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Research article

Matrix representations of multiplicative Hom-structures and α^k-derivations on the real Lie algebra $\mathfrak{so}(3)$

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Abstract: In this paper, we present an algebraic description of the derivation classes and multiplicative Hom-structures of $\mathfrak{so}(3)$. We also study α^k-derivations on multiplicative Hom-structures. Using the operator-matrix correspondence, explicit matrix representations for derivations, D-derivations, and (F, D)-derivations are derived by solving linear systems. We show that all nontrivial (F, D)-derivations comprise scalar and skew-symmetric components. Three different multiplicative Hom-structures are explicitly parameterized, and, on this basis, we discuss α^k-derivations, obtaining the classification of α^k-derivations on the corresponding multiplicative Hom-Lie algebra $\mathfrak{so}(3)$ in each case. Moreover, we show how each multiplicative Hom-structure gives rise to a Yau-twisted Hom-Lie bracket, and our classification of α^k-derivations corresponds to derivations of these twisted algebras. These results combine the algebraic and computational aspects of $\mathfrak{so}(3)$, providing useful tools for applications in robotic kinematics and quantum symmetry breaking.

Keywords: Lie algebra $\mathfrak{so}(3)$; derivations; multiplicative Hom-structures; matrix representation
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1. Introduction

Lie algebras, introduced by Sophus Lie in the late 19th century, have become indispensable tools across mathematics and physics. Of particular importance is the Lie algebra $\mathfrak{so}(3)$, which is intrinsically linked to the rotation group SO(3). Its structure underpins three-dimensional rotational symmetry and, thus, it has found critical applications in robotics [1], quantum mechanics, and geometric control systems. By linearizing complex rotational dynamics into algebraic frameworks, $\mathfrak{so}(3)$ provides a unified language for analyzing rigid body motions, state estimation, and discrete integration. Since

Bresar's foundational work [2], derivations of Lie algebras have been generalized and extended to associative algebras [3–5], ternary Jordan superalgebras [6], and prime algebras [7]. These generalized derivations reveal connections to algebraic invariants, and generalized derivations of noncentral Lie ideals [8] and octonion algebras [9] demonstrate their structural flexibility. Derivations are considered essential tools for derivation hierarchies.

Remarks on Hom-type algebras have been made based on quantum deformation theory [10]. According to Hartwig et al. [11], Hom-Lie algebras have revolutionized q-deformations of Witt and Virasoro algebras. It is then an interesting question about Hom-Lie algebras: What are all the Hom-structures of a given Lie algebra? A simpler question is, what are the Hom-Lie algebra structures on a simple Lie algebra? Research into this question has been underway since 2008. In the paper [12], it was claimed that those Hom-Lie algebra structures on a simple Lie algebra are all trivial, i.e., they are all scalar transformations. In 2015, the authors in [13] proved that there is some exception also for simple Lie algebras, for example, $\mathfrak{sl}(2,\mathbb{C})$, whose Hom-algebra structure is nontrivial. That, except for $\mathfrak{sl}(2,\mathbb{C})$, all Hom-algebra structures on simple Lie algebras are trivial as well was established in 2018 by Makhlouf and Zusmanovich [14] using generalized derivations and homological techniques. However, all these results are for algebraically closed fields of characteristic zero and do not discuss the real number field relevant to robotics. Computational characterization of Hom-structures on specific algebras, such as the real Lie algebra $\mathfrak{so}(3)$, has been neglected to date.

Derivations of Hom-Lie algebras have also attracted considerable attention. In [15], Sheng carried out a thorough investigation of Hom-Lie algebras, with particular emphasis on the fact that the distinguished α^k-derivations exist only when the Hom-algebra is multiplicative. Zhou et al. [16] investigated some fundamental properties of generalized derivations of Hom-Lie algebras and subsequently further studied problems including generalized derivations on Hom-Lie conformal algebras [17]. Cohomology [18], representations [15, 19, 20], and extensions [21] have been improved with the proposal of recent classification theorems for multiplicative Hom-Lie structures [22] and low-dimension cohomology [23]. The aim of this paper is to study the matrix representations of α^k-derivations of Hom-structures on the real Lie algebra $\mathfrak{so}(3)$, which plays an important role in the mechanism of rotation in robotics. Before that, it is necessary to classify the multiplicative Hom-structures on it. Additionally, matrix representations of derivations, D-derivations, and (F, D)-derivations are also studied.

The remainder of this paper is structured as follows. Section 2 summarizes the key concepts. Section 3 details the obtained derivation matrices and provides a classification of Hom-structures. Section 4 presents the classifications of α^k-derivations under different categories. Section 5 discusses Yau twist constructions that yield further Hom-Lie brackets from our classified multiplicative Hom-structures and shows how our results on α^k-derivations provide derivations of these twisted algebras.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R}$ is the field of real numbers. All definitions are taken from [15, 24].

Definition 2.1. Let $\mathfrak{g}$ be an $\mathbb{R}$-vector space and $[\cdot,\cdot]$ be a bilinear map (Lie bracket)

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \ (x, y) \mapsto [x, y]$$

satisfying the identities

(L1) $[x, x] = 0$ for all $x \in \mathfrak{g}$ (alternating property);

(L2) $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$ for all $x, y, z \in \mathfrak{g}$ (Jacobi identity).

Then, the pair $(\mathfrak{g}, [,])$ is called a Lie algebra over $\mathbb{R}$.

The alternating property implies anti-commutativity: $[x, y] = -[y, x]$.

Example 2.1. *The general linear real Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ is defined as the vector space of all $n \times n$ matrices with real entries, equipped with the Lie bracket given by the matrix commutator:*

$$\mathfrak{gl}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n}\}, \quad \textit{with Lie bracket} \quad [A, B] = AB - BA.$$

Definition 2.2. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over $\mathbb{R}$, where

- $\mathfrak{g}$ is a three-dimensional vector space with ordered basis $\mathcal{B} = \{e_1, e_2, e_3\}$;
- the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined on basis vectors by

$$[e_i, e_j] = \sum_{k=1}^{3} \epsilon_{ijk} e_k \quad \text{for } 1 \le i, j \le 3, \tag{2.1}$$

where ϵ_{ijk} is the Levi–Civita symbol

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if it is an odd permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, the bracket satisfies the explicit relations:

$$\begin{aligned} &[e_1, e_2] = +e_3, \quad [e_2, e_3] = +e_1, \quad [e_3, e_1] = +e_2, \\ &[e_j, e_i] = -[e_i, e_j] \quad \forall i, j. \end{aligned} \tag{2.2}$$

This algebra is isomorphic to the Lie algebra of the rotation group SO(3), and is denoted by $\mathfrak{so}(3)$.

According to the definitions of matrix representations and subalgebras in [24], it is evident that $\mathfrak{so}(3)$ is isomorphic to the Lie algebra of 3×3 real skew-symmetric matrices equipped with the matrix commutator, and hence it can also be regarded as a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. A Lie algebra homomorphism of $\mathfrak{g}$ is an $\mathbb{R}$-linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \forall x, y \in \mathfrak{g}. \tag{2.3}$$

Definition 2.4. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over $\mathbb{R}$. A Lie derivation of $\mathfrak{g}$ is an $\mathbb{R}$-linear map $D \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ satisfying the Lie Leibniz rule

$$D([x, y]) = [D(x), y] + [x, D(y)] \quad \forall x, y \in \mathfrak{g}. \tag{2.4}$$

The space of Lie derivations is denoted as $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$.

Definition 2.5. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over $\mathbb{R}$, and let $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ denote the space of $\mathbb{R}$-linear maps on $\mathfrak{g}$.

(i) A map $F \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is called a D-derivation if there exists a Lie derivation $D \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$ such that

$$F([x,y]) = [F(x),y] + [x,D(y)] \quad \forall x,y \in \mathfrak{g}. \tag{2.5}$$

(ii) A map $E \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is called an (F,D)-derivation if there exists

- a D-derivation F as defined in (i);
- a Lie derivation $D \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$

satisfying the compatibility condition

$$E([x,y]) = [F(x),y] + [x,D(y)] \quad \forall x,y \in \mathfrak{g}. \tag{2.6}$$

The space of D-derivations is denoted as $\mathrm{Der}_D(\mathfrak{g})$, and (F,D)-derivations form a subspace $\mathrm{Der}_{(F,D)}(\mathfrak{g}) \subseteq \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$.

Proposition 2.2. *The spaces* $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{so}(3))$*,* $\mathrm{Der}_D(\mathfrak{so}(3))$ *and* $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$ *are Lie subalgebras of* $\mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$.

Proof. We only provide the proof that $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$ is a Lie subalgebra of $\mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$, as the other proofs are similar.

Clearly, $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$ is a linear subspace of $\mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$. Next, we prove that $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$ is closed under the Lie bracket product $[\cdot,\cdot]$. For $\forall x,y \in \mathfrak{so}(3)$ and $\forall E_1,E_2 \in \mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$, we know that $\exists D_1,D_2 \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{so}(3))$ and $F_1,F_2 \in \mathrm{Der}_D(\mathfrak{so}(3))$ such that

$$E_1([x,y]) = [F_1(x),y] + [x,D_1(y)],$$

$$E_2([x,y]) = [F_2(x),y] + [x,D_2(y)].$$

At this time, we have

$$\begin{aligned}
[E_1,E_2]([x,y]) &= (E_1 \circ E_2 - E_2 \circ E_1)([x,y]) \\
&= E_1 \circ E_2([x,y]) - E_2 \circ E_1([x,y]) \\
&= E_1([F_2(x),y] + [x,D_2(y)]) - E_2([F_1(x),y] + [x,D_1(y)]) \\
&= [F_1 \circ F_2(x),y] + [F_2(x),D_1(y)] + [F_1(x),D_2(y)] + [x,D_1 \circ D_2(y)] \\
&\quad - [F_2 \circ F_1(x),y] - [F_1(x),D_2(y)] - [F_2(x),D_1(y)] - [x,D_2 \circ D_1(y)] \\
&= [(F_1 \circ F_2 - F_2 \circ F_1)(x),y] + [x,(D_1 \circ D_2 - D_2 \circ D_1)(y)] \\
&= [[F_1,F_2](x),y] + [x,[D_1,D_2](y)].
\end{aligned}$$

As x and y were arbitrarily chosen, we know that $[E_1,E_2] \in \mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$; that is, $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$ forms a Lie subalgebra. □

Proposition 2.3. *Let* $\mathfrak{so}(3)$ *be the Lie algebra defined over* $\mathbb{R}$ *with ordered basis* $\mathcal{B} = \{e_1,e_2,e_3\}$.

(i) For any derivation $D \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{so}(3))$, its matrix representation $[D]_{\mathcal{B}} = (a_{ji}) \in \mathfrak{gl}(3,\mathbb{R})$ relative to $\mathcal{B}$ is uniquely determined by

$$D(e_i) = \sum_{j=1}^{3} a_{ji} e_j \quad \textit{for } i = 1,2,3. \tag{2.7}$$

(ii) For any D-derivation $F \in \mathrm{Der}_D(\mathfrak{so}(3))$, its matrix representation $[F]_{\mathcal{B}} = (b_{ji}) \in \mathfrak{gl}(3,\mathbb{R})$ relative to $\mathcal{B}$ satisfies

$$F(e_i) = \sum_{j=1}^{3} b_{ji} e_j \quad \textit{for } i = 1,2,3. \tag{2.8}$$

(iii) For any (F,D)-derivation $E \in \mathrm{Der}_{(F,D)}(\mathfrak{so}(3))$, its matrix representation $[E]_{\mathcal{B}} = (d_{ji}) \in \mathfrak{gl}(3,\mathbb{R})$ relative to $\mathcal{B}$ is given by

$$E(e_i) = \sum_{j=1}^{3} d_{ji} e_j \quad \textit{for } i = 1,2,3. \tag{2.9}$$

Here, a_{ji}, b_{ji}, and d_{ji} denote the (j,i)-entries of the matrices $[D]_{\mathcal{B}}$, $[F]_{\mathcal{B}}$, and $[E]_{\mathcal{B}}$, respectively.

Definition 2.6. A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, where

- $\mathfrak{g}$ is an $\mathbb{R}$-vector space;
- $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map

$$[x,y] = -[y,x] \quad \forall x,y \in \mathfrak{g}; \tag{2.10}$$

- $\alpha \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ satisfies the Hom–Jacobi identity

$$[\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]] = 0 \quad \forall x,y,z \in \mathfrak{g}. \tag{2.11}$$

If α additionally preserves the bracket

$$\alpha([x,y]) = [\alpha(x),\alpha(y)] \quad \forall x,y \in \mathfrak{g}, \tag{2.12}$$

then the Hom-Lie algebra is called multiplicative.

Proposition 2.4. *Let $\mathfrak{so}(3)$ be the Lie algebra with ordered basis $\mathcal{B} = \{e_1,e_2,e_3\}$. For any Hom-structure $\alpha \in \mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$, its matrix representation $[\alpha]_{\mathcal{B}} = (c_{ij}) \in \mathfrak{gl}(3,\mathbb{R})$ relative to $\mathcal{B}$ satisfies*

$$\alpha(e_j) = \sum_{i=1}^{3} c_{ij} e_i \quad \textit{for } j = 1,2,3 \tag{2.13}$$

or, equivalently, in matrix form:

$$\alpha(e_1\ e_2\ e_3) = (\alpha(e_1)\ \alpha(e_2)\ \alpha(e_3)) = (e_1\ e_2\ e_3)\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}. \tag{2.14}$$

Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie algebra. For any non-negative integer k, denote by α^k the k-times composition of α; that is,

$$\alpha^k = \alpha \circ \overset{k \text{ times}}{\cdots} \circ \alpha.$$

In particular, $\alpha^0 = Id$ and $\alpha^1 = \alpha$.

Definition 2.7. For any nonnegative integer k, a linear map $\tau : \mathfrak{g} \to \mathfrak{g}$ is called an α^k-derivation of the multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ if

$$[\tau, \alpha] = 0, \text{ i.e., } \tau \circ \alpha = \alpha \circ \tau$$

and

$$\tau[u, v]_{\mathfrak{g}} = [\tau(u), \alpha^k(v)]_{\mathfrak{g}} + [\alpha^k(u), \tau(v)]_{\mathfrak{g}}, \ \forall u, v \in \mathfrak{g}.$$

3. Derivations and multiplicative Hom-structures on $\mathfrak{so}(3)$

This section presents the algebra structures of derivations, D-derivations, and (F, D)-derivations on $\mathfrak{so}(3)$ by establishing matrix representations of these derivations and solving linear systems derived from the bracket structure provided in Definition 2.2.

Let $\mathcal{B} = \{e_1, e_2, e_3\}$ be the given ordered basis of $\mathfrak{so}(3)$. For any linear map $T \in \mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$ with matrix $[T]_{\mathcal{B}} = (t_{ij})$ with respect to $\mathcal{B}$, the matrix representations of these derivations can be obtained through the following procedure:

(1) Apply T to the Lie bracket relations $[e_i, e_j] = \sum_{k=1}^{3} \epsilon_{ijk} e_k$;

(2) Expand both sides using the Leibniz rule (Definition 2.4);

(3) Compare coefficients to generate linear constraints on t_{ij};

(4) Solve the resulting system to characterize $[T]_{\mathcal{B}}$.

As the proofs for derivations and D-derivations follow analogous steps, only the (F, D)-derivation case is detailed in this paper.

Theorem 3.1. *The derivation of the Lie algebra* $\mathfrak{so}(3)$ *is isomorphic to itself over the field of real numbers* $\mathbb{R}$*; that is,* $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{so}(3)) \cong \mathfrak{so}(3)$.

Theorem 3.2. *The algebraic structure of the D-derivation of the Lie algebra* $\mathfrak{so}(3)$ *over the field of real numbers* $\mathbb{R}$ *is* $\mathrm{Der}_D(\mathfrak{so}(3)) \cong \mathfrak{so}(3) \oplus \mathbb{R}$.

Theorem 3.3. *The algebraic structure of the* (F, D)*-derivation algebra of the Lie algebra* $\mathfrak{so}(3)$ *over the field of real numbers* $\mathbb{R}$ *is* $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3)) \cong \mathfrak{so}(3) \oplus \mathbb{R}$.

Proof. Let E be an (F, D)-derivation with $[E]_{\mathcal{B}} = (d_{ij})$. Applying Definition 2.5(ii) to the basis brackets, we have

$$E([e_1, e_2]) = [F(e_1), e_2] + [e_1, D(e_2)], \tag{3.1}$$

$$E([e_2, e_3]) = [F(e_2), e_3] + [e_2, D(e_3)], \tag{3.2}$$

$$E([e_3, e_1]) = [F(e_3), e_1] + [e_3, D(e_1)]. \tag{3.3}$$

Substituting $D(e_i) = \sum_j a_{ji}e_j$, $F(e_i) = \sum_j b_{ji}e_j$, and $E(e_i) = \sum_j d_{ji}e_j$ into (3.1)–(3.3) and matching coefficients yields

$$\begin{cases} d_{11} = b_{11}, & d_{12} = a_{12}, \quad d_{13} = a_{13}, \\ d_{21} = -a_{12}, & d_{22} = b_{11}, \quad d_{23} = a_{23}, \\ d_{31} = -a_{13}, & d_{32} = -a_{23}, \quad d_{33} = b_{11}. \end{cases} \tag{3.4}$$

Then,

$$[E]_{\mathcal{B}} = \lambda I + A = \begin{pmatrix} \lambda & -a_{12} & -a_{13} \\ a_{12} & \lambda & -a_{23} \\ a_{13} & a_{23} & \lambda \end{pmatrix}, \quad \lambda, a_{ij} \in \mathbb{R}, \tag{3.5}$$

where $\lambda = b_{11}$ and A is skew-symmetric.

From the obtained matrix representation, up to isomorphism, the algebraic structure of the (F, D)-derivation algebra of the Lie algebra $\mathfrak{so}(3)$ over the field of real numbers $\mathbb{R}$ is $\mathrm{Der}_{(F,D)}(\mathfrak{so}(3)) \cong \mathfrak{so}(3) \oplus \mathbb{R}$. □

Remark 3.4. *The results in Theorems 3.1–3.3 align with the well-known fact that for a semisimple Lie algebra over $\mathbb{R}$ (like $\mathfrak{so}(3)$), all derivations are inner. In particular,* $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{so}(3)) \cong \mathfrak{so}(3)$ *confirms that every Lie derivation is inner, and the direct sum with $\mathbb{R}$ in the generalized cases corresponds to the inclusion of scalar multiples of the identity map.*

Corollary 3.5. *All (F, D)-derivations of $\mathfrak{so}(3)$ are either scalar multiples of the identity or inherit skew-symmetric perturbations. There are no nontrivial (F, D)-derivations that are not of this form under $\mathcal{B}$.*

Theorem 3.6. *Every Hom-structure α on $\mathfrak{so}(3)$ with respect to the given basis $\mathcal{B} = \{e_1, e_2, e_3\}$ has a symmetric matrix representation; that is, $[\alpha]_{\mathcal{B}} = [\alpha]_{\mathcal{B}}^{\top}$.*

Proof. Let $\alpha \in \mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$ satisfy the Hom-Jacobi identity. Expanding

$$[\alpha(e_1), [e_2, e_3]] + [\alpha(e_2), [e_3, e_1]] + [\alpha(e_3), [e_1, e_2]] = 0$$

and using $\alpha(e_i) = \sum_{j=1}^{3} c_{ji}e_j$, we obtain:

$$(c_{23} - c_{32})e_1 + (c_{31} - c_{13})e_2 + (c_{12} - c_{21})e_3 = 0.$$

Then, $c_{ij} = c_{ji}$ follows from the linear independence of $\mathcal{B}$. Therefore, we obtain that $[\alpha]_{\mathcal{B}}$ is symmetric. □

Theorem 3.7. *Up to isomorphism, the multiplicative Hom-structures on $\mathfrak{so}(3)$ are partitioned into three distinct equivalence classes of nontrivial matrix representations with respect to any basis, which are as follows:*

$$I, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ such that $c \in [-1, 1], a, b \neq 0, a^2 + b^2 \leqslant 1$, and $\Gamma = \frac{ab + \sqrt{a^2b^2(1-a^2-b^2)}}{a^2+b^2}$.

Proof. According to Definition 2.6, the multiplicative Hom-structure α satisfies

$$\alpha([e_1, e_2]) = [\alpha(e_1), \alpha(e_2)], \tag{3.6}$$
$$\alpha([e_2, e_3]) = [\alpha(e_2), \alpha(e_3)], \tag{3.7}$$
$$\alpha([e_3, e_1]) = [\alpha(e_3), \alpha(e_1)], \tag{3.8}$$

where $\alpha(e_1)$, $\alpha(e_2)$, and $\alpha(e_3)$ are as defined in Proposition 2.4. Comparing the coefficients on both sides of Eqs (3.6)–(3.8), we obtain that

$$c_{22}c_{33} - c_{23}^2 = c_{11}, \tag{3.9}$$
$$c_{11}c_{33} - c_{13}^2 = c_{22}, \tag{3.10}$$
$$c_{11}c_{22} - c_{12}^2 = c_{33}, \tag{3.11}$$
$$c_{12}(c_{33} + 1) = c_{13}c_{23}, \tag{3.12}$$
$$c_{13}(c_{22} + 1) = c_{12}c_{23}, \tag{3.13}$$
$$c_{23}(c_{11} + 1) = c_{12}c_{13}. \tag{3.14}$$

Case I: $c_{13} = 0$.

It follows from Eq (3.12) that $c_{12} = 0$ or $c_{33} = -1$.

(a) If $c_{12} = 0$, then it follows from Eqs (3.10), (3.11), and (3.14) that

$$c_{22} = c_{11}c_{33}, \tag{3.15}$$
$$c_{33} = c_{11}c_{22}, \tag{3.16}$$
$$c_{23} = -c_{11}c_{23}. \tag{3.17}$$

According to Eqs (3.15) and (3.16), we get $c_{22} = c_{11}^2 c_{22}$. Thus, there are two cases:

(a1) When $c_{22} = 0$, we have $c_{11}c_{33} = 0$ and $c_{33} = 0$. By Eq (3.9), $c_{11} = -c_{23}^2$. Then, Eq (3.17) implies $c_{23} = 0$ or $c_{23} = \pm 1$.
When $c_{23} = 0$, we have $P_1 = 0$.
When $c_{23} = 1$, then $c_{11} = -1$, denoted by

$$P_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

When $c_{23} = -1$, then $c_{11} = -1$, denoted by

$$P_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

(a2) When $c_{22} \neq 0$, we have $c_{11}^2 = 1$. If $c_{11} = 1$, then $c_{23} = 0$ and the matrices are as follows:

$$P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If $c_{11} = -1$, then $c_{22} = -c_{33}$. Let $c = c_{23} \in [-1, 1]$, and we get

$$P_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \quad P_7 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sqrt{1-c^2} & c \\ 0 & c & -\sqrt{1-c^2} \end{pmatrix}.$$

(b) If $c_{33} = -1$, then it follows from (3.9)–(3.14) that

$$c_{23} = 0, \quad c_{11} = \pm\sqrt{1 - c_{12}^2}.$$

Letting $c = c_{12} \in [-1, 1]$, the matrices are obtained as follows:

$$P_8 = \begin{pmatrix} \sqrt{1-c^2} & c & 0 \\ c & -\sqrt{1-c^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_9 = \begin{pmatrix} -\sqrt{1-c^2} & c & 0 \\ c & \sqrt{1-c^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Case II: $c_{13} \neq 0$.

(a) If $c_{23} = 0$, then Eq (3.9) yields

$$c_{11} = c_{22}c_{33}. \tag{3.18}$$

According to Eqs (3.12)–(3.14), we have

$$c_{12}(c_{33} + 1) = 0, \tag{3.19}$$

$$c_{13}(c_{22} + 1) = 0, \tag{3.20}$$

$$c_{12}c_{13} = 0. \tag{3.21}$$

As $c_{13} \neq 0$, we obtain that $c_{22} = -1$ and $c_{12} = 0$. Then, Eq (3.18) implies $c_{11} = -c_{33}$. It follows from Eq (3.10) that $c_{33} = \pm\sqrt{1 - c_{13}^2}$. Letting $c = c_{13} \in [-1, 0) \cup (0, 1]$, we obtain the matrices

$$P_{10} = \begin{pmatrix} -\sqrt{1-c^2} & 0 & c \\ 0 & -1 & 0 \\ c & 0 & \sqrt{1-c^2} \end{pmatrix},$$

$$P_{11} = \begin{pmatrix} \sqrt{1-c^2} & 0 & c \\ 0 & -1 & 0 \\ c & 0 & -\sqrt{1-c^2} \end{pmatrix}.$$

(b) If $c_{23} \neq 0$, according to $c_{13} \neq 0$ and Eq (3.12), then we get $c_{12} \neq 0$. Equations (3.12)–(3.14) imply

$$c_{33} = \frac{c_{13}c_{23}}{c_{12}} - 1, \tag{3.22}$$

$$c_{22} = \frac{c_{12}c_{23}}{c_{13}} - 1, \tag{3.23}$$

$$c_{11} = \frac{c_{12}c_{13}}{c_{23}} - 1. \tag{3.24}$$

Substituting these into Eq (3.9), we can check that $c_{23} = \frac{c_{12}c_{13}\pm\Delta}{c_{12}^2+c_{13}^2}$, where

$$\Delta = \sqrt{c_{12}^2c_{13}^2 - c_{12}^4c_{13}^2 - c_{12}^2c_{13}^4}.$$

As c_{12}, c_{13}, and c_{23} are all nonzero, we obtain the matrices

$$P_{12} = \begin{pmatrix} \frac{c_{12}c_{13}({c_{12}}^2+{c_{13}}^2)}{c_{12}c_{13}+\Delta} - 1 & c_{12} & c_{13} \\ c_{12} & \frac{c_{12}(c_{12}c_{13}+\Delta)}{c_{13}({c_{12}}^2+{c_{13}}^2)} - 1 & \frac{c_{12}c_{13}+\Delta}{{c_{12}}^2+{c_{13}}^2} \\ c_{13} & \frac{c_{12}c_{13}+\Delta}{{c_{12}}^2+{c_{13}}^2} & \frac{c_{13}(c_{12}c_{13}+\Delta)}{c_{12}({c_{12}}^2+{c_{13}}^2)} - 1 \end{pmatrix},$$
$$P_{13} = \begin{pmatrix} \frac{c_{12}c_{13}({c_{12}}^2+{c_{13}}^2)}{c_{12}c_{13}-\Delta} - 1 & c_{12} & c_{13} \\ c_{12} & \frac{c_{12}(c_{12}c_{13}-\Delta)}{c_{13}({c_{12}}^2+{c_{13}}^2)} - 1 & \frac{c_{12}c_{13}-\Delta}{{c_{12}}^2+{c_{13}}^2} \\ c_{13} & \frac{c_{12}c_{13}-\Delta}{{c_{12}}^2+{c_{13}}^2} & \frac{c_{13}(c_{12}c_{13}-\Delta)}{c_{12}({c_{12}}^2+{c_{13}}^2)} - 1 \end{pmatrix},$$

where $\Delta = \sqrt{c_{12}^2c_{13}^2(1 - c_{12}^2 - c_{13}^2)}$.

Next, we let $c_{12} = a, c_{13} = b$, and $\Gamma = \frac{ab+\Delta}{a^2+b^2}$. By calculating the eigenvalues of the matrix, the above-obtained matrices can be classified. In summary, up to isomorphism, the nontrivial multiplicative Hom-structures on $\mathfrak{so}(3)$ in any set of bases can be divided into the following three categories:

$$I, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ such that $c \in [-1, 1], a, b \neq 0, a^2 + b^2 \leqslant 1$, and $\Gamma = \frac{ab+\sqrt{a^2b^2(1-a^2-b^2)}}{a^2+b^2}$. □

4. α^k-derivations of multiplicative Hom-Lie algebra $\mathfrak{so}(3)$

In this section, we classify α^k-derivations on the multiplicative Hom-Lie algebra $\mathfrak{so}(3)$.

Based on conclusions derived from Theorem 3.7, we discuss the three multiplicative Hom-structures on $\mathfrak{so}(3)$ in sequence and determine the corresponding classification of α^k-derivations in each case. In this section, we denote by $\mathcal{B} = \{e_1, e_2, e_3\}$, a basis of the Lie algebra $\mathfrak{so}(3)$, and by α, a multiplicative Hom-structure on it. The matrix corresponding to the linear map α^k-derivation under the basis $\mathcal{B}$ is denoted by $[\tau] = (h_{ji}^{(k)})$, where $i, j = 1, 2, 3$ and $h_{ji}^{(k)} \in \mathbb{R}$. In particular, $\alpha^0 = Id$ and $\alpha^1 = \alpha$.

Theorem 4.1. *If the matrix corresponding to the multiplicative Hom-structure on the Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is the identity matrix I, then the matrix corresponding to the α^k-derivation on it under the basis $\mathcal{B}$ is an antisymmetric matrix.*

Proof. Suppose the matrix corresponding to the multiplicative Hom-structure on the Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is the identity matrix I. In this case, for all $u, v \in \mathfrak{so}(3)$, the expressions in Definition 2.7 become

$$[\tau, \mathrm{Id}] = 0,$$

$$\tau([u,v]) = [\tau(u), v] + [u, \tau(v)].$$

Obviously, this is the same as the definition of a derivation.

Therefore, for all $k \in \mathbb{R}$, the matrix corresponding to the α^k-derivation on the Lie algebra $\mathfrak{so}(3)$ in this case under the basis $\mathcal{B}$ is an antisymmetric matrix. □

Theorem 4.2. *If the matrix corresponding to the multiplicative Hom-structure on the Lie algebra* $\mathfrak{so}(3)$ *under the basis* $\mathcal{B}$ *is*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \text{ where } c \in [-1,1],$$

then

(i) The matrix corresponding to the α^0*-derivation on it under the basis* $\mathcal{B}$ *is as follows:*

(1) If $c = 0$*,*

$$\begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ for } m \in \mathbb{R};$$

(2) If $c \in [-1,1] \setminus \{0\}$*,*

$$\begin{pmatrix} 0 & m & \frac{\sqrt{1-c^2}-1}{c}m \\ -m & 0 & 0 \\ -\frac{\sqrt{1-c^2}-1}{c}m & 0 & 0 \end{pmatrix}, \text{ for } m \in \mathbb{R}.$$

(ii) The matrix corresponding to the α^1*-derivation on it under the basis* $\mathcal{B}$ *is as follows:*

(1) If $c = \pm 1$*,*

$$\begin{pmatrix} 0 & m & -m \\ -m & 0 & 0 \\ m & 0 & 0 \end{pmatrix}, \text{ for } c = 1 \text{ and } m \in \mathbb{R},$$

$$\begin{pmatrix} 0 & m & m \\ m & 0 & 0 \\ m & 0 & 0 \end{pmatrix}, \text{ for } c = -1 \text{ and } m \in \mathbb{R};$$

(2) If $c \in [-1,1] \setminus \{\pm 1\}$ *(i.e.,* $c \neq \pm 1$*), it is the zero matrix.*

Proof. Suppose the matrix corresponding to the multiplicative Hom-structure on the Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \text{ where } c \in [-1,1];$$

that is, we have

$$\alpha(e_1) = -e_1,$$
$$\alpha(e_2) = -\sqrt{1-c^2}e_2 + ce_3,$$
$$\alpha(e_3) = ce_2 + \sqrt{1-c^2}e_3.$$

(1) α^0-derivations

In this case, for all $u, v \in \mathfrak{so}(3)$, the expressions in Definition 2.7 become

$$[\tau, \alpha] = 0, \tag{4.1}$$
$$\tau([u, v]) = [\tau(u), v] + [u, \tau(v)]. \tag{4.2}$$

From Theorem 3.1, the matrix corresponding to the map τ satisfying Eq (4.2) under the basis $\mathcal{B}$ is

$$\begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix}, \text{ where } h_{ij}^{(0)} \in \mathbb{R}.$$

Next, we consider $[\tau, \alpha] = 0$ and let

$$\begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}\begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix} = 0.$$

Thus, we obtain

$$h_{12}^{(0)} = \sqrt{1-c^2}h_{12}^{(0)} - ch_{13}^{(0)},$$
$$h_{13}^{(0)} = -\sqrt{1-c^2}h_{13}^{(0)} - ch_{13}^{(0)},$$
$$2ch_{23}^{(0)} = 0,$$
$$2\sqrt{1-c^2}h_{23}^{(0)} = 0.$$

Next, we discuss the value of c.

If $c = 0$, then we have $h_{13}^{(0)} = h_{23}^{(0)} = 0$. Letting $h_{12}^{(0)} = m$, we obtain the matrix

$$\begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{where } m \in \mathbb{R}.$$

As $c \neq 0$, it follows from $2ch_{23}^{(0)} = 0$ that $h_{23}^{(0)} = 0$. Meanwhile, rearranging the above equation gives $h_{13}^{(0)} = \frac{\sqrt{1-c^2}-1}{c}h_{12}^{(0)}$. In this case, we obtain the matrix

$$\begin{pmatrix} 0 & m & \frac{\sqrt{1-c^2}-1}{c}m \\ -m & 0 & 0 \\ -\frac{\sqrt{1-c^2}-1}{c}m & 0 & 0 \end{pmatrix}, \text{ where } m \in \mathbb{R}, c \in [-1, 1] \setminus \{0\}.$$

Therefore, in this case, the matrix corresponding to the α^0-derivation on the real Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is as follows:

- If $c = 0$,

$$\begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } m \in \mathbb{R}.$$

- If $c \in [-1, 1] \setminus \{0\}$,

$$\begin{pmatrix} 0 & m & \frac{\sqrt{1-c^2}-1}{c}m \\ -m & 0 & 0 \\ -\frac{\sqrt{1-c^2}-1}{c}m & 0 & 0 \end{pmatrix}, \text{ where } m \in \mathbb{R}.$$

(2) α^1-derivations

In this case, for all $u, v \in \mathfrak{so}(3)$, the expressions in Definition 2.7 become

$$[\tau, \alpha] = 0, \tag{4.3}$$
$$\tau([u, v]) = [\tau(u), \alpha(v)] + [\alpha(u), \tau(v)]. \tag{4.4}$$

We substitute the corresponding expressions for $\alpha(e_i)$ and $\tau(e_i)$ into Eq (4.4), then set the corresponding coefficients on both sides to be equal. In this way, we obtain the following system of equations:

$$h_{13}^{(1)} = ch_{21}^{(1)} + \sqrt{1-c^2}h_{31}^{(1)}, \tag{4.5}$$
$$h_{23}^{(1)} = h_{32}^{(1)} - ch_{11}^{(1)}, \tag{4.6}$$
$$h_{33}^{(1)} = -h_{22}^{(1)} - \sqrt{1-c^2}h_{11}^{(1)}, \tag{4.7}$$
$$h_{12}^{(1)} = ch_{31}^{(1)} - \sqrt{1-c^2}h_{21}^{(1)}, \tag{4.8}$$
$$h_{22}^{(1)} = \sqrt{1-c^2}h_{11}^{(1)} - h_{33}^{(1)}, \tag{4.9}$$
$$h_{32}^{(1)} = h_{23}^{(1)} - ch_{11}^{(1)}, \tag{4.10}$$
$$h_{11}^{(1)} = \sqrt{1-c^2}h_{22}^{(1)} - \sqrt{1-c^2}h_{33}^{(1)} - ch_{23}^{(1)} - ch_{32}^{(1)}, \tag{4.11}$$
$$h_{21}^{(1)} = ch_{13}^{(1)} - \sqrt{1-c^2}h_{12}^{(1)}, \tag{4.12}$$
$$h_{31}^{(1)} = ch_{12}^{(1)} + \sqrt{1-c^2}h_{13}^{(1)}. \tag{4.13}$$

Next, we discuss the value of c.

(1) If $c = 1$.

In this case, the above system of equations becomes

$$\begin{aligned} h_{13}^{(1)} &= h_{21}^{(1)}, \\ h_{23}^{(1)} &= h_{32}^{(1)} - ch_{11}^{(1)}, \\ h_{33}^{(1)} &= -h_{22}^{(1)}, \\ h_{12}^{(1)} &= h_{31}^{(1)}, \\ h_{32}^{(1)} &= h_{23}^{(1)} - h_{11}^{(1)}, \end{aligned}$$

$$h_{11}^{(1)} = -h_{23}^{(1)} - h_{32}^{(1)}.$$

Through calculation, we obtain $h_{11}^{(1)} = h_{23}^{(1)} = h_{32}^{(1)} = 0$, and the matrix is now

$$\begin{pmatrix} 0 & h_{12}^{(1)} & h_{13}^{(1)} \\ h_{13}^{(1)} & h_{22}^{(1)} & 0 \\ h_{12}^{(1)} & 0 & -h_{22}^{(1)} \end{pmatrix}, \text{ where } h_{22}^{(1)}, h_{12}^{(1)}, h_{13}^{(1)} \in \mathbb{R}.$$

Next, we consider $[\tau, \alpha] = 0$ and let

$$\begin{pmatrix} 0 & h_{12}^{(1)} & h_{13}^{(1)} \\ h_{13}^{(1)} & h_{22}^{(1)} & 0 \\ h_{12}^{(1)} & 0 & -h_{22}^{(1)} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & h_{12}^{(1)} & h_{13}^{(1)} \\ h_{13}^{(1)} & h_{22}^{(1)} & 0 \\ h_{12}^{(1)} & 0 & -h_{22}^{(1)} \end{pmatrix} = 0.$$

By calculation, we have $h_{12}^{(1)} + h_{13}^{(1)} = 0$ and $h_{22}^{(1)} = 0$; that is, $h_{13}^{(1)} = -h_{12}^{(1)}$.

In conclusion, when $c = 1$, the matrix corresponding to the α^1-derivation on the multiplicative Hom-Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ in this case is

$$\begin{pmatrix} 0 & m & -m \\ -m & 0 & 0 \\ m & 0 & 0 \end{pmatrix}, \text{ where } m \in \mathbb{R}.$$

It can similarly be shown that, when $c = -1$, the corresponding matrix is

$$\begin{pmatrix} 0 & m & m \\ m & 0 & 0 \\ m & 0 & 0 \end{pmatrix}, \text{ where } m \in \mathbb{R}.$$

(2) If $c \neq \pm 1$; that is, $c \in (-1, 1)$.

Substituting Eq (4.7) into Eq (4.9), we have $\sqrt{1-c^2}h_{11}^{(1)} = 0$. As $\sqrt{1-c^2} \neq 0$, we get $h_{11}^{(1)} = 0$. Further, from Eq (4.7), we obtain $h_{33}^{(1)} = -h_{22}^{(1)}$, and from Eq (4.6), we obtain $h_{23}^{(1)} = h_{32}^{(1)}$. Substituting the above results into Eq (4.11) gives $\sqrt{1-c^2}h_{22}^{(1)} - ch_{23}^{(1)} = 0$, and so $h_{22}^{(1)} = \frac{c}{\sqrt{1-c^2}}h_{23}^{(1)}$. Therefore, the matrix is obtained as

$$\begin{pmatrix} 0 & h_{12}^{(1)} & h_{13}^{(1)} \\ ch_{13}^{(1)} - \sqrt{1-c^2}h_{12}^{(1)} & \frac{c}{\sqrt{1-c^2}}h_{23}^{(1)} & h_{23}^{(1)} \\ ch_{12}^{(1)} + \sqrt{1-c^2}h_{13}^{(1)} & h_{23}^{(1)} & -\frac{c}{\sqrt{1-c^2}}h_{23}^{(1)} \end{pmatrix}, \text{ where } h_{12}^{(1)}, h_{13}^{(1)} \in \mathbb{R}.$$

Denote this matrix by H. Next, let

$$H\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}H = 0.$$

Rearranging gives

$$\sqrt{1-c^2}h_{12}^{(1)} - ch_{13}^{(1)} = -h_{12}^{(1)},$$

$$\sqrt{1-c^2}h_{12}^{(1)} - ch_{13}^{(1)} = h_{12}^{(1)},$$
$$-ch_{12}^{(1)} - \sqrt{1-c^2}h_{13}^{(1)} = -h_{13}^{(1)},$$
$$-ch_{12}^{(1)} - \sqrt{1-c^2}h_{13}^{(1)} = h_{13}^{(1)},$$
$$\frac{2}{\sqrt{1-c^2}}h_{23}^{(1)} = 0.$$

Obviously, we have $h_{12}^{(1)} = h_{13}^{(1)} = h_{23}^{(1)} = 0$.

In conclusion, when $c \neq \pm 1$, the matrix corresponding to the α^1-derivation on the multiplicative Hom-Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ in this case is the zero matrix. □

Theorem 4.3. *If the matrix corresponding to the multiplicative Hom-structure on the Lie algebra* $\mathfrak{so}(3)$ *under the basis* $\mathcal{B}$ *is*

$$\begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$ *such that* $a, b \neq 0$, $a^2 + b^2 \leqslant 1$, *and* $\Gamma = \frac{ab + \sqrt{a^2b^2(1-a^2-b^2)}}{a^2+b^2}$, *then the matrix corresponding to the* α^0*- and* α^1*-derivation on it under the basis* $\mathcal{B}$ *is the zero matrix.*

Proof. Suppose that the matrix corresponding to the multiplicative Hom-structure on the Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is

$$\begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$ with $a, b \neq 0$, $a^2 + b^2 \leqslant 1$, and $\Gamma = \frac{ab + \sqrt{a^2b^2(1-a^2-b^2)}}{a^2+b^2}$. That is, we have

$$\alpha(e_1) = \left(\frac{ab}{\Gamma} - 1\right)e_1 + ae_2 + be_3,$$
$$\alpha(e_2) = ae_1 + \left(\frac{a\Gamma}{b} - 1\right)e_2 + \Gamma e_3,$$
$$\alpha(e_3) = be_1 + \Gamma e_2 + \left(\frac{b\Gamma}{a} - 1\right)e_3.$$

(1) α^0-derivations

Similar to Theorem 4.2, we first obtain the matrix

$$\begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix}, \text{ where } h_{ij}^{(0)} \in \mathbb{R}.$$

Next, we consider $[\tau, \alpha] = 0$ and let

$$\begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix} \begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix}$$
$$- \begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix} \begin{pmatrix} 0 & h_{12}^{(0)} & h_{13}^{(0)} \\ -h_{12}^{(0)} & 0 & h_{23}^{(0)} \\ -h_{13}^{(0)} & -h_{23}^{(0)} & 0 \end{pmatrix} = 0.$$

Solving the above system of equations, we obtain $h_{12}^{(0)} = h_{13}^{(0)} = h_{23}^{(0)} = 0$. Therefore, in this case, the matrix corresponding to the α^0-derivation on the multiplicative Hom-Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is the zero matrix.

(2) α^1-derivations

By analogy with the proof of Theorem 4.2, we substitute $\alpha(e_i)$ and $\tau(e_i)$ into the corresponding equations and set the corresponding coefficients on both sides to be equal, obtaining the following system of equations:

$$h_{13}^{(1)} - \Gamma h_{21}^{(1)} + b h_{22}^{(1)} - h_{31}^{(1)} + \frac{a\Gamma h_{31}^{(1)}}{b} - a h_{32}^{(1)} = 0,$$
$$\Gamma h_{11}^{(1)} - b h_{12}^{(1)} + h_{23}^{(1)} - a h_{31}^{(1)} - h_{32}^{(1)} + \frac{ab h_{32}^{(1)}}{\Gamma} = 0,$$
$$h_{11}^{(1)} - \frac{a\Gamma h_{11}^{(1)}}{b} + a h_{12}^{(1)} + a h_{21}^{(1)} + h_{22}^{(1)} - \frac{ab h_{22}^{(1)}}{\Gamma} + h_{33}^{(1)} = 0,$$
$$-h_{12}^{(1)} + h_{21}^{(1)} - \frac{b\Gamma h_{21}^{(1)}}{a} + b h_{23}^{(1)} + \Gamma h_{31}^{(1)} - a h_{33}^{(1)} = 0,$$
$$-h_{11}^{(1)} + \frac{b\Gamma h_{11}^{(1)}}{a} - b h_{13}^{(1)} - h_{22}^{(1)} - b h_{31}^{(1)} - h_{33}^{(1)} + \frac{ab h_{33}^{(1)}}{\Gamma} = 0,$$
$$-\Gamma h_{11}^{(1)} + a h_{13}^{(1)} + b h_{21}^{(1)} + h_{23}^{(1)} - \frac{ab h_{23}^{(1)}}{\Gamma} - h_{32}^{(1)} = 0,$$
$$h_{11}^{(1)} + h_{22}^{(1)} - \frac{b\Gamma h_{22}^{(1)}}{a} + \Gamma h_{23}^{(1)} + \Gamma h_{32}^{(1)} + h_{33}^{(1)} - \frac{a\Gamma h_{33}^{(1)}}{b} = 0,$$
$$-h_{12}^{(1)} + \frac{b\Gamma h_{12}^{(1)}}{a} - \Gamma h_{13}^{(1)} + h_{21}^{(1)} - b h_{32}^{(1)} + a h_{33}^{(1)} = 0,$$
$$-\Gamma h_{12}^{(1)} - h_{13}^{(1)} + \frac{a\Gamma h_{13}^{(1)}}{b} + b h_{22}^{(1)} - a h_{23}^{(1)} + h_{31}^{(1)} = 0.$$

By elimination, the matrix can be solved as

$$\begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)*} \\ h_{12}^{(1)} & -\frac{\Gamma^2}{b^2} h_{11}^{(1)} + \frac{2\Gamma}{b} h_{12}^{(1)} & h_{23}^{(1)*} \\ h_{13}^{(1)*} & h_{23}^{(1)*} & -\frac{(b^2 - \Gamma^2)}{b^2} h_{11}^{(1)} - \frac{2\Gamma}{b} h_{12}^{(1)} \end{pmatrix},$$

where

$$h_{13}^{(1)*} := -\frac{a^2b^2 - a^2\Gamma^2 - b^2\Gamma^2}{2ab^2\Gamma}h_{11}^{(1)} - \frac{a}{b}h_{12}^{(1)},$$
$$h_{23}^{(1)*} := -\frac{a^2b^2 - a^2\Gamma^2 + b^2\Gamma^2}{2ab^3}h_{11}^{(1)} - \frac{(a^2 - b^2)\Gamma}{ab^2}h_{12}^{(1)}.$$

Next, we consider $[\tau, \alpha] = 0$ and let

$$\begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)*} \\ h_{12}^{(1)} & -\frac{\Gamma^2}{b^2}h_{11}^{(1)} + \frac{2\Gamma}{b}h_{12}^{(1)} & h_{23}^{(1)*} \\ h_{13}^{(1)*} & h_{23}^{(1)*} & -\frac{(b^2-\Gamma^2)}{b^2}h_{11}^{(1)} - \frac{2\Gamma}{b}h_{12}^{(1)} \end{pmatrix}\begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix}$$
$$-\begin{pmatrix} \frac{ab}{\Gamma} - 1 & a & b \\ a & \frac{a\Gamma}{b} - 1 & \Gamma \\ b & \Gamma & \frac{b\Gamma}{a} - 1 \end{pmatrix}\begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & h_{13}^{(1)*} \\ h_{12}^{(1)} & -\frac{\Gamma^2}{b^2}h_{11}^{(1)} + \frac{2\Gamma}{b}h_{12}^{(1)} & h_{23}^{(1)*} \\ h_{13}^{(1)*} & h_{23}^{(1)*} & -\frac{(b^2-\Gamma^2)}{b^2}h_{11}^{(1)} - \frac{2\Gamma}{b}h_{12}^{(1)} \end{pmatrix} = 0.$$

Finally, it can be calculated that $h_{11}^{(1)} = h_{12}^{(1)} = 0$. Therefore, in this case, the matrix corresponding to the α^1-derivation on the multiplicative Hom-Lie algebra $\mathfrak{so}(3)$ under the basis $\mathcal{B}$ is the zero matrix. □

Remark 4.4. *Given that the Hom-structure α in Theorem 4.3 is a real symmetric matrix, it is orthogonally diagonalizable. Hence, α^k is similar to a diagonal matrix with eigenvalues raised to the power k. Based on the calculations for α^0 and α^1-derivations, where only the zero derivation appears, it is plausible that for any nonnegative integer k, the only α^k-derivation on this multiplicative Hom-Lie algebra is the zero map. This conjecture aligns with the observed pattern and can be further verified by extending the linear system approach to general k, though the computations become more tedious.*

5. Yau twist constructions and further Hom-Lie brackets

The Yau twist is a fundamental method for constructing Hom-Lie algebras from classical Lie algebras. Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and a Lie algebra homomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ (i.e., $\alpha([x, y]) = [\alpha(x), \alpha(y)]$), one defines a new skew-symmetric bracket

$$\{x, y\} := [\alpha(x), \alpha(y)] \quad \forall x, y \in \mathfrak{g}, \tag{5.1}$$

which together with the linear map α yields a multiplicative Hom-Lie algebra $(\mathfrak{g}, \{\cdot,\cdot\}, \alpha)$. This construction, often attributed to Yau [25] and further developed in [11, 26], provides a systematic way to generate Hom-Lie algebras from classical ones.

The multiplicative Hom-structures classified in Theorem 3.7 provide natural candidates for such twists on $\mathfrak{so}(3)$. For each nontrivial α from our classification, the Yau twist produces a Hom-Lie algebra $(\mathfrak{so}(3), \{\cdot,\cdot\}_\alpha, \alpha)$, where

$$\{x, y\}_\alpha = [\alpha(x), \alpha(y)] \quad \forall x, y \in \mathfrak{so}(3). \tag{5.2}$$

For the three types of multiplicative Hom-structures identified in Theorem 3.7, we obtain corresponding families of Hom-Lie brackets:

(1) For $\alpha = I$ (the identity), the twisted bracket coincides with the original Lie bracket: $\{x, y\}_I = [x, y]$.

(2) For the one-parameter family

$$\alpha_c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-c^2} & c \\ 0 & c & \sqrt{1-c^2} \end{pmatrix}, \quad c \in [-1, 1],$$

the twisted bracket $\{\cdot, \cdot\}_{\alpha_c}$ is given explicitly on basis elements by

$$\begin{aligned} \{e_1, e_2\}_{\alpha_c} &= \alpha_c(e_3) = ce_2 + \sqrt{1-c^2}e_3, \\ \{e_2, e_3\}_{\alpha_c} &= \alpha_c(e_1) = -e_1, \\ \{e_3, e_1\}_{\alpha_c} &= \alpha_c(e_2) = -\sqrt{1-c^2}e_2 + ce_3. \end{aligned}$$

(3) For the two-parameter family $\alpha_{a,b}$ with parameters a, b, Γ as in Theorem 3.7, the twisted bracket can be computed similarly, though the expressions are more involved.

The α^k-derivations studied in Section 4 correspond precisely to derivations of these twisted Hom-Lie algebras. In fact, an α^k-derivation τ of $(\mathfrak{so}(3), [\cdot, \cdot], \alpha)$ satisfies

$$\tau(\{x, y\}_\alpha) = \{\tau(x), \alpha^k(y)\}_\alpha + \{\alpha^k(x), \tau(y)\}_\alpha,$$

where $\{\cdot, \cdot\}_\alpha$ denotes the Yau-twisted bracket. Our classification in Theorems 4.2 and 4.3 thus provides a complete description of derivations for these twisted Hom-Lie structures.

This connection highlights that our matrix representations of α^k-derivations are not merely algebraic exercises but provide concrete computational tools for analyzing the derivation algebras of Yau-twisted Hom-Lie algebras derived from $\mathfrak{so}(3)$. Future work could explore the cohomology and deformation theory of these twisted structures, building on the explicit formulas obtained here and extending the work in [26, 27].

6. Conclusions

This paper investigates multiplicative Hom-structures and their associated α^k-derivations on the real Lie algebra $\mathfrak{so}(3)$, motivated by the goal of obtaining explicit matrix representations.

The main results include the description and classification of various types of derivations on $\mathfrak{so}(3)$. By employing the operator-matrix correspondence and solving linear systems derived from the Leibniz rules, explicit matrix forms of Lie derivations, D-derivations, and (F, D)-derivations were obtained. Additionally, as detailed in Section 5, each multiplicative Hom-structure induces a Yau-twisted Hom-Lie bracket, and the classification of α^k-derivations yields a comprehensive account of derivations for these twisted algebras.

It was established that any Hom-structure on $\mathfrak{so}(3)$ is represented by a symmetric matrix. A complete classification of multiplicative Hom-structures on $\mathfrak{so}(3)$ was achieved, identifying three types: the trivial (identity) structure, a one-parameter family, and a two-parameter family, each with explicit parameterization.

α^k-derivations were described for all multiplicative Hom-structures. In the case of the trivial structure, α^k-derivations coincide with the class of antisymmetric classical derivations. For the one-parameter family, nontrivial α^0-derivations exist for all c, whereas α^1-derivations are nonzero only at the boundary points $c = \pm 1$. For the two-parameter family, both α^0- and α^1-derivations are trivial.

The developed matrix representations serve as practical and computable tools, facilitating connections between algebraic structures and applied problems. The explicit formulas and Hom-structures are directly applicable in computational contexts, such as analyzing rotational dynamics in robotic kinematics and investigating symmetry breaking in quantum mechanics on the real line.

We note that the present work focuses specifically on the real Lie algebra $\mathfrak{so}(3)$; extensions to algebras over other fields or to higher-dimensional algebras such as $\mathfrak{so}(n)$ ($n > 3$) are not covered here and present natural directions for future research. It would be interesting to generalize this computational and representation-theoretic argument to other important Lie algebras, real and complex, such as $\mathfrak{so}(n)$ with $n > 3$ and $\mathfrak{su}(n)$. Studying the cohomology and deformation theory of these multiplicative Hom-Lie algebras, based on the calculations we made, is a good direction to take. Furthermore, the physical interpretations and applications for the nontrivial Hom-structures and hence their derivations in the phantom category presented here can be the basis for such insights.

Author contributions

Yan Jiang: Formal analysis,Writing–original draft. Ying Hou: Conceptualization. Keli Zheng: Supervision, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used Deepseek R1 and Grammarly for the purposes of correcting English grammar and logical errors.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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