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*Research article*

# Integral representation of the fundamental system of solutions for third order differential operators and completeness of the system of root functions of one irregular boundary value problem

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**Abstract:** This paper addresses two interrelated problems: the integral representation of solutions to third-order linear differential equations and the completeness of the root function system of the corresponding differential operator under irregular boundary conditions. In the first part, an integral representation for the fundamental system of solutions of a third-order differential equation with a complex spectral parameter is constructed. Unlike the classical approach by Marchenko, the obtained representations remain valid even when the coefficients are not holomorphic. The method is based on reducing the problem to Volterra integral equations of the second kind, which are solved using Picard's iterative method. Special representations are established for the initial terms of the iteration sequence, and a universal integral form is derived for the higher-order terms. The second part of the work focuses on a third-order differential operator on a finite interval with general irregular boundary conditions. The aim is to establish the completeness of the system of eigenfunctions and associated functions of this operator in the space $L_2$. To achieve this, properties of the characteristic determinant, its asymptotic behavior, and its relation to root functions are analyzed. It is proved that the root function system is complete even under boundary conditions that do not satisfy Birkhoff regularity. The results generalize known theorems for second-order operators and significantly extend the class of boundary conditions for which completeness holds. The proposed methods and results are of interest for the spectral theory of differential operators and the theory of transmutation operators and can be applied further to the study of inverse problems and problems with more general boundary conditions.

**Keywords:** differential operators; completeness; entire functions; integral equations; self-adjoint differential operators
**Mathematics Subject Classification:** 47E07, 47E05

## 1. Introduction and formulation of the main result

When solving inverse scattering problems, Marchenko introduced a method using transformation operators, which furthered the study and theory of inverse problems. Let us recall the theorem on transformation operators from Marchenko's monographs [1].

**Theorem 1.** *Let $q(x)$ be an arbitrary complex-valued function from the space $L_2(0,1)$ and $\lambda$- be any complex number. Solutions $y_1(x,\lambda), y_2(x,\lambda)$ of the equation $-y''(x)+q(x)y(x)=\lambda y(x),\ \ 0<x<1$ given initial data $y_1(0)=y_2'(0)=1,\ \ y_1'(0)=y_2(0)=0$ can be represented in the form of*

$$y_1(x,\lambda)=y_{10}(x,\lambda)+\int\limits_0^x K_1(x,t)y_{10}(t,\lambda)dt,\ \ y_2(x,\lambda)=y_{20}(x,\lambda)+\int\limits_0^x K_2(x,t)y_{20}(t,\lambda)dt,$$

*at some continuous functions $K_1(x,t), K_2(x,t)$. Here, the continuous functions $K_1(x,t), K_2(x,t)$ are independent of $\lambda$, and $y_{10}(x,\lambda)=\cos\sqrt{\lambda}x, y_{20}(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$.*

*This paper gives an integral representation of solutions of third-order linear differential equations.*

*Let us denote by $y_1(x), y_2(x), y_3(x)$ solutions of a homogeneous equation*

$$y'''(x)+p_1(x)y'(x)+p_0(x)y(x)=\lambda y(x),\quad 0<x<1 \tag{1.1}$$

*subject to the initial Cauchy conditions at $x=0$*

$$y_j^{(k-1)}(0)=\delta_{kj},\ \ k,j=1,2,3. \tag{1.2}$$

*Note that the functions $y_k(x,\lambda),\ k=1,2,3$ represent entire functions of $\lambda$. Consider the case when $p_1(x)=p_0(x)=0$. In this case, the fundamental system of solutions has a simpler form. To write it out, we introduce the following notations:*

$$\theta_0=1,\ \theta_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\ \ \theta_2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

*Then, when $p_1(x)=p_0(x)=0$, we will reduce the fundamental system of solutions to the form*

$$y_{30}(x,\lambda)=\frac{1}{3\sqrt[3]{\lambda^2}}\Big(\theta_0e^{\theta_0\sqrt[3]{\lambda}x}+\theta_1e^{\theta_1\sqrt[3]{\lambda}x}+\theta_2e^{\theta_2\sqrt[3]{\lambda}x}\Big),$$
$$y_{20}(x,\lambda)=y_{30}'(x,\lambda),\quad y_{10}(x,\lambda)=y_{30}''(x,\lambda).$$

Similar solutions were introduced and used in the works of Zolotarev [2, 3]. Referring to the above work, we conclude the main assertion of the article.

**Theorem 2.** *Let $p_1(x)\in C^1[0,1], p_0(x)\in C[0,1]$. Then for $k=2,3$ there exist such functions $R_k(x,\tau,t), S_k(x,\tau,t)$, such that the integral representation is valid for all complex $\lambda$:*

$$y_k(x,\lambda)=y_{k0}(x,\lambda)+\int\limits_0^x d\tau\int\limits_0^{x-\tau}R_k(x,\tau,t)y_{k0}(t+\tau,\lambda)dt+\int\limits_0^x d\tau\int\limits_0^{x-\tau}S_k(x,\tau,t)y_{k0}(t+\theta_1\tau,\lambda)dt, \tag{1.3}$$

*where $R_k(x,\tau,t), S_k(x,\tau,t)$-independent of $\lambda$.*

*Also, there exists a pair of functions $R_1(x,\tau,t), S_1(x,\tau,t)$ such that the following representation is true:*

$$y_1(x,\lambda) = y_{10}(x,\lambda) + p_1(0)y_{30}(x,\lambda) + \int_0^x d\tau \int_0^{x-\tau} R_1(x,\tau,t)y_{10}(t+\tau,\lambda)dt + \int_0^x d\tau \int_0^{x-\tau} S_1(x,\tau,t)y_{10}(t+\theta_1\tau,\lambda)dt, \quad (1.4)$$

*where we have $R_1(x,\tau,t), S_1(x,\tau,t)$-independent of $\lambda$.*

If in the representations (1.3) and (1.4) of the functions $S_k(x,\tau,t)$, $k = 1,2,3$ are identically zero, then the representations from Theorem 2 coincide with the representations from Theorem 1. This is possible only when $p_1(x), p_0(x)$ holomorphically continues from the segment $[0,1]$ to the complex plane.

In the case of nonholomorphism of the coefficients $p_1(x)$ and $p_0(x)$, the representations (1.3) and (1.4) are essentially different from those of Theorem 1. For ordinary differential equations of order greater than two, the transformation operator was first constructed by Fage [4] and also independently by Delsarte and Lions [5]. Then, other derivations of the transformation operator of the indicated type were proposed by [6–8]. The formulas obtained in these works have a more complex structure. In this case, the transformation operator was first obtained by Sakhnovich [9], and then a more accurate result belongs to Khachatryan [10], Khachatryan proved that if the coefficients of a differential equation are holomorphic in some quadrangle, then the corresponding integral representation holds. The question of the necessity of the condition of holomorphy of the coefficients of a differential equation of order $n$ for the existence of transformation operators was discussed in [11–13]. In particular, Malamud proved that if there is a transformation operator of a certain type and some of the coefficients $q_k(x)$ are holomorphic, then the remaining coefficients also necessarily possess the holomorphic property. The search for a Lax pair for the nonlinear Camassa–Holm and Degasperis–Procesi equations leads to the study of the spectral properties of third-order linear differential operators. A fairly detailed study of third-order operators based on the special mathematical apparatus of $p$-hyperbolic functions can be found in the works of Zolotarev [2, 3, 14]. In contrast to our work, in his works, the presence of an imaginary unit in the highest third-order derivative allows one to identify classes of nonlocal self-adjoint operators. Self-adjoint third-order operators have various physical applications. In particular, their systems of eigenfunctions are always complete in the original space. In our case, we specifically study non-self-adjoint boundary value problems, because the completeness of the root functions of such operators is of particular interest.

## 2. Supporting statements and proof of Theorem 2

Let us recall some simple statements.

**Lemma 1.** *The identity at $k = 1,2,3$ is*

$$y_{10}(\tau,\lambda)y_{k0}(t,\lambda) = \frac{1}{3}(y_{k0}(t+\theta_0\tau,\lambda) + y_{k0}(t+\theta_1\tau,\lambda) + y_{k0}(t+\theta_2\tau,\lambda)),$$

$$y_{10}(x) = y_{10}(\theta_1 x) = y_{10}(\theta_2 x),\ \ y_{20}(\theta_1 x) = \theta_1 y_{20}(x),\ \ y_{20}(\theta_k x) = \theta_k y_{20}(x),$$
$$y_{20}(\theta_2 x) = \theta_2 y_{20}(x),\ \ y_{30}(\theta_1 x) = \theta_1^2 y_{30}(x),\ \ y_{30}(\theta_2 x) = \theta_2^2 y_{30}(x).$$

*In the works of Zolotarev [2, 3], one can find additional formulas about the solutions of* $y_{k0}(x,\lambda)$ *under* $k = 1, 2, 3$. *Lemma 1 contains only those formulas that we will use in the future.*

**Lemma 2.** *The general solution of the inhomogeneous linear differential equation of the third order*

$$y'''(x) - \lambda y(x) = F(x)$$

*is defined by the following formula:*

$$y(x,\lambda) = C_1 y_{10}(x,\lambda) + C_2 y_{20}(x,\lambda) + C_3 y_{30}(x,\lambda) + \int_0^x \begin{vmatrix} y_{10}(t,\lambda) & y_{20}(t,\lambda) & y_{30}(t,\lambda) \\ y'_{10}(t,\lambda) & y'_{20}(t,\lambda) & y'_{30}(t,\lambda) \\ y_{10}(x,\lambda) & y_{20}(x,\lambda) & y_{30}(x,\lambda) \end{vmatrix} F(t)dt,$$

*where* $F(x)$ *– is a given function.*

*Thus, substituting instead of* $F(x)$ *the expression* $-p_1(t)y'(x,\lambda) - p_0(t)y(t,\lambda)$ *in Lemma 2, the general solution of the problem* (1.1)-(1.2) *can be written in the following form:*

$$y(x,\lambda) = C_1 y_{10}(x,\lambda) + C_2 y_{20}(x,\lambda) + C_3 y_{30}(x,\lambda) + C_1 p_1(0) y_{30}(x,\lambda)$$
$$+ \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) y(t,\lambda) dt, \tag{2.1}$$

*where* $T(x-\tau,t) = \frac{x-t-\tau}{1!}\Big(p'_1(t) - p_0(t)\Big) - p_1(t)$.

In this paper, we discover the dependence of $y_k(x,\lambda)$, $k = 1, 2, 3$ on the spectral parameter. It follows from relations (1.3) that they are solutions of the Volterra integral equations of the second kind,

$$y_1(x,\lambda) = y_{10}(x,\lambda) + p_1(0) y_{30}(x,\lambda) + \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) y_1(t,\lambda) dt,$$

$$y_2(x,\lambda) = y_{20}(x,\lambda) + \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) y_2(t,\lambda) dt,$$

$$y_3(x,\lambda) = y_{30}(x,\lambda) + \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) y_3(t,\lambda) dt.$$

We solve the Volterra integral equations by the Picard method. Let us introduce the functions

$$\chi_{10}(x,\lambda) = y_{10}(x,\lambda) + p_1(0) y_{30}(x,\lambda),\ \ \chi_{k0}(x,\lambda) = y_{k0}(x,\lambda),\ \ k = 2, 3.$$

From the general theorems on integral equations, it follows that at $k = 1, 2, 3$,

$$y_k(x,\lambda) = \chi_{k0}(x,\lambda) + \sum_{j=1}^{\infty} \chi_{kj}(x,\lambda).$$

The elements of the series are calculated by the formulas

$$\chi_{kj}(x,\lambda) = \int\limits_0^x d\tau \int\limits_0^{x-\tau} T(x-\tau,t)y_{10}(\tau,\lambda)\chi_{kj-1}(t,\lambda)dt, \;\; j>0.$$

Let $\lambda \in K$, where $K$ is a compact set in the complex plane. Then, for all $x \in [0,1]$, the estimate $|\chi_{kj}(x,\lambda)| \le M \cdot N \cdot m \cdot \frac{x^{j+1}}{(j+1)!}$ holds for $j \ge 1$, where

$$M = \max_{0\le x\le 1} \left(|p_1'(x)| + |p_0(x)| + |p_1(x)|\right),$$

$$N = \max_{\substack{0\le x\le 1\\ \lambda\in K}} |y_{10}(x,\lambda)|,$$

$$m_k = \max_{\substack{0\le x\le 1\\ \lambda\in K}} |\chi_{k0}(x,\lambda)|.$$

From the given estimates, it follows that the given series converge uniformly on the corresponding compact sets. Our goal is to obtain a unified integral representation for iterations $\chi_{kj}(x,\lambda)$, $j>1$. It becomes apparent that the first two terms, $\chi_{k0}(x,\lambda), \chi_{k1}(x,\lambda)$, can have individual representations. However, starting from the term $\chi_{k2}(x,\lambda)$, there is a single universal form of writing for all $\chi_{kj}(x,\lambda)$, $j>1$.

**Lemma 3.** *At $k = 1,2,3$, the identity*

$$\begin{aligned}
&\int\limits_0^x d\tau \int\limits_0^{x-\tau} T(x-\tau,t)y_{10}(\tau,\lambda)y_{k0}(t,\lambda)dt \\
&= \int\limits_0^x d\tau \int\limits_0^{x-\tau} \frac{1}{3}T(x-\tau,t)y_{k0}(t+\tau,\lambda)dt \\
&\quad \times \int\limits_0^x d\tau \int\limits_0^{x-\tau} \frac{1}{3}\Big(T(x-\tau,t) + \theta_1^{2k+1}T(x-\tau,t)\Big)y_{k0}(y+\theta_1\tau,\lambda)dt
\end{aligned}$$

*is valid.*

In the derivation of Lemma 3, the relations from Lemma 1 are taken into account.

According to Lemma 3, when $k = 2,3$, we have

$$\chi_{k1}(x,\lambda) = \int\limits_0^x d\tau \int\limits_0^{x-\tau} R_{k1}(x,\tau,t)y_{k0}(t+\tau,\lambda)dt + \int\limits_0^x d\tau \int\limits_0^{x-\tau} S_{k1}(x,\tau,t)y_{k0}(t+\theta_1\tau,\lambda)dt,$$

where

$$R_{k1}(x,\tau,t) = \frac{1}{3}T(x-\tau,t),$$

$$S_{k1}(x,\tau,t) = \frac{1}{3}\Big(T(x-\tau,t) + \theta_1^{2k+1}T(x-t,\tau)\Big).$$

Now, at $k = 2, 3$, we find representations for subsequent iterations,

$$\chi_{k2}(x,\lambda) = \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) dt \int_0^t d\sigma \int_0^{t-\sigma} R_{k1}(x,\sigma,\omega) y_{k0}(\omega+\sigma,\lambda) d\omega$$
$$+ \int_0^x d\tau \int_0^{x-\tau} T(x-\tau,t) y_{10}(\tau,\lambda) dt \int_0^t d\sigma \int_0^{t-\sigma} S_{k1}(x,\sigma,\omega) y_{k0}(\omega+\sigma,\lambda) d\omega.$$

Changing the order of integration, we obtain the representation

$$\chi_{k2}(x,\lambda) = \int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} R_{k2}(x,\tau,\sigma,\omega) y_{10}(\tau,\lambda) y_{k0}(\omega+\sigma,\lambda) d\omega$$
$$+ \int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} S_{k2}(x,\tau,\sigma,\omega) y_{10}(\tau,\lambda) y_{k0}(\omega+\theta_1\sigma,\lambda) d\omega,$$

where

$$R_{k2}(x,\tau,\sigma,\omega) = \int_{\sigma+\omega}^{x-\tau} T(x-\tau,t) \tilde{R}_{k1}(x,\sigma,\omega) dt,$$

$$S_{k2}(x,\tau,\sigma,\omega) = \int_{\sigma+\omega}^{x-\tau} T(x-\tau,t) \tilde{S}_{k1}(t,\sigma,\omega) dt.$$

According to Lemma 1, we have

$$y_{10}(\tau,\lambda) y_{k0}(\omega+\theta_1\sigma,\lambda) = \frac{1}{3}\Big( y_{k0}(\omega+\theta_1\sigma+\tau,\lambda) + y_{k0}(\omega+\theta_1\sigma+\theta_1\tau,\lambda) + y_{k0}(\omega+\theta_1\sigma+\theta_2\tau,\lambda)\Big).$$

Let us transform the integral

$$\int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega) \Big[ y_{k0}(\omega+\theta_1\sigma+\tau,\lambda) + y_{k0}(\omega+\theta_1\sigma+\theta_1\tau,\lambda) + y_{k0}(\omega+\theta_1\sigma+\theta_2\tau,\lambda)\Big] d\omega.$$

Let us separately calculate the integral of

$$\int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega) y_{k0}(\omega+\tau+\theta_1\sigma,\lambda) d\omega$$
$$= \int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega-\tau) y_{k0}(\omega+\theta_1\sigma,\lambda) d\omega$$
$$= \int_0^x d\omega \int_0^{x-\omega} y_{k0}(\omega+\theta_1,\lambda) d\sigma \int_0^{\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega-\tau) d\tau.$$

Similarly, we obtain

$$\int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega) y_{k0}(\omega+\theta_1(\sigma+\tau),\lambda) d\omega$$
$$= \int_0^x d\tau \int_0^{x-\tau} d\omega \int_\tau^{x-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma-\tau,\omega) y_{k0}(\omega+\theta_1\sigma,\lambda) d\sigma$$
$$= \int_0^x d\omega \int_0^{x-\omega} y_{k0}(\omega+\theta_1\sigma,\lambda) d\sigma \int_0^{\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma-\tau,\omega) d\tau.$$

Now, let us calculate a more complicated integral,

$$I = \int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\sigma} \frac{1}{3} S_{k2}(x,\tau,\sigma,\omega) y_{k0}(\omega+\theta_2\sigma+\theta_4\tau,\lambda) d\omega$$
$$= \int_0^x d\tau \int_0^{x-\tau} d\sigma \int_0^{x-\tau-\omega} \frac{1}{3} S_2(x,\tau,\sigma,\omega) y_{k0}\Big(\omega - \frac{1}{2}(\sigma+\tau) + i\frac{\sqrt{3}}{2}(\sigma-\tau),\lambda\Big) d\omega.$$

We introduce new variables $(X, Y)$ instead of variables $(\tau, \sigma)$. Also, $X = \omega - \frac{1}{2}(\sigma+\tau)$, $Y = \frac{\sqrt{3}}{2}(\sigma-\tau)$. Then, $\tau = \omega - X - \frac{1}{\sqrt{3}}Y$, $\sigma = \omega - X + \frac{1}{\sqrt{3}}Y$, $d\tau d\sigma = -\frac{2}{\sqrt{3}}dXdY$.

$$I = \iiint_{P(x)} \frac{1}{3} S_{k2}\Big(x, \omega - X - \frac{1}{\sqrt{3}}Y, \omega - X + \frac{1}{\sqrt{3}}Y, \omega\Big) y_{k0}(X+iY,\lambda)\Big(-\frac{2}{\sqrt{3}}\Big) dXdYd\omega.$$

Here, $P(x)$ is the image of the pyramid $\{(\tau,\sigma,\omega) : \tau \geq 0, \sigma \geq 0, \omega \geq 0, \sigma+\tau+\omega \leq x\}$ in the new variables $(X, Y, \omega)$. Let us represent the pyramid $P(x)$ as the union of three nonintersecting pyramids, $P_1(x), P_2(x), P_3(x)$. Then,

$$I = -\frac{2}{\sqrt{3}} \iiint_{P_1(x)} \frac{1}{3} S_{k2}\Big(x, \omega - X - \frac{1}{\sqrt{3}}Y, \omega - X + \frac{1}{\sqrt{3}}Y, \omega\Big) y_{k0}(X+iY,\lambda) dXdYd\omega$$
$$-\frac{2}{\sqrt{3}} \iiint_{P_2(x)} \frac{1}{3} S_{k2}\Big(x, \omega - X - \frac{1}{\sqrt{3}}Y, \omega - X + \frac{1}{\sqrt{3}}Y, \omega\Big) y_{k0}(X+iY,\lambda) dXdYd\omega$$
$$-\frac{2}{\sqrt{3}} \iiint_{P_3(x)} \frac{1}{3} S_{k2}\Big(x, \omega - X - \frac{1}{\sqrt{3}}Y, \omega - X + \frac{1}{\sqrt{3}}Y, \omega\Big) y_{k0}(X+iY,\lambda) dXdYd\omega.$$

Let us denote the first integral by $I_1$ and transform it as follows. We introduce new variables $(\tau, \sigma)$ instead of variables $(X, Y)$. Now, $X = -\frac{1}{2}\tau - \frac{1}{2}\sigma$, $Y = \frac{\sqrt{3}}{2}\tau - \frac{\sqrt{3}}{2}\sigma$, then $dXdY = \frac{\sqrt{3}}{2}d\tau d\sigma$. As a result, we have

$$I_1 = -\int_0^x d\tau \int_0^{x-\tau} y_{k0}(\theta_2\sigma+\theta_1\tau,\lambda) d\sigma \int_0^{\frac{x}{3}+\frac{\sigma+\tau}{2}} \frac{1}{3} S_{k2}(x,\omega+\sigma,\omega+\tau,\omega) d\omega.$$

Thus, the integral representation is valid:

$$I_1 = \int_0^x d\tau \int_0^{x-\tau} S_{k3}(x,\tau,\sigma) y_{k0}(\theta_2\sigma + \theta_1\tau, \lambda) d\sigma = \int_0^x d\tau \int_0^{x-\tau} S_{k3}(x,\tau,\sigma) y_{k0}(\theta_2(\sigma + \theta_2\tau), \lambda) d\sigma,$$

where we have $S_{k3}(x,\tau,\sigma)$-independent of $\lambda$. From Lemma 1, a new integral representation follows:

$$I_1 = \int_0^x d\tau \int_0^{x-\tau} \theta_2^{2k+1} S_{k3}(x,\tau,\sigma) y_{k0}(\sigma + \theta_2\tau, \lambda) d\sigma.$$

In the second integral, $I_2$, the variables $(\tau,\sigma)$ and $(X,Y)$ are related as follows: $X = \tau - \frac{1}{2}\sigma,\ Y = -\frac{\sqrt{3}}{2}\sigma$. Hence,

$$\begin{aligned} I_2 &= \int_0^x d\tau \int_0^{x-\tau} S_{k4}(x,\tau,\sigma) y_{k0}(\theta_2\sigma + \tau, \lambda) d\sigma \\ &= \int_0^x d\tau \int_0^{x-\tau} S_{k4}(x,\tau,\sigma) y_{k0}(\theta_2(\sigma + \theta_1\tau), \lambda) d\sigma \\ &= \int_0^x d\tau \int_0^{x-\tau} \theta_2^{2k+1} S_{k4}(x,\tau,\sigma) y_{k0}(\sigma + \theta_1\tau), \lambda) d\sigma, \end{aligned}$$

where we have $S_{k4}(x,\tau,\sigma)$-independent of $\lambda$.

In the third integral, $I_3$, the variables $(\tau,\sigma)$ and $(X,Y)$ are related as follows: $X = \tau - \frac{1}{2}\sigma,\ Y = \frac{\sqrt{3}}{2}\sigma$. Hence,

$$I_3 = \int_0^x d\tau \int_0^{x-\tau} S_{k5}(x,\tau,\sigma) y_{k0}(\theta_1\sigma + \tau, \lambda) d\sigma = \int_0^x d\tau \int_0^{x-\tau} S_{k5}(x,\sigma,\tau) y_{k0}(\sigma + \theta_2\tau, \lambda) d\sigma,$$

where we have $S_{k5}(x,\tau,\sigma)$-independent of $\lambda$.

Thus, the integral $I$ has an integral representation

$$I = \int_0^x d\tau \int_0^{x-\tau} S_{k6}(x,\tau,\sigma) y_{k0}(\sigma + \theta_2\tau, \lambda) d\sigma,$$

where we have $S_{k6}(x,\tau,\sigma)$-independent of $\lambda$.

Ultimately, for the $\chi_{k2}(x,\lambda)$, we obtain the integral representation

$$\chi_{k2}(x,\lambda) = \int_0^x d\tau \int_0^{x-\tau} \tilde{R}_{k2}(x,\tau,t) y_{k0}(t+\tau,\lambda) dt + \int_0^x d\tau \int_0^{x-\tau} \tilde{S}_{k2}(x,\tau,t) y_{k0}(t + \theta_2\tau, \lambda) dt,$$

where we have $\tilde{R}_{kj}(x,\tau,t), \tilde{S}_{kj}(x,\tau,t)$-independent of $\lambda$. Note that the functions $\tilde{R}_{kj}(x,\tau,t), \tilde{S}_{kj}(x,\tau,t)$ depend on the coefficients $p_0(x), p_1(x)$ and can be defined recursively. Their specific form is not specified in this paper due to space limitations.

The statement of Theorem 2 follows from the latter notion.

## 3. Theorems on the completeness of the system of root functions of a third-order differential operator on a segment

Differential operators with a discrete spectrum are an important source of root function systems. Thus, reversible operators give rise to minimal systems in the corresponding functional g-spaces. The root function systems of self-adjoint operators are orthogonal complete systems. However, self-adjoint differential operators are a rather narrow class of operators with complete root function systems. In this section, we study the completeness of root functions of third-order differential operators on a segment with general boundary conditions, which are not self-adjoint and sometimes irreversible.

In the function space $L_2(0,1)$, consider a differential operator $K$ given by the linear differential expression

$$Ky \equiv y^{(3)}(x) + p_1(x)y^{(1)}(x) + p_0(x)y(x), \quad 0 < x < 1,$$

in the field of definition

$$y \in D(K) \equiv \left\{ y \in W_2^3[0,1] : U_j(y) = 0, \quad j = \overline{1,3} \right\},$$

where $y \in W_2^3[0,1]$–Sobolev space. Here, $p_k(x)$ are the coefficients of the differential expression of $k$ over times continuously differentiable $[0,1]$ functions. The boundary forms $U_1(y), U_2(y), U_3(y)$ are defined as follows:

$$U_1(y) = y^{(2)}(0) + \alpha_{11}y(0),$$
$$U_2(y) = y^{(1)}(0) + \alpha_{21}y(0),$$
$$U_3(y) = y^{(2)}(1) + \beta_{32}y^{(1)}(1) + \beta_{31}y(1) + \alpha_{31}y(0),$$

where $\alpha_{11}, \alpha_{21}, \alpha_{31}, \beta_{31}, \beta_{32}$-are arbitrary complex numbers.

The purpose of this paper is to investigate the completeness of the system of root functions of the operator $K$ in the function space $L_2(0,1)$. The most complete results in this direction are obtained in the case of second-order differential operators. In [1], the completeness of root functions in the case of nondegenerate boundary conditions in the sense of Marchenko was proved. Moreover, the completeness of root functions for nondegenerate boundary conditions does not depend on the coefficients of the differential expression generating the operator. In [4], the completeness of the system of root functions of linear differential operators of the second order in the case of degenerate boundary conditions was investigated. It was shown that in this case, the completeness of the system of root functions depends not only on the J matrix of boundary coefficients, but also on the coefficients of the differential expression. Similar questions for second-order differential operators were studied in [5].

The completeness of the system of root functions of linear differential operators of higher orders is guaranteed in the case of regular boundary conditions in the sense of Birchhoff [6]. Moreover, the completeness of the root functions for regular boundary conditions in the sense of Birchhoff does not depend on $p_k(x)$ of the coefficients of the differential expression. The case of irregular and decaying boundary conditions for the completeness of the system of root functions was studied in the work of Shkalikov [7].

We are interested in finding an analog of nondegenerate boundary conditions for the operator $K$, extending the class of regular Birchhoff boundary conditions and preserving the completeness of the system of eigenfunctions and adjoint functions in the space $L_2(0,1)$ [15].

Now let us formulate the main conclusion of this section.

**Theorem 3.** *Let $p_1(x) \in C^1[0,1]$, $p_0(x) \in C[0,1]$, and $\alpha_{11}, \alpha_{21}, \alpha_{31}, \beta_{31}, \beta_{32}$ – be arbitrary complex numbers. Then, the system of eigenfunctions and adjoint functions of the operator K is complete in the space $L_2(0,1)$.*

*Let us denote by $y_1(x), y_2(x), y_3(x)$ the solutions of the homogeneous equation*

$$l(y) \equiv y^{(3)}(x) + p_1(x)y^{(1)}(x) + p_0(x)y(x) = \lambda y(x), \quad 0 < x < 1,$$

*subject to the Cauchy initial conditions at $x = 0$:*

$$y_j^{(k-1)}(0) = \delta_{kj}, \quad k, j = 1,2,3.$$

*Note that the functions $y_n(x,\lambda)$, $n = 1,2,3$ represent integer functions of $\lambda$.*

*We denote the formally conjugate differential expression by*

$$l^+(z) \equiv -z^{(3)}(x) - \overline{p_1(x)}z^{(1)}(x) + \Big( -\overline{p_1^{(1)}(x)} + \overline{p_0(x)}\Big)z(x).$$

*By $z_k(x)$, $k = 1,2,3$, we denote the solutions of the homogeneous conjugate equation*

$$l^+(z) = \overline{\lambda}z(x), \quad 0 < x < 1 \tag{3.1}$$

*with Cauchy conditions at $x = 0$*

$$\begin{array}{lll} z_1(0) = 0, & z_2(0) = 0, & z_3(0) = 1, \\ z_1^{(1)}(0) = 0, & z_2^{(1)}(0) = -1, & z_3^{(1)}(0) = 0, \\ z_1^{(2)}(0) = 1, & z_2^{(2)}(0) = 0, & z_3^{(2)}(0) = -\overline{p_1(0)}. \end{array} \tag{3.2}$$

*Note that the solutions $z_k(x,\overline{\lambda})$, $k = 1,2,3$ depend on the spectral parameter $\overline{\lambda}$. The functions $z_k(x,\overline{\lambda})$, $k = 1,2,3$ represent entire functions of $\overline{\lambda}$.*

*We denote the Wronski matrix of the fundamental system of solutions $y_1(x), y_2(x), y_3(x)$ by*

$$Y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) & y_3'(x,\lambda) \\ y_1''(x,\lambda) & y_2''(x,\lambda) & y_3''(x,\lambda) \end{bmatrix}.$$

*Because $\det Y(x,\lambda) = 1$ for all $x \in [0,1]$, the Wronski matrix is invertible. By the fundamental system of solutions $z_k(x,\overline{\lambda})$, $k = 1,2,3$, we introduce the Wronski matrix $Z(x,\overline{\lambda})$. Then, the matrix identity*

$$\Big(Y(x,\lambda)\Big)^T \overline{Q(x)}\,\overline{Z(x,\overline{\lambda})} = \Big(Y(0,\lambda)\Big)^T \overline{Q(0)}\,\overline{Z(0,\overline{\lambda})},$$

*follows from the Lagrange formula [6], where* $Q(x) = \begin{bmatrix} \overline{p_1(0)} & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. *Given a special choice of systems of fundamental solutions $z_k(x,\overline{\lambda}), y_k(x,\lambda), k = 1,2,3$ from the last identity, we have*

$$\Big(Y(x,\lambda)\Big)^{-1} = \Big(\overline{Z(x,\overline{\lambda})}\Big)^T \Big(\overline{Q(x)}\Big)^T.$$

*Thus, the elements of the inverse matrix can be written out through the solutions of the conjugate equation. In fact, the elements of the inverse matrix are written through the second-order minors of the Wronski matrix. Therefore, the above formula allows us to write out the second-order minors of the Wronski matrix through the solutions of the conjugate equation,*

$$\begin{vmatrix} y_2'(x) & y_3'(x) \\ y_2''(x) & y_3''(x) \end{vmatrix} = \overline{z_1''(x,\overline{\lambda})} + p_1(x)\overline{z_1(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_2(x) & y_3(x) \\ y_2''(x) & y_3''(x) \end{vmatrix} = \overline{z_1'(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1'(x) & y_3'(x) \\ y_1''(x) & y_3''(x) \end{vmatrix} = -\overline{z_2''(x,\overline{\lambda})} - p_1(x)\overline{z_2(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1(x) & y_3(x) \\ y_1''(x) & y_3''(x) \end{vmatrix} = -\overline{z_2'(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1'(x) & y_2'(x) \\ y_1''(x) & y_2''(x) \end{vmatrix} = -\overline{z_3''(x,\overline{\lambda})} - p_1(x)\overline{z_3(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1''(x) & y_2''(x) \end{vmatrix} = \overline{z_3'(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_2(x) & y_3(x) \\ y_2'(x) & y_3'(x) \end{vmatrix} = \overline{z_1(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1(x) & y_3(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = -\overline{z_2(x,\overline{\lambda})},$$
$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \overline{z_3(x,\overline{\lambda})}.$$

*Now, consider the characteristic determinant*

$$\Delta(\lambda) = \begin{vmatrix} \alpha_{11} & 0 & 1 \\ \alpha_{21} & 1 & 0 \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix}.$$

The following auxiliary statements are useful. For this purpose, let us introduce the following notations.

**Lemma 4.** *For any $p_1 \in C^1[0,1], p_0 \in C[0,1]$, the characteristic determinant $\Delta(\lambda)$ has the following representation:*

$$\Delta(\lambda) = -\alpha_{31} + \alpha_{11}\beta_{31}y_3(1,\lambda) + \alpha_{11}\beta_{32}y_3'(1,\lambda) + \alpha_{11}y_3''(1,\lambda) + \alpha_{21}\beta_{31}y_2(1,\lambda) + \alpha_{21}\beta_{32}y_2'(1,\lambda) +$$
$$+\alpha_{21}y_2''(1,\lambda) - \beta_{31}y_1(1,\lambda) - \beta_{32}y_1'(1,\lambda) - y_1''(1,\lambda).$$

*Thus, we have found a general representation of the characteristic determinant. So far, we have considered that the coefficients $p_1(x), p_0(x)$ are arbitrary smooth functions. Now, consider the case where $p_1(x), p_0(x) = 0$. In this case, the characteristic determinant $\Delta_0(\lambda)$ has a simpler form. Given the above relations and the result of Lemma 4, we obtain the following representation:*

$$\Delta_0(\lambda) = -\alpha_{31} + (\alpha_{11}\beta_{31} + \lambda\alpha_{21} - \lambda\beta_{32})y_{30}(1,\lambda) \\ + (\alpha_{11}\beta_{32} + \alpha_{21}\beta_{31} - \lambda)y_{20}(1,\lambda) + (\alpha_{11} + \alpha_{21}\beta_{32} - \beta_{31})y_{10}(1,\lambda).$$

*Note [6] that the zeros of the characteristic determinant $\Delta(\lambda)$ represent all eigenvalues of the operator K. Moreover, the multiplicity of zero coincides with the algebraic multiplicity of the eigenvalue. Let us introduce the function $\varphi_3(x,\lambda)$ as a determinant, which is obtained from the characteristic determinant*

$$\varphi_3(x,\lambda) = \begin{vmatrix} \alpha_{11} & 0 & 1 \\ \alpha_{21} & 1 & 0 \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}.$$

*Let $\lambda = \lambda_0$- be an eigenvalue of the operator K with algebraic multiplicity $m+1$. It is known [6] that the function $\varphi_3(x,\lambda)$ at fixed x is an integer function of the spectral parameter $\lambda$. Then, in the ordered set $\left\{\varphi_3(x,\lambda_0), \frac{\partial\varphi_3(x,\lambda_0)}{\partial\lambda} \dots \frac{1}{m!}\frac{\partial^m\varphi_3(x,\lambda_0)}{\partial\lambda^m}\right\}$, the first nonzero function is an eigenvalue, and the subsequent ones give a chain of attached functions corresponding to the eigenvalue $\lambda_0$. These sets of functions give rise to a complete set of chains of eigenfunctions and adjoint functions corresponding to the eigenvalue $\lambda_0$. Therefore, the functions $\varphi_3(x,\lambda)$ introduced by us play the role of an interpolating function. The facts given by us are known [6].*

**Lemma 5.** *For all $f(x) \in L_2(0,1)$, the following limit equality is true:*

*(1) At $-\frac{\pi}{3} < \sqrt[3]{\lambda} < \frac{\pi}{3}$ : $\lim\limits_{|\lambda|\to\infty} e^{-Re\sqrt[3]{\lambda}} \int\limits_0^1 f(x)y_{10}(x,\lambda)dx = 0$;*

*(2) At $\frac{\pi}{3} < \sqrt[3]{\lambda} < \pi$ : $\lim\limits_{|\lambda|\to\infty} e^{-Re\theta_2\sqrt[3]{\lambda}} \int\limits_0^1 f(x)y_{10}(x,\lambda)dx = 0$;*

*(3) At $\pi < \sqrt[3]{\lambda} < \frac{5\pi}{3}$ : $\lim\limits_{|\lambda|\to\infty} e^{-Re\theta_1\sqrt[3]{\lambda}} \int\limits_0^1 f(x)y_{10}(x,\lambda)dx = 0$.*

*Proof of Lemma 5.* The set $C^1(0,1)$ of continuously differentiable functions on the segment $[0,1]$ is dense in the space $L_2(0,1)$. For any function $f(x) \in L_2(0,1)$ and arbitrary $\varepsilon > 0$, there exists a function $g_\varepsilon(x) \in C^1(0,1)$, such that

$$\int_0^1 \left|f(x) - g_\varepsilon(x)\right| dx < \varepsilon.$$

Because

$$\int_0^1 f(x)y_{10}(x,\lambda)dx = \int_0^1 \{f(x) - g(x)_\varepsilon\}y_{10}(x,\lambda)dx + \int_0^1 g_\varepsilon(x)y_{10}(x,\lambda)dx$$
$$= \int_0^1 \{f(x) - g_\varepsilon(x)\}y_{10}(x,\lambda)dx + g_\varepsilon(1)\frac{1}{3}\Big(\frac{1}{\theta_0\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_0} + \frac{1}{\theta_1\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_1} + \frac{1}{\theta_2\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_2}\Big)$$
$$- \int_0^1 g'_\varepsilon(1)\frac{1}{3}\Big(\frac{1}{\theta_0\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_0 x} + \frac{1}{\theta_1\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_1 x} + \frac{1}{\theta_2\sqrt[3]{\lambda}}e^{\sqrt[3]{\lambda}\theta_2 x}\Big)dx$$

$$= \int_0^1 \Big\{f(x) - g(x)_\varepsilon\Big\} y_{10}(x,\lambda)dx$$

$$+ \sqrt[3]{\lambda}^{-1}\Big\{g_\varepsilon(1)\frac{1}{3}\Big(\theta_0 e^{\sqrt[3]{\lambda}\theta_0} + \theta_2 e^{\sqrt[3]{\lambda}\theta_1} + \theta_1 e^{\sqrt[3]{\lambda}\theta_2}\Big)$$

$$- \int_0^1 g'_\varepsilon(1)\frac{1}{3}\Big(\theta_0 e^{\sqrt[3]{\lambda}\theta_0} + \theta_2 e^{\sqrt[3]{\lambda}\theta_1} + \theta_1 e^{\sqrt[3]{\lambda}\theta_2}\Big)dx\Big\},$$

if at $-\frac{\pi}{3} < \sqrt[3]{\lambda} < \frac{\pi}{3}$, then $Re\theta_1 < Re\theta_2 < Re\theta_0$, and for all $x \in [0,1]$, the estimates $|y_{10}(x,\lambda)| \le e^{Re\sqrt[3]{\lambda}}$, are satisfied, then

$$\Big|\int_0^1 f(x)y_{10}(x,\lambda)dx\Big| \le e^{Re\sqrt[3]{\lambda}}\Big[\int_0^1 |f(x) - g_\epsilon(x)|dx + |\sqrt[3]{\lambda}|^{-1}\Big\{|g_\epsilon(1)| + \int_0^1 |g_\epsilon(x)|dx\Big\}\Big].$$

Therefore,

$$\overline{\lim_{|\lambda|\to\infty}} e^{-Re\sqrt[3]{\lambda}} \int_0^1 f(x)y_{10}(x,\lambda)dx = 0.$$

Similar estimates are proved in the remaining sectors. Therefore, Lemma 5 is proved. □

Lemmas 6 and 7 are proved in the same way.

**Lemma 6.** *For all $f(x) \in L_2(0,1)$, the following limit equality is true:*

*(1) At* $-\frac{\pi}{3} < arg\sqrt[3]{\lambda} < \frac{\pi}{3}$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda}e^{-Re\sqrt[3]{\lambda}} \int_0^1 f(x)y_{20}(x,\lambda)dx = 0$;

*(2) At* $\frac{\pi}{3} < arg\sqrt[3]{\lambda} < \pi$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda}e^{-Re\theta_2\sqrt[3]{\lambda}} \int_0^1 f(x)y_{20}(x,\lambda)dx = 0$;

*(3) At* $\pi < arg\sqrt[3]{\lambda} < \frac{5\pi}{3}$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda}e^{-Re\theta_1\sqrt[3]{\lambda}} \int_0^1 f(x)y_{20}(x,\lambda)dx = 0.$

**Lemma 7.** *For all $f(x) \in L_2(0,1)$, the following limit equality is true:*

*(1) At* $-\frac{\pi}{3} < \sqrt[3]{\lambda} < \frac{\pi}{3}$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda^2}e^{-Re\sqrt[3]{\lambda^2}} \int_0^1 f(x)y_{30}(x,\lambda)dx = 0$;

*(2) At* $\frac{\pi}{3} < \sqrt[3]{\lambda} < \pi$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda^2}e^{-Re\theta_2\sqrt[3]{\lambda^2}} \int_0^1 f(x)y_{30}(x,\lambda)dx = 0$;

*(3) At* $\pi < \sqrt[3]{\lambda} < \frac{5\pi}{3}$ : $\lim_{|\lambda|\to\infty} \sqrt[3]{\lambda^2}e^{-Re\theta_1\sqrt[3]{\lambda^2}} \int_0^1 f(x)y_{30}(x,\lambda)dx = 0.$

Now, we will show how the following statement follows from Lemmas 5–7.

**Lemma 8.** *For all $f(x) \in L_2(0,1)$, the limit equality is valid for $k = 1,2,3$:*

*(1) At* $-\frac{\pi}{3} < \sqrt[3]{\lambda} < \frac{\pi}{3}$ : $\lim_{|\lambda|\to\infty} (\sqrt[3]{\lambda})^{k-1}e^{-Re\sqrt[3]{\lambda}} \int_0^1 f(x)y_k(x,\lambda)dx = 0$;

*(2) At* $\frac{\pi}{3} < \sqrt[3]{\lambda} < \pi$ : $\lim\limits_{|\lambda|\to\infty} (\sqrt[3]{\lambda})^{k-1} e^{-Re\theta_2 \sqrt[3]{\lambda}} \int\limits_0^1 f(x) y_k(x,\lambda)dx = 0$*;*

*(3) At* $\pi < \sqrt[3]{\lambda} < \frac{5\pi}{3}$ : $\lim\limits_{|\lambda|\to\infty} (\sqrt[3]{\lambda})^{k-1} e^{-Re\theta_1 \sqrt[3]{\lambda}} \int\limits_0^1 f(x) y_k(x,\lambda)dx = 0.$

*Proof of Lemma 8.* For $k = 3$ and for $-\frac{\pi}{3} < \sqrt[3]{\lambda} < \frac{\pi}{3}$, the set $C^1(0,1)$ of continuously differentiable functions on the segment $[0,1]$ is dense in the space $L_2(0,1)$. For any function $f(x) \in L_2(0,1)$ and arbitrary $\varepsilon > 0$, there exists a function $g_\varepsilon(x) \in C^1(0,1)$ such that

$$\int\limits_0^1 |f(x) - g_\varepsilon(x)| dx < \varepsilon.$$

Because

$$\int\limits_0^1 f(x) y_3(x,\lambda)dx = \int\limits_0^1 \{f(x) - g_\varepsilon(x)\} y_3(x,\lambda)dx + \int\limits_0^1 g_\varepsilon(x) y_3(x,\lambda)dx, \tag{3.3}$$

Theorem 2 implies the representation

$$y_3(x,\lambda) = y_{30}(x,\lambda) + \int\limits_0^x d\tau \int\limits_0^{x-\tau} R_3(x,\tau,t) y_{30}(t+\tau,\lambda)dt + \int\limits_0^x d\tau \int\limits_0^{x-\tau} S_3(x,\tau,t) y_{30}(t+\theta_1\tau,\lambda)dt. \tag{3.4}$$

Substitute expression (3.3) into Eq (3.4). As a result, we obtain

$$\begin{aligned}\int\limits_0^1 f(x) y_3(x,\lambda)dx = &\int\limits_0^1 \{f(x) - g_\varepsilon(x)\} y_{30}(x,\lambda)dx + \int\limits_0^1 g_\varepsilon(x) y_{30}(x,\lambda)dx + \int\limits_0^1 \{f(x) - g_\varepsilon(x)\} \\ &\times \left( \int\limits_0^x d\tau \int\limits_0^{x-\tau} R_3(x,\tau,t) y_{30}(t+\tau,\lambda)dt + \int\limits_0^x d\tau \int\limits_0^{x-\tau} S_3(x,\tau,t) y_{30}(t+\theta_1\tau,\lambda)dt \right) dx \\ &+ \int\limits_0^1 g_\varepsilon(x) \left( \int\limits_0^x d\tau \int\limits_0^{x-\tau} R_3(x,\tau,t) y_{30}(t+\tau,\lambda)dt + \int\limits_0^x d\tau \int\limits_0^{x-\tau} S_3(x,\tau,t) y_{30}(t+\theta_1\tau,\lambda)dt \right) dx.\end{aligned}$$

In proving Lemma 7 for the expression $\int\limits_0^1 \{f(x) - g_\varepsilon(x)\} y_{30}(x,\lambda)dx + \int\limits_0^1 g_\varepsilon(x) y_{30}(x,\lambda)dx$, an upper bound was obtained for $\lambda \to \infty$. Now, we will separately estimate the modulus of the following expression:

$$\begin{aligned}|I_1| = &\left| \int\limits_0^1 \{f(x) - g_\varepsilon(x)\} \left( \int\limits_0^x d\tau \int\limits_0^{x-\tau} R_3(x,\tau,t) y_{30}(t+\tau,\lambda)dt + \int\limits_0^x d\tau \int\limits_0^{x-\tau} S_3(x,\tau,t) y_{30}(t+\theta_1\tau,\lambda)dt \right) dx \right| \\ \le &\int\limits_0^1 |f(x) - g_\varepsilon(x)| dx \int\limits_0^x d\tau \int\limits_0^{x-\tau} |R_3(x,\tau,t)| |y_{30}(t+\tau,\lambda)| dt\end{aligned}$$

$$+\int_0^1 |f(x)-g_\varepsilon(x)|dx\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)||y_{30}(t+\theta_1\tau,\lambda)|dt.$$

Let $-\frac{\pi}{3}<arg\sqrt[3]{\lambda}<\frac{\pi}{3}$ and $\lambda\to\infty$. Let us estimate the modulus of the expression

$$\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)||y_{30}(t+\theta_1\tau,\lambda)|dt$$
$$\le\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)|\frac{1}{3|\sqrt[3]{\lambda^2}|}\Big(|e^{\theta_0\sqrt{\lambda}(t+\theta_1\tau)}|+|e^{\theta_1\sqrt{\lambda}(t+\theta_1\tau)}|+|e^{\theta_2\sqrt{\lambda}(t+\theta_1\tau)}|\Big)dt$$
$$=\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)|\frac{1}{3|\sqrt[3]{\lambda^2}|}\Big(e^{Re(\theta_0\sqrt{\lambda}(t+\theta_1\tau))}+e^{Re(\theta_1\sqrt{\lambda}(t+\theta_1\tau))}+e^{Re(\theta_2\sqrt{\lambda}(t+\theta_1\tau))}\Big)dt \tag{3.5}$$

from above. At $-\frac{\pi}{3}<arg\sqrt[3]{\lambda}<\frac{\pi}{3}$, $Re\rho\theta_1\le Re\rho\theta_2\le Re\rho\theta_0$ holds; therefore,

$$e^{Re(\theta_0\rho(t+\theta_1\tau))}=e^{Re(\theta_0\rho t)}e^{Re(\theta_0\rho(\theta_1\tau)}\le e^{tRe\rho}e^{\tau Re\rho}\le e^{(t+\tau)Re\rho}.$$

Because $0\le t+\tau\le x$, the following inequality holds:

$$e^{(t+\tau)Re\rho}\le e^{xRe\rho}.$$

From the last inequality and inequality (3.5), the final inequality follows:

$$\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)||y_{30}(t+\theta_1\tau,\lambda)|dt\le\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)|dt. \tag{3.6}$$

Let $\rho^3=\lambda$. It is assessed in the same way:

$$\int_0^x d\tau\int_0^{x-\tau}|R_3(x,\tau,t)||y_{30}(t+\tau,\lambda)|dt\le\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int_0^x d\tau\int_0^{x-\tau}|R_3(x,\tau,t)|dt. \tag{3.7}$$

Using inequalities (3.6) and (3.7), inequality (3.5) implies that

$$|I_1|\le\int_0^1|f(x)-g_\varepsilon(x)|dx\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int_0^x d\tau\int_0^{x-\tau}|R_3(x,\tau,t)|dt$$
$$+\int_0^1|f(x)-g_\varepsilon(x)|dx\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int_0^x d\tau\int_0^{x-\tau}|S_3(x,\tau,t)|dt$$
$$\le\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int_0^1|f(x)-g_\varepsilon(x)|dx\int_0^x d\tau\int_0^{x-\tau}|R_3(x,\tau,t)|dt$$

$$+\frac{e^{xRe\rho}}{|\sqrt[3]{\lambda^2}|}\int\limits_0^1|f(x)-g_\varepsilon(x)|dx\int\limits_0^x d\tau\int\limits_0^{x-\tau}|S_3(x,\tau,t)|dt.$$

To evaluate $I_2$, we need

$$I_2=\int\limits_0^1 g_\varepsilon(x)\left(\int\limits_0^x d\tau\int\limits_0^{x-\tau}R_3(x,\tau,t)y_{30}(t+\tau,\lambda)dt+\int\limits_0^x d\tau\int\limits_0^{x-\tau}S_3(x,\tau,t)y_{30}(t+\theta_1\tau,\lambda)dt\right)dx$$

preliminarily integrate by parts and then perform estimates similar to (3.5)–(3.7).

Thus, Lemma 8 is completely proven for $k=3$ and for $-\frac{\pi}{3}<\sqrt[3]{\lambda}<\frac{\pi}{3}$. □

The remaining cases are proven similarly. We will need one useful statement.

**Lemma 9.** *Let* $\Omega=\{(\tau,t):\tau,t\geq 0,\tau+t<1\}$ *and* $N(\tau,t)\in L_2(\Omega)$*:*

*(1) If* $-\frac{\pi}{3}<\sqrt[3]{\lambda}<\frac{\pi}{3}$*, then* $\lim\limits_{|\lambda|\to\infty}\rho^{k-1}e^{-Re\rho}\int\limits_0^1 d\tau\int\limits_0^{1-\tau}N(\tau,t)y_{k0}(t+\theta_1\tau,\lambda)dt=0.$

*(2) If* $\frac{\pi}{3}<\sqrt[3]{\lambda}<\pi$*, then* $\lim\limits_{|\lambda|\to\infty}\rho^{k-1}e^{-Re\theta_2\rho}\int\limits_0^1 d\tau\int\limits_0^{1-\tau}N(\tau,t)y_{k0}(t+\theta_1\tau,\lambda)dt=0.$

*(3) If* $\pi<\sqrt[3]{\lambda}<\frac{5\pi}{3}$*, then* $\lim\limits_{|\lambda|\to\infty}\rho^{k-1}e^{-Re\theta_1\rho}\int\limits_0^1 d\tau\int\limits_0^{1-\tau}N(\tau,t)y_{k0}(t+\theta_1\tau,\lambda)dt=0.$

The proof of Lemma 9 is similar to the proof of Lemma 8. Now, let us estimate the modulus of the characteristic determinant from below as $\lambda\to\infty$.

**Lemma 10.** *When* $\lambda\to\infty$*, the characteristic determinant has asymptotic representations*

$$\Delta(\lambda)=-\lambda y_{20}(1,\lambda)(1+o(1)).$$

*Proof of Lemma 10.* According to Lemma 4, the characteristic determinant has the form

$$\begin{aligned}\Delta(\lambda)=&-\alpha_{31}+\alpha_{11}\beta_{31}y_3(1,\lambda)+\alpha_{11}\beta_{32}y_3'(1,\lambda)+\alpha_{11}y_3''(1,\lambda)+\alpha_{21}\beta_{31}y_2(1,\lambda)\\&+\alpha_{21}\beta_{32}y_2'(1,\lambda)+\alpha_{21}y_2''(1,\lambda)-\beta_{31}y_1(1,\lambda)-\beta_{32}y_1'(1,\lambda)-y_1''(1,\lambda).\end{aligned}$$

From Theorem 2, it follows that

$$\begin{aligned}y_3(x,\lambda)=&y_{30}(x,\lambda)+\int\limits_0^x d\tau\int\limits_0^{x-\tau}R_3(x,\tau,t)y_{30}(t+\tau,\lambda)dt+\int\limits_0^x d\tau\int\limits_0^{x-\tau}S_3(x,\tau,t)y_{30}(t+\theta_1\tau,\lambda)dt,\\y_3'(x,\lambda)=&y_{30}'(x,\lambda)+\int\limits_0^x d\tau\int\limits_0^{x-\tau}\frac{\partial}{\partial x}R_3(x,\tau,t)y_{30}(t+\tau,\lambda)dt+y_{30}(x,\lambda)\int\limits_0^x R_3(x,\tau,x-\tau)d\tau+\\&+\int\limits_0^x d\tau\int\limits_0^{x-\tau}\frac{\partial}{\partial x}S_3(x,\tau,t)y_{30}(t+\theta_1\tau,\lambda)dt+\int\limits_0^x S_3(x,\tau,x-\tau)y_{30}(x-\tau+\theta_1\tau,\lambda)d\tau,\end{aligned}$$

$$y_3''(x,\lambda) = y_{30}''(x,\lambda) + \int_0^x d\tau \int_0^{x-\tau} \frac{\partial^2}{\partial x^2} R_3(x,\tau,t) y_{30}(t+\tau,\lambda)dt + y_{30}(x,\lambda)\int_0^x \frac{\partial}{\partial x} R_3(x,\tau,x-\tau)d\tau$$
$$+ y_{30}(x,\lambda)\frac{\partial}{\partial x}\left(\int_0^x R_3(x,\tau,x-\tau)d\tau\right) + y_{30}'(x,\lambda)\int_0^x R_3(x,\tau,x-\tau)d\tau$$
$$+ \int_0^x d\tau \int_0^{x-\tau} \frac{\partial^2}{\partial x^2} S_3(x,\tau,t) y_{30}(t+\theta_1\tau,\lambda)dt + \int_0^x \frac{\partial}{\partial x} S_3(x,\tau,x-\tau)\Big|_{t=x-\tau} y_{30}(x-\tau+\theta_1\tau,\lambda)d\tau$$
$$\times \int_0^x \left[\frac{\partial}{\partial x}\big(S_3(x,\tau,x-\tau)\big) y_{30}(x-\tau+\theta_1\tau,\lambda) + S_3(x,\tau,x-\tau) y_{30}'(x-\tau+\theta_1\tau,\lambda)\right]d\tau$$
$$+ S_3(x,x,0)y_{30}(\theta_1 x,\lambda).$$

Hence, using Lemma 9, as $\lambda \to \infty$, we obtain asymptotic representations of

$$y_3(1,\lambda) = y_{30}(1,\lambda) + |y_{30}(1,\lambda)|\varepsilon_{31}(\lambda),$$

$$y_3'(1,\lambda) = y_{20}(1,\lambda) + |y_{20}(1,\lambda)|\varepsilon_{32}(\lambda),$$

$$y_3''(1,\lambda) = y_{10}(1,\lambda) + |y_{10}(1,\lambda)|\varepsilon_{33}(\lambda),$$

where $\lim_{\lambda\to\infty} \varepsilon_{3k}(\lambda) = 0, k = 1,2,3.$

Similarly, when $\lambda \to \infty$, the following asymptotic representations are obtained:

$$\begin{aligned}
y_2(1,\lambda) &= y_{20}(1,\lambda) + |y_{20}(1,\lambda)|\varepsilon_{21}(\lambda),\\
y_2'(1,\lambda) &= y_{10}(1,\lambda) + |y_{10}(1,\lambda)|\varepsilon_{22}(\lambda),\\
y_2''(1,\lambda) &= \lambda y_{30}(1,\lambda) + |\lambda||y_{30}(1,\lambda)|\varepsilon_{23}(\lambda),\\
y_1(1,\lambda) &= y_{10}(1,\lambda) + |y_{10}(1,\lambda)|\varepsilon_{11}(\lambda),\\
y_1'(1,\lambda) &= \lambda y_{30}(1,\lambda) + |\lambda||y_{30}(1,\lambda)|\varepsilon_{12}(\lambda),\\
y_1''(1,\lambda) &= \lambda y_{20}(1,\lambda) + |\lambda||y_{20}(1,\lambda)|\varepsilon_{13}(\lambda).
\end{aligned}$$

Now, using the values found, when $\lambda \to \infty$, we rewrite the characteristic determinant as

$$\begin{aligned}
\Delta(\lambda) = &\Delta_0(\lambda) + \alpha_{11}\beta_{31}y_{30}(1,\lambda)\varepsilon_{31}(\lambda) + \alpha_{11}\beta_{32}y_{20}(1,\lambda)\varepsilon_{32}(\lambda) + \alpha_{11}y_{10}(1,\lambda)\varepsilon_{33}(\lambda)\\
&+ \alpha_{21}\beta_{31}y_{20}(1,\lambda)\varepsilon_{21}(\lambda) + \alpha_{21}\beta_{32}y_{10}(1,\lambda)\varepsilon_{22}(\lambda) + \alpha_{21}\lambda y_{30}(1,\lambda)\varepsilon_{23}(\lambda)\\
&- \beta_{31}y_{10}(1,\lambda)\varepsilon_{11}(\lambda) - \beta_{32}\lambda y_{30}(1,\lambda)\varepsilon_{12}(\lambda) - \lambda y_{20}(1,\lambda)\varepsilon_{13}(\lambda).
\end{aligned}$$

When $\lambda \to \infty$, the characteristic determinant has asymptotic representations

$$\Delta(\lambda) = -\lambda y_{20}(1,\lambda)(1 + o(1)).$$

Thus, Lemma 10 is completely proven. □

*Proof of Theorem 3.* Let $M$ be the spectrum of operator $K$, that is the set of all eigenvalues $\lambda_n$ and $m_n$ be their multiplicity. According to the above properties of function $\varphi_3(x,\lambda)$, the results of the zeta function

$$\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\varphi_3(x,\lambda)\Big|_{\lambda=\lambda_n}\ (0\le k\le m_n-1,\ \lambda_n\in M)$$

are either identically equal to zero or are root functions of the operator $K$.

Therefore, to prove the completeness of the root functions of the operator $K$, it suffices to show that if $f(x)\in L_2(0,1)$ and

$$\int_0^1 \frac{\partial^k}{\partial\lambda^k}\varphi_3(x,\lambda)f(x)dx\Big|_{\lambda=\lambda_n}=0\ \ (0\le k\le m_n-1,\ \lambda_n\in M), \tag{3.8}$$

then $f(x)=0$ almost everywhere. The characteristic function $\Delta(\lambda)$ and the function

$$\omega_3(\lambda,f)=\int_0^1\varphi_3(x,\lambda)f(x)dx$$

are entire functions of $\lambda$. If equalities (3.8) hold, then each $m_n$, – a multiple root of $\lambda_n$ of the function $\Delta(\lambda)$, will also be a root of at least the same multiplicity of the function $\omega_3(\lambda,f)$. Therefore, $\frac{\omega_3(\lambda,f)}{\Delta(\lambda)}$ is an entire function of $\lambda$. According to Lemmas 8 and 10, there exist a constant $C$ and a sequence of unbounded expanding contours $\gamma_n$ such that for $\rho\in\gamma_n$

$$\frac{\omega_3(\rho^3,f)}{\Delta(\rho^3)}\le\frac{C}{|\rho^2|}.$$

This assessment implies the equality

$$\lim_{n\to\infty}\max\left|\frac{\omega_3(\rho^3,f)}{\Delta(\rho^3)}\right|=0.$$

Next, using standard methods, applying Mittag-Leffler's theorems, the maximum principle, and Liouville's theorem, we obtain that

$$\omega_3(\lambda,f)\equiv 0\quad\text{in all}\quad\lambda\in\mathbb{C}.$$

Thus, for all $\lambda\in\mathbb{C}$, the identity

$$\int_0^1 f(x)\varphi_3(x,\lambda)dx=0 \tag{3.9}$$

holds, where $\varphi_3(x,\lambda)$ is the solution to the Cauchy problem

$$\varphi_3'''(x)+p_1(x)\varphi_3'(x)+p_0(x)\varphi_3(x)=\lambda\varphi_3(x),\ \ 0<x<1,$$

$$\varphi_3(0,\lambda)=-1,\varphi_3'(0,\lambda)=-\alpha_{21},\ \ \varphi_3''(0,\lambda)=-\alpha_{11}.$$

Because the initial data does not depend on $\lambda$, the following statement is true:

**Lemma by Shkalikov [16].** If, for some summable function $f(x)$, the following identity holds:

$$\int_0^1 f(x)\varphi_3(x,\lambda)dx = 0,$$

where the function $\varphi_3(x,\lambda)$ is the solution to the Cauchy problem with nonzero initial data that do not depend on $\lambda$, then $f(x) = 0$ almost everywhere on $(0,1)$.

From Shkalikov's lemma, it follows that $f(x) = 0$ almost everywhere on $(0,1)$. Thus, the completeness of the system of root functions of the operator $K$ in the space $L_2(0,1)$ is proven. □

## 4. Conclusions

In the first part of the article, integral representations of solutions to a single-valued third-order linear differential equation with variable coefficients are obtained. Then, Birchhoff-irregular boundary conditions are considered, and the completeness of the system of root functions of the corresponding boundary value problem for a linear differential equation with variable coefficients is proven. The proof is based on a detailed asymptotic analysis of the characteristic determinant and interpolation solutions of a homogeneous third-order linear differential equation. The interpolation solutions are entire functions of the spectral parameter, and they interpolate the system of root functions of the boundary value problem.

## Author contributions

Introduction and statement of the problem, B.K. and Z.K.; proof theorem, Z.K. and M.M.; writing—review and editing, B.K., Z.K., and M.M.; project administration, B.K. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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