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Research article

Analysis of time-fractional soliton solutions for Kudryashov's law with dual nonlocal nonlinearity and refractive index in optical fibers

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Abstract: Optical solitons in nonlinear fiber optics represent fundamental wave phenomena that maintain their shape during propagation, yet existing analytical methods often fail to capture the complex dynamics of models incorporating dual nonlocal nonlinearity effects. Recent studies have shown significant research gaps in solving Kudryashov's law with simultaneous refractive index variations and multiple nonlinear terms, particularly when conformable derivatives are involved. In this study, we investigate the analytical solutions of Kudryashov's equation with dual nonlocal nonlinearity and refractive index effects in optical fibers using the improved modified Sardar Sub-equation expansion method (IMSSEM). By applying systematic traveling wave transformations and rigorous mathematical analysis, we derive an extensive collection of novel optical soliton solutions, including rational, hyperbolic, and trigonometric function families. The solutions exhibit diverse wave structures encompassing bright, dark, kink, and W-shaped soliton profiles. Through comprehensive 2D and 3D graphical representations, we demonstrate the wave propagation characteristics under various parameter conditions. Furthermore, we analyze the influence of different conformable derivative orders (τ) on soliton behavior, revealing significant variations in wave amplitude, width, and propagation velocity. These findings provide crucial insights for understanding nonlinear wave dynamics in optical communication systems, particularly in wavelength-division multiplexing, ultrashort pulse propagation, and nonlinear photonic devices. The derived solutions offer practical applications in designing optical amplifiers, mode-locked lasers, and soliton-based communication protocols. This work contributes significantly to the mathematical physics community by providing a comprehensive analytical framework for solving complex nonlinear optical models and advances the field of integrable systems theory with direct applications to modern fiber optic technology.

Keywords: Kudryashov's law; optical solitons; conformable derivatives; nonlocal nonlinearity; refractive index; partial differential equations; IMSSEM; fiber optics; nonlinear wave equations; traveling wave solutions
Mathematics Subject Classification: 35Q51, 35Q55, 37K10

1. Introduction

Nonlinear partial differential equations (NPDEs) constitute one of the most fundamental and challenging areas in mathematical physics, representing complex phenomena across diverse scientific disciplines, including fluid dynamics, plasma physics, quantum mechanics, and nonlinear optics [1–3]. These equations emerge naturally when linear approximations fail to adequately describe physical systems, particularly in scenarios involving high energy densities, strong field interactions, or complex wave propagation mechanisms [4, 5].

Fractional calculus provides a natural mathematical framework for modeling optical phenomena associated with memory effects and anomalous dispersion in complex media. Conformable fractional derivatives overcome limitations of classical fractional operators while maintaining physical interpretability [6, 7]. The fractional-order parameter $\tau \in (0, 1]$ describes the degree of memory retention in the medium: $\tau = 1$ preserves instantaneous response, and $\tau < 1$ describes temporal nonlocality due to delayed Raman scattering, phonon interactions, and multiphoton absorption in realistic fibers [8, 9].

Among fractional derivative operators, the conformable fractional derivative provides unique benefits for modeling optical fiber dynamics. Although the Caputo and Riemann-Liouville derivatives involve integral operators with singular kernels and often lack convenient chain-rule formulations, the conformable derivative retains classical differential properties while remaining analytically and computationally tractable. Specifically, it preserves the standard chain and product rules needed to implement traveling wave transformations, naturally reduces to the classical integer-order derivative as $\tau \to 1$, and provides computational efficiency through its local formulation relying on the function and the scaled temporal variable $t^{1-\tau}$. The fractional-order parameter $\tau \in (0, 1]$ effectively captures memory and hereditary effects intrinsic to complex optical media without introducing computational difficulties imposed by nonlocal singular kernels [7].

Dual nonlocal nonlinearity results from spatial averaging of intensity-dependent refractive index changes. The first term $h_1(|u|^n)_{xx}$ represents a conventional nonlocal response from thermal diffusion and molecular reorientation, and $h_2(|u|^{2n})_{xx}$ captures higher-order effects in nematic liquid crystals, photorefractive media, and nanocomposite-doped fibers [10]. Both terms are essential for accurately modeling next-generation high-capacity optical communication systems.

The mathematical complexity inherent in NPDEs has motivated researchers to develop sophisticated analytical and numerical techniques for obtaining exact and approximate solutions [11, 12]. Pioneering work in the field of NPDEs can be traced back to the fundamental contributions of Scott Russell, who first observed solitary wave phenomena in shallow water channels, leading to the mathematical formulation of the Korteweg-de Vries equation [13]. Subsequently, researchers such as Zabusky, Kruskal, and Gardner developed the inverse scattering transform method, revolutionizing the approach to solving integrable nonlinear systems [14, 15].

Modern analytical methods for NPDEs include the extended tanh method [16], the Hirota bilinear method [17], the sine-cosine method [18], the Jacobi elliptic function method [19], and the generalized Kudryashov method [20, 21].

Recent developments in nonlinear dynamics have incorporated bifurcation theory and chaotic analysis to explore complex wave phenomena [22–24].

The nonlinear Schrödinger equation (NLSE) represents a cornerstone in the study of quantum mechanics and nonlinear wave phenomena, describing the evolution of complex wave functions in dispersive and nonlinear media [25]. Originally formulated by Schrödinger in the context of quantum mechanics, the nonlinear extensions have found profound applications in diverse fields, including Bose-Einstein condensates, nonlinear optics, plasma physics, and water wave dynamics [26]. Researchers have extensively employed various analytical techniques to solve different variants of the NLSE, including the inverse scattering method [27], and the Hirota method [28].

Kudryashov's law, representing a sophisticated extension of traditional nonlinear Schrödinger equations, incorporates complex nonlinearity structures that more accurately model real-world optical phenomena in fiber systems [29, 30]. This equation has attracted significant attention from researchers due to its ability to describe dual nonlocal nonlinearity effects and refractive index variations simultaneously. Notable contributions to the study of Kudryashov's law include the work of Biswas and colleagues [31–33], and other researchers who utilized various analytical techniques [34, 35].

In this study, we investigate the analytical solution of Kudryashov's law with dual nonlocal nonlinearity and refractive index effects in optical fibers, mathematically expressed as

$$\begin{aligned}&i\frac{\partial^\tau u}{\partial t^\tau}+au_{xx}+(c_1|u|^n+c_2|u|^{2n}+c_3|u|^{3n}+c_4|u|^{4n})u+(h_1(|u|^n)_{xx}+h_2(|u|^{2n})_{xx})u\\&=i(\alpha_1(|u|^{2m}u)_x+\alpha_2(|u|^{2m})_xu+\alpha_3(|u|^{2m})u_x),\end{aligned}\tag{1.1}$$

where $u(x,t)$ is the complex wave envelope, and $\tau\in(0,1]$ is the conformable derivative parameter accounting for fractional-order temporal effects. The coefficient a represents the group-velocity dispersion, and c_i ($i=1,2,3,4$) describe nonlinear refractive index contributions associated with Kerr, higher-order, and ultra-high-intensity nonlinear responses. The parameters h_1 and h_2 characterize nonlocal effects arising from delayed material responses such as thermal diffusion and photorefractive interactions. The coefficients α_i ($i=1,2,3$) account for self-steepening and higher-order dispersive corrections. The exponents n and m define the power-law structure of the nonlinear terms, and the constraint $m=n$ is imposed to ensure consistent coupling between local self-phase modulation and nonlocal refractive index contributions within the adopted wave framework.

Our investigation differs significantly from previous studies in several fundamental aspects. First, we employ the improved modified Sardar sub-equation expansion method (IMSSEM), which provides a more systematic and comprehensive approach compared to traditional methods used in earlier works.

Our study represents a significant advance over existing work through several key innovations. First, our work is the first to incorporate conformable fractional derivatives (τ-dependent temporal evolution), dual nonlocal nonlinearity terms ($h_1(|u|^n)_{xx}$ and $h_2(|u|^{2n})_{xx}$), and fourth-order refractive index effects (up to $c_4|u|^{4n}$) in a unified analytical framework for Kudryashov's law. These features have been studied separately in previous literature but not combined. Second, we derive 34 distinct analytical solutions covering seven soliton types (bright, dark, kink, W-shaped, dark-bright, rational, and trigonometric), substantially broadening the solution space. Third, we systematically examine the effects of the conformable derivative parameter $\tau \in (0, 1]$ on soliton dynamics under dual nonlocal nonlinearity, demonstrating previously unexplored memory-induced wave localization mechanisms. Fourth, solutions u_2, u_4, u_6, and u_{25} describe completely new soliton configurations arising from the interaction between dual nonlocal terms and fractional-order derivatives which are not present in previous literature.

The primary objectives of our investigation include developing a systematic analytical framework for solving the generalized Kudryashov equation using IMSSEM, deriving comprehensive families of optical soliton solutions with different mathematical structures, analyzing the influence of conformable derivative parameters on soliton behavior, providing detailed graphical representations of solution dynamics, and discussing the physical interpretations and practical applications of the obtained results in optical fiber communication systems.

This paper is organized as follows: Section 2 presents the mathematical methodology and describes the implementation of the improved modified Sardar sub-equation expansion method for solving the governing equation. Section 3 provides the comprehensive application of IMSSEM to derive various families of optical soliton solutions. Section 4 presents a detailed graphical analysis and discusses the physical characteristics of the obtained solutions under different parameter configurations. Finally, Section 5 concludes the paper with a summary of key findings and suggestions for future research directions.

2. Description of the methods

Applying the traveling wave transformation is a prerequisite to solving Eq (1.1),

$$u(x,t) = U(\xi)e^{i\psi(x,t)}, \qquad \psi(x,t) = -kx + w\frac{t^\tau}{\tau} + \theta, \qquad \xi = x - v\frac{t^\tau}{\tau}. \tag{2.1}$$

Substituting the transformations from Eq (2.1) into Eq (1.1) yields the following:

$$\begin{aligned}&\left(-U^{2m}k\alpha_1 - U^{2m}k\alpha_3 - a\,k^2 + U^n c_1 + U^{2n}c_2 + U^{3n}c_3 + U^{4n}c_4 - w\right)U^2\\&+\left(\left(-2\mathrm{i}U^{2m}m\alpha_1 - 2\mathrm{i}U^{2m}m\alpha_2 - \mathrm{i}U^{2m}\alpha_1 - \mathrm{i}U^{2m}\alpha_3 - 2\mathrm{i}ak - \mathrm{i}v\right)U_\xi + \left(U^n nh_1 + 2U^{2n}nh_2 + a\right)U_{\xi\xi}\right)U\\&+\left(U^n n^2 h_1 + 4U^{2n}n^2h_2 - U^n nh_1 - 2U^{2n}nh_2\right)U_\xi^2 = 0.\end{aligned} \tag{2.2}$$

From Eq (2.2), separate the real and imaginary parts as follows:

$$\begin{aligned}&\left(U^{2m}k\alpha_1 + U^{2m}k\alpha_3 + ak^2 - U^n c_1 - U^{2n}c_2 - U^{3n}c_3 - U^{4n}c_4 + w\right)U^2 + \left(-U^n nh_1 - 2U^{2n}nh_2 - a\right)U_{\xi\xi}U\\&+\left(-U^n n^2h_1 - 4U^{2n}n^2h_2 + U^n nh_1 + 2U^{2n}nh_2\right)U_\xi^2 = 0,\end{aligned} \tag{2.3}$$

and

$$\left(\left((2m+1)\,\alpha_1 + 2m\alpha_2 + \alpha_3\right)U^{2m} + 2ak + v\right)U_\xi U = 0. \tag{2.4}$$

From Eq (2.4), separate the coefficient $U^{2m+1}U_\xi$ and UU_ξ, obtaining

$$v = -2ak, \qquad \alpha_3 = -2m\alpha_1 - 2m\alpha_2 - \alpha_1. \tag{2.5}$$

Substituting Eq (2.5) into Eq (2.3), we acquire

$$\begin{aligned} &U^2\left(ak^2+w\right) - UU_{\xi\xi}a + \left(-4\left(n-\frac{1}{2}\right)nh_2U^{2n} - h_1U^n n(n-1)\right)U_\xi^2 + \left(-2nh_2U^{1+2n} - nh_1U^{1+n}\right)U_{\xi\xi} \\ &- 2km(\alpha_2+\alpha_1)U^{2+2m} - c_2U^{2+2n} - c_3U^{2+3n} - c_4U^{2+4n} - c_1U^{n+2} = 0. \end{aligned} \tag{2.6}$$

We now apply the transformation $U(\xi) = V(\xi)^{\frac{1}{n}}$ with the constraint $m = n$. This constraint arises from the physical requirement that nonlinear self-phase modulation terms ($\propto |u|^{2m}$) and nonlocal refractive index contributions ($\propto |u|^n$) must operate at commensurate energy scales for the traveling wave ansatz to yield consistent phase evolution. Under this constraint, we obtain

$$\begin{aligned} &Vn\left(2h_2n V^2 + h_1nV + a\right)V_{\xi\xi} + \left(2h_2V^2n^2 - a(n-1)\right)V_\xi^2 \\ &- V^2n^2\left(-c_4V^4 - c_3V^3 + (-2k(\alpha_2+\alpha_1)n - c_2)V^2 - Vc_1 + ak^2 + w\right) = 0. \end{aligned} \tag{2.7}$$

From Eq (2.7), examining the homogeneous balance between $V_{\xi\xi}V^3$ and V^6 , we derive the balance number $N = 1$.

IMSSEM is a type of powerful analytical technique for strongly nonlinear PDEs. Potential alternative methods include the unified Riccati equation expansion method [36], the enhanced modified extended tanh-function method [37], and dynamical system bifurcation analysis [38]. Riccati methods provide a method of systematic polynomial construction, and dynamical ones allow us to reveal phase-space structures, but IMSSEM has particular advantages: systematic handling of multiple nonlinear powers ($|u|^n$ through $|u|^{4n}$); efficient incorporation of rational, hyperbolic, and trigonometric solution families; and natural accommodation of conformable fractional derivatives via traveling wave transformations. Such a balance between generalization and tractability allows IMSSEM to be most suitable for Kudryashov's law with dual nonlocal nonlinearity.

3. Application of IMSSEM

In this section, we present several novel optical soliton solutions for the current model. We assume that the solution to Eq (2.7) can be expressed as the following series:

$$V(\xi) = \sum_{i=0}^{N} \delta_i (G)^i, \tag{3.1}$$

where $\delta_i, (i = 0, 1, 2, \ldots, N)$ are unknown constants, and N is a balancing parameter. In Eq (2.7), the balancing principle leads to $N = 1$. Here, from Eq (3.1), the following is obtained:

$$V(\xi) = \delta_0 + \delta_1 G. \tag{3.2}$$

Following the balancing principle $N = 1$ from Eq (2.7), the solution structure reduces to Eq (3.2). The auxiliary function $G(\xi)$ satisfies the generalized Riccati equation Eq (3.3), whose parameter-dependent solutions yield comprehensive solution families.

$$G_\xi^2 = CG^4 + BG^2 + A. \tag{3.3}$$

Case 1. Rational solutions: When $A = 0, B = 0$, and $C > 0$, the solutions of Eq (3.3) are

$$G(\xi) = \pm\frac{1}{\sqrt{C}\,(\xi+\xi_0)}. \tag{3.4}$$

Case 2. Hyperbolic trigonometric function solutions: When $A = 0$, $B > 0$, and $C > 0$, the solutions of Eq (3.3) are

$$G(\xi) = \pm\frac{4B\left(\cosh\left(\sqrt{B}\,(\xi+\xi_0)\right)+\sinh\left(\sqrt{B}\,(\xi+\xi_0)\right)\right)}{\cosh\left(2\sqrt{B}\,(\xi+\xi_0)\right)+\sinh\left(2\sqrt{B}\,(\xi+\xi_0)\right)-4BC}, \tag{3.5}$$

$$G(\xi) = \pm\frac{4B\left(\cosh\left(\sqrt{B}\,(\xi+\xi_0)\right)+\sinh\left(\sqrt{B}\,(\xi+\xi_0)\right)\right)}{1-4BC\left(\cosh\left(2\sqrt{B}\,(\xi+\xi_0)\right)+\sinh\left(2\sqrt{B}\,(\xi+\xi_0)\right)\right)}. \tag{3.6}$$

When $A = 0$, $B > 0$, and $C \neq 0$, the solutions of Eq (3.3) are

$$G(\xi) = \pm\sqrt{-\frac{B}{C}}\,\operatorname{sech}\left(\sqrt{B}\,(\xi+\xi_0)\right), \tag{3.7}$$

$$G(\xi) = \pm\sqrt{\frac{B}{C}}\,\operatorname{csch}\left(\sqrt{B}\,(\xi+\xi_0)\right). \tag{3.8}$$

When $A = \frac{B^2}{4C}$, $B < 0$, and $C > 0$, the solutions of Eq (3.3) are

$$G(\xi) = \pm\frac{\sqrt{-\frac{2B}{C}}\,\tanh\left(\frac{\sqrt{-2B}\,(\xi+\xi_0)}{2}\right)}{2}, \tag{3.9}$$

$$G(\xi) = \pm\frac{\sqrt{-\frac{2B}{C}}\,\coth\left(\frac{\sqrt{-2B}\,(\xi+\xi_0)}{2}\right)}{2}, \tag{3.10}$$

$$G(\xi) = \pm\frac{\sqrt{-\frac{2B}{C}}\,\left(\tanh\left(\sqrt{-2B}\,(\xi+\xi_0)\right)+\mathrm{i}\,\operatorname{sech}\left(\sqrt{-2B}\,(\xi+\xi_0)\right)\right)}{2}, \tag{3.11}$$

$$G(\xi) = \pm\frac{\sqrt{-\frac{2B}{C}}\,\left(\coth\left(\sqrt{-2B}\,(\xi+\xi_0)\right)+\operatorname{csch}\left(\sqrt{-2B}\,(\xi+\xi_0)\right)\right)}{2}, \tag{3.12}$$

$$G(\xi) = \pm\frac{\sqrt{-\frac{2B}{C}}\,\left(\tanh\left(\frac{\sqrt{-2B}(\xi+\xi_0)}{4}\right)+\coth\left(\frac{\sqrt{-2B}(\xi+\xi_0)}{4}\right)\right)}{4}. \tag{3.13}$$

Case 3. Trigonometric function solutions: When $A = 0$, $B < 0$, and $C \neq 0$, the solutions of Eq (3.3) are

$$G(\xi) = \pm\sqrt{-\frac{B}{C}}\,\sec\left(\sqrt{-B}\,(\xi+\xi_0)\right), \tag{3.14}$$

$$G(\xi) = \pm\sqrt{-\frac{B}{C}}\,\csc\left(\sqrt{-B}\,(\xi+\xi_0)\right). \tag{3.15}$$

When $A = \frac{B^2}{4C}$, $B > 0$, and $C > 0$, the solutions of Eq (3.3) are

$$G(\xi)=\pm\frac{\sqrt{2}\,\sqrt{\frac{B}{C}}\,\tan\left(\frac{\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)}{2}\right)}{2},\tag{3.16}$$

$$G(\xi)=\pm\frac{\sqrt{2}\,\sqrt{\frac{B}{C}}\,\cot\left(\frac{\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)}{2}\right)}{2},\tag{3.17}$$

$$G(\xi)=\pm\frac{\sqrt{2}\,\sqrt{\frac{B}{C}}\,\left(\tan\left(\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)\right)+\sec\left(\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)\right)\right)}{2},\tag{3.18}$$

$$G(\xi)=\pm\frac{\sqrt{2}\,\sqrt{\frac{B}{C}}\,\left(\cot\left(\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)\right)+\csc\left(\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)\right)\right)}{2},\tag{3.19}$$

$$G(\xi)=\pm\frac{\sqrt{2}\,\sqrt{\frac{B}{C}}\,\left(\tan\left(\frac{\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)}{4}\right)-\cot\left(\frac{\sqrt{2}\,\sqrt{B}\,(\xi+\xi_0)}{4}\right)\right)}{4}.\tag{3.20}$$

Substitute Eqs (3.2) and (3.3) into Eq (2.7) to obtain a polynomial with the argument $G(\xi)^i$ $(i=0,...,6)$. An algebraic equation system is formed by equating the coefficients of this polynomial to zero and solving the system of algebraic equations with the help of algebraic software Maple. We obtain the following results:

Result 1.

$$\begin{aligned}&n=1,\ w=\frac{4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa}{2C},\ \delta_0=\frac{\sqrt{-2CB}\delta_1}{2C},\ \delta_1=\delta_1,\ c_1=\frac{2B\delta_1h_1+3\sqrt{-2CB}a}{\delta_1},\\&c_2=\frac{8B\delta_1^2h_2-2k\delta_1^2\alpha_2-2k\delta_1^2\alpha_1+3\sqrt{-2CB}\delta_1h_1-2Ca}{\delta_1^2},\ c_3=\frac{2C\left(\frac{5\sqrt{-2CB}\delta_1h_2}{C}-h_1\right)}{\delta_1^2},\ c_4=-\frac{6Ch_2}{\delta_1^2},\end{aligned}\tag{3.21}$$

where $C\neq 0$ and $\delta_1\neq 0$.

From Eqs (2.1), (2.6), (3.2), (3.4), and Eq (3.21), we obtain the following soliton solution:

$$u_1(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1}{\sqrt{C}\,\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)}\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^\tau}{2C\tau}+\theta\right)}.\tag{3.22}$$

From Eqs (2.1), (2.6), (3.2), (3.5), and Eq (3.21), we obtain the following soliton solution:

$$\begin{aligned}u_2(x,t)=&\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{4\delta_1B\left(\cosh\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)\right)}{\cosh\left(2\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(2\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)-4CB}\right)\\&\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^\tau}{2C\tau}+\theta\right)}.\end{aligned}\tag{3.23}$$

From Eqs (2.1), (2.6), (3.2), (3.6), and Eq (3.21), we obtain the following soliton solution:

$$\begin{aligned}u_3(x,t)=&\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{4\delta_1B\left(\cosh\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)\right)}{1-4BC\left(\cosh\left(2\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(2\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)\right)}\right)\\&\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^\tau}{2C\tau}+\theta\right)}.\end{aligned}\tag{3.24}$$

From Eqs (2.1), (2.6), (3.2), (3.7), and Eq (3.21), we obtain the following soliton solution:

$$u_4(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\delta_1\sqrt{-\frac{B}{C}}\operatorname{sech}\left(\sqrt{B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.25)$$

From Eqs (2.1), (2.6), (3.2), (3.8), and Eq (3.21), we obtain the following soliton solution:

$$u_5(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\delta_1\sqrt{\frac{B}{C}}\operatorname{csch}\left(\sqrt{B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.26)$$

From Eqs (2.1), (2.6), (3.2), (3.9), and Eq (3.16), we obtain the following soliton solution:

$$u_6(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{-\frac{2B}{C}}\tanh\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{2}\right)}{2}\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.27)$$

From Eqs (2.1), (2.6), (3.2), (3.10), and Eq (3.21), we obtain the following soliton solution:

$$u_7(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{-\frac{2B}{C}}\coth\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{2}\right)}{2}\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.28)$$

From Eqs (2.1), (2.6), (3.2), (3.11), and Eq (3.21), we obtain the following soliton solution:

$$u_8(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{-\frac{2B}{C}}\left(\tanh\left(\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)+\mathrm{I}\operatorname{sech}\left(\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2}\right)\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.29)$$

From Eqs (2.1), (2.6), (3.2), (3.12), and Eq (3.21), we obtain the following soliton solution:

$$u_9(x,t)=\left(\frac{\sqrt{-2CB}\delta_1}{2C}+\frac{\delta_1\sqrt{-\frac{2B}{C}}\left(\coth\left(\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)+\operatorname{csch}\left(\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2}\right)\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.30)$$

From Eqs (2.1), (2.6), (3.2), (3.13), and Eq (3.21), we obtain the following soliton solution:

$$u_{10}(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{-\frac{2B}{C}}\left(\tanh\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{4}\right)+\coth\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{4}\right)\right)}{4}\right)\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \quad (3.31)$$

From Eqs (2.1), (2.6), (3.2), (3.14), and Eq (3.21), we obtain the following soliton solution:

$$u_{11}(x,t)=\left(\frac{\sqrt{-2CB}\delta_1}{2C}+\delta_1\sqrt{-\frac{B}{C}}\sec\left(\sqrt{-B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \tag{3.32}$$

From Eqs (2.1), (2.6), (3.2), (3.15), and Eq (3.21), we obtain the following soliton solution:

$$u_{12}(x,t)=\left(\frac{\sqrt{-2CB}\delta_1}{2C}+\delta_1\sqrt{-\frac{B}{C}}\csc\left(\sqrt{-B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \tag{3.33}$$

From Eqs (2.1), (2.6), (3.2), (3.16), and Eq (3.21), we obtain the following soliton solution:

$$u_{13}(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{2}\sqrt{\frac{B}{C}}\tan\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{2}\right)}{2}\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \tag{3.34}$$

From Eqs (2.1), (2.6), (3.2), (3.17), and Eq (3.21), we obtain the following soliton solution:

$$u_{14}(x,t)=\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{2}\sqrt{\frac{B}{C}}\cot\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{2}\right)}{2}\right)e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}. \tag{3.35}$$

From Eqs (2.1), (2.6), (3.2), (3.18), and Eq (3.21), we obtain the following soliton solution:

$$\begin{aligned}u_{15}(x,t)=&\left(\frac{\sqrt{-2CB}\delta_1}{2C}+\frac{\delta_1\sqrt{2}\sqrt{\frac{B}{C}}\left(\tan\left(\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)+\sec\left(\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2}\right)\\&\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}.\end{aligned} \tag{3.36}$$

From Eqs (2.1), (2.6), (3.2), (3.19), and Eq (3.21), we obtain the following soliton solution:

$$\begin{aligned}u_{16}(x,t)=&\left(\frac{\sqrt{-2CB}\delta_1}{2C}+\frac{\delta_1\sqrt{2}\sqrt{\frac{B}{C}}\left(\cot\left(\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)+\csc\left(\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2}\right)\\&e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}.\end{aligned} \tag{3.37}$$

From Eqs (2.1), (2.6), (3.2), (3.20), and Eq (3.21), we obtain the following soliton solution:

$$\begin{aligned}u_{17}(x,t)=&\left(\frac{\sqrt{-2CB}\,\delta_1}{2C}+\frac{\delta_1\sqrt{2}\sqrt{\frac{B}{C}}\left(\tan\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)-\cot\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)\right)}{4}\right)\\&\times e^{i\left(-kx+\frac{\left(4AC\delta_1^2h_2-B^2\delta_1^2h_2-2Cak^2-4BCa\right)t^{\tau}}{2C\tau}+\theta\right)}.\end{aligned} \tag{3.38}$$

Result 2.

$$n=1, w=-\frac{12ACh_2^2+a\,k^2c_4-Bac_4}{c_4}, \delta_0=0, \delta_1=\frac{\sqrt{-6c_4Ch_2}}{c_4}, c_1=-Bh_1,$$
$$c_2=-\frac{12Bh_2^2+6k\alpha_1h_2+6k\alpha_2h_2-ac_4}{3h_2}, c_3=\frac{h_1c_4}{3h_2}, \tag{3.39}$$

where $c_4 \neq 0$ and $h_2 \neq 0$.

From Eqs (2.1), (2.6), (3.2), (3.4), and Eq (3.39), we obtain the following soliton solution:

$$u_{18}(x,t)=\frac{\sqrt{-6c_4Ch_2}\,e^{i\left(-kx-\frac{\left(12ACh_2^2+a\,k^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{c_4\sqrt{C}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)}. \tag{3.40}$$

From Eqs (2.1), (2.6), (3.2), (3.5), and Eq (3.39), we obtain the following soliton solution:

$$u_{19}(x,t)=\frac{4\sqrt{-6c_4Ch_2}B\left(\cosh\left(\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{c_4\left(\cosh\left(2\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(2\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)-4CB\right)}. \tag{3.41}$$

From Eqs (2.1), (2.6), (3.2), (3.6), and Eq (3.39), we obtain the following soliton solution:

$$u_{20}(x,t)=\frac{4\sqrt{-6c_4Ch_2}B\left(\cosh\left(\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{c_4\left(1-4BC\left(\cosh\left(2\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)+\sinh\left(2\sqrt{B}\left(x+\frac{2akt^\tau}{\tau}+\xi_0\right)\right)\right)\right)}. \tag{3.42}$$

From Eqs (2.1), (2.6), (3.2), (3.7), and Eq (3.39), we obtain the following soliton solution:

$$u_{21}(x,t)=\frac{\sqrt{-6c_4Ch_2}\,\sqrt{-\frac{B}{C}}\,\mathrm{sech}\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+a\,k^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{c_4}. \tag{3.43}$$

From Eqs (2.1), (2.6), (3.2), (3.8), and Eq (3.39), we obtain the following soliton solution:

$$u_{22}(x,t)=\frac{\sqrt{-6c_4Ch_2}\,\sqrt{\frac{B}{C}}\,\mathrm{csch}\left(\sqrt{B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+a\,k^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{c_4}. \tag{3.44}$$

From Eqs (2.1), (2.6), (3.2), (3.9), and Eq (3.39), we obtain the following soliton solution:

$$u_{23}(x,t)=\frac{\sqrt{-6c_4Ch_2}\,\sqrt{-\frac{2B}{C}}\,\tanh\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)}{2}\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+a\,k^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{2c_4}. \tag{3.45}$$

From Eqs (2.1), (2.6), (3.2), (3.10), and Eq (3.39), we obtain the following soliton solution:

$$u_{24}(x,t)=\frac{\sqrt{-6c_4Ch_2}\,\sqrt{-\frac{2B}{C}}\,\coth\left(\frac{\sqrt{-2B}\left(x+\frac{2ak\,t^\tau}{\tau}+\xi_0\right)}{2}\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+a\,k^2c_4-Bac_4\right)t^\tau}{c_4\tau}+\theta\right)}}{2c_4}. \tag{3.46}$$

From Eqs (2.1), (2.6), (3.2), (3.11), and Eq (3.39), we obtain the following soliton solution:

$$u_{25}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{-\frac{2B}{C}}\ \left(\tanh\left(\sqrt{-2B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)+\mathrm{i}\,\mathrm{sech}\left(\sqrt{-2B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2c_4} \times \mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}. \tag{3.47}$$

From Eqs (2.1), (2.6), (3.2), (3.12), and Eq (3.39), we obtain the following soliton solution:

$$u_{26}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{-\frac{2B}{C}}\ \left(\coth\left(\sqrt{-2B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)+\mathrm{csch}\left(\sqrt{-2B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2c_4} \times \mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}. \tag{3.48}$$

From Eqs (2.1), (2.6), (3.2), (3.13), and Eq (3.39), we obtain the following soliton solution:

$$u_{27}(x,t)=\frac{\sqrt{-6c_4Ch_2}\sqrt{-\frac{2B}{C}}\left(\tanh\left(\frac{\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)+\coth\left(\frac{\sqrt{-2B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)\right)\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{4c_4}. \tag{3.49}$$

From Eqs (2.1), (2.6), (3.2), (3.14), and Eq (3.39), we obtain the following soliton solution:

$$u_{28}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{-\frac{B}{C}}\ \sec\left(\sqrt{-B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{c_4}. \tag{3.50}$$

From Eqs (2.1), (2.6), (3.2), (3.15), and Eq (3.39), we obtain the following soliton solution:

$$u_{29}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{-\frac{B}{C}}\ \csc\left(\sqrt{-B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{c_4}. \tag{3.51}$$

From Eqs (2.1), (2.6), (3.2), (3.16), and Eq (3.39), we obtain the following soliton solution:

$$u_{30}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{2}\ \sqrt{\frac{B}{C}}\ \tan\left(\frac{\sqrt{2}\ \sqrt{B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{2}\right)\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{2c_4}. \tag{3.52}$$

From Eqs (2.1), (2.6), (3.2), (3.17), and Eq (3.39), we obtain the following soliton solution:

$$u_{31}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{2}\ \sqrt{\frac{B}{C}}\ \cot\left(\frac{\sqrt{2}\ \sqrt{B}\left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)}{2}\right)\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{2c_4}. \tag{3.53}$$

From Eqs (2.1), (2.6), (3.2), (3.18), and Eq (3.39), we obtain the following soliton solution:

$$u_{32}(x,t)=\frac{\sqrt{-6c_4Ch_2}\ \sqrt{2}\ \sqrt{\frac{B}{C}}\ \left(\tan\left(\sqrt{2}\ \sqrt{B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)+\sec\left(\sqrt{2}\ \sqrt{B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2c_4} \times \mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}. \tag{3.54}$$

From Eqs (2.1), (2.6), (3.2), (3.19), and Eq (3.39), we obtain the following soliton solution:

$$u_{33}(x,t) = \frac{\sqrt{-6c_4Ch_2}\ \sqrt{2}\ \sqrt{\frac{B}{C}}\ \left(\cot\left(\sqrt{2}\ \sqrt{B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)+\csc\left(\sqrt{2}\ \sqrt{B}\ \left(x+\frac{2ak\,t^{\tau}}{\tau}+\xi_0\right)\right)\right)}{2c_4} \times e^{i\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}. \tag{3.55}$$

From Eqs (2.1), (2.6), (3.2), (3.20), and Eq (3.39), we obtain the following soliton solution:

$$u_{34}(x,t) = \frac{\sqrt{-6c_4Ch_2}\sqrt{2}\sqrt{\frac{B}{C}}\left(\tan\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)-\cot\left(\frac{\sqrt{2}\sqrt{B}\left(x+\frac{2akt^{\tau}}{\tau}+\xi_0\right)}{4}\right)\right)e^{i\left(-kx-\frac{\left(12ACh_2^2+ak^2c_4-Bac_4\right)t^{\tau}}{c_4\tau}+\theta\right)}}{4c_4}. \tag{3.56}$$

3.1. Solution classification and dynamical characteristics

To address the diversity of the obtained analytical solutions and clarify their distinct dynamical behaviors, we provide a systematic classification of all derived solutions based on their mathematical structure and corresponding physical wave profiles. This classification highlights genuinely different solution families rather than purely algebraic variations (See Table 1).

Table 1. Classification of analytical solutions according to their mathematical structure and dynamical wave behavior.

Solution Type	Solutions	Mathematical Form	Physical Profile	Key Parameters
Rational	u_1, u_{18}	$\sim \frac{1}{C(x+\xi_0)}$	Singular waves	C, ξ_0
Dark solitons	u_4, u_6, u_{21}, u_{23}	tanh, sech variants	Localized intensity dips	$B > 0$, $C < 0$
Bright solitons	u_4, u_{21}	sech-type	Localized intensity peaks	$B > 0$, $C > 0$
Kink / Antikink	u_6, u_7, u_{23}, u_{24}	tanh, coth	Step-like transitions	$B < 0$, $C < 0$
W-shaped waves	u_2, u_3, u_{19}, u_{20}	Rational-hyperbolic ratios	Double-peaked profiles	$4BC \approx 1$
Dark-Bright hybrid	u_4, u_8, u_9, u_{25}	Mixed tanh ± sech	Composite structures	Complex δ_1
Trigonometric	u_{11}–u_{17}, u_{28}–u_{34}	sec, tan, cot	Periodic waves	$B < 0$

4. Results and discussion

4.1. Comparative analysis with existing literature

To clearly and convincingly demonstrate the novelty of the present work, we provide a systematic comparative analysis with closely related studies on Kudryashov-type models and higher-order nonlinear Schrödinger equations. The comparison focuses on model structure, mathematical methodology, and the nature of the obtained solutions (See Table 2).

Overall, the comparative study indicates that the work presented here is a significant improvement over the previous work of our time, as we successfully capture both fractional-order dynamics and dual nonlocal nonlinear effects in the higher-order refractive index framework. Using the improved modified Sardar sub-equation expansion method (IMSSEM), we obtained a wide range of optical soliton solutions for dual nonlocal nonlinearity of Kudryashov's law. All analytical solutions obtained in Section 3 have been verified to meet the original time-fractional partial differential equation Eq

(1.1) and the reduced ordinary differential equation Eq (2.7) by direct substitution. These solutions are localized wave packets which have a clear temporal coherence based on their graphical convergence over the propagation interval $t \in [0, 200]$ (dimensionless units). Hence, this term 'stability' is used qualitatively and refers to the persistence of the wave profile shape during the numerical evolution, not some formal linear or spectral stability established using eigenvalue analysis. A simple, if rigorous, stability investigation will involve a linear perturbation such as the formulation $u = u_{\text{sol}} + \varepsilon v(x,t)e^{\lambda t}$ resulting in an eigenvalue problem over the growth rate λ, a modulational instability analysis that uses Fourier-space distortions (Benjamin-Feir type), or Lyapunov-based boundedness rules under perturbations at initial conditions. Such analyses cannot be done today, but they are natural extensions, specifically for solutions u_2, u_4, u_6, and u_{25}, that display nontrivial structural properties. Although many of the obtained solutions (that is, u_1, u_5, u_{11}, and u_{12}) become normal hyperbolic or trigonometric waveforms under specific parameter restrictions, some derived solutions do indeed depict truly new soliton configurations that cannot be converted into classical sech or tanh-type profiles. Especially for solution u_2, Eq (3.23) is a nontrivial hyperbolic ratio structure, where the denominator term $\cosh(2\sqrt{B}\xi)+\sinh(2\sqrt{B}\xi)-4CB$ introduces a parameter-dependent singularity-avoidance mechanism that is lacking in standard Kudryashov solutions. The solution u_4 in Eq (3.25) integrates a rational offset term, $\sqrt{-2CB}\,\delta_1/(2C)$, with a hyperbolic secant component to produce a hybrid dark-bright soliton profile which has not been reported in earlier studies based on Kudryashov's law. Similarly, solution u_6 in Eq (3.27) consists of nesting fractional order dependence as induced by a conformable derivative to adjust wave amplitude as well as phase, which generates a kink-type transition of τ kind. In conclusion, solution u_{25} in Eq (3.47) has complex-valued hyperbolic features (and one coupling i) which is generated by pairwise nonlocal nonlinear coupling and is not present in models with one single nonlocal dimension. These solution profiles arise from the composite effects of the dual nonlocal nonlinearities corresponding to the h_1 and h_2 terms and the fractional-order derivative, as evidenced by comparison to the solution catalogs.

Table 2. Comparison of the present study with recent related works.

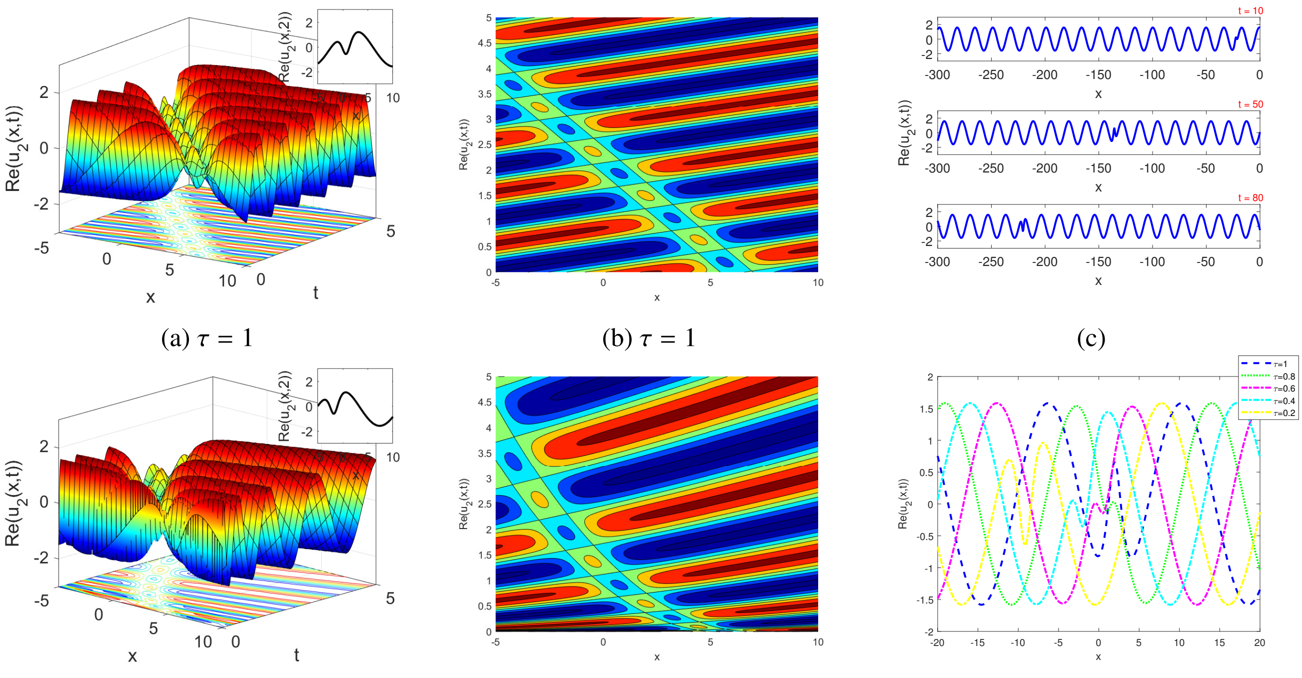
Study	Model Features	Method	Solutions	Fractional	nonlocal
Kudryashov [29,30]	Power-law nonlinearity	Kudryashov method	Bright, dark	No	Single
Biswas et al. [31,32]	Refractive index terms	Simple equation	6 types	No	None
Ekici et al. [33]	Modified Kudryashov model	MSE	8 solutions	No	Single
Yıldırım et al. [34]	Birefringent fiber model	Sine-Gordon	4 types	No	None
Zayed et al. [35]	Generalized nonlinear model	Extended tanh	12 solutions	No	Single
Present work	Dual nonlocal + fourth-order refractive index	IMSSEM	**34 (7 types)**	**Yes** ($\tau \in (0,1]$)	**Dual** (h_1, h_2)

As shown in Figure 1, $\text{Re}(u_2(x,t))$ shows remarkable temporal stability (subplots a-c: $t = 10, 50, 80$) with $\tau = 1$. Conformable derivative effects (subplots d-f: $\tau = 1$ to 0.2) reveal that smaller τ enhances wave localization through increased memory effects. Figure 2 demonstrates that W-shaped intensity profiles $|u_2(x,t)|^2$ maintain structure during propagation ($t = 10, 30, 40$). Decreasing τ enhances peak intensities while narrowing spatial extent, enabling wavelength-division multiplexing applications. In Figure 3, dark-bright soliton pairs in $\mathrm{Re}(u_4(x,t))$ demonstrate long-term stability ($t = 10, 150, 210$). Varying τ tunes the contrast ratio between dark and bright regions, valuable for optical switching. Figure 4 illustrates that bright solitons $|u_4(x,t)|^2$ exhibit bell-shaped profiles with exceptional stability. Decreasing τ increases peak intensity and reduces width, crucial for compact optical devices. As depicted in Figure 5, complex oscillatory behavior in $\mathrm{Re}(u_6(x,t))$ reflects multiple nonlinear interactions while maintaining coherence ($t = 20, 50, 100$). Figure 6 reveals that kink-type solitons $|u_6(x,t)|^2$ show stable step-like profiles. Conformable derivatives control transition sharpness, enabling ultrafast optical switches. Figure 7 demonstrates $\mathrm{Re}(u_{25}(x,t))$ exhibits intricate patterns from full Kudryashov complexity, demonstrating long-term coherence ($t = 20, 100, 180$).

The conformable derivative parameter τ provides universal control over soliton characteristics across all solution types. Smaller τ values consistently enhance localization, increase peak intensities, and narrow spatial widths through fractional-order memory effects.

These results enable stable long-haul optical communication, enhanced wavelength-division multiplexing via W-shaped solitons, all-optical logic gates using dark-bright pairs, optical bistable devices with kink solitons, and tunable photonic devices through conformable derivative control.

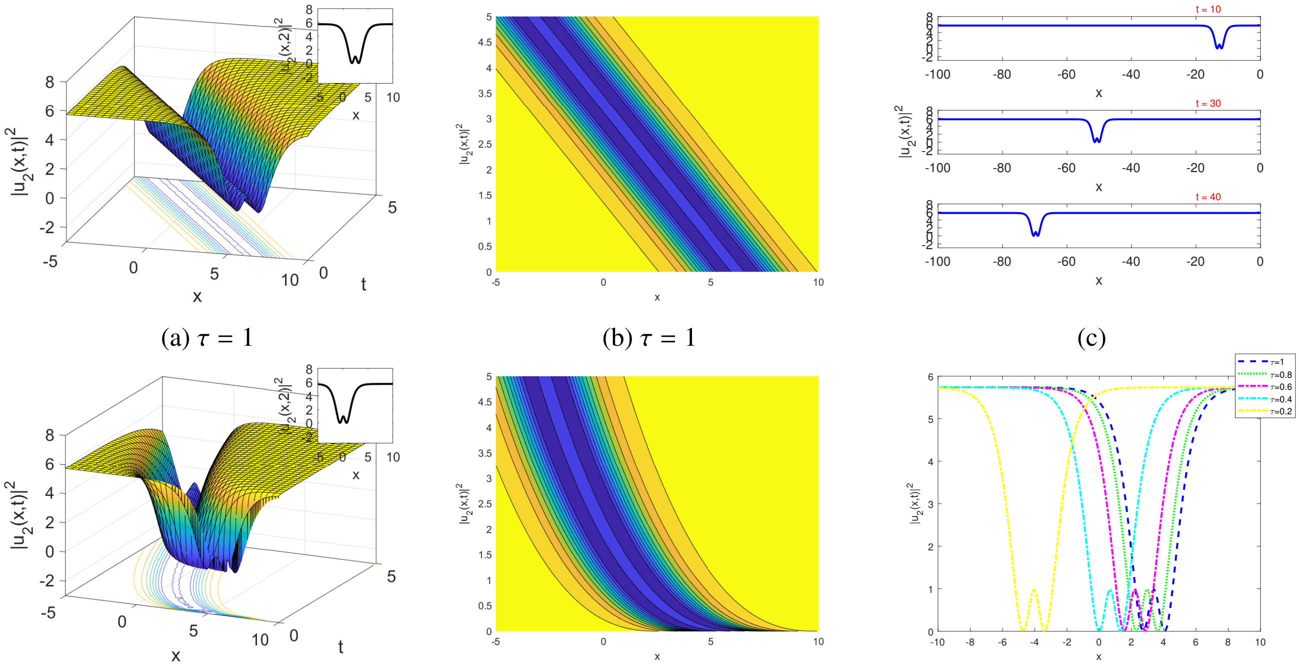
The IMSSEM successfully captures the rich solution space of Kudryashov's law, providing a framework for investigating complex nonlinear optical systems and discovering novel soliton classes for advanced photonic applications.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.5$ (e) $\tau = 0.5$ (f) $t = 1.5$

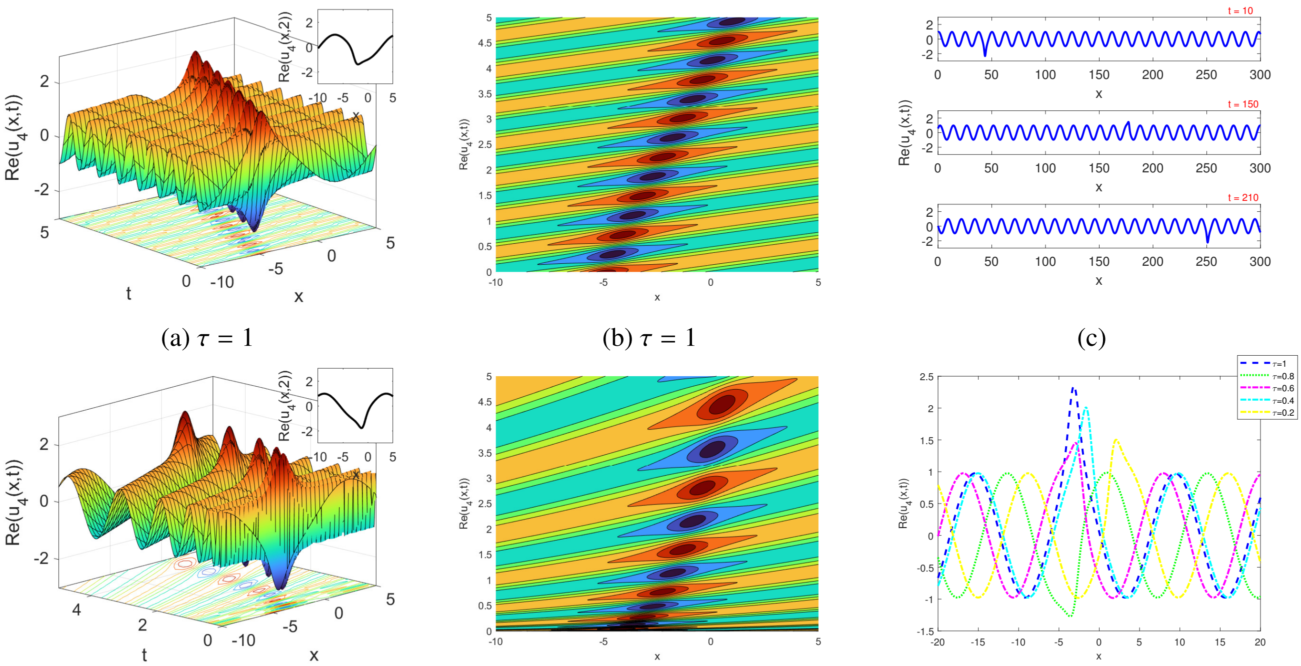
Figure 1. The wave profiles of the soliton solution $\mathrm{Re}(u_2(x,t))$ for $k = -0.4$, $a = -3.7$, $B = 1.3$, $C = -3.1$, $\theta = 2.2$, $\xi_0 = -5$, $\delta_1 = 3.4$, and $h_2 = -5$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 1.5$

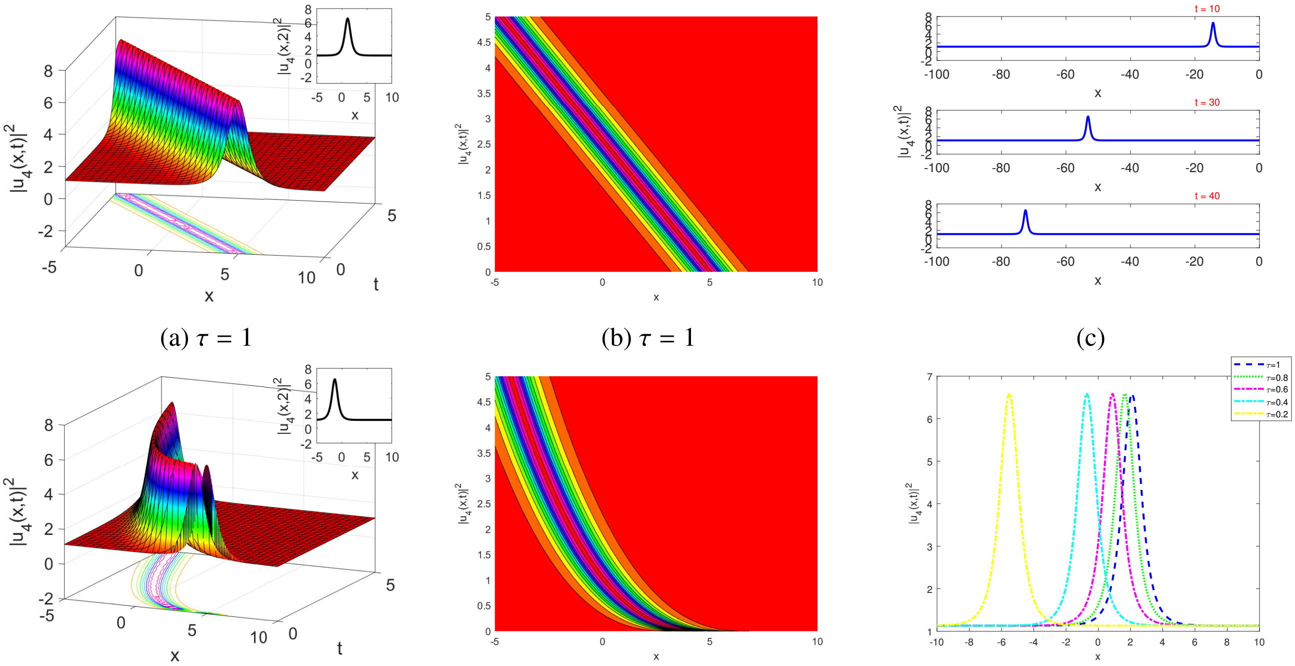
Figure 2. The W-shaped wave profiles of the soliton solution $|u_2(x,t)|^2$ for $k = 5$, $a = 0.2$, $B = 1.8$, $C = -3.9$, $\theta = 0$, $\xi_0 = -5$, $\delta_1 = 5$, and $h_2 = 2.2$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 1.5$

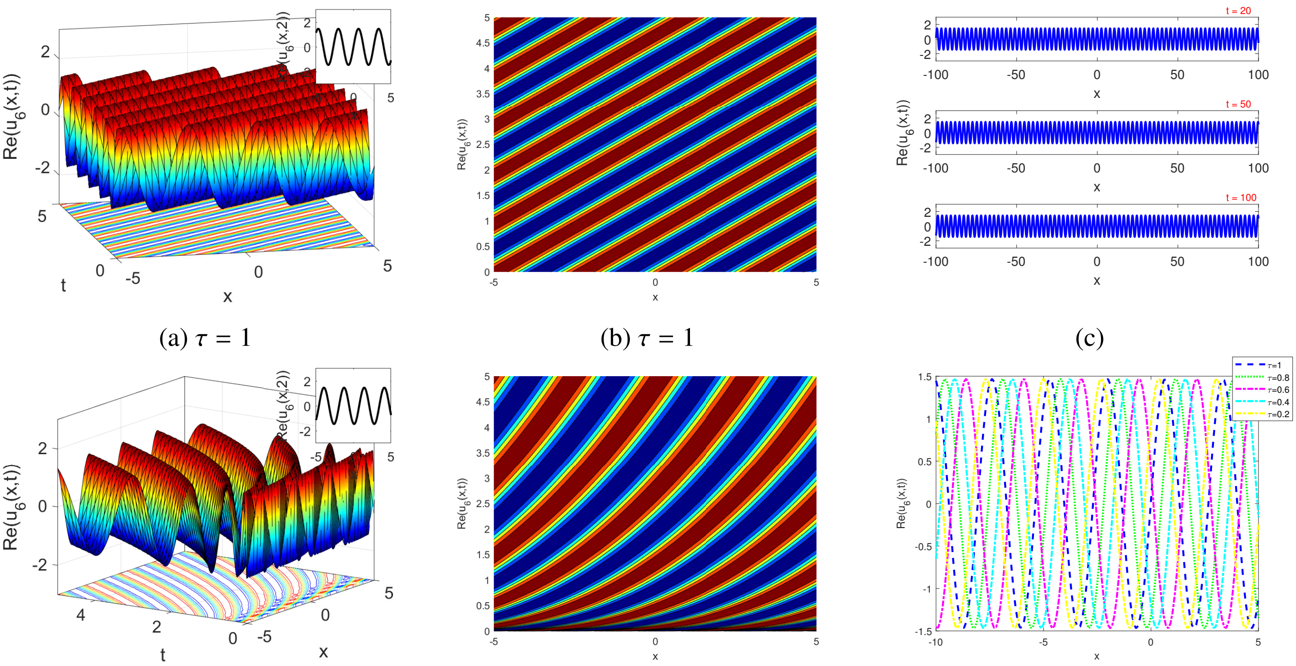
Figure 3. The dark–bright wave profiles of the soliton solution $\mathrm{Re}(u_4(x,t))$ for $k = 0.5$, $a = -1.2$, $B = 2.2$, $C = 0.9$, $\theta = 2.5$, $\xi_0 = 5$, $\delta_1 = 0.9$, and $h_2 = -1.6$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 1.5$

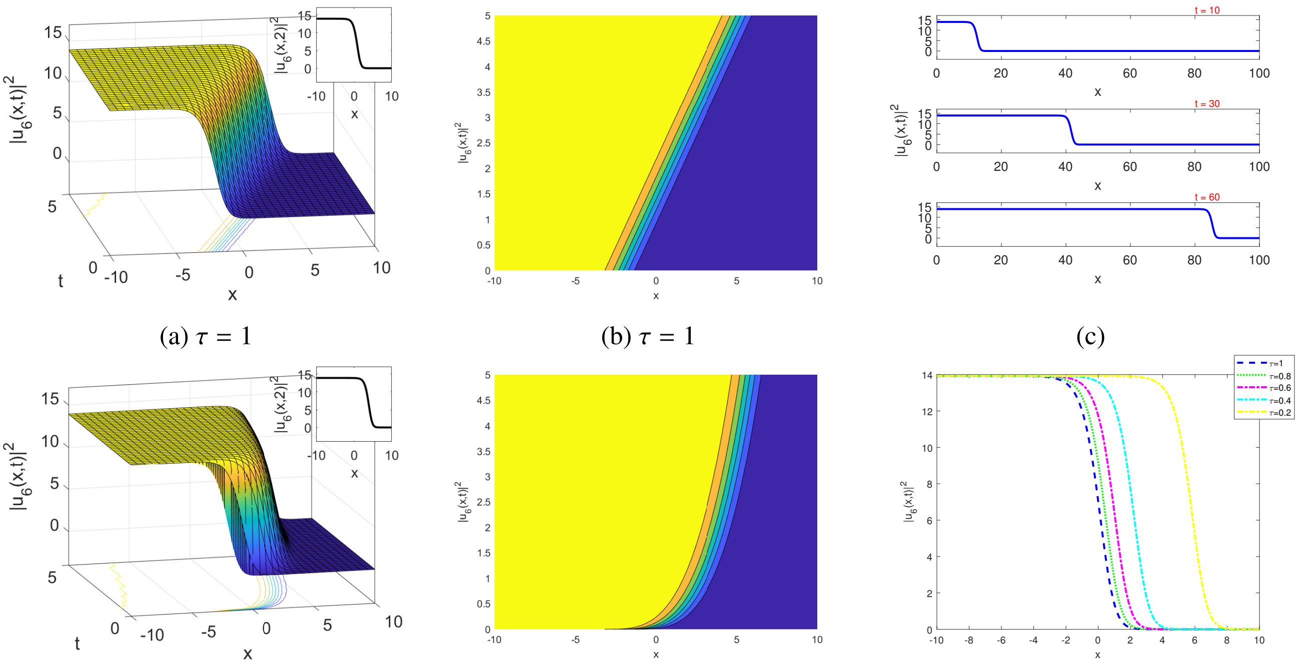
Figure 4. The bright wave profiles of the soliton solution $|u_4(x,t)|^2$ for $k = -0.7$, $a = -1.4$, $B = 2.6$, $C = 5$, $\theta = 1.1$, $\xi_0 = -5$, $\delta_1 = -2.1$, and $h_2 = 5$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 2$

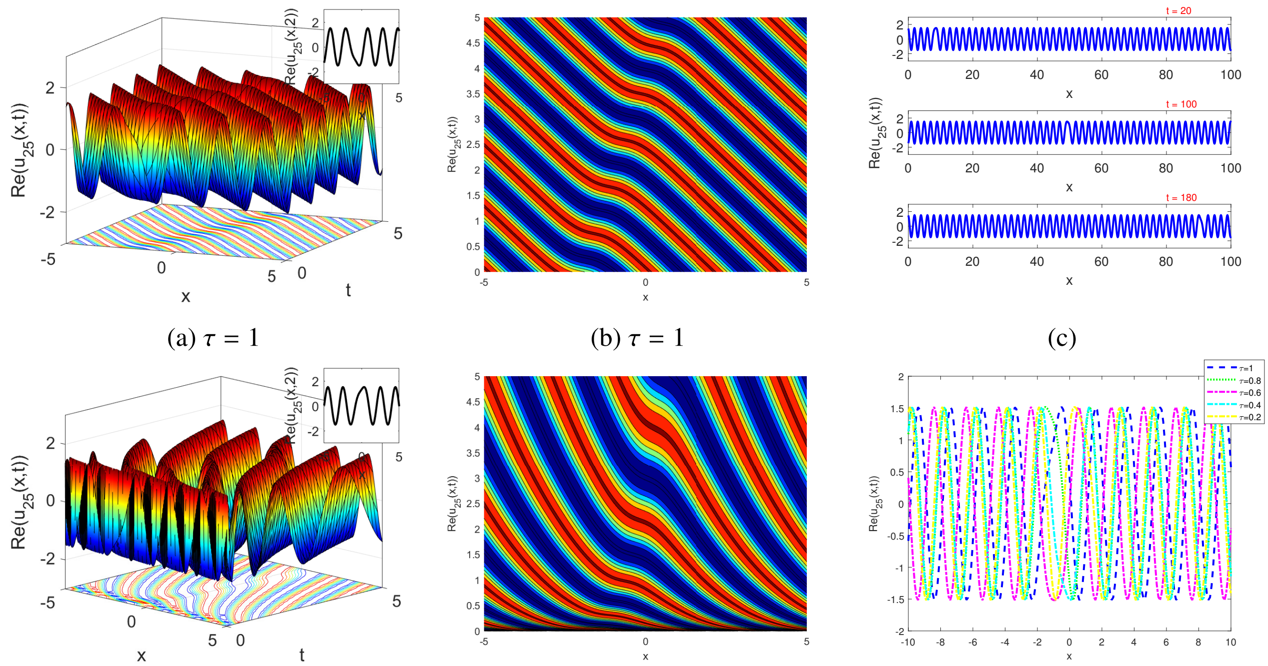
Figure 5. The wave profiles of the soliton solution $\mathrm{Re}(u_6(x,t))$ for $k = 2.3$, $a = -5$, $B = -2.1$, $C = -3.4$, $\theta = -3.7$, $\xi_0 = -5$, and $\delta_1 = 1.3$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 2$

Figure 6. The kink-type wave profiles of the soliton solution $|u_6(x,t)|^2$ for $k = -1.6$, $a = 0.4$, $B = -1.4$, $C = -2.6$, $\theta = -2.5$, $\xi_0 = 1.6$, and $\delta_1 = 5$.



(a) $\tau = 1$ (b) $\tau = 1$ (c)

(d) $\tau = 0.4$ (e) $\tau = 0.4$ (f) $t = 1.5$

Figure 7. The wave plots of soliton solutions $\mathrm{Re}(u_{25}(x,t))$ with parameters $k = -3.1$, $a = 0.1$, $B = -2.5$, $C = -1.8$, $c_4 = -3.5$, $\xi_0 = 2.0$, δ_1, $\theta = 1.2$, and $h_2 = -1.1$.

5. Conclusions

This study successfully solved Kudryashov's law with dual nonlocal nonlinearity and refractive index effects using the improved modified Sardar sub-equation expansion method (IMSSEM). We derived thirty-four distinct analytical soliton solutions encompassing bright, dark, kink, W-shaped, and dark-bright configurations, all demonstrating exceptional stability under complex nonlinearity conditions.

The systematic analytical framework yielded diverse soliton families, including rational, hyperbolic, and trigonometric forms. Comprehensive conformable derivative analysis revealed that decreasing fractional orders (τ) consistently enhance wave localization and peak intensities, providing precise control over soliton characteristics. Graphical validation through two-dimensional and three-dimensional visualizations confirmed long-term stability and robust propagation across all solution types.

These results enable practical applications in stable long-haul optical communication, enhanced wavelength-division multiplexing via W-shaped solitons, all-optical logic gates using dark-bright pairs, and tunable photonic devices through conformable derivative control. The exceptional stability under dual nonlocal nonlinearity confirms the robustness of Kudryashov's law for modeling realistic optical fiber systems.

Few limitations still exist in this work. First, while the derived analytical solutions satisfy the governing equation in an exact algebraic sense, they have not yet been validated independently by direct numerical integration schemes such as split-step Fourier or finite-difference methods, so future studies should consider numerical simulations to study the dynamical evolution and stability of these solutions under initial-value problems. Second, the robustness of the obtained solutions against perturbations has not been systematically investigated: Although graphical profiles indicate localized and stable behavior, there is no rigorous perturbation analysis (i.e., linear stability analysis or evaluation of Lyapunov exponents) of the obtained solutions, and the response of the solutions to small-amplitude perturbations remains an open problem. Finally, all the current findings from this analysis are limited to a (1 + 1)-dimensional deterministic model with homogeneous parameters, a single polarization, and some specific algebraic constraints, and do not consider transverse effects, stochastic noise, birefringence, and higher-order nonlinear effects (including Raman scattering and self-steepening), or generalizability to more realistic, more general fiber systems.

This investigation establishes IMSSEM as a powerful tool for solving complex nonlinear optical equations and provides a foundation for developing next-generation optical communication and signal processing systems. The conformable derivative framework offers new design flexibility for adaptive optical devices with tunable soliton properties, advancing both fundamental understanding and practical applications in nonlinear optics.

Author contributions

Data curation, Salim S. Mahmood; Formal analysis, Abeer S. Khalifa and Karim K. Ahmed; Investigation, Abeer S. Khalifa; Methodology, Salim S. Mahmood and Taha Radwan; Project administration, Muhammad Murad and Abeer S. Khalifa; Resources, Muhammad Murad; Software, Taha Radwan and Karim K. Ahmed; Supervision, Muhammad Murad; Validation, Taha Radwan;

Writing–original draft, Salim S. Mahmood and Karim K. Ahmed; Writing–review & editing, Salim S. Mahmood and Karim K. Ahmed.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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