)^2} \left(\frac{x_1^2}{1 + |x_1|} \right) + \frac{\tan^{-1} x_1^2}{4\pi} \sin|x_2|. \tag{4.7}$$

A straightforward estimation shows that

$$|\nu_1(t, x_1, x_2)| \le 1 + \frac{1}{10}|x_1| + \frac{1}{11}|x_2|,$$

and

$$|\nu_2(t, x_1, x_2)| \le \frac{1}{6} + \frac{1}{7}|x_1| + \frac{1}{8}|x_2|.$$

Accordingly, selecting the constants $f_0 = 1$, $f_1 = \frac{1}{10}$, $f_2 = \frac{1}{11}$, $g_0 = \frac{1}{6}$, $g_1 = \frac{1}{7}$, and $g_2 = \frac{1}{8}$ from the set of admissible parameters, we obtain

$$\Omega_1 f_1 + \Omega_2 g_1 \approx 0.872006 < 1 \text{ and } \Omega_1 f_2 + \Omega_2 g_2 \approx 0.779407 < 1. \tag{4.8}$$

These inequalities confirm that condition (3.13) is fulfilled. Hence, all assumptions of Theorem 3.2 are satisfied. As a result, Theorem 3.2 ensures that the coupled system (4.1) with nonlinearities ν_1 and ν_2 specified in (4.6) and (4.7), admits at least one solution on the interval $\left[0, \frac{1}{2}\right]$.

5. Conclusions

This article addresses the solvability of a coupled system of (k, ψ)-Caputo fractional differential equations subject to multipoint fractional closed boundary conditions. The existence and uniqueness of the results are established through the use of classical fixed-point techniques, including Banach's contraction principle and the Leray–Schauder alternative. Numerical examples are presented to validate the theoretical analysis. The findings are original and provide a meaningful extension of the current literature on coupled systems involving (k, ψ)-Hilfer-type fractional derivative operators. In our next work, we will extend the present work to multivalued maps and nonlinear contractions. Extensions to more general fractional operators, variable-order derivatives, and higher-dimensional systems also remain topics for future research.

Author contributions

Furkan Erkan: Conceptualization, methodology, writing–original draft preparation. Nuket Aykut Hamal: Conceptualization, methodology, writing–original draft preparation. Sotiris K. Ntouyas: Conceptualization, methodology, writing–review suggestions and editing, supervision. Bashir Ahmad: Conceptualization, methodology, writing–review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Sotiris K. Ntouyas is a Guest Editor of special issue "Advances in Fractional Boundary Value Problems and Their Applications" for AIMS Mathematics. Prof. Sotiris K. Ntouyas was not involved in the editorial review and the decision to publish this article.

The authors declare no conflicts of interest.

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