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Research article

Results on Coincidence and common fixed point theorems for $L_{\mathcal{R}}$ -contraction

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Abstract: In this paper, we establish coincidence and common fixed point theorems for a pair of mappings (T, S) that utilize the binary relation in a metric space employing the concept of a locally finitely T -transitive relation together with an $L_{\mathcal{R}}$ contraction. We also provide appropriate added suitable examples to establish the genuineness of our newly established results over the corresponding earlier known results.

Keywords: common fixed point and coincidence point; binary relation; $L_{\mathcal{R}}$ -contraction
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1. Introduction

The classical Banach contraction concept, proclaimed by Banach [1] in 1922, sparked the development of metric fixed point theory. The Banach contraction principle (BCP) is a cornerstone of nonlinear functional analysis, providing a fundamental tool for establishing the existence and uniqueness of fixed points in metric spaces. Subsequently, several authors have generalized this result in various ways, (cf. [2, 3]), among others.

During the 1970s, there was a notable expansion in the literature aimed at broadening the class of contraction mappings. In 1977, Rhoades conducted a comprehensive comparison of different types of contractive mappings, analyzing and contrasting distinct conditions associated with such mappings [4]. This extensive analysis provided valuable insight into on the diversity of contraction conditions and their implications. In 1979, Browder [5] subsumed a major part of the work of Rhoades [4] under an intuitive and simple mode of argument. In 2015, Alam and Imdad [6] obtained a relation-theoretic analogue of the BCP under an arbitrary binary relation, thereby unifying several well-known order-theoretic fixed-point results. Historically, BCP is a fundamental theorem in fixed point theory, asserting that a contraction mapping on a complete metric space possesses a unique fixed point. The novelty of Alam and Imdad's work lies in their use of an amorphous binary relation instead of the more commonly employed partial order. Subsequently, several researchers proposed various relation-theoretic results [7, 8]. These results involve weak contraction conditions that are applicable only to pairs of comparable elements [9, 10].

In particular, Jheli and Samet [11] introduced the concept of a Θ-contraction and extended the BCP to a generalized metric space. Thereafter, Ahmad et al. [12] modified the conditions imposed on the auxiliary function Θ and obtained a natural analogue of this result in a metric space. On the other hand, by employing a family of control functions (known as simulations), Khojasteh et al. [13] developed the concept of a $\mathcal{Z}$-contraction, which unified various linear and nonlinear contractions discussed in the existing literature. Subsequently, several researchers ([14, 15] and references therein) extended and generalized the results presented in [13]. Motivated by these developments, Cho [16] proposed a new type of contraction, termed an L-contraction, and established several fixed point results in generalized metric spaces for this class of contractions.

The notions of coincidence and common fixed points build upon the concept of fixed points by considering multiple mappings within a given space. A fixed point of a self-mapping $\mathcal{T}$ in a nonempty set Ω is a point $\zeta \in \Omega$ such that $\mathcal{T}(\zeta) = \zeta$. This can be seen as $\mathcal{T}(\zeta) = I(\zeta)$, where I denotes the identity mapping on Ω. This observation naturally raises the question of whether the identity mapping I can be replaced by another self-mapping S on Ω. Accordingly, given two self-mapping $\mathcal{T}$ and S on a non-empty set Ω, we are interested in finding points $\zeta, \zeta' \in \Omega$ such that

$$\mathcal{T}(\zeta) = \mathsf{S}(\zeta) = \zeta'.$$

A point $\zeta \in \Omega$ satisfying $\mathcal{T}(\zeta) = \mathsf{S}(\zeta)$ is called a coincidence point. If ζ is also a fixed point of both $\mathcal{T}$ and S, it is referred to as a common fixed point. The study of coincidence theorems was initiated by Goebel [17] and Jungck [18], who extended the Banach Contraction Principle (BCP) to the framework of two mappings. Since then, substantial research has been conducted on the theory of common and coincidence fixed points [19, 20].

In this manuscript, we establish the coincidence and common fixed point theorems for a pair of mappings $(\mathcal{T}, \mathsf{S})$ utilizing a binary relation in the setting of a metric space with a locally finite $\mathcal{T}$-transitive relation involving the $L_{\mathcal{R}}$ -contraction. Furthermore, an illustrative example is provided to demonstrate the validity and applicability of the obtained results.

2. Preliminaries

Throughout this manuscript $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{R}$, denote the set of whole numbers, natural numbers, and real numbers respectively.

Following [10], let Θ be the set of all the functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is nondecreasing;

(θ_2) For each sequence $\{\zeta_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(\zeta_n) = 1 \iff \lim_{n\to\infty} \zeta_n = 0; \tag{2.1}$$

(θ_3) $\exists\, k \in (0,1)$ and $\gamma \in (0,\infty)$:

$$\lim_{n\to\infty} \frac{\theta(\zeta) - 1}{\zeta^k} = \gamma. \tag{2.2}$$

After that, Ahmad et al. [12] modified the condition and replaced it (θ_3) with the following:

(θ_4) θ is continuous .

In the recent past, Cho [16] initiated the idea of L-simulation function if the following conditions are satisfied:

(ξ_1) $\xi(1,1) = 1$;
(ξ_2) $\xi(\zeta,\nu) < \frac{\nu}{\zeta}$ $\forall \zeta,\nu > 1$;
(ξ_3) if $\{\zeta_n\}$, $\{\nu_n\}$ are sequences in $(1,\infty)$ such that $\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}\nu_n > 1$, then $\lim_{n\to\infty}\sup\xi(\zeta_n,\nu_n) < 1$.

The final of L-simulation functions will be denoted by L. A few examples of L-simulation functions are as follows:

Example 2.1. *[16] We define the mapping $\xi_i : [1,\infty)\times[1,\infty) \to \mathbb{R}$ for $i = 1,2,3,$ as follows:*

- $\xi_1(\zeta,\nu) = \frac{\nu^k}{\zeta}$ $\forall\, \zeta,\nu \in [1,\infty)$ *where* $k \in (0,1)$
- $\xi_2(\zeta,\nu) = \frac{\nu}{\zeta\phi(\nu)}$ $\forall\, \zeta,\nu \in [1,\infty)$*, where* $\phi : [1,\infty) \to [1,\infty)$ *is a lower semi-continuous and non-decreasing functions such that* $\phi^{-}(\{1\}) = \{1\}$.
- $$\xi_3(\zeta,\nu) = \begin{cases} 1 & if (\nu,\zeta) = (1,1); \\ \frac{\nu}{2\zeta} & if \nu < \zeta; \\ \frac{\nu^k}{\zeta} & elsewhere, \end{cases}$$

  $\forall\, \zeta,\nu \in [1,\infty)$ *and* $k \in (0,1)$.

Then ξ_i are L-simulation functions for $i = 1,2,3$.

Definition 2.1. *[21] Let (Ω,d) be a metric space and $\mathcal{T},\mathsf{S} : \Omega \to \Omega$. Then $\mathcal{T}$ is called L-contraction with respect to ξ if there exist $\xi \in L$ and $\theta \in \Theta$ such that $\forall\, \zeta,\nu \in \Omega$,*

$$\xi(\theta(d(\mathcal{T}\zeta,\mathcal{T}\nu)),\theta(d(\mathsf{S}\zeta,\mathsf{S}\nu))) \geq 1.$$

If we take $\xi(\zeta,\nu) = \frac{\nu^k}{\zeta}$ $\forall\, \zeta,\nu \in [0,\infty)$ with $k \in (0,1)$, then L-contraction takes the form of θ-contraction, which was extensively used in getting many results in the literature.

Definition 2.2. *[6,22]. Let $\mathcal{R}$ a binary relation on Ω. Then, for $\zeta,\nu \in \Omega$,*

(i) The inverse relation of $\mathcal{R}^{-1} = \{(\zeta,\nu) \in \Omega^2 : (\nu,\zeta) \in \mathcal{R}\}$ and symmetric closure $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$.
(ii) $\zeta,\nu \in \Omega$, we say that ζ & ν are $\mathcal{R}$-comparative if either $(\zeta,\nu) \in \mathcal{R}$ or $(\nu,\zeta) \in \mathcal{R}$.
We denote it by $[\zeta,\nu] \in \mathcal{R}$.
(iii) $[\zeta,\nu] \in \mathcal{R} \iff (\nu,\zeta) \in \mathcal{R}^s$.
(iv) A sequence $\{\zeta_n\} \subset \Omega$ is called $\mathcal{R}$- preserving if $(\zeta_n,\zeta_{n+1}) \in \mathcal{R}\ \forall\, n \in \mathbb{N}_0$.

Definition 2.3. *[23] A relation $\mathcal{R}$ defined on a nonempty set Ω, $G \subseteq \Omega$, such that the restriction of $\mathcal{R}$ to G is defined by $\mathcal{R}|_G$ and is defined to be the set $\mathcal{R} \cap G^2$, i.e.,*

$$\mathcal{R}|_G := \mathcal{R} \cap G^2.$$

In fact, $\mathcal{R}|_G$ is a relation on G induced by $\mathcal{R}$.

Definition 2.4. *[24] $\mathcal{R}$ is termed as $(\mathcal{T}, \mathsf{S})$ - closed if it satisfies $\forall\ \zeta, \nu \in \Omega$,*

$$(\mathsf{S}\zeta, \mathsf{S}\nu) \in \mathcal{R} \implies (\mathcal{T}\zeta, \mathcal{T}\nu) \in \mathcal{R}.$$

Proposition 2.1. *[24] If $\mathcal{R}$ remains $(\mathcal{T}, \mathsf{S})$-closed, then $\mathcal{R}^s$ also remains $(\mathcal{T}, \mathsf{S})$-closed.*

Definition 2.5. *[25] $\mathcal{R}$ is called $(\mathcal{T}, \mathsf{S})$-compatible if it satisfies $\forall\ \zeta, \nu \in \Omega$*

$$(\mathsf{S}\zeta, \mathsf{S}\nu) \in \mathcal{R} \text{ and } \mathsf{S}(\zeta) = \mathsf{S}(\nu) \implies \mathcal{T}(\zeta) = \mathcal{T}(\nu).$$

Definition 2.6. *[24] (Ω, d) is called $\mathcal{R}$-complete metric space if every $\mathcal{R}$-preserving Cauchy sequence in Ω remains convergent.*

Definition 2.7. *[24] $\mathcal{T}$ is termed as $\mathcal{R}$-continuous at $\zeta \in \Omega$ if it satisfies*

$$\mathcal{T}(\zeta_n) \xrightarrow{d} \mathcal{T}(\zeta),$$

for any $\mathcal{R}$-preserving sequence $\{\zeta_n\} \subset \Omega$ with $\zeta_n \xrightarrow{d} \zeta$. Naturally, if a mapping remains $\mathcal{R}$-continuous at all points of Ω, it is known as $\mathcal{R}$-continuous.

Definition 2.8. *[24] $\mathcal{T}$ is termed as $(\mathsf{S}, \mathcal{T})$-continuous at $\zeta \in \Omega$ if it satisfies*

$$\mathcal{T}(\zeta_n) \xrightarrow{d} \mathcal{T}(\zeta),$$

for any sequence $\{\zeta_n\} \subset \Omega$, whereas $\{\mathsf{S}\zeta_n\}$ remains $\mathcal{R}$-preserving satisfying $S(\zeta_n) \xrightarrow{d} \mathsf{S}(\zeta)$. Naturally, if a mapping remains $(\mathsf{S}, \mathcal{R})$-continuous at all points of Ω, it is known as $(\mathsf{S}, \mathcal{R})$- continuous.

Definition 2.9. *[24] $\mathcal{T}$ & S are known as $\mathcal{R}$-compatible if they satisfy*

$$\lim_{n\to\infty} d(\mathsf{S}\mathcal{T}\zeta_n, \mathcal{T}\mathsf{S}\zeta_n) = 0,$$

for any sequence $\{\zeta_n\} \subset \Omega$, whereas $\{\mathcal{T}\zeta_n\}$ and $\{\mathsf{S}\zeta_n\}$ remain $\mathcal{R}$-preserving and

$$\lim_{n\to\infty} \mathcal{T}(\zeta_n) = \lim_{n\to\infty} \mathsf{S}(\zeta_n).$$

Definition 2.10. *[24] $\mathcal{T}$ and S are known as weakly compatible if*

$$\mathcal{T}(\zeta) = \mathsf{S}(\zeta) \implies \mathcal{T}(\mathsf{S}\zeta) = \mathsf{S}(\mathcal{T}\zeta), \quad \forall \zeta \in \Omega.$$

Compatibility $\implies$ Weak compatibility but reverse implication not possible.

Definition 2.11. *[24] $\mathcal{R}$ is called $(\mathsf{S},\mathcal{R})$-self-closed if every $\mathcal{R}$-preserving sequence $\{\zeta_n\} \subset \Omega$ satisfying $\zeta_n \xrightarrow{d} \zeta$ admits a subsequence $\{\zeta_{n_k}\}$ verifying*

$$[\mathsf{S}\zeta_{n_k}, \mathsf{S}\zeta] \in \mathcal{R}.$$

Definition 2.12. *[23] Given $\zeta, \nu \in \Omega$ a finite sequence $\{\varrho_0, \varrho_1, \ldots, \varrho_l\} \subset \Omega$ is referred to as a path of length $\varpi \in \mathbb{N}$ in $\mathcal{R}$ from ζ to ν if the following ones hold:*

(i) $\varrho_0 = \alpha$ *and* $\varrho_l = \beta$,
(ii) $(\varrho_\vartheta, \varrho_{\vartheta+1}) \in \mathcal{R}$, $0 \le \vartheta \le \varpi - 1$.

Definition 2.13. *[26] A subset $G \subseteq \Omega$ is called $\mathcal{R}$-connected if, between any pair of elements of G, $\exists$ a path in $\mathcal{R}$ from ζ to ν.*

Definition 2.14. *[27] Given $Q \in \mathbb{N}$, $Q \le 2$, $\mathcal{R}$ is called Q-transitive if, for any $\zeta_0, \zeta_1, \ldots, \zeta_\varsigma \in \Omega$,*

$$(\zeta_{\vartheta-1}, \zeta_\vartheta) \in \mathcal{R}, \quad for\ each\ \ \vartheta(0 \le \vartheta \le Q) \implies (\zeta_0, \zeta_\varsigma) \in \mathcal{R}.$$

Definition 2.15. *[28] $\mathcal{R}$ is known as locally finitely $\mathcal{T}$-transitive if, countable subset $G \subseteq \mathcal{T}(\Omega)$, $\exists$ $Q = Q(G) \ge 2$, such that $\mathcal{R}|_G$ remains Q-transitive.*

Lemma 2.1. *[29] Let (Ω, d) be a metric space and $\{\zeta_n\} \subset \Omega$ be a sequence. If $\{\zeta_n\}$ remains not Cauchy, then $\exists$ $\epsilon > 0$ and a pair of subsequences $\{\zeta_{n_\varsigma}\}$ and $\{\zeta_{m_\varsigma}\}$ of $\{\zeta_n\}$ verifying*

(i) $\varsigma \le m_\varsigma < n_\varsigma$, $\forall\ \varsigma \in \mathbb{N}$,
(ii) $d(\zeta_{m_\varsigma}, \zeta_{n_\varsigma}) \ge \epsilon$,
(iii) $d(\zeta_{m_\varsigma}, \zeta_{n_\varsigma}) < \epsilon$, $\forall\ p_\varsigma \in \{m_\varsigma + 1, m_\varsigma + 2, \ldots, n_\varsigma - 2, n_\varsigma - 1\}$.

Moreover, if $\{\zeta_n\}$ satisfies $\lim_{n\to\infty} d(\zeta_n, \zeta_{n+1}) = 0$, then

$$\lim_{\varsigma\to\infty} d(\zeta_{m_\varsigma}, \zeta_{n_\varsigma+p}) = \epsilon, \quad \forall\ \ p \in \mathbb{N}_0.$$

Lemma 2.2. *[30] Let $\Omega \ne \emptyset$ and $\{q_n\} \subset \Omega$ be an $\mathcal{R}$-preserving sequence. If $\mathcal{R}$ remains Q-transitive on $G = \{q_n : n \in \mathbb{N}_0\}$ for some $Q \ge 2$, then*

$$(q_n, q_{n+1+N(Q-1)}) \in \mathbb{N}_0.$$

Lemma 2.3. *[31] If $\mathcal{T}$ is a self-mapping on a non-empty set Ω, then $\exists$ a subset $\mathcal{H}$ of Ω such that $\mathcal{T}(\mathcal{H}) = \mathcal{T}(\Omega)$ and $\mathcal{T} : \mathcal{H} \to \Omega$ is injective.*

Lemma 2.4. *[25] Let $\mathcal{T}$ and S be two self-mappings on a non-empty set Ω such that $\mathcal{T}$ and S have a unique point of coincidence. If*

(i) *$\mathcal{T}$ and S remain weakly compatible, then the point of coincidence is also a unique common fixed point.*
(ii) *Either $\mathcal{T}$ or S be injective, then $\mathcal{T}$ and S have a unique coincidence point.*

Definition 2.16. *Let $\mathcal{R}$ be a binary relation on a metric space (Ω, d) and $\mathcal{T}$ and S self-mappings on Ω. We say that $\mathcal{T}$ and S are $L_{\mathcal{R}}$ with respect to $\xi \in L$, if there exist $\theta \in \Theta$ and $\xi \in L$, such that the following conditions hold:*

$$1 \leq \xi(\theta(d(\mathcal{T}\zeta, \mathcal{T}\nu)), \theta(d(\mathsf{S}\zeta, \mathsf{S}\nu))), \forall \zeta, \nu \in \Omega \ \text{with} \ \zeta, \nu \in \mathcal{R}$$

where $(\zeta, \nu) \in \mathcal{R}^* := \{(\zeta, \nu) \in \mathcal{R} : \mathcal{T}\zeta \neq \mathcal{T}\nu,\ \mathsf{S}\zeta \neq \mathsf{S}\nu\}$.

Proposition 2.2. *Suppose that (Ω, d) is a metric space endowed with a relation $\mathcal{R}$, and $\mathcal{T}$ and S are self-mappings on Ω. For a given $\theta \in \Theta$ and $\xi \in L$, then the following conditions are equivalent:*

(i) $\forall\ \zeta, \nu \in \Omega$ *with* $(\zeta, \nu) \in \mathcal{R}^* \implies \xi(\theta(d(\mathcal{T}\zeta, \mathcal{T}\nu)), \theta(d(\mathsf{S}\zeta, \mathsf{S}\nu))) \geq 1$,
(ii) $\forall\ \zeta, \nu \in \Omega$ *with* $[\zeta, \nu] \in \mathcal{R}^* \implies \xi(\theta(d(\mathcal{T}\zeta, \mathcal{T}\nu)), \theta(d(\mathsf{S}\zeta, \mathsf{S}\nu))) \geq 1$.

Proof. Obviously $(ii) \implies (i)$. We claim that $(i) \implies (ii)$, choose $(\zeta, \nu \in \Omega)$ such that $[\zeta, \nu] \in \mathcal{R}^*$, then (ii) immediately follows from (i). otherwise, if $(\nu, \zeta) \in \mathcal{R}^*$, then by (i) and owing to the symmetry of d (metric), we conclude the claim. □

Proposition 2.3. *[32] Let (Ω, d) be a metric space endowed with a relation $\mathcal{R}$, and $\mathcal{T}$ and S be a set of self-mappings on Ω. Then*

(i) $\mathcal{R}$ *is* $\mathcal{T}$*-transitive* $\iff \mathcal{R}|_{\mathcal{T}(X)}$ *is transitive,*
(ii) $\mathcal{R}$ *is locally* $\mathcal{T}$*-transitive* $\iff \mathcal{R}|_{\mathcal{T}(X)}$ *is locally transitive,*
(iii) $\mathcal{R}$ *is transitive* $\implies \mathcal{R}$ *is locally transitive* $\implies \mathcal{R}$ *is locally* $\mathcal{T}$*- transitive,*
(iv) $\mathcal{R}$ *is transitive* $\implies \mathcal{R}$ *is* $\mathcal{T}$*- transitive* $\implies \mathcal{R}$ *is locally* $\mathcal{T}$*- transitive,*
(iv) if $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$*, then* S*-transitivity of* $\mathcal{R} \implies \mathcal{R}$ *is* $\mathcal{T}$*-transitive and locally* S*-transitivity of* $\mathcal{R}$ $\implies \mathcal{R}$ *is locally* $\mathcal{T}$*- transitive but not conversely.*

3. Results

Theorem 3.1. *Let (Ω, d) be a metric space, $\mathcal{T}$ and S be the self-mappings on Ω, while $\mathcal{R}$ remains a relation on Ω. In addition,*

(i) $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$*,*
(ii) $\mathcal{R}$ *remains* $(\mathcal{T}, \mathsf{S})$*- closed and locally finitely* $\mathcal{T}$*-transitive,*
(iii) $\exists\ \zeta_0 \in \Omega$ *verifying* $(\mathsf{S}\zeta_0, \mathcal{T}\zeta_0) \in \mathcal{R}$*,*
(iv) $\exists\ \theta \in \Theta$ *and* $\xi \in L$ *such that, for all* $\zeta, \nu \in \Omega$ *with* $d(\mathcal{T}\zeta, \mathcal{T}\nu) > 0$*,*

$$1 \leq \xi(\theta(d(\mathcal{T}\zeta, \mathcal{T}\nu)), \theta(d(\mathsf{S}\zeta, \mathsf{S}\nu)))$$

(v) (v1) (Ω, d) *remains* $\mathcal{R}$*-complete,*
(v2) $\mathcal{T}$ *and* S *remains* $\mathcal{R}$*-compatible,*
(v3) S *remains* $\mathcal{R}$*-continuous,*
(v4) $\mathcal{T}$ *remains* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *remains* $(\mathcal{T}, \mathsf{S})$*-compatible and* (S, d)*-self-closed, or, alternatively,*
(v')(v'1) $\exists\ \mathcal{R}$*-complete subspace* G *of* Ω *verifying* $\mathcal{T}(\Omega) \subseteq G \subseteq \mathsf{S}(\Omega)$*,*
(v'2) $\mathcal{T}$ *remains* $(\mathsf{S}, \mathcal{R})$*-continuous or* $\mathcal{T}$ *and* S *remain continuous or* $\mathcal{R}$ *and* $\mathcal{R}|_G$ *remain* $(\mathcal{T}, \mathsf{S})$*-compatible and* d*-self-closed, respectively.*

Then, $\mathcal{T}$ and S have a coincidence point.

Proof. By condition (iii), if $\mathsf{S}(\zeta_0) = \mathcal{T}(\zeta_0)$, then ζ_0 remains a coincidence point of $\mathcal{T}$ and S. Otherwise, if $\mathsf{S}(\zeta_0) \neq \mathcal{T}\zeta_0$, then owing to $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$, we choose $\zeta_1 \in \Omega$ satisfying $\mathsf{S}(\zeta_1) = \mathcal{T}(\zeta_0)$. Again, using $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$, we can choose $\zeta_2 \in \Omega$ satisfying $\mathsf{S}(\zeta_2) = \mathcal{T}(\zeta_1)$. Thus, we construct a sequence $\{\zeta_n\} \subset \Omega$ verifying

$$\mathsf{S}(\zeta_{n+1}) = \mathcal{T}(\zeta_n), \quad \forall n \in \mathbb{N}_0. \tag{3.1}$$

Now, by induction, we need to demonstrate that $\{\mathsf{S}\zeta_n\}$ remains to $\mathcal{R}$-preserving sequence, i.e.,

$$(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1}) \in \mathcal{R}, \quad \forall\ n \in \mathbb{N}_0. \tag{3.2}$$

By assumption (*iii*) and (3.1) (for n = 0), one obtains

$$(\mathsf{S}\zeta_0, \mathsf{S}\zeta_1) \in \mathcal{R}.$$

Thus, (3.2) is true for $n = 0$. Assume that (3.2) holds for $n = \varsigma > 0$, i.e.,

$$(\mathsf{S}\zeta_\varsigma, \mathsf{S}\zeta_{\varsigma+1}) \in \mathcal{R}.$$

By $(\mathcal{T}, \mathsf{S})$ - closedness of $\mathcal{R}$, one has

$$(\mathcal{T}\zeta_\varsigma, \mathcal{T}\zeta_{\varsigma+1}) \in \mathcal{R},$$

using (3.1), gives rise to

$$(\mathsf{S}\zeta_{\varsigma+1}, \mathsf{S}\zeta_{\varsigma+2}) \in \mathcal{R}.$$

Hence, by (3.2) holds $\forall\ n \in \mathbb{N}_0$.
From (3.1) and (3.2), $\{\mathcal{T}x_n\}$ remains also $\mathcal{R}$-preserving so that

$$(\mathcal{T}\zeta_n, \mathcal{T}\zeta_{n+1}) \in \mathcal{R}, \ \ \forall n \in \mathbb{N}_0. \tag{3.3}$$

If there exists $n_0 \in \mathbb{N}_0$ satisfying $d(\mathsf{S}\zeta_{n_0}, \mathsf{S}\zeta_{n_0+1}) = 0$, then by (3.1), we conclude that ζ_{n_0} remains a coincidence point of $\mathcal{T}$ and S. Otherwise, we get

$$d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1}) > 0, \ \ d(\mathcal{T}\zeta_n, \mathcal{T}\zeta_{n+1}) > 0 \ \ \forall\ \ n \in \mathbb{N}_0.$$

By (3.1), (3.2) and hypothesis (*iv*), we obtain

$$\begin{aligned} 1 &\leq \xi(\theta(d(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_n)), \theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n)) \\ &= \theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})), \theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n) \\ &< \frac{\theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n))}{\theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1}))} \end{aligned}$$

so that

$$\theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})) < \theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n) \tag{3.4}$$

then due to (θ_1), we deduce $d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1}) < d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n)$. Therefore, $\{d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})\}_{n=0}^{\infty}$ is a monotonic decreasing sequence of non-negative real numbers, and hence there exists $a \geq 0$, such that

$\lim_{n\to\infty} d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1}) = a.$
Now, we show that $a = 0$ Suppose that $a > 0$ then by using (θ_4), we get

$$\lim_{n\to\infty} \theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})) = \lim_{n\to\infty} \theta(d(\mathsf{S}\zeta_{n+1}, \mathsf{S}\zeta_{n+2})) = \theta(a).$$

Now, if we set $p_n = \theta(d(\mathcal{T}\zeta_n, \mathcal{T}\zeta_{n+1}))$ and $q_n = \theta(d(\mathcal{T}\zeta_{n+1}, \mathcal{T}\zeta_{n+2}))$, then $q_n < p_n$, for all $n \in \mathbb{N}$ by (3.3) and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 1$. Then by (ξ_3) we obtain

$$\begin{aligned} 1 &\leq \lim_{n\to\infty} \sup \xi(\theta(d(\mathcal{T}\zeta_{n-1}, \mathcal{T}\zeta_n)), \theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n)) \\ &= \lim_{n\to\infty} \sup \theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})), \theta(d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n) \\ &< 1 \end{aligned}$$

which is a contradiction to the definition of $L_{\mathcal{R}}$-contraction and hence $a = 0$, i.e.,

$$\lim_{n\to\infty} d(\mathsf{S}\zeta_{n-1}, \mathsf{S}\zeta_n) = 0, \tag{3.5}$$

by (θ_2), we also have

$$\lim_{n\to\infty} \theta(d(\mathsf{S}\zeta_n, \mathsf{S}\zeta_{n+1})) = 1.$$

Now, we show that $\{\mathsf{S}\zeta_n\}$ is a Cauchy sequence. Consider $\{\mathsf{S}\zeta_n\}$ is not Cauchy. Using Lemma 2.1, $\exists$ $\epsilon > 0$ and $\{\mathsf{S}\zeta_{m_\varsigma}\}, \{\mathsf{S}\zeta_{n_\varsigma}\} \subset \{\mathsf{S}\zeta_n\}$ with $n_\varsigma > m_\varsigma > \varsigma \geq \varsigma_0$, such that

$$d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}) \geq \epsilon \quad \text{and} \quad d(\mathsf{S}\zeta_{m_\varsigma-1}, \mathsf{S}\zeta_{n_\varsigma}) < \epsilon.$$

Thus, we can have

$$\epsilon \leq d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}) \leq d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{m_\varsigma-1}) + d(\mathsf{S}\zeta_{m_\varsigma-1}, \mathsf{S}\zeta_{n_\varsigma}) < d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{m_\varsigma-1}) + \epsilon,$$

taking $\varsigma \to \infty$ and using (3.3), we get

$$\lim_{\varsigma\to\infty} d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}) = \epsilon \ \ or \ \ \lim_{\varsigma\to\infty} \theta(d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma})) = \theta(\epsilon), \tag{3.6}$$

and hence

$$\lim_{\varsigma\to\infty} d(\mathsf{S}\zeta_{m_\varsigma+1}, \mathsf{S}\zeta_{n_\varsigma+1}) = \epsilon \ \ or \ \ \lim_{\varsigma\to\infty} \theta(d(\mathsf{S}\zeta_{m_\varsigma+1}, \mathsf{S}\zeta_{n_\varsigma+1})) = \theta(\epsilon). \tag{3.7}$$

As the sequence $\{\mathsf{S}\zeta_n\}$ is $\mathcal{R}$-preserving and $\mathcal{R}$ is S-transitive, therefore $(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}) \in \mathcal{R}$, on using (3.4), (3.5) and property (ξ_3) of a simulation function with $\zeta_n = d(\mathcal{T}\zeta_{m_\varsigma+1}, \mathcal{T}\zeta_{n_\varsigma+1})$ and $\nu_n = d(\mathcal{T}\zeta_{m_\varsigma}, \mathcal{T}\zeta_{n_\varsigma})$ we obtain

$$\begin{aligned} 1 &\leq \lim_{\varsigma\to\infty} \sup \xi(\theta(d(\mathcal{T}\zeta_{m_\varsigma}, \mathcal{T}\zeta_{n_\varsigma})), \theta(d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}))) \\ &= \lim_{\varsigma\to\infty} \sup \xi(\theta(d(\mathsf{S}\zeta_{m_\varsigma+1}, \mathsf{S}\zeta_{n_\varsigma+1})), \theta(d(\mathsf{S}\zeta_{m_\varsigma}, \mathsf{S}\zeta_{n_\varsigma}))) < 1, \end{aligned}$$

which is a contradiction, so that $\{\mathsf{S}\zeta_n\}$ remains Cauchy.
Now, we take assumption (v) and (v'). Let (v) hold. As $\{\mathsf{S}\zeta_n\}$ remains an $\mathcal{R}$-preserving Cauchy in Ω and Ω remains $\mathcal{R}$-complete, $\exists\ p \in \Omega$ verifying

$$\lim_{n\to\infty} \mathsf{S}(\zeta_n) = p. \tag{3.8}$$

By (3.1) and (3.8), we obtain

$$\lim_{n\to\infty} \mathcal{T}(\zeta_n) = p. \tag{3.9}$$

By (3.2), (3.6) and condition $(v2)$, one gets

$$\lim_{n\to\infty} \mathsf{S}(\mathsf{S}\zeta_n) = \mathsf{S}(\lim_{n\to\infty} \mathsf{S}\zeta_n) = \mathsf{S}(p). \tag{3.10}$$

By (3.3), (3.7) and assumption $(v2)$, we obtain

$$\lim_{n\to\infty} \mathsf{S}(\mathcal{T}\zeta_n) = \mathsf{S}(\lim_{n\to\infty} \mathcal{T}\zeta_n) = \mathsf{S}(p). \tag{3.11}$$

Now, both $\{\mathcal{T}\zeta_n\}$ and $\{\mathsf{S}\zeta_n\}$ remains $\mathcal{R}$-preserving (due to (3.2) and (3.3)) and $\lim_{n\to\infty} \mathcal{T}(\zeta_n) = \lim_{n\to\infty} \mathsf{S}(\zeta_n) = p$ (due to (3.8) and (3.9). By condition $(v1)$, one obtains

$$\lim_{n\to\infty} d(\mathsf{S}\mathcal{T}\zeta_n, \mathcal{T}\mathsf{S}\zeta_n) = 0. \tag{3.12}$$

Under assumptions $(v4)$, first consider that $\mathcal{T}$ remains $\mathcal{R}$-continuous. Utilizing (3.2), (3.8) and $\mathcal{R}$-continuity of $\mathcal{T}$, we obtains

$$\lim_{n\to\infty} \mathcal{T}(\mathsf{S}\zeta_n) = \mathcal{T}(\lim_{n\to\infty} \mathsf{S}\zeta_n) = \mathcal{T}(p). \tag{3.13}$$

Using (3.11)–(3.13) and continuity of d, we get

$$\begin{aligned} d(\mathsf{S}p, \mathcal{T}p) &= d(\lim_{n\to\infty} \mathsf{S}\mathcal{T}\zeta_n, \lim_{n\to\infty} \mathcal{T}\mathsf{S}\zeta_n) \\ &= \lim_{n\to\infty} d(\mathsf{S}\mathcal{T}\zeta_n, \mathcal{T}\mathsf{S}\zeta_n) \\ &= 0 \end{aligned}$$

so that

$$\mathsf{S}(p) = \mathcal{T}(p).$$

Now, we consider that $\mathcal{R}$ remains $(\mathcal{T}, \mathsf{S})$-compatible as well as (S, d) - self-closed. Since remains $\mathcal{R}$-preserving (due to (3.2)) and $\mathsf{S}(\zeta_n) \xrightarrow{d} p$ (due to (3.8)), therefore, using (S, d)-self-closedness of $\mathcal{R}$. We find a subsequence $\{\mathsf{S}\zeta_{n_\varsigma}\}$ of $\{\mathsf{S}\zeta_n\}$ satisfying

$$[\mathsf{S}\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}_p] \in \mathcal{R}, \quad \forall\, \varsigma \in \mathbb{N}. \tag{3.14}$$

As $\mathsf{S}(\zeta_{n_\varsigma}) \xrightarrow{d} p$, equation (3.8) – (3.12) also holds for $\{\zeta_{n_\varsigma}\}$ instead of $\{\zeta_n\}$. One claims that

$$d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p) \leq d(\mathsf{S}\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}p), \quad \forall\, \varsigma \in \mathbb{N}. \tag{3.15}$$

Let $\{\mathbb{N}^0, \mathbb{N}^+\}$ denote a partition of $\mathbb{N}$ satisfy the following cases:

(a) $d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p) = 0, \ \forall\, \varsigma \in \mathbb{N}^0$,

(b) $d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p) > 0, \ \forall\, \varsigma \in \mathbb{N}^+$.

In case (a), making use of (3.14) and $(\mathcal{T}, \mathsf{S})$-compatibility of $\mathcal{R}$, we get $d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p) = 0, \forall\, \varsigma \in \mathbb{N}^0$ so that (3.15) holds $\forall\, \varsigma \in \mathbb{N}^0$. In case (b), by (3.14), condition (iv) and Proposition 2.2, we obtain

$$\begin{aligned} 1 &\leq \xi(\theta(d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p)), \theta(d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p)) \\ &< \frac{\theta(d(\mathsf{SS}_{n_\varsigma}, \mathsf{S}p))}{\theta(d(\mathcal{T}\mathsf{S}_{n_\varsigma}, \mathsf{S}_P))} \end{aligned}$$

so that

$$\theta(d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p)) \leq \theta(d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p), \tag{3.16}$$

then due to (θ_1), we deduce $d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p) \leq d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p)\ \forall\, \varsigma \in \mathbb{N}^+$ so that (3.15) holds $\forall\, \varsigma \in \mathbb{N}^+$. Thus, (3.15) holds $\forall\, \varsigma \in \mathbb{N}$. By the triangular inequality, (3.10), (3.11), (3.12), and (3.15), we get

$$\begin{aligned} d(\mathsf{S}p, \mathcal{T}p) &\leq d(\mathsf{S}p, \mathsf{S}\mathcal{T}\zeta_{n_\varsigma}) + d(\mathsf{S}\mathcal{T}\zeta_{n_\varsigma}, \mathcal{T}\mathsf{S}\zeta_{n_\varsigma}) + d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}p) \\ &\leq d(\mathsf{S}p, \mathsf{S}\mathcal{T}\zeta_{n_\varsigma}) + d(\mathsf{S}\mathcal{T}\zeta_{n_\varsigma}, \mathcal{T}\mathsf{S}x_{n_\varsigma}) + d(\mathsf{SS}\zeta_{n_\varsigma}, \mathsf{S}p) \\ &\to 0 \quad as \ \varsigma \to \infty \end{aligned}$$

so that

$$\mathsf{S}(p) = \mathcal{T}(p).$$

Therefore, p remains a coincidence point of $\mathcal{T}$ and S. Alternatively, if assumption (v') holds, considering $G \subseteq \mathsf{S}(\Omega)$, $\exists\, \zeta \in \Omega$ s.t. $p = \mathsf{S}(\zeta)$. Consequently, (3.8) and (3.9) respectively become

$$\lim_{n\to\infty} \mathsf{S}(\zeta_n) = \mathsf{S}(\zeta). \tag{3.17}$$

$$\lim_{n\to\infty} \mathcal{T}(\zeta_n) = \mathcal{T}(\zeta). \tag{3.18}$$

Now, we have to demonstrate that ζ remains a coincidence point of $\mathcal{T}$ and S under condition $(v'2)$. Initially, consider that $\mathcal{T}$ remains $(\mathsf{S}, \mathcal{R})$- continuous. Utilizing (3.17), we get

$$\lim_{n\to\infty} \mathcal{T}(\zeta_n) = \mathcal{T}(\zeta). \tag{3.19}$$

On using (3.18) and (3.19), we obtain

$$\mathsf{S}(\zeta) = \mathcal{T}(\zeta).$$

Secondly, consider that $\mathcal{T}$ and S remain continuous, and from Lemma 2.2, there exists a subset $\mathcal{H} \subseteq \Omega$ verifying $\mathsf{S}(\mathcal{H}) = \mathsf{S}(\Omega)$ and $\mathsf{S} : \mathcal{H} \to \Omega$ remains injective.
Define a function $l : \mathsf{S}(\mathcal{H}) \to \mathsf{S}(\Omega)$ by

$$l(\mathsf{S}\mathfrak{G}) \to \mathcal{T}(\mathfrak{G}), \quad \forall\ \mathsf{S}(\mathfrak{G}) \in \mathsf{S}(\mathcal{H}). \tag{3.20}$$

As $\mathsf{S} : \mathcal{H} \to \Omega$ remains injective and $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$, '$l$' remains well defined. Continuities of $\mathcal{T}$ and S guarantee that 'l' remains continuous. By $\mathsf{S}(\Omega) = \mathsf{S}(\mathcal{H})$, condition (i) reduces to $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\mathcal{H})$. Thus, we construct $\{\zeta_n\}_{n=1}^{\infty} \subset \mathcal{H}$ satisfying (3.1) and enabling us to choose $\zeta \in \mathcal{H}$. By (3.17),(3.18), (3.20), and continuity of 'l', we have

$$\mathcal{T}(\zeta) = \mathsf{S}(\mathsf{S}\zeta) = l(\lim_{n\to\infty} \mathsf{S}\zeta_n) = \lim_{n\to\infty} l(\mathsf{S}\zeta_n) = \lim_{n\to\infty} \mathcal{T}(\zeta_n) = \mathsf{S}(\zeta).$$

Hence, ζ remains a coincidence point of $\mathcal{T}$ and S. Finally, consider that $\mathcal{R}$ and $\mathcal{R}|_G$ remain $(\mathcal{T}, \mathsf{S}$)-compatible and d-self-closed, respectively. Since $\{\mathsf{S}_{\zeta_n}\}$ remains $\mathcal{R}|_G$- preserving (as per (3.2)) and $\mathsf{S}(\zeta_n) \xrightarrow{d} \mathsf{S}(\zeta) \in G$ (as per (3.17)), by d-self-closeness of $\mathcal{R}|_G$, there exists a subsequence $\{\mathsf{S}_{\zeta_{n_\varsigma}}\}$ of $\{\mathsf{S}_{\zeta_n}\}$ satisfying

$$[\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta] \in \mathcal{R}|_G \quad \forall\ \varsigma \in \mathbb{N}_0. \tag{3.21}$$

One claims that

$$d(\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta) \le d(\mathsf{S}_{\zeta_{n_\varsigma}}, \mathsf{S}\zeta) \quad \forall\ \varsigma \in\ \mathbb{N}. \tag{3.22}$$

Let $\{\mathbb{N}^0, \mathbb{N}^+\}$ denote a partition of $\mathbb{N}$ satisfy the following cases:

(a) $d(\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta) = 0, \ \ \forall\, \varsigma \in \mathbb{N}^0,$

(b) $d(\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta) > 0, \ \ \forall\, \varsigma \in \mathbb{N}^+.$

In case (a) holds, then by (3.21) and $(\mathcal{T}, \mathsf{S})$-compatibility of $\mathcal{R}$, we obtain $d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta) = 0, \forall\, \varsigma \in \mathbb{N}^0$ which in the presence of (3.1) gives rise to $d(\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta) = 0, \forall\, \varsigma \in \mathbb{N}^0$ and hence (22) holds $\forall\, \varsigma \in \mathbb{N}^0$. In case (b), by (3.1), (3.21), condition (iv) and Proposition 2.2, one obtains

$$\begin{aligned} 1 &\le \xi(\theta(d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta)), \theta(d(\mathsf{S}\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta)) \\ &< \frac{\theta(d(\mathsf{S}\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta))}{\theta(d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta))} \end{aligned}$$

so that

$$\theta(d(\mathcal{T}\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta)) \le \theta(d(\mathsf{S}\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta) \tag{3.23}$$

then due to (θ_1), we deduce $d(\mathsf{S}\zeta_{n_\varsigma+1}, \mathcal{T}\zeta) < d(\mathsf{S}\zeta_{n_\varsigma}, \mathsf{S}\zeta)$, $\forall\, \varsigma \in \mathbb{N}^+$ and hence (3.22) holds $\forall\, \varsigma \in \mathbb{N}^+$. Therefore, (3.22) holds $\varsigma \in \mathbb{N}$. By (3.17), (3.22) and continuity of d, we obtain

$$\begin{aligned} d(\mathsf{S}\zeta, \mathcal{T}\zeta) &= d(\lim_{\varsigma\to\infty} \mathsf{S}\zeta_{n_\varsigma+1}, \mathcal{T}\zeta) \\ &= \lim_{\varsigma\to\infty} d(\mathsf{S}\zeta_{n_\varsigma+1}, \mathcal{T}\zeta) \\ &\le \lim_{\varsigma\to\infty} d(\mathsf{S}\zeta_{n_\varsigma}, \mathcal{T}\zeta) \\ &= 0 \end{aligned}$$

so that

$$\mathsf{S}(\zeta) = \mathcal{T}(\zeta).$$

Therefore, ζ be a coincidence point of $\mathcal{T}$ and S.

□

Theorem 3.2. *In addition to Theorem (3.1), if the following assumptions hold:*

($vi1$) $\mathcal{T}(\Omega)$ *remains* $\mathcal{R}|^{s}_{\mathsf{S}(\Omega)}$*-connected,*
($vi2$) $\mathcal{R}$ *remains* $(\mathcal{T}, \mathsf{S})$*-compatible,*

then $\mathcal{T}$ *and* S *have a unique point of coincidence. Furthermore, if*

(a) $\mathcal{T}$ *and* S *remain weakly compatible, then the point of coincidence is also a unique common fixed point and*
(b) either $\mathcal{T}$ *or* S *remains injective, then* $\mathcal{T}$ *and* S *have a unique coincidence point.*

Proof. By Theorem 3.1, if $\bar{u}$ and $\bar{\zeta}$ are two points of coincidence of $\mathcal{T}$ and S, then $\exists\, u, \zeta \in \Omega$ satisfying

$$\bar{u} = \mathcal{T}(u) = \mathsf{S}(u) \ \ \& \ \ \bar{\zeta} = \mathcal{T}(\zeta) = \mathsf{S}(\zeta). \tag{3.24}$$

we prove that $\bar{u} = \bar{\zeta}$. As $\mathcal{T}(u), \mathcal{T}(\zeta) \in \mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$, by hypothesis ($vi1$), there exists a path $\{\mathsf{S}\varrho_0, \mathsf{S}\varrho_1, \mathsf{S}\varrho_2, \ldots, \mathsf{S}\varrho_\varpi\}$ in $d|^{\varrho}_{\mathsf{S}(\Omega)}$ from $\mathcal{T}(u)$ to $\mathcal{T}(\zeta)$, whereas $\varrho_0, \varrho_1, \varrho_2, \ldots, \varrho_\varpi \in \Omega$. By (3.24), we can set $\varrho_0 = u$ and $\varrho_\varpi = \zeta$. Therefore, we obtain

$$[\mathsf{S}\varrho_\vartheta, \mathsf{S}\varrho_{\vartheta+1}] \in d|_{\mathsf{S}(\Omega)}, \quad \forall \ \vartheta(0 \le \vartheta \le \varpi - 1). \tag{3.25}$$

Define the constant sequence $\varrho^0_n = u$ and $\varrho^0_n = \zeta$. Using (3.24), we have $\mathsf{S}(\varrho^0_{n+1}) = \mathcal{T}\varrho^0_n = \bar{u}$ and $\mathsf{S}(\varrho^\varpi_{n+1}) = \mathcal{T}\varrho^\varpi_n = \bar{\zeta}$, $\forall\ n \in \mathbb{N}_0$. Put $\varrho^1_0 = \varrho_1$, $\varrho^2_0 = \varrho_2, \ldots, \varrho^{\varpi-1}_0 = \varrho_{\varpi-1}$. Since $\mathcal{T}(\Omega) \subseteq \mathsf{S}(\Omega)$, therefore similar to Theorem 3.1, we construct sequence $\{\varrho^1_n\}, \{\varrho^2_n\}, \ldots, \{\varrho^{\varpi-1}_n\}$ in Ω verifying $\mathsf{S}(\varrho^1_{n+1}) = \mathcal{T}(s^1_n), \mathsf{S}(\varrho_{n+1}) = \mathcal{T}(\varrho^2_n), \ldots, \mathsf{S}(\varrho^{\varpi-1}_{n+1}) = \mathcal{T}(\varrho^{\varpi-1}_n)$, $\forall\ n \in \mathbb{N}_0$. Hence, we obtain

$$\mathsf{S}(\varrho^\vartheta_{n+1}) = \mathcal{T}(\varrho^\vartheta_n), \ \ \forall n \in \mathbb{N}_0 \ \ \& \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi). \tag{3.26}$$

Now, one claim that

$$[\mathsf{S}\varrho^\vartheta_n, \mathsf{S}\varrho^{\vartheta+1}_n] \in \mathcal{R}, \ \ \forall n \in \mathbb{N}_0 \ \ \& \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi - 1). \tag{3.27}$$

Our claim will be justified by induction. Using (3.25), (3.27) true for $n = 0$. Consider that (3.27) holds for $n = \varsigma > 0$ so that

$$[\mathsf{S}\varrho^\vartheta_n, \mathsf{S}\varrho^{\vartheta+1}_n] \in \mathcal{R}, \ \ \forall n \in \mathbb{N}_0 \ \ \& \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi - 1).$$

As $\mathcal{R}$ remains $(\mathcal{T}, \mathsf{S})$-closed, by Proposition 2.1, we obtain

$$[\mathcal{T}\varrho^\vartheta_\varsigma, \mathcal{T}\varrho^{\vartheta+1}_\varsigma] \in \mathcal{R}, \ \ \forall \varsigma \in \mathbb{N}_0 \ \ \& \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi - 1),$$

by using (3.26), gives rise to

$$[\mathsf{S}\varrho^\vartheta_{\varsigma+1}, \mathsf{S}\varrho^{\vartheta+1}_{\varsigma+1}] \in \mathcal{R}, \ \ \forall \varsigma \in \mathbb{N}_0 \ \ \& \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi - 1).$$

Therefore, by induction, (3.27) holds $\forall\ n \in \mathbb{N}_0$ and $\forall\ \vartheta(0 \le \vartheta \le \varpi - 1)$, define $\eta^\vartheta_n := d(\mathsf{S}\varrho^\vartheta_n, \mathsf{S}\varrho^{\vartheta+1}_n)$. One claim that

$$\lim_{n\to\infty} \eta^\vartheta_n = 0, \ \ \forall \ \ \vartheta(0 \le \vartheta \le \varpi - 1). \tag{3.28}$$

Let ϑ be fixed and consider two cases. First, suppose $\eta^{\vartheta}_{n_0} = d(\mathsf{S}\varrho^{\vartheta}_{n_0}, \mathsf{S}\varrho^{\vartheta+1}_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$. By assumption (vi2), one obtains $d(\mathcal{T}\varrho^{\vartheta}_{n_0}, \mathcal{T}\varrho^{\vartheta+1}_{n_0}) = 0$. Consequently, from (3.26), we have $\eta^{\vartheta}_{n_0+1} = d(\mathsf{S}\varrho^{\vartheta}_{n_0+1}, \mathsf{S}\varrho^{\vartheta+1}_{n_0+1}) = d(\mathcal{T}\varrho^{\vartheta}_{n_0}, \mathcal{T}\varrho^{\vartheta+1}_{n_0}) = 0$. Hence, by induction, we deduce $\eta^{\vartheta}_n = 0$, $\forall\ n \geq n_0$ implying $\lim_{n\to\infty} \eta^{\vartheta}_n = 0$.
If $\eta^{\vartheta}_n > 0$, $\forall\ n \in \mathbb{N}_0$, then by (3.26), (3.27), condition (iv) and Proposition 2.2, we obtain

$$\begin{aligned} 1 &\leq \xi(\theta(d(\mathcal{T}\varrho^{\vartheta}_{n+1}, \mathcal{T}\varrho^{\vartheta+1}_{n+1})), \theta(d(\mathsf{S}\varrho^{\vartheta}_{n+1}, \mathsf{S}\varrho^{\vartheta+1}_{n+1})) \\ &= \theta(d(\mathsf{S}\varrho^{\vartheta}_{n}, \mathsf{S}\varrho^{\vartheta+1}_{n})), \theta(d(\mathsf{S}\varrho^{\vartheta}_{n+1}, \mathsf{S}\varrho^{\vartheta+1}_{n+1}) \\ &< \frac{\theta(d(\mathsf{S}\varrho^{\vartheta}_{n}, \mathsf{S}\varrho^{\vartheta+1}_{n}))}{\theta(d(\mathsf{S}\varrho^{\vartheta}_{n+1}, \mathsf{S}\varrho^{\vartheta+1}_{n+1}))} \end{aligned}$$

so that

$$\theta(\eta^{\vartheta}_{n+1}) < \theta(\eta^{\vartheta}_n) \tag{3.29}$$

then due to (θ_1), we deduce $\eta^{\vartheta}_{n+1} < \eta^{\vartheta}_n$. Hence, $\lim_{n\to\infty} \eta^{\vartheta}_n = 0$. Therefore, in both cases, (3.28) is proven $\forall\ \vartheta(0 \leq \vartheta \leq \varpi - 1)$. Applying triangle inequality and (3.28), we obtain

$$\begin{aligned} d(\bar{u}, \bar{\zeta}) &\leq \eta^0_n + \eta^1_n + \cdots + \eta^{\varpi-1}_n \to 0, \ \ as \ \ n \to \infty \\ &\Longrightarrow \bar{u} = \bar{\zeta}. \end{aligned}$$

Hence, $\mathcal{T}$ and S have a unique point of coincidence. Finally, conclusions (*a*) and (*b*) are immediate in the presence of Lemma 2.4.

□

Corollary 3.1. *Theorem 3.1 remains valid if the locally finitely $\mathcal{T}$-transitivity condition is substituted with any one of the following assumptions:*

- *$\mathcal{R}$ is transitive,*
- *$\mathcal{R}$ is finitely transitive,*
- *$\mathcal{R}$ is $\mathcal{T}$-transitive,*
- *$\mathcal{R}$ is locally finitely transitive.*

Corollary 3.2. *In addition to the hypotheses of Theorem 3.2, if anyone of the following assumptions is satisfied,*

(i) $\mathcal{R}|_{\mathcal{T}(\Omega)}$ is complete binary relation,
(ii) $\mathcal{T}(\Omega)$ is $\mathcal{R}|^s_{\mathsf{S}(\Omega)}$- directed,

then $\mathcal{T}$ admits a unique fixed point.

Proof. If using (i), then for any $(\zeta, \nu) \in \mathcal{T}(\Omega)$, $[\zeta, \nu] \in \mathcal{R}$, which yields that $\{\zeta, \nu\}$ is a path of length 1 in $\mathcal{R}|^s_{\mathsf{S}(\Omega)}$ from ζ to ν. So, $\mathcal{T}(\Omega)$ is $\mathcal{R}|^s_{\mathsf{S}(\Omega)}$- connected and by Theorem 3.2, the conclusion follows immediately.
Else (ii) holds, then for each $(\zeta, \nu) \in \mathcal{T}(\Omega)$, $\exists\ \omega \in \Omega$ such that $[\zeta, \omega] \in \mathcal{R}$ and $[\omega, \nu] \in \mathcal{R}$, which means that $\{\zeta, \omega, \nu\}$ is a path of length 2 in $\mathcal{R}|^s_{\mathsf{S}(\Omega)}$ from ζ to ν. Thus, $\mathcal{T}(\Omega)$ is $\mathcal{R}|^s_{\mathsf{S}(\Omega)}$-connected and then by Theorem 3.2, the conclusion follows immediately. □

Here we provide an example in support of our results.

Example 3.1. *Suppose $\Omega = [0,1]$ is endowed with its Euclidian metric $d(\zeta,\nu) = |\zeta - \nu|$. Define binary relation $\mathcal{R}$ by*

$$\mathcal{R} = \{(\zeta,\nu) \in \Omega^2 : 0 \le \zeta < \nu \le \frac{1}{2}\},$$

then, $\mathcal{R}$ remains locally finitely $\mathcal{T}$-transitive, (Ω, d) is an $\mathcal{R}$-complete metric space.
Take $\mathcal{T}, \mathsf{S} : \Omega \to \Omega$ defined by

$$\mathcal{T}(\zeta) = \begin{cases} \frac{1}{5} + \frac{\zeta}{3}, & if\ \zeta \in [0, \frac{1}{2}] \\ \frac{1}{4} & if\ \zeta \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\mathsf{S}(\zeta) = \sqrt{\frac{3\zeta}{10}}.$$

Then, $\mathcal{R}$ has $(\mathcal{T},\mathsf{S})$- closed while $\mathcal{T}$ and S remain $\mathcal{R}$-compatible. Moreover, $\mathcal{T}$ and S remain $\mathcal{R}$-continuous. Let $\zeta, \nu \in \mathcal{R}$ verify $(\mathsf{S}\zeta, \mathsf{S}\nu) \in \mathcal{R}$.

Now, we choose $\theta(\beta) = e^{\sqrt{\beta}}$ $\forall\ \beta > 0$ and if we take $\xi(\zeta,\nu) = \frac{\nu^k}{\zeta}$ $\forall\ \zeta,\nu \in [1,\infty)$ and $k \in (0,1)$. Then the following cases arise:
Case (I): we take $\zeta = 0$ and $\nu = \frac{1}{10}$, then we get

$$\begin{aligned}\xi(\theta(d(\mathcal{T}(0),\mathcal{T}(\frac{1}{10}))),\theta(d(\mathsf{S}(0),\mathsf{S}(\frac{1}{10})))) &= \xi(\theta(d(\frac{1}{5},\frac{7}{30}),\theta(d(0,\sqrt{\frac{3}{100}})))) = \xi(\theta(\frac{1}{30}),\theta(\sqrt{\frac{3}{100}})) \\ &= \xi(e^{\sqrt{\frac{1}{30}}}, e^{\sqrt{\sqrt{\frac{3}{100}}}}) = \frac{e^{\sqrt{\sqrt{\frac{3}{100}}k}}}{e^{\sqrt{\frac{1}{30}}}} = e^{(\frac{3}{100})^{\frac{1}{4}}k - \sqrt{\frac{1}{30}}} > 1.\end{aligned}$$

Case (II): we take $\zeta = \frac{1}{10}$ and $\nu = \frac{49}{100}$, then we have

$$\begin{aligned}\xi(\theta(d(\mathcal{T}(\frac{1}{10}),\mathcal{T}(\frac{49}{100}))),\theta(d(\mathsf{S}(\frac{1}{10}),\mathsf{S}(\frac{49}{100})))) &= \xi(\theta(d(\frac{7}{30},\frac{109}{300}),\theta(d(\sqrt{\frac{3}{100}},\sqrt{\frac{147}{1000}})))) \\ = \xi(\theta(\frac{39}{300}),\theta(\left|\sqrt{\frac{3}{100}} - \sqrt{\frac{147}{1000}}\right|)) &= \xi(e^{\sqrt{\frac{39}{300}}}, e^{\left|\sqrt{\frac{39}{100}} - \sqrt{\frac{147}{1000}}\right|}) = \frac{e^{\sqrt{\left|\sqrt{\frac{3}{100}} - \sqrt{\frac{147}{1000}}\right|k}}}{e^{\sqrt{\frac{1}{30}}}} \\ &= e^{\sqrt{\left|\sqrt{\frac{3}{100}} - \sqrt{\frac{147}{1000}}\right|k} - \sqrt{\frac{39}{100}}} > 1.\end{aligned}$$

Therefore, all the assumptions of Theorem 3.1 are fulfilled, and hence $\mathcal{T}$ and S have a coincidence point. Similarly, all the requirements of Theorem 3.2 are satisfied. Consequently, $\mathcal{T}$ and S admit a common fixed point for $\zeta = \frac{3}{10}$.

4. Conclusions

In this paper, we have established coincidence and common fixed point theorems for a pair of mappings (T, S) in a metric space endowed with a binary relation. The results are obtained by employing the notion of a locally finitely T -transitive relation together with an $L_{\mathcal{R}}$ contraction. To illustrate the effectiveness and originality of our results, we have provided suitable examples showing that the obtained theorems properly generalize and improve several existing results in the literature.

As directions for future research, these results may be extended to other generalized metric structures such as b-metric spaces, quasi-metric spaces, and cone metric spaces. Furthermore, we can generalize the present results to pairs of self-mappings or apply the established theorems to investigate the existence and uniqueness of solutions for certain boundary value problems.

Author contributions

All authors have read and finalized the manuscript with equally contribution. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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