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Research article

Strong and weak approximation for a two-time-scale stochastic quasi-geostrophic flow equation driven by Lévy processes

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Abstract: This work investigates the strong and weak approximation for a stochastic quasi-geostrophic flow equation with two time scales, where the slow component is coupled with a fast oscillation governed by a stochastic reaction-diffusion equation, and both are driven by Lévy noises. Employing Khasminskii's time discretization, we first prove that the slow component of the slow-fast system converges to the solution of the averaged equation in a strong sense with the help of an auxiliary process in small subintervals. Based on an asymptotic expansion of solutions for the Kolmogorov equation associated with the slow-fast system through a discontinuous path, we then decompose the weak solution with respect to the small parameter. By means of the components being determined recursively, we further establish the weak convergence from the original to the averaged dynamics.

Keywords: stochastic quasi-geostrophic flow equation; fast oscillation; averaging principle; weak and strong convergence; asymptotic expansion
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1. Introduction

The geophysical flows play a crucial role in both scientific research and engineering applications [8, 12, 22, 26]. The quasi-geostrophic flow equation, as a simplified model of geophysical flows, captures the essential features of large scale phenomena in the geophysical flows and has consequently attracted significant attention (see [2, 4, 7, 9] and their references).

More specifically speaking, the quasi-geostrophic (Q-G) equations model large-scale, rotating geophysical flows in the atmosphere and ocean, where the Coriolis force nearly balances the pressure gradient force. They describe the evolution of potential vorticity and capture phenomena such as oceanic eddies, Rossby waves, and weather systems at mid-latitudes. The key assumptions in the Q-G approximation are small Rossby number (slow evolution compared to planetary rotation), hydrostatic balance, and incompressibility.

In real geophysical systems, unresolved sub-grid processes (e.g., small-scale turbulence, convective bursts, and wind stress fluctuations) act as random forcings. Adding stochastic terms accounts for model uncertainty and unresolved dynamics, and represents intermittent or extreme events (especially with Lévy noise). The inclusion of jumps in both scales extends existing models and better captures real-world geophysical intermittency and impulsive forcing.

Thus, in this paper, we investigate a stochastic quasi-geostrophic flow equation driven by Lévy noise

$$\begin{cases} du^\varepsilon = [\nu\Delta u^\varepsilon - ru^\varepsilon - J(\psi(u^\varepsilon), u^\varepsilon) - \beta\psi_x(u^\varepsilon) \\ \qquad + f(u^\varepsilon, v^\varepsilon)]dt + \sigma_1 dW^{Q_1} + \int_{\mathbb{Z}} h_1(u^\varepsilon_{t-}, z)\tilde{N}_1(dt, dz), \quad \text{in} \quad D, \\ u^\varepsilon = 0, \quad \text{on} \quad \partial D, \\ u^\varepsilon(0) = u, \end{cases} \tag{1.1}$$

with a fast oscillation v^ε governed by a stochastic reaction-diffusion equation

$$\begin{cases} dv^\varepsilon = \frac{1}{\varepsilon}[\Delta v^\varepsilon + g(u^\varepsilon, v^\varepsilon)]dt + \frac{\sigma_2}{\sqrt{\varepsilon}} dW^{Q_2} + \int_{\mathbb{Z}} h_2(u^\varepsilon_{t-}, v^\varepsilon_{t-}, z)\tilde{N}^\varepsilon_2(dt, dz), \quad \text{in} \quad D, \\ v^\varepsilon = 0, \quad \text{on} \quad \partial D, \\ v^\varepsilon(0) = v, \end{cases} \tag{1.2}$$

where $\nu > 0$ is the viscous dissipation constant, $r > 0$ is the Ekman dissipation constant, and $\beta \geq 0$ is the meridional gradient of the Coriolis parameter. The streamfunction $\psi(x, y, t)$ satisfies $\Delta\psi(x, y, t) = u^\varepsilon(x, y, t)$, and the Jacobian operator $J(u, v)$ meets $J(u, v) = u_x v_y - u_y v_x$. We denote ε as the small singular perturbing parameter satisfying $0 < \varepsilon \ll 1$. The bounded planar domain D is supplemented by a homogenous Dirichlet boundary condition in $\mathbb{R}^2$. The functions f, g, h_1, h_2, and the mutually independent Wiener processes W^{Q_1} and W^{Q_2} will be specified in the next section. Moreover, $\tilde{N}_1$ is a scalar Poisson process with the intensity ν_1 and $\tilde{N}^\varepsilon_2$ is a scalar Poisson process with the intensity $\frac{\nu_2}{\varepsilon}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Physically, the small parameter ε indicates that the fast dynamics reach a statistical equilibrium much quicker than the slow dynamics change. The slow variable u^ε in (1.1) represents the Laplacian of the streamfunction, the vorticity in the quasi-geostrophic framework. The fast variable v^ε in (1.2) models rapidly evolving, small-scale processes coupled to the slow variable u^ε, such as fast temperature or salinity fluctuations (in oceanography). The reaction-diffusion form arises from diffusive transport and local source/sink terms (heating) acting on much faster time scales than the slow variable.

Function f represents coupling from fast processes to the slow vorticity dynamics. It can model the wind stress curl effects modulated by small-scale turbulent momentum fluxes. The function g describes the intrinsic dynamics and slow-fast interactions within the fast variable v^ε, which may represent nonlinear damping or saturation in small-scale turbulence.

There are numerous works related the quasi-geostrophic flow equation with oscillating external forcing (see the references and those therein [1, 3, 5, 10, 18, 21]). The quasi-geostrophic flow equation with a fast oscillation (1.1) and (1.2) is also called a slow-fast system, since the stochastic process u^ε and v^ε evolve with different rates as the parameter ε tends to zero. Under some dissipative assumptions, the fast process v^ε will be averaged at large time scales. The averaged equation is established as

$$\begin{cases} d\bar{u} = [\nu\Delta\bar{u} - r\bar{u} - J(\psi(\bar{u}),\bar{u}) - \beta\psi_x(\bar{u}) + \bar{f}(\bar{u})]dt \\ \qquad\qquad + \sigma_1 dW^{Q_1} + \int_{\mathbb{Z}} h_1(\bar{u}_{t-},z)\tilde{N}_1(dt,dz), \qquad \text{in} \quad D, \\ \bar{u} = 0, \qquad \text{on} \quad \partial D, \\ \bar{u}(0) = u, \end{cases} \tag{1.3}$$

where $\bar{f}(u) = \int_{L^2(D)} f(u,v)\mu^u(dv)$, and μ^u is the unique invariant measure associated with (3.1).

The averaging principle is an effective tool to analyze qualitative behaviors of the multisclale systems (1.1) and (1.2). There are a few works about the asymptotic behavior of the multiscale systems by the strong averaging principle, which provides a strong approximation in pathwise sense between the original solution of the slow equation and the effective solution of the corresponding averaged equation [15, 16, 25]. The issue of the weak averaging principle, Bréhier [6] showed a averaging result for stochastic evolution equations of parabolic type with slow and fast time scales in a weak sense. Fu et al. [14] studied the weak error in the averaging principle for a stochastic wave equation with a fast oscillation. Sun [23, 24] obtained the weak averaging principle of multiscale stochastic partial differential equations driven by α-stable process with α in the interval $(1,2)$.

In this paper, we are especially interested in the strong and weak order in averaging principle of (1.1) and (1.2). Here, the strong averaging principle refers to the pathwise (or mean-square) convergence of the slow component u^ε to the averaged solution $\bar{u}$ in (1.4), while the weak averaging principle concerns the convergence of expectations of $\phi(u^\varepsilon)$ in (1.5). More precisely, we prove that for any $T > 0$, $t \in [0,T]$, real-valued function ϕ with bounded and continuous Fréchet derivatives up to the third order, and a positive parameter $\kappa \in (0,\frac{1}{4})$, that

$$\mathbb{E}\|u^\varepsilon(t) - \bar{u}(t)\| \leq C\varepsilon^{\frac{1-\kappa}{2}}, \tag{1.4}$$

and

$$|\mathbb{E}\phi(u^\varepsilon(t,u,v)) - \mathbb{E}\phi(\bar{u}(t,u))| \leq C\varepsilon^{1-\kappa}, \tag{1.5}$$

where C is a positive constant independent of ε.

In order to derive the strong convergence (1.4), we reduce the systems (1.1) and (1.2) into an effective equation with the help of the averaging principle. Based on the Khasminskii's time discretization [17], we employ the skill of partitioning the time interval into small subintervals to establish an auxiliary process for which the slow component of the fast variable is frozen on small intervals of a subdivision. Furthermore, we can provide an intermediate errors between the processes and arrive at the strong averaging principle with the help of the auxiliary process.

As to the weak convergence (1.5), we adopt an asymptotic expansion with respect to ε of the solution for the Kolmogorov equation. We introduce the Kolmogorov operators associated with the multiscale systems (5.1) and the averaged system (5.2). Since the Kolmogorov equation involves the unbounded operator and there is no general analytic approach to regularity properties in the infinite-dimensional space, we apply the Galerkin approximation and reduce the infinite dimension space into a finite dimension space to estimate the Kolmogorov equation. In addition, due to the Lévy noise not only in the fast motion but also in the slow motion, we use the Itô formula to derive the explicit expression for the derivative of function $\phi(\bar{u})$, and borrow the argument of Bréhier [6] to improve the regularities of the effective equation driven by Lévy noise.

This paper is organized as follows. In the next section, we state some basic sets. Section 3 derives the stationary measure of the fast oscillation with the frozen slow component. Section 4 obtains the strong averaging principle (Theorem 4.2). In section 5, we prove the weak averaging result (Theorem 5.1) of this paper. And in the appendix, we will state some a priori estimates applied in proving the strong and weak result.

2. Preliminaries

For the domain $D = [0, K] \times [0, K]$, let $L^2(D) := \{u(x) \mid u(x)\text{measurable and } \int_D |u(x)|^2 dx < \infty\}$ be the usual Hilbert space on D, whose scalar product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Also, let $H^s(D)$ be the usual Sobolev space on D with $s \geq 0$. Especially, the scalar product and norm of $H^1(D)$ are $\langle \cdot, \cdot \rangle_1$ and $\|\cdot\|_1$, respectively. For any positive integer k, we denote by $C_b^k(L^2(D), \mathbb{R})$ the space of all k-times differentiable functions on $L^2(D)$ with bounded and uniformly continuous derivatives up to k-th order.

Let A be the Laplacian operator $-\Delta$ with Dirichlet boundary condition which generates a strongly continuous semigroup $\{S_t\}_{t\geq 0}$. For a complete orthonormal system of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$, $Ae_k = \lambda_k e_k$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots$. It holds that

$$\|u\|_1^2 \geq \lambda \|u\|^2, \quad \text{for any} \quad u \in H^1(D),$$

where $\lambda = \frac{\pi^2}{K^2}$ is the first eigenvalue of the operator A on D.

For $\theta \in (0, 1)$, the fractional power operator A^θ is defined as $A^\theta e_k = \lambda_k^\theta e_k$ with domain

$$\mathcal{D}(A^\theta) = \{u : \|u\|_{D(A^\theta)} = \sum_{k\in\mathbb{N}} \lambda_k^\theta \langle u, e_k \rangle e_k \leq +\infty\}.$$

The semigroup $\{S_t\}_{t\geq 0}$ satisfies the following properties for any $0 \leq \tau < t \leq T$ and $u \in \mathcal{D}(A^\theta)$ (see [6]):

$$\begin{aligned}
&\|S_t u\|_{D(A^\theta)} \leq C t^{-\theta} e^{-\frac{\lambda}{2}t} \|u\|, \\
&\|S_t u - S_\tau u\| \leq C \frac{|t-\tau|^\theta}{\tau^\theta} e^{-\frac{\lambda}{2}\tau} \|u\|, \\
&\|S_t u - S_\tau u\| \leq C |t-\tau|^\theta e^{-\frac{\lambda}{2}\tau} \|u\|_{D(A^\theta)}.
\end{aligned}$$

Hypothesis (H1) *The nonlinear terms f, g, h_1, and h_2 satisfy the following Lipschitz conditions: there exist positive constants L_f, L_g, C_{h_1}, and C_{h_2} such that $C_{h_1} + L_f, C_{h_2} + L_g \in (0, \lambda)$, and for any $u_1, u_2, v_1, v_2 \in L^2(D)$,*

$$\begin{aligned}
\|f(u_1, v_1) - f(u_2, v_2)\|^2 &\leq L_f(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \\
\|g(u_1, v_1) - g(u_2, v_2)\|^2 &\leq L_g(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \\
\int_{\mathbb{Z}} \|h_1(u_1, z) - h_1(u_2, z)\|^2 \nu_1(dz) &\leq C_{h_1} \|u_1 - u_2\|^2, \\
\int_{\mathbb{Z}} \|h_2(u_1, v_1, z) - h_2(u_2, v_2, z)\|^2 \nu_2(dz) &\leq C_{h_2}(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2).
\end{aligned}$$

Hypothesis (H2) *The nonlinearities $f(u,v)$, $g(u,v)$, $h_1(u)$, and $h_2(u,v)$ are of class C^2 and have the bounded first and second derivatives.*

Let W_1 and W_2 be two mutually independent $L^2(D)$-value Wiener processes with covariance operators Q_1 and Q_2, respectively. These operators are nonnegative, symmetric, and of trace class, i.e., $TrQ_i = \sum_{k\in\mathbb{N}} \alpha_{i,k} < \infty$ for $i = 1, 2$, where $Q_i e_k = \sum_{k\in\mathbb{N}} \alpha_{i,k} e_k$.

Throughout the paper, we denote by C a generic positive constant whose value may change from line to line.

3. Ergodicity of a frozen equation

This section is devoted to the study of the stationary measure and the asymptotic behavior of the fast process with a frozen slow component u. With the slow process u fixed, we introduce the "fast" variable $V^{u,v}$, which satisfies the following equation

$$\begin{cases} dV^{u,v} = [\Delta V^{u,v} + g(u, V^{u,v})]dt + \sigma_2 dW^{Q_2} + \int_{\mathbb{Z}} h_2(u_{t-}, V^{u,v}, z)\tilde{N}_2(dt, dz), & \text{in} \quad D, \\ V^{u,v} = 0, \qquad \text{on} \quad \partial D, \\ V^{u,v}(0) = v. \end{cases} \tag{3.1}$$

Here, the superscript u of $V^{u,v}$ means the frozen u in (1.2), and the superscript v of $V^{u,v}$ denotes the initial data of (1.2) with $\varepsilon = 1$ at $t = 0$.

Lemma 3.1. *Under the Hypotheses (H1) and (H2), (3.1) admits a unique solution $V^{u,v}$ such that*

$$\mathbb{E}\|V^{u,v}\|^2 \le C(e^{-\eta_1 t}\|v\|^2 + \|u\|^2),$$
$$\mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 \le Ce^{-\eta_1 t}\|v - \tilde{v}\|^2),$$

where $\eta_1 = \lambda - Lg - C_{h_2} > 0$.

Proof. Applying Itô's formula to $\|V^{u,v}\|^2$, it has

$$\begin{aligned} \|V^{u,v}\|^2 =&\|v\|^2 + 2\int_0^t \langle \Delta V^{u,v}, V^{u,v}\rangle ds + 2\int_0^t \langle g(u, V^{u,v}), V^{u,v}\rangle ds \\ &+ \sigma_2 T_r Q_2 t + 2\int_0^t \langle \sigma_2 dW^{Q_2}, V^{u,v}\rangle \\ &+ \int_0^t \int_{\mathbb{Z}} [\|V^{u,v} + h_2(u, V^{u,v}, z)\|^2 - \|V^{u,v}\|^2 - 2\langle h_2(u, V^{u,v}, z), V^{u,v}\rangle]\nu_2(dz)ds \\ &+ \int_0^t \int_{\mathbb{Z}} [\|V^{u,v} + h_2(u, V^{u,v}, z)\|^2 - \|V^{u,v}\|^2]\tilde{N}_2(ds, dz). \end{aligned}$$

Taking expectations and using (H1), we obtain

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\|V^{u,v}\|^2 \le& - 2\lambda\mathbb{E}\|V^{u,v}\|^2 + 2[C\|u\|\|V^{u,v}\| + Lg\|V^{u,v}\|^2] \\ &+ \mathbb{E}[\int_{\mathbb{Z}} \|h_2(u, V^{u,v}, z)\|^2 \nu_2(dz)] \\ \le& - \eta_1\mathbb{E}\|V^{u,v}\|^2 + C\|u\|^2, \end{aligned}$$

which implies from the Gronwall inequality that

$$\mathbb{E}\|V^{u,v}\|^2 \le e^{-\eta_1 t}\|v\|^2 + C\|u\|^2.$$

Note that

$$\begin{aligned} d(V^{u,v} - V^{u,\tilde{v}}) =& [\Delta(V^{u,v} - V^{u,\tilde{v}}) + g(u, V^{u,v}) - g(u, V^{u,\tilde{v}})]dt \\ &+ \int_{\mathbb{Z}} [h_2(u, V^{u,v}, z) - h_2(u, V^{u,\tilde{v}}, z)]\tilde{N}_2(dt, dz). \end{aligned}$$

By the Itô formula, it infers

$$\begin{aligned} d\|V^{u,v} - V^{u,\tilde{v}}\|^2 =& \langle \Delta(V^{u,v} - V^{u,\tilde{v}}), V^{u,v} - V^{u,\tilde{v}}\rangle dt + \langle g(u, V^{u,v}) - g(u, V^{u,\tilde{v}}), V^{u,v} - V^{u,\tilde{v}}\rangle dt \\ &+ \int_{\mathbb{Z}} \|h_2(u, V^{u,v}, z) - h_2(u, V^{u,\tilde{v}}, z)\|^2 \nu_2(dz)dt \\ &- 2\int_{\mathbb{Z}} \langle h_2(u, V^{u,v}, z) - h_2(u, V^{u,\tilde{v}}, z), V^{u,v} - V^{u,\tilde{v}}\rangle \tilde{N}_2(dt, dz), \end{aligned}$$

which implies that

$$\begin{aligned} d\mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 \le& -\lambda\mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 dt + Lg\mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 dt \\ &+ C_{h_2}\|V^{u,v} - V^{u,\tilde{v}}\|^2 dt \\ \le& -\eta_1\|V^{u,v} - V^{u,\tilde{v}}\|^2 dt. \end{aligned}$$

Therefore,

$$\mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 \le \|v - \tilde{v}\|^2 e^{-\eta_1 t}.$$

The proof is completed. ■

Let P_t^u be the transition semigroup associated with (3.1). Lemma 3.1 implies the existence of a unique invariant measure μ^u for P_t^u. Define the averaging term as follows:

$$\bar{f}(u) := \int_{L^2(D)} f(u, v)\mu^u(dv).$$

Then

$$\begin{aligned} \|\mathbb{E}f(u, V^{u,v}) - \bar{f}(u)\|^2 =& \|\int_{L^2(D)} \mathbb{E}(f(u, V^{u,v}) - f(u, V^{u,\tilde{v}}))\mu^u(d\tilde{v})\|^2 \\ \le& \int_{L^2(D)} \mathbb{E}\|V^{u,v} - V^{u,\tilde{v}}\|^2 \mu^u(d\tilde{v}) \\ \le& Ce^{-\eta_1 t}\int_{L^2(D)} \|v - \tilde{v}\|^2 \mu^u(d\tilde{v}) \\ \le& Ce^{-\eta_1 t}(\|u\|^2 + \|v\|^2). \end{aligned} \tag{3.2}$$

4. Strong averaging principle

In this section, we will state the result of the strong averaging principle for a two-time-scale stochastic quasi-geostrophic flow equation driven by Lévy noises. In other words, we will show that the slow component of the solutions (1.1) and (1.2) converges towards the solution of (1.3) in a strong sense.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces. C ompact operator $\mathcal{R}$ maps bounded subsets of $\mathcal{X}_1$ to relatively compact subsets of $\mathcal{X}_2$.

Lemma 4.1. *[20] Suppose $\mathcal{S}$ a bounded subset of $L^1(0,T;\mathcal{X}_1)$ for any $T>0$ such that $\Lambda := \mathcal{R}\mathcal{S}$ is a subset of $C(0,T;\mathcal{X}_2)$ bounded in $L^q(0,T;\mathcal{X}_2)$ with $q>1$. If*

$$\lim_{\sigma\to 0}\|u(\cdot+\sigma)-u(\cdot)\|_{C(0,T;\mathcal{X}_2)} = 0 \text{ uniformly for } u\in\Lambda,$$

then Λ is relatively compact in $C(0,T;\mathcal{X}_2)$.

Denote $\{\mathcal{L}(u^\varepsilon)\}_\varepsilon$ as the law of the slow component u^ε.

Lemma 4.2. *(Prohorov theorem) [11] Assume that $\mathcal{X}$ is a separable Banach space. The set of probability measures $\{\mathcal{L}(u^\varepsilon)\}_\varepsilon$ on $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ is relatively compact if and only if $\{u^\varepsilon\}_\varepsilon$ is tight.*

Theorem 4.1. *Assume that Hypothesis (H1) and (H2) hold. For any $T>0$, $\{u^\varepsilon\}_\varepsilon$ is tight in $C(0,T;L^2(D))$.*

Proof. With the help of Lemmas A.2 and A.3,

$$\mathbb{E}\int_0^T \|u^\varepsilon\|_1 ds \le C_T,$$

and

$$\sup_{0\le s<t\le T}\frac{\mathbb{E}\|u^\varepsilon(t)-u^\varepsilon(s)\|}{|t-s|^{1-\kappa}} \le C_T,$$

with some constant $C_T>0$.

Then by the Markov inequality for any $\varrho>0$, there exist constants $K_1,K_2>0$ such that

$$\mathbb{P}\{\int_0^T \|u^\varepsilon\|_1 ds \le K_1\} \ge 1-\frac{C_T}{K_1} \ge 1-\frac{\varrho}{2}, \tag{4.1}$$

and

$$\mathbb{P}\{\sup_{0\le s<t\le T}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|}{|t-s|^{1-\kappa}} \le K_2\} \ge 1-\frac{C_T}{K_2} \ge 1-\frac{\varrho}{2}. \tag{4.2}$$

Define the sets

$$\mathcal{S}_1 := \{u^\varepsilon\in L^1(0,T;H^1(D)) : \int_0^T \|u^\varepsilon\|_1 ds \le K_1\},$$

and

$$\mathcal{S}_2 := \{u^\varepsilon\in \mathcal{S}_1 : \sup_{0\le s<t\le T}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|}{|t-s|^{1-\kappa}} \le K_2\}.$$

Then it follows from (4.1) and (4.2) that

$$\mathbb{P}\{u^\varepsilon \in \mathcal{S}_2\} > 1 - \varrho.$$

By the definition of $\mathcal{S}_2$, it has

$$\lim_{\sigma\to 0} \sup_{0<s<T} \|u^\varepsilon(s+\sigma) - u^\varepsilon(s)\| = 0,$$

where $\sigma = t - s$.

Therefore, the set $\mathcal{S}_2$ is compact in $C(0,T;H^1(D))$, which means that the $\{\mathcal{L}(u^\varepsilon)\}_\varepsilon$ is relatively compact in $C(0,T;L^2(D))$ by Lemma 4.1. Furthermore, due to Lemma 4.2, it derives that $\{u^\varepsilon\}_\varepsilon$ is tight in $C(0,T;L^2(D))$, which completes the proof. ■

To prove the strong convergence, we partition the interval $[0,T]$ into subintervals of length δ and introduce an auxiliary process $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)$. For $t \in [k\delta, (k+1)\delta)$, this process is defined by:

$$\begin{cases} \tilde{u}^\varepsilon(t) = e^{\mathcal{A}(t-k\delta)}\tilde{u}^\varepsilon(k\delta) + \int_{k\delta}^t e^{\mathcal{A}(t-r)} J(\psi(\tilde{u}^\varepsilon(k\delta)), \tilde{u}^\varepsilon(k\delta))dr \\ \qquad + \int_{k\delta}^t e^{\mathcal{A}(t-r)} f(\tilde{u}^\varepsilon(k\delta), \tilde{v}^\varepsilon)dr + \int_{k\delta}^t e^{\mathcal{A}(t-r)}\sigma_1 dW^{Q_1} \\ \qquad + \int_{k\delta}^t \int_{\mathbb{Z}} e^{\mathcal{A}(t-r)} h_1(\tilde{u}^\varepsilon(k\delta), z)\tilde{N}_1(dr,dz), \qquad \text{in } D, \\ d\tilde{v}^\varepsilon(t) = \dfrac{1}{\varepsilon}[\Delta\tilde{v}^\varepsilon(t) + g(u^\varepsilon(k\delta), \tilde{v}^\varepsilon(t))]dt + \dfrac{\sigma_2}{\sqrt{\varepsilon}}dW^{Q_2} + \int_{\mathbb{Z}} h_2(\tilde{u}^\varepsilon_{t-}, \tilde{v}^\varepsilon_{t-}, z)\tilde{N}_2^\varepsilon(dt,dz), \qquad \text{in } D, \\ \tilde{u}^\varepsilon(0) = u, \tilde{v}^\varepsilon(k\delta) = v^\varepsilon(k\delta), \\ \tilde{u}^\varepsilon = 0, \tilde{v}^\varepsilon = 0, \qquad \text{on} \quad \partial D. \end{cases} \tag{4.3}$$

Lemma 4.3. *For $\varepsilon \in (0,1)$, let u be in $H^1(D)$, and let v be in $L^2(D)$. Then $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)$ is the unique solution to (4.3), and there exists a positive constant C such that*

$$\sup_{t\in[0,T]} \mathbb{E}\|\tilde{u}^\varepsilon\| \le C, \quad \textit{and} \quad \sup_{t\in[0,T]} \mathbb{E}\|\tilde{v}^\varepsilon\| \le C.$$

Using the same method as in the Appendix, it is easy to prove Lemma 4.3. Here we omit it.

Lemma 4.4. *For $t \in [k\delta, (k+1)\delta)$, there exists a positive constant C such that*

$$\mathbb{E}\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\| \le C\delta^{1-\kappa}.$$

Proof. Using the mild formulations of $v^\varepsilon(t)z$ and $\tilde{v}^\varepsilon(t)$, and Hypothesis (H1), we obtain an inequality of the form

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|^2 \le\ & -\tfrac{\lambda}{\varepsilon}\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|^2 + \tfrac{1}{\varepsilon}\langle g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon, \tilde{v}^\varepsilon), v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\rangle \\ & + \tfrac{1}{\varepsilon}\langle g(u^\varepsilon, v^\varepsilon) - g(u^\varepsilon(k\delta), \tilde{v}^\varepsilon), v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\rangle \\ & + \tfrac{1}{\varepsilon}\int_{\mathbb{Z}} \|h_2(u^\varepsilon, v^\varepsilon, z) - h_2(u^\varepsilon(k\delta), \tilde{v}^\varepsilon, z)\|^2 \nu_2(dz)dt \\ & - \tfrac{2}{\varepsilon}\int_{\mathbb{Z}} \langle h_2(u^\varepsilon, v^\varepsilon, z) - h_2(u^\varepsilon(k\delta), \tilde{v}^\varepsilon, z), v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\rangle \tilde{N}_2(dt,dz), \\ \le\ & -\tfrac{\eta_1}{\varepsilon}\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|^2 + \tfrac{C\delta^{2(1-\kappa)}}{\varepsilon}, \end{aligned} \tag{4.4}$$

which implies from the Gronwall inequality that

$$\begin{aligned}\|v^{\varepsilon}(t)-\tilde{v}^{\varepsilon}(t)\|^2 &\le \frac{C}{\varepsilon}\int_0^t e^{-\frac{\eta_1}{\varepsilon}(t-r)}\delta^{2(1-\kappa)}dr\\ &\le C\delta^{2(1-\kappa)}\int_0^{\frac{t}{\varepsilon}} e^{-\eta_1\sigma}d\sigma\\ &\le C\delta^{2(1-\kappa)}.\end{aligned}\tag{4.5}$$

The proof is completed. ■

We now state and prove the main result of this section.

Theorem 4.2. *(Strong averaging principle) Assume that Hypothesis (H1) and (H2) hold. For any $T>0$ and $t\in[0,T]$, then*

$$\mathbb{E}\|u^{\varepsilon}(t)-\bar{u}(t)\|\le C\varepsilon^{\frac{1-\kappa}{2}},\tag{4.6}$$

where $\bar{u}$ is the solution of the following effective equation:

$$\begin{cases} d\bar{u}=[\nu\Delta\bar{u}-r\bar{u}-J(\psi(\bar{u}),\bar{u})-\beta\psi_x(\bar{u})+\bar{f}(\bar{u})]dt\\ \qquad\qquad +\sigma_1 dW^{Q_1}+\displaystyle\int_{\mathbb{Z}}h_1(\bar{u}_{t-},z)\tilde{N}_1(dt,dz), \qquad in \quad D,\\ \bar{u}=0, \qquad on \quad \partial D,\\ \bar{u}(0)=u.\end{cases}\tag{4.7}$$

Proof. Considering the mild solution with $t\in[k\delta,(k+1)\delta)$, we have

$$\begin{aligned}u^{\varepsilon}(t)=&e^{\mathcal{A}(t-k\delta)}u^{\varepsilon}(k\delta)+\int_{k\delta}^t e^{\mathcal{A}(t-r)}J(\psi(u^{\varepsilon}),u^{\varepsilon})dr+\int_{k\delta}^t e^{\mathcal{A}(t-r)}f(u^{\varepsilon},v^{\varepsilon})dr\\ &+\int_{k\delta}^t e^{\mathcal{A}(t-r)}\sigma_1 dW^{Q_1}+\int_{k\delta}^t\int_{\mathbb{Z}}e^{\mathcal{A}(t-r)}h_1(u^{\varepsilon}_{t-},z)\tilde{N}_1(dr,dz).\end{aligned}\tag{4.8}$$

It is easy to obtain that

$$\begin{aligned}\|u^{\varepsilon}(t)-\tilde{u}^{\varepsilon}(t)\|\le&C\int_{k\delta}^t\|v^{\varepsilon}(r)-\tilde{v}^{\varepsilon}(r)\|dr+C\int_{k\delta}^t\|u^{\varepsilon}(r)-u^{\varepsilon}(k\delta)\|dr\\ \le&C\delta^{1-\kappa}.\end{aligned}\tag{4.9}$$

Then, the mild solution of (4.7) is

$$\begin{aligned}\bar{u}(t)=&e^{\mathcal{A}t}u_0-\int_0^t e^{\mathcal{A}(t-r)}J(\psi(\bar{u}),\bar{u})dr+\int_0^t e^{\mathcal{A}(t-r)}\bar{f}(\bar{u})dr\\ &+\int_0^t e^{\mathcal{A}(t-r)}\sigma_1 dW^{Q_1}+\int_0^t\int_{\mathbb{Z}}e^{\mathcal{A}(t-r)}h_1(\bar{u}_{t-},z)\tilde{N}_1(dr,dz).\end{aligned}$$

Denote $\lfloor a\rfloor$ as the largest positive integer less than a. The difference $\tilde{u}^{\varepsilon}(t)-\bar{u}(t)$ can be estimated as:

$$\begin{aligned}\mathbb{E}\|\tilde{u}^{\varepsilon}(t)-\bar{u}(t)\|\le&\int_0^t e^{\mathcal{A}(t-r)}\mathbb{E}\|J(\psi(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta)),u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-J(\psi(\bar{u}),\bar{u})\|dr\\ &+\int_0^t e^{\mathcal{A}(t-r)}\mathbb{E}\|f(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon})-\bar{f}(\bar{u})\|dr\\ &+\int_0^t\int_{\mathbb{Z}}e^{\mathcal{A}(t-r)}\mathbb{E}\|h_1(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-h_1(\bar{u})\|\nu_1(dz)dr.\end{aligned}\tag{4.10}$$

It follows from Lemma A.2 that

$$\begin{aligned}&\int_0^t e^{\mathcal{A}(t-r)}\|J(\psi(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta)),u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-J(\psi(\bar{u}),\bar{u})\|dr\\ \leq&\int_0^t e^{\mathcal{A}(t-r)}\|J(\psi(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta)),u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-J(\psi(u^{\varepsilon}(r)),u^{\varepsilon}(r))\|dr\\ &+\int_0^t e^{\mathcal{A}(t-r)}\|J(\psi(u^{\varepsilon}(r)),u^{\varepsilon}(r))-J(\psi(\bar{u}),\bar{u})\|dr\\ \leq& C\delta^{1-\kappa}+C\int_0^t\|u^{\varepsilon}-\bar{u}\|dr,\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbb{Z}} e^{\mathcal{A}(t-r)}\mathbb{E}\|h_1(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-h_1(\bar{u})\|\nu_1 dzdr\\ \leq& C\delta^{1-\kappa}+C\int_0^t\|u^{\varepsilon}-\bar{u}\|dr.\end{aligned}\tag{4.12}$$

Then,

$$\begin{aligned}&\int_0^t e^{\mathcal{A}(t-r)}\|f(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon})-\bar{f}(\bar{u})\|dr\\ \leq&\int_0^t e^{\mathcal{A}(t-r)}\|f(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon})-\bar{f}(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))\|dr\\ &+\int_0^t e^{\mathcal{A}(t-r)}\|\bar{f}(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))-\bar{f}(u^{\varepsilon}(r))\|dr\\ &+\int_0^t e^{\mathcal{A}(t-r)}\|\bar{f}(u^{\varepsilon})-\bar{f}(\bar{u})\|dr\\ :=& L_1+L_2+L_3.\end{aligned}\tag{4.13}$$

Considering a time shift transformation [17], there is

$$(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon}(r))=(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon}(s+\lfloor r/\delta\rfloor\delta))\simeq(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),V^{u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),v^{\varepsilon}(\lfloor r/\delta\rfloor\delta)}(\frac{s}{\varepsilon})).$$

It follows from Lemma 3.2 that

$$\begin{aligned}L_1=&\int_0^t e^{\mathcal{A}(t-r)}\|f(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta),\tilde{v}^{\varepsilon}(r))-\bar{f}(u^{\varepsilon}(\lfloor r/\delta\rfloor\delta))\|dr\\ \leq&\frac{\varepsilon}{\delta}\delta^{1-\kappa}\int_0^{\frac{t}{\varepsilon}}e^{-\rho s}ds\\ \leq& C\varepsilon\delta^{-\kappa},\end{aligned}\tag{4.14}$$

where ρ is a positive number.

Then from the ergodicity, it infers

$$\begin{aligned} L_2 =& \int_0^t e^{\mathcal{A}(t-r)} \|\bar{f}(u^\varepsilon(\lfloor r/\delta \lfloor \delta)) - \bar{f}(u^\varepsilon(r))\| dr \\ =& \int_0^t e^{\mathcal{A}(t-r)} \| \int_{L^2(D)} [f(u^\varepsilon(\lfloor r/\delta \rfloor \delta), v) - f(u^\varepsilon(r), v)] \mu^u(dv) \| dr \\ \le& \int_0^t e^{\mathcal{A}(t-r)} \|u^\varepsilon(\lfloor r/\delta \rfloor \delta) - u^\varepsilon(r)\| dr \\ \le& C\delta^{1-\kappa}. \end{aligned} \tag{4.15}$$

Similarly to L_2, it is easy to get

$$\begin{aligned} L_3 =& \int_0^t e^{\mathcal{A}(t-r)} \|\bar{f}(u^\varepsilon(r)) - \bar{f}(\bar{u}(r))\| dr \\ \le& C \int_0^t \|u^\varepsilon(r) - \bar{u}(r)\| dr. \end{aligned} \tag{4.16}$$

Therefore, for $t \in [0, T]$, it follows from (4.9) and (4.10) that

$$\mathbb{E}\|u^\varepsilon(t) - \bar{u}(t)\| \le C[\varepsilon\delta^{-\kappa} + \delta^{1-\kappa} + \int_0^T \mathbb{E}\|u^\varepsilon(r) - \bar{u}(r)\| dr].$$

Using the Gronwall inequality, it follows that

$$\mathbb{E}\|u^\varepsilon(t) - \bar{u}(t)\| \le C(\varepsilon\delta^{-\kappa} + \delta^{1-\kappa}).$$

In particular taking $\delta = \sqrt{\varepsilon}$, there is

$$\mathbb{E}\|u^\varepsilon(t) - \bar{u}(t)\| \le C\varepsilon^{\frac{1-\kappa}{2}},$$

which completes the proof. ■

5. Weak-order convergence

This section is devoted to establishing the weak convergence order (1.5). Due to the presence of Lévy noises and the infinite-dimensional setting, we employ the Galerkin approximation and an asymptotic expansion of the Kolmogorov equation.

Let P_m be the projection onto the first m eigenfunctions $\{e_1, e_2, \cdots, e_m\}$ of the operator A. Define the finite-dimensional space $L^2_m(D) := P_m L^2(D)$ and the approximated operator $A_m = P_m A$. The semigroup generated by A_m is denoted by $\{S_{t,m}\}_{t\ge 0}$. The Galerkin approximation of the slow-fast systems (1.1) and (1.2) is

$$\begin{cases} du^\varepsilon_m = [\nu A_m u^\varepsilon_m - r u^\varepsilon_m - J(\psi(u^\varepsilon_m), u^\varepsilon_m) - \beta\psi_x(u^\varepsilon_m) + f_m(u^\varepsilon, v^\varepsilon)]dt \\ \qquad + \sigma_1 P_m dW^{Q_1} + \int_{\mathbb{Z}} h_{1,m}(u^\varepsilon_{t-}, z)\tilde{N}_1(dt, dz), \qquad \text{in} \quad D, \\ dv^\varepsilon_m = \frac{1}{\varepsilon}[A_m v^\varepsilon_m + g_m(u^\varepsilon, v^\varepsilon)]dt + \frac{\sigma_2}{\sqrt{\varepsilon}} P_m dW^{Q_2} + \int_{\mathbb{Z}} h_{2,m}(u^\varepsilon_{t-}, v^\varepsilon_{t-}, z)\tilde{N}^\varepsilon_2(dt, dz), \qquad \text{in} \quad D, \\ u^\varepsilon_m = v^\varepsilon_m = 0, \qquad \text{on} \quad \partial D, \\ u^\varepsilon_m(0) = u, v^\varepsilon_m(0) = v, \end{cases} \tag{5.1}$$

where $f_m = P_m f$, $g_m = P_m g$, $h_{1,m} = P_m h_1$, and $h_{2,m} = P_m h_2$.

Moreover, the approximation system of Eq (1.3) is also defined as

$$\begin{cases} d\bar{u}_m = [\nu A_m \bar{u}_m - r\bar{u}_m - J(\psi(\bar{u}_m), \bar{u}_m) - \beta\psi_x(\bar{u}_m) + \bar{f}_m(u_m^\varepsilon, v_m^\varepsilon)]dt \\ \qquad\qquad + \sigma_1 P_m dW^{Q_1} + \int_{\mathbb{Z}} h_{1,m}(\bar{u}_m, z)\tilde{N}_1(dt, dz), \quad \text{in} \quad D, \\ \bar{u}_m = 0, \quad \text{on} \quad \partial D, \\ \bar{u}_m(0) = u, \end{cases} \tag{5.2}$$

where $\bar{f}_m = \int_{L_m^2(D)} f(u,v)\mu_m^u(dv)$, and μ_m^u is the unique invariant measure associated with the following equation:

$$\begin{cases} dV_m^{u,v} = [A_m V_m^{u,v} + g_m(u, V_m^{u,v})]dt + \sigma_2 P_m dW^{Q_2} + \int_{\mathbb{Z}} h_{2,m}(u, V_m^{u,v}, z)\tilde{N}_2(dt, dz), \\ V_m^{u,v}(0) = v. \end{cases}$$

Observe that

$$\begin{aligned} \|\mathbb{E}\phi(u^\varepsilon) - \mathbb{E}\phi(\bar{u})\| \leq & \|\mathbb{E}\phi(u^\varepsilon) - \mathbb{E}\phi(u_m^\varepsilon)\| + \|\mathbb{E}\phi(u_m^\varepsilon) - \mathbb{E}\phi(\bar{u}_m)\| \\ & + \|\mathbb{E}\phi(\bar{u}_m) - \mathbb{E}\phi(\bar{u})\|. \end{aligned} \tag{5.3}$$

From the Galerkin approximation and the strong averaging result, we have

$$\begin{aligned} & \lim_{m\to\infty} \|\mathbb{E}\phi(u^\varepsilon) - \mathbb{E}\phi(u_m^\varepsilon)\| = 0, \\ & \lim_{m\to\infty} \|\mathbb{E}\phi(\bar{u}_m) - \mathbb{E}\phi(\bar{u})\| = 0. \end{aligned} \tag{5.4}$$

Remark 5.1. *For any $T > 0$ and $\phi \in C_b^3(L^2(D), \mathbb{R})$, it follows from (5.3) and (5.4) that*

$$\lim_{m\to\infty} \|\mathbb{E}\phi(u^\varepsilon) - \mathbb{E}\phi(\bar{u})\| \leq \lim_{m\to\infty} \|\mathbb{E}\phi(u_m^\varepsilon) - \mathbb{E}\phi(\bar{u}_m)\|. \tag{5.5}$$

From now on, we consider the deviation $\|\mathbb{E}\phi(u_m^\varepsilon) - \mathbb{E}\phi(\bar{u}_m)\|$ which is independent of m and further proves Theorem 5.1 as the dimension goes to infinity.

Consider the two Kolmogorov operators associated with Eq (5.1) as follows:

$$\begin{aligned} \mathcal{L}_1\Phi(u) := & \langle \nu A_m u - ru - J(\psi, u) - \beta\psi_x(u) + f_m(u, v), D_u\Phi(u)\rangle \\ & + \frac{1}{2}\sigma_1^2 Tr(D_{uu}^2\Phi(u))(Q_{1,m}^{\frac{1}{2}})(Q_{1,m}^{\frac{1}{2}})^* + \int_{\mathbb{Z}} [\Phi(u + h_{1,m}(u,\Phi)) - \Phi(u) \\ & - \langle D_u\Phi(u), h_{1,m}(u,z)\rangle]\nu_1(dz), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_2\Phi := & \langle A_m v + g_m(u,v), D_v\Phi(v)\rangle + \frac{1}{2}\sigma_2^2 Tr(D_{vv}^2\Phi(v))(Q_{2,m}^{\frac{1}{2}})(Q_{2,m}^{\frac{1}{2}})^* \\ & + \int_{\mathbb{Z}} [\Phi(v + h_{2,m}(v,\Phi)) - \Phi(v) - \langle D_v\Phi(v), h_{2,m}(v,z)\rangle]\nu_2(dz). \end{aligned}$$

Then $U^\varepsilon := \mathbb{E}\phi(u_m^\varepsilon)$ is the solution for the following Kolmogorov equation:

$$\begin{cases} \dfrac{\partial}{\partial t} U^\varepsilon = \mathcal{L}^\varepsilon U^\varepsilon, \\ U^\varepsilon(0) = \phi(u), \end{cases} \tag{5.6}$$

where $\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_1 + \mathcal{L}_2$.

Similarly, define the Kolmogorov operator of Eq (5.2) as

$$\begin{aligned} \bar{\mathcal{L}}\bar{\Phi} :=& \langle \nu A_m u - ru - J(\psi, u) - \beta\psi_x(u) + \bar{f}_m(u), D_u\bar{\Phi}(u)\rangle \\ &+ \frac{1}{2}\sigma_1^2 Tr(D_{uu}^2\bar{\Phi}(u))(Q_{1,m}^{\frac{1}{2}})(Q_{1,m}^{\frac{1}{2}})^* + \int_{\mathbb{Z}} [\bar{\Phi}(u + h_{1,m}(u, \bar{\Phi})) \\ &- \bar{\Phi}(u) - \langle D_u\bar{\Phi}(u), h_{1,m}(u, z)\rangle]\nu_1(dz). \end{aligned}$$

Then, the solution $\bar{U} := \mathbb{E}\phi(\bar{u}_m)$ satisfies the Komolgorov equation as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{U} = \bar{\mathcal{L}}\bar{U}, \\ \bar{U}(0) = \phi(u). \end{cases} \tag{5.7}$$

Now, we will introduce an asymptotic expansion of solutions for the Kolmogorov equation associated with Eq (5.1) as follows:

$$U^\varepsilon = U_0 + \varepsilon U_1 + R^\varepsilon,$$

where $U_0 = \bar{U}$, the term U_1, and the remainder R^ε will be introduced in next subsections.

With the help of Komolgorov Eq (5.6), it follows that

$$\begin{aligned} \varepsilon\frac{\partial U_0}{\partial t} + \varepsilon^2\frac{\partial U_1}{\partial t} + \varepsilon\frac{\partial R^\varepsilon}{\partial t} =& \mathcal{L}_1 U_0 + \varepsilon\mathcal{L}_1 U_1 + \mathcal{L}_1 R^\varepsilon \\ &+ \varepsilon\mathcal{L}_2 U_0 + \varepsilon^2\mathcal{L}_2 U_1 + \varepsilon\mathcal{L}_2 R^\varepsilon. \end{aligned}$$

By comparing the powers of ε, it implies that

$$\mathcal{L}_1 U_0 = 0, \text{ and } \frac{\partial U_0}{\partial t} = \mathcal{L}_1 U_1 + \mathcal{L}_2 U_0, \tag{5.8}$$

which implies

$$U_0(t, u, v) = U_0(t, u), \tag{5.9}$$

where $U_0(t, u, v) = \int_{L_m^2(D)} U_0(t, u, v)\mu(dv)$ is independent of v, and μ is the invariant measure of the Markov process with generator $\mathcal{L}_1$. Thus,

$$\int_{L_m^2(D)} \mathcal{L}_1 U_1\mu(dv) = 0. \tag{5.10}$$

Lemma 5.1. *Under the Hypotheses (H1) and (H2), the processes U_0 and $\bar{U}$ satisfy the same evolution equation with the initial condition (u, v).*

Proof. It follows from (5.9) and (5.10) that

$$
\begin{aligned}
\frac{\partial}{\partial t}U_0(t,u) =& \int_{L_m^2(D)} \frac{\partial}{\partial t}U_0(t,u)\mu(dv)\\
=& \int_{L_m^2(D)} (\mathcal{L}_1 U_1 + \mathcal{L}_2 U_0)\mu(dv)\\
=& \int_{L_m^2(D)} \mathcal{L}_2 U_0 \mu(dv)\\
=& \langle \nu A_m u - ru - J(\psi,u) - \beta\psi_\xi(u) + \bar{f}_m(u), D_u\bar{\Phi}(u)\rangle\\
&+ \frac{1}{2}\sigma_1^2 Tr(D_{uu}^2\bar{\Phi}(u))(Q_{1,m}^{\frac{1}{2}})(Q_{1,m}^{\frac{1}{2}})^* + \int_{\mathbb{Z}}[\bar{\Phi}(u+h_{1,m}(u,\bar{\Phi}))\\
&- \bar{\Phi}(u) - \langle D_u\bar{\Phi}(u), h_{1,m}(u,z)\rangle]\nu_1(dz)\\
=& \bar{\mathcal{L}}U_0.
\end{aligned}
$$

Further, using the uniqueness arguments as in [19], it follows that U_0 coincides with $\bar{U}$. The proof is completed. ■

5.1. *The term of* U_1

From Lemma 5.1, the function U_1 is determined by the following equation:

$$
\begin{aligned}
\mathcal{L}_2 U_1(u,v) &= \bar{\mathcal{L}}\bar{U} - \mathcal{L}_1\bar{U}\\
&= \langle \bar{f}_m(u) - f_m(u,v), D_u\bar{U}\rangle \qquad (5.11)\\
&:= -\varrho(t,u,v).
\end{aligned}
$$

From the Hypothesis (H2) and Lemma B.3, it is obvious that the first and second derivatives of $\varrho(t,u,v)$ are bounded and

$$
\int_{L_m^2(D)} \varrho(t,u,v)\mu(dv) = \int_{L_m^2(D)} \langle \bar{f}_m(u) - f_m(u,v), D_u\bar{U}\rangle\mu(dv) = 0,
$$

where $t \in [0,T]$ with $T > 0$.

Lemma 5.2. *Under the Hypotheses (H1) and (H2), the processes* U_1 *admits that*

$$
U_1(T,u,v) = \int_0^\infty \mathbb{E}\varrho(t,u,v(s))ds.
$$

Proof. Considering the term $\mathbb{E}\varrho(t,u,v(s))$, it infers

$$
\begin{aligned}
&|\mathbb{E}\varrho(t,u,v) - \int_{L_m^2(D)} \varrho(t,u,\tilde{v})\mu(d\tilde{v})|\\
=&|\int_{L_m^2(D)} \mathbb{E}[\varrho(t,u,v) - \varrho(t,u,\tilde{v})]\mu(d\tilde{v})|\\
=&\int_{L_m^2(D)} |\mathbb{E}\langle f(u,v) - f(u,v(\tilde{v}))|\mu(d\tilde{v})\\
\leq& C(\|v\| + \|\tilde{v}\|)e^{-\eta_1 s},
\end{aligned}
$$

which implies that

$$\lim_{s\to+\infty} \mathbb{E}\varrho(t,u,v(s)) = \int_{L_m^2(D)} \varrho(t,u,\tilde{v})\mu(d\tilde{v}) = 0.$$

Furthermore,

$$\mathcal{L}_2 \int_0^\infty \mathbb{E}\varrho(t,u,v(s))ds = \frac{tial}{tials} \int_0^\infty \mathbb{E}\varrho(t,u,v(s))ds, \tag{5.12}$$

and

$$\begin{aligned}\int_0^\infty \frac{tial}{tials}\mathbb{E}\varrho(t,u,v)ds &= \lim_{s\to+\infty} \mathbb{E}\varrho(t,u,v(s)) - \varrho(t,u,v)\\ &= \int_{L_m^2(D)} \varrho(t,u,v)\mu(dv) - \varrho(t,u,v)\\ &= -\varrho(t,u,v).\end{aligned} \tag{5.13}$$

It follows from (5.12) and (5.13) that

$$\mathcal{L}_2(\int_0^\infty \mathbb{E}\varrho(t,u,v(s))ds) = -\varrho(t,u,v),$$

which immediately implies that

$$U_1(T,u,v) = \int_0^\infty \mathbb{E}\varrho(t,u,v(s))ds.$$

The proof is completed. ■

Remark 5.2. *Assume that the Hypotheses (H1) and (H2) hold. From (3.2) and Lemma B.3, using the same argument as in [6], for any $t \in [0,T]$, it holds that*

$$\begin{aligned}|U_1(t,u,v)| &\leq \int_0^\infty \mathbb{E}\|\bar{f}_m(u) - f_m(u,v)\|\mathbb{E}\|D_u\bar{U}\|ds\\ &\leq C(\|u\| + \|v\|)\int_0^\infty e^{-\eta_1 s}ds\\ &\leq C(\|u\| + \|v\|).\end{aligned}$$

5.2. The remainder R^ε

The remainder term R^ε satisfies that

$$\begin{aligned}(\partial t - \mathcal{L}^\varepsilon)R^\varepsilon &= -(\partial t - \mathcal{L}^\varepsilon)U_0 - \varepsilon(\partial t - \mathcal{L}^\varepsilon)U_1\\ &= -(\partial t - \mathcal{L}_1 - \frac{1}{\varepsilon}\mathcal{L}_2)U_0 - \varepsilon(\partial t - \mathcal{L}_1 - \frac{1}{\varepsilon}\mathcal{L}_2)U_1\\ &= \varepsilon(\mathcal{L}_1 U_1 - \partial_t U_1).\end{aligned}$$

By the variation of constant formula, we have

$$\begin{aligned}R^\varepsilon(T,u,v) =&\mathbb{E}[R^\varepsilon(\rho^\varepsilon, u^\varepsilon(T-\rho^\varepsilon), v^\varepsilon(T-\rho^\varepsilon))]\\ &+ \varepsilon\mathbb{E}[\int_{\rho^\varepsilon}^T ((\mathcal{L}_1 U_1 - \partial_t U_1)(s, u^\varepsilon(T-s), v^\varepsilon(T-s))ds],\end{aligned} \tag{5.14}$$

with $0 < \rho^\varepsilon \leq \varepsilon$.

Lemma 5.3. *Assume that the Hypotheses (H1) and (H2) hold. Then,*

$$|\mathcal{L}_1 U_1(t,u,v)| \le C(\|u\| + \|v\|),$$

with any $t \in [0,T]$.

Proof. For any $u \in L^2_m(D)$, it holds that

$$\begin{aligned}\mathcal{L}_1 U_1 =& \langle \nu A_m u - ru - J(\psi,u) - \beta\psi_x(u) + f_m(u,v), D_u U_1\rangle \\ &+ \frac{1}{2}\sigma_1^2 Tr(D^2_{uu}U_1)(Q^{\frac{1}{2}}_{1,m})(Q^{\frac{1}{2}}_{1,m})^* + \int_{\mathbb{Z}}[U_1(u + h_{1,m}(u,U_1)) - U_1 \\ &- \langle D_u U_1, h_{1,m}(u,z)\rangle]\nu_1(dz) \\ :=& I_1 + I_2 + I_3.\end{aligned}$$

For I_1, it follows Lemma A.1 and Lemma A.3 that

$$\begin{aligned}|I_1| \le& \|\nu A_m u - ru - J(\psi,u) - \beta\psi_x(u) + f_m(u,v)\| + |D_u U_1| \\ \le& C + \int_0^\infty |D_u(\bar{f}_m - \mathbb{E}f_m)| \cdot |\eta^h| ds + \int_0^\infty |\bar{f}_m - \mathbb{E}f_m| \cdot |\varsigma| ds.\end{aligned}$$

Then, with the help of the result of Lemma 6.8 in [13], Lemmas B.1–B.3, we have

$$|I_1| \le C(\|u\| + \|v\|).$$

Notice that

$$\begin{aligned}D^2_{uu}U_1 \cdot (h,l) =& \int_0^\infty \langle D_{uu}(\bar{f}_m - \mathbb{E}f_m)(h,l), \eta^h\rangle ds + \int_0^\infty \langle D_u(\bar{f}_m - \mathbb{E}f_m)(h), \varsigma\rangle ds \\ &+ \int_0^\infty \langle D_u(\bar{f}_m - \mathbb{E}f_m)(l), \varsigma\rangle ds + \int_0^\infty \langle \bar{f}_m - \mathbb{E}f_m, D_{uuu}\bar{U}\rangle ds.\end{aligned}$$

Therefore,

$$|I_2| \le C(\|u\| + \|v\|).$$

Moreover, if the Hypothesis (H2) holds, it implies from Lemma 5.2 that

$$|I_3| \le C(\|u\| + \|v\|).$$

Consequently,

$$|\mathcal{L}_1 U_1(t,u,v)| \le C(\|u\| + \|v\|).$$

The proof is completed. ■

Lemma 5.4. *Assume that the Hypotheses (H1) and (H2) hold. Then,*

$$|\frac{\partial}{\partial t}U_1(t,u,v)| \le C(\|v\| + \|u\|),$$

with any $t \in [0,T]$.

Proof. It follows from Lemma 5.2 that

$$\frac{\partial U_1}{\partial t} = \int_0^\infty \mathbb{E}\langle \bar{f}_m(u) - f_m(u,v), \frac{\partial}{\partial t} D_u \bar{U}(t)\rangle ds.$$

For any $h \in L_m^2(D)$, it infers

$$\begin{aligned} D_u\bar{U} \cdot h &= \mathbb{E}[\phi'(\bar{u}_m) \cdot D_u \bar{u}_m] \\ &= \mathbb{E}[\phi'(\bar{u}_m) \cdot \eta^h], \end{aligned}$$

where η^h is the solution of Eq (B.2).

For the solution $\bar{u}$ of Eq (5.2), applying the Itô formula in the finite dimensional space, it gets

$$\begin{aligned} \phi'(\bar{u}) =& \phi'(u) + \int_0^t \phi''(\bar{u}_m) \cdot [\nu A_m \bar{u}_m - r\bar{u}_m - J(\psi(\bar{u}_m), \bar{u}_m) - \beta\psi_x(\bar{u}_m) + \bar{f}_m(\bar{u}_m)]dt \\ &+ \sigma_1(\int_0^t \phi''(\bar{u}_m) dW^{Q_1} + \frac{1}{2}\int_0^t \phi'''(\bar{u}_m) TrQ_{1,m} ds) \\ &+ \int_0^t \int_{\mathbb{Z}} [\phi'(\bar{u}_m + h_{1,m}) - \phi'(\bar{u}_m) - 2\phi''(\bar{u}_m) h_{1,m}]\nu_1(dz)ds \\ &+ \int_0^t \int_{\mathbb{Z}} [\phi'(\bar{u}_m + h_{1,m}) - \phi'(\bar{u}_m)]\tilde{N}_1(dz)ds. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{\partial}{\partial t}(D_u\bar{U} \cdot h) =& \langle \phi'(u), h\rangle + \mathbb{E}\frac{\sigma_1}{2}\langle \phi'''(\bar{u}_m) TrQ_{1,m}, \eta^h\rangle \\ &+ \mathbb{E}\langle \phi''(\bar{u}) \cdot [\nu A_m \bar{u}_m - r\bar{u}_m - J(\psi(\bar{u}_m, \bar{u}_m) - \beta\psi_x(\bar{u}_m) + \bar{f}_m(\bar{u})], \eta^h\rangle \\ &+ \mathbb{E}\int_{\mathbb{Z}} \langle \phi'(\bar{u}_m + h_{1,m}) - \phi'(\bar{u}_m) - 2\phi''(\bar{u}_m) h_{1,m}, \eta^h\rangle \nu_1(dz) \\ &+ \mathbb{E}\langle \phi'(\bar{u}_m), \nu A_m \eta^h - r\eta^h - J(\psi(\eta^h), \bar{u}_m) - J(\psi(\bar{u}_m), \eta^h) - \beta\psi_x(\eta^h) + \bar{f}_m'(\bar{u}_m)\eta^h\rangle, \end{aligned}$$

which implies from Lemmas A.3 and B.1 that

$$|\frac{\partial}{\partial t}(D_u\bar{U} \cdot h)| \leq C(\|v\| + \|u\|),$$

in the finite dimensional space. Furthermore,

$$\begin{aligned} |\frac{\partial U_1}{\partial t}| &= |\int_0^\infty \mathbb{E}\langle \bar{f}_m(u) - f_m(u,v), \frac{\partial}{\partial t} D_u\bar{U}(t)\rangle ds| \\ &\leq C\int_0^\infty \mathbb{E}\|\bar{f}_m(u) - f_m(u,v)\| ds \\ &\leq C(\|v\| + \|u\|). \end{aligned}$$

The proof is completed. ■

Lemma 5.5. *Assume that the Hypotheses (H1) and (H2) hold. Then,*

$$|R^\varepsilon(\rho^\varepsilon, u, v)| \leq C\varepsilon^{1-\kappa}(\|v\| + \|u\|),$$

with any $\kappa \in (0, \frac{1}{4})$.

Proof. It is easy to obtain

$$\begin{aligned}\mathbb{E}[R^\varepsilon(\rho^\varepsilon, u, v)] =& U^\varepsilon(\rho^\varepsilon, u, v) - U_0(\rho^\varepsilon, u, v) - \varepsilon U_1(\rho^\varepsilon, u, v)\\ =& U^\varepsilon(\rho^\varepsilon, u, v) - \bar{U}(\rho^\varepsilon, u, v) - \varepsilon U_1(\rho^\varepsilon, u, v)\\ =& [U^\varepsilon(\rho^\varepsilon, u, v) - U^\varepsilon(0, u, v)] - [\bar{U}(\rho^\varepsilon, u, v) - \bar{U}(0, u, v)]\\ &- \varepsilon U_1(\rho^\varepsilon, u, v)\\ :=& G_1 + G_2 + G_3.\end{aligned}$$

For G_1, it follows from the Itô formula that

$$\begin{aligned}G_1 =& \mathbb{E}\int_0^{\rho^\varepsilon} \phi'(u_m^\varepsilon) \cdot [\nu A_m u_m^\varepsilon - r u_m^\varepsilon - J(\psi(u_m^\varepsilon), u_m^\varepsilon) - \beta\psi_x(u_m^\varepsilon) + f_m(u_m^\varepsilon, v_m^\varepsilon)]dt\\ &+ \sigma_1(\int_0^{\rho^\varepsilon} \phi'(u_m^\varepsilon)dW^{Q_1} + \frac{1}{2}\int_0^{\rho^\varepsilon} \phi''(u_m^\varepsilon)TrQ_{1,m}ds)\\ &+ \int_0^{\rho^\varepsilon}\int_{\mathbb{Z}}[\phi(u_m^\varepsilon + h_{1,m}) - \phi(u_m^\varepsilon) - 2\phi'(u_m^\varepsilon)h_{1,m}]\nu_1(dz)ds\\ &+ \int_0^{\rho^\varepsilon}\int_{\mathbb{Z}}[\phi(u_m^\varepsilon + h_{1,m}) - \phi(u_m^\varepsilon)]\tilde{N}_1(dz)ds.\end{aligned}$$

And for G_2, we deduce that

$$\begin{aligned}G_2 =& \mathbb{E}\int_0^{\rho^\varepsilon} \phi'(\bar{u}_m) \cdot [\nu A_m \bar{u}_m - r\bar{u}_m - J(\psi(\bar{u}_m), \bar{u}_m) - \beta\psi_x(\bar{u}_m) + \bar{f}_m(\bar{u}_m)]dt\\ &+ \sigma_1(\int_0^{\rho^\varepsilon} \phi'(\bar{u}_m)dW^{Q_{1,m}} + \frac{1}{2}\int_0^{\rho^\varepsilon} \phi''(\bar{u}_m)TrQ_{1,m}ds)\\ &+ \int_0^{\rho^\varepsilon}\int_{\mathbb{Z}}[\phi(\bar{u}_m + h_{1,m}) - \phi(\bar{u}_m) - 2\phi'(\bar{u}_m)h_{1,m}]\nu_1(dz)ds\\ &+ \int_0^{\rho^\varepsilon}\int_{\mathbb{Z}}[\phi(\bar{u}_m + h_{1,m}) - \phi(\bar{u}_m)]\tilde{N}_1(dz)ds.\end{aligned}$$

With the help of Lemmas A.1, A.2, and A.4, we infer that

$$|G_1| + |G_2| \leq C\varepsilon^{1-\kappa}(\|u\| + \|v\|),$$

with $\kappa \in (0, \frac{1}{4})$.

For G_3, it follows from Lemma 5.2 that

$$|G_3| = |\varepsilon U_1(\rho^\varepsilon, u, v)| \leq C\varepsilon(\|u\| + \|v\|).$$

For any $u_m^\varepsilon(T-\rho^\varepsilon,u,v)\in L_m^2(D)$ and $v_m^\varepsilon(T-\rho^\varepsilon,u,v)\in L_m^2(D)$, it implies that

$$\begin{aligned}&|R^\varepsilon(\rho^\varepsilon,u_m^\varepsilon(T-\rho^\varepsilon),v_m^\varepsilon(T-\rho^\varepsilon))|\\ \leq& C(\varepsilon+\frac{1}{\varepsilon^\kappa})(\|u_m^\varepsilon(T-\rho^\varepsilon)\|+\|v_m^\varepsilon(T-\rho^\varepsilon)\|)\\ \leq& C\varepsilon^{1-\kappa}(\|u\|+\|v\|).\end{aligned}$$

The proof is completed. ■

Lemma 5.6. *Under Hypotheses (H1) and (H2), the remainder term satisfies*

$$|R^\varepsilon(T,u,v)|\leq C\varepsilon^{1-\kappa}(\|u\|+\|v\|),\qquad \kappa\in(0,\frac{1}{4}).$$

Proof. Recall that

$$\begin{aligned}R^\varepsilon(T,u,v)=&\mathbb{E}[R^\varepsilon(\rho^\varepsilon,u_m^\varepsilon(T-\rho^\varepsilon),v_m^\varepsilon(T-\rho^\varepsilon))]\\ &+\varepsilon\mathbb{E}[\int_{\rho^\varepsilon}^T((\mathcal{L}_1U_1-\partial_tU_1))(s,u_m^\varepsilon(T-s),v_m^\varepsilon(T-s))ds].\end{aligned}\tag{5.15}$$

Thus from Lemmas 5.3–5.5, we have

$$\begin{aligned}|R^\varepsilon(T,u,v)|\leq&|\mathbb{E}[R^\varepsilon(\rho^\varepsilon,u_m^\varepsilon(T-\rho^\varepsilon),v_m^\varepsilon(T-\rho^\varepsilon))]|\\ &+\varepsilon\mathbb{E}[\int_{\rho^\varepsilon}^T(|(\mathcal{L}_1U_1|+|\partial_tU_1|)ds]\\ \leq& C\varepsilon^{1-\kappa}(\|u\|+\|v\|),\end{aligned}\tag{5.16}$$

where $\rho^\varepsilon\in(0,\varepsilon)$. The proof is completed. ■

5.3. Weak approximation

Theorem 5.1. *(Weak averaging principle) Under the Hypotheses (H1) and (H2), for any $\kappa\in(0,\frac{1}{4})$, $T>0$ and $\phi\in C_b^3(L^2(D),\mathbb{R})$, there exists a constant $C>0$ such that*

$$|\mathbb{E}\phi(u^\varepsilon(T,u,v))-\mathbb{E}\phi(\bar{u}(T,u))|\leq C\varepsilon^{1-\kappa}.$$

Proof. From the asymptotic expansion, we have

$$\mathbb{E}\phi(u_m^\varepsilon)=U_0+\varepsilon U_1+R^\varepsilon,$$

where $U_0=\bar{U}=\mathbb{E}\phi(\bar{u}_m)$. It follows from Lemma 5.1 that

$$|\mathbb{E}\phi(u_m^\varepsilon(T,u,v))-\mathbb{E}\phi(\bar{u}_m(T,u))|\leq\varepsilon|U_1|+|R^\varepsilon|.$$

Moreover, combining Remark 5.2 and Lemma 5.6, it can be deduced that

$$|\mathbb{E}\phi(u_m^\varepsilon(T,u,v))-\mathbb{E}\phi(\bar{u}_m(T,u))|\leq C\varepsilon^{1-\kappa}.$$

Then, it follows from Remark 5.1 that

$$|\mathbb{E}\phi(u^\varepsilon(T,u,v))-\mathbb{E}\phi(\bar{u}(T,u))|\leq C\varepsilon^{1-\kappa},\qquad \kappa\in(0,\frac{1}{4}),$$

which completes the proof of the weak convergence result. ■

6. Conclusions

This work investigates a stochastic quasi-geostrophic flow equation with two time scales, where the slow component is coupled with a fast oscillation governed by a stochastic reaction-diffusion equation, and both are driven by Lévy noises. Under the Hypothesis (H1) and (H2), we obtain the strong averaging principle as in Theorem 4.2 and the weak averaging principle as in Theorem 5.1. These results will further enrich the theoretical framework of the averaging principle. And they also provide a more concise form of stochastic quasi-geostrophic equations in application.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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Appendix

This appendix collects some essential a priori estimates used in the proofs.

Appendix A. The high-order estimates

Lemma A.1. *Under the Hypotheses (H1) and (H2), the solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) and (1.2) satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}(\|u^\varepsilon\|^2 + \|v^\varepsilon\|^2) \le C_T(\|u\|^2 + \|v\|^2),$$

where $(u^\varepsilon, v^\varepsilon)$ is the solution of Eqs (1.1) and (1.2) with the initial date (u, v), and the positive constant C_T only depends on T.

Proof. By the Itô formula, it can be inferred that

$$\begin{aligned}\|u^\varepsilon\|^2 =& \|u\| + 2\int_0^t \langle \Delta u^\varepsilon, u^\varepsilon\rangle ds + 2\int_0^t \langle -ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon), u^\varepsilon\rangle ds\\ &+ 2\int_0^t \langle \sigma_1 dW^{Q_1}, u^\varepsilon\rangle + \sigma_1^2 TrQ_1 t\\ &+ \int_0^t\int_{\mathbb{Z}} (\|u^\varepsilon + h_1\|^2 - \|u^\varepsilon\|^2)\tilde{N}_1(dt, dz)\\ &+ \int_0^t\int_{\mathbb{Z}} (\|u^\varepsilon + h_1\|^2 - \|u^\varepsilon\|^2 - 2\langle h_1, u^\varepsilon\rangle)\nu_1(dz)ds,\end{aligned}$$

which implies

$$\mathbb{E}\|u^\varepsilon\|^2 \le e^{Ct}\|u\|^2 + C\int_0^t e^{C(t-s)}\mathbb{E}\|v^\varepsilon\|^2 ds. \tag{A.1}$$

Applying the Itô formula to $\|v^\varepsilon\|^2$, it can be deduced that

$$\begin{aligned}\|v^\varepsilon\|^2 =& \|v\|^2 + \frac{2}{\varepsilon}\int_0^t \langle \Delta v^\varepsilon, v^\varepsilon\rangle ds + \frac{2}{\varepsilon}\int_0^t \langle g(u^\varepsilon, v^\varepsilon), v^\varepsilon\rangle ds\\ &+ \frac{2}{\sqrt{\varepsilon}}\int_0^t \langle \sigma_2 dW^{Q_2}, v^\varepsilon\rangle + \frac{\sigma_2^2}{\varepsilon} TrQ_2 t\\ &+ \int_0^t\int_{\mathbb{Z}} (\|v^\varepsilon + h_2\|^2 - \|v^\varepsilon\|^2)\tilde{N}_2^\varepsilon(dt, dz)\\ &+ \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{Z}} (\|v^\varepsilon + h_2\|^2 - \|v^\varepsilon\|^2 - 2\langle h_2, v^\varepsilon\rangle)\nu_2(dz)ds,\end{aligned}$$

then

$$\mathbb{E}\|v^\varepsilon\|^2 \leq \|v\|^2 + \mathbb{E}\int_0^t (-\frac{\eta_1}{\varepsilon}\|v^\varepsilon\|^2 + \frac{C}{\varepsilon}\|u^\varepsilon\|^2)ds, \tag{A.2}$$

which implies from the Gronwall inequality that

$$\mathbb{E}\|v^\varepsilon\|^2 \leq e^{-\frac{\eta_1}{\varepsilon}t}\|v\|^2 + \frac{C}{\varepsilon}\int_0^t e^{-\frac{\eta_1}{\varepsilon}(t-s)}\|u^\varepsilon\|^2 ds. \tag{A.3}$$

Moreover, it implies from (A.1) and (A.3) that

$$\begin{aligned}\mathbb{E}\|v^\varepsilon\|^2 \leq& e^{-\frac{\eta_1}{\varepsilon}t}\|v\|^2 + \frac{C}{\varepsilon}\int_0^t e^{-\frac{\eta_1}{\varepsilon}(t-s)}\int_0^s \mathbb{E}\|v^\varepsilon\|^2 d\tau ds\\ \leq& C\int_0^t \mathbb{E}\|v^\varepsilon\|^2 ds.\end{aligned} \tag{A.4}$$

Applying the Gronwall inequality again, from (A.1)–(A.4), we obtain

$$\sup_{t\in[0,T]} \mathbb{E}(\|u^\varepsilon\|^2 + \|v^\varepsilon\|^2) \leq C_T(\|u\|^2 + \|v\|^2),$$

which completes the proof. ∎

Lemma A.2. *Under the Hypotheses (H1) and (H2), for Eqs (1.1) and (1.2), there exists a positive constant C such that for any* $0 < s < t \leq T$ *and any* $\kappa \in (0, \frac{1}{4})$,

$$\mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(s)\|^2 \leq C(\frac{|t-s|^{2(1-\kappa)}}{s^{2(1-\kappa)}} + |t-s|^{2(1-\kappa)} + |t-s|^2),$$

$$\mathbb{E}\|v^\varepsilon(t) - v^\varepsilon(s)\|^2 \leq C(\frac{|t-s|^{2\kappa}}{s^{2\kappa}} + \frac{|t-s|^{2\kappa}}{\varepsilon^{2\kappa}}).$$

Proof. Consider the mild solutions of Eq (1.1)

$$\begin{aligned}u^\varepsilon =& S_t u + \int_0^t S_{t-s}(-ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon))ds\\ &+ \sigma_1\int_0^t S_{t-s}dW^{Q_1} + \int_0^t\int_{\mathbb{Z}} S_{t-s}h_1(u^\varepsilon, z)\tilde{N}_1(ds, dz).\end{aligned} \tag{A.5}$$

Then,

$$\begin{aligned}u^\varepsilon(t) - u^\varepsilon(s) =& (S_t - S_s)u + \int_s^t S_t(-ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon))d\tau\\ &+ \int_0^s (S_{t-\tau} - S_{s-\tau})(-ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon))d\tau\\ &+ \sigma_1\int_s^t S_{t-\tau}dW^{Q_1}(\tau) + \sigma_1\int_0^s (S_{t-\tau} - S_{s-\tau})dW^{Q_1}(\tau)\\ &+ \int_s^t\int_{\mathbb{Z}} S_{t-\tau}h_1(u^\varepsilon, z)\tilde{N}_1(d\tau, dz)\\ &+ \int_0^s\int_{\mathbb{Z}} (S_{t-\tau} - S_{s-\tau})h_1(u^\varepsilon, z)\tilde{N}_1(d\tau, dz)\\ =& J_1 + J_2 + \cdots + J_7.\end{aligned} \tag{A.6}$$

Next, we will estimate them in turn:

$$
\begin{aligned}
\mathbb{E}\|J_1\|^2 &= \mathbb{E}\|(S_t - S_s)u\|^2 \le C\frac{|t-s|^{2(1-\kappa)}}{s^{2(1-\kappa)}}\|u\|, \text{with } \kappa \in (0, \frac{1}{4}),\\
\mathbb{E}\|J_2\|^2 &\le |t-s|\int_s^t \mathbb{E}\|S_{t-\tau}(-ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon))\|^2 d\tau\\
&\le C|t-s|^2,\\
\mathbb{E}\|J_3\|^2 &\le [\int_0^s \|(S_{t-\tau} - S_{s-\tau})(-ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon))\|d\tau]^2\\
&\le C[\int_0^s \frac{(t-s)^{1-\kappa}}{(s-\tau)^{1-\kappa}} e^{-\frac{\lambda}{2}(s-\tau)}\| - ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon)\|^2 d\tau]^2\\
&\le C|t-s|^{2(1-\kappa)},\\
\mathbb{E}\|J_4\|^2 &\le \sigma_1 TrQ_1|t-s|^2,\\
\mathbb{E}\|J_5\|^2 &\le C|t-s|^{2\kappa},
\end{aligned}
$$

$$
\mathbb{E}\|J_6\|^2 \le |t-s|\int_s^t\int_{\mathbb{Z}} \mathbb{E}\|h_1\|^2 \nu_1(dz)d\tau \le C|t-s|^2,
$$

and

$$
\begin{aligned}
\mathbb{E}\|J_7\|^2 &\le \mathbb{E}\int_0^s \|(S_{t-\tau} - S_{s-\tau})h_1\|^2 \nu_1(dz)d\tau\\
&\le C\int_0^s \frac{(t-s)^{1-\kappa}}{(s-\tau)^{1-\kappa}} e^{-\frac{\lambda}{2}(s-\tau)} d\tau\\
&\le C|t-s|^{2(1-\kappa)}.
\end{aligned}
$$

Combining Eq (A.6) with the estimates from J_1 to J_7, we have

$$
\mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(s)\|^2 \le C(\frac{|t-s|^{2(1-\kappa)}}{s^{2(1-\kappa)}} + |t-s|^{2(1-\kappa)} + |t-s|^2).
$$

Using a similar argument, for Eq (1.1), it is easy to get

$$
\mathbb{E}\|v^\varepsilon(t) - v^\varepsilon(s)\|^2 \le C(\frac{|t-s|^{2\kappa}}{s^{2\kappa}} + \frac{|t-s|^{2\kappa}}{\varepsilon^{2\kappa}}).
$$

The proof is completed. ∎

Lemma A.3. *Assume that the initial value u is in $\mathcal{D}(A)^\theta$ with $\theta \in (0, 1]$. Under the Hypotheses (H1) and (H2), for Eq (1.1), there exists a positive constant C such that for any $0 < s < t \le T$ and $\varepsilon > 0$, it holds that*

$$
\mathbb{E}\|Au^\varepsilon\|^2 \le \frac{C}{\varepsilon^\kappa},
$$

with any $\kappa \in (0, \frac{1}{4})$.

Proof. Put $M(t) := -ru^\varepsilon - J(\psi, u^\varepsilon) - \beta\psi_x(u^\varepsilon) + f(u^\varepsilon, v^\varepsilon)$. From (A.5), it can be deduced that

$$\begin{aligned}&\mathbb{E}\|AS_t u\| + \mathbb{E}\|A\int_0^t S_{t-\tau}M(\tau)d\tau\| + \mathbb{E}\|A\int_0^t\int_{\mathbb{Z}} h_1(u^\varepsilon, z)\tilde{N}_1(d\tau, dz))\|\\ \leq& C\|u\|_{D(A)} + \mathbb{E}\|(S_t - I)M(t)\| + \mathbb{E}\|(S_t - I)\int_{\mathbb{Z}} h_1(u^\varepsilon, z)\nu_1(\cdot, dz))\| + K_1 + K_2\\ \leq& C\|u\|_{D(A)} + C,\end{aligned}$$

where $K_1 := \|A\int_0^t S_{t-\tau}[M(\tau) - M(t)d\tau]\|$ and $K_2 := \mathbb{E}\|A\int_0^t\int_{\mathbb{Z}} S_{t-\tau}(h_1(u^\varepsilon(t), z) - h_1(u^\varepsilon(\tau), z))\tilde{N}_1(d\tau, dz)\|$. Then, from Lemma A.2, we have

$$\begin{aligned}K_1 =& \|A\int_0^t S_{t-\tau}[M(\tau) - M(t)d\tau]\|\\ \leq& C\int_0^t \frac{e^{-\frac{\lambda}{2}}}{t-\tau}[\|u^\varepsilon(\tau) - u^\varepsilon(t)\| + \|v^\varepsilon(\tau) - v^\varepsilon(t)\|]d\tau\\ \leq& \frac{C}{\varepsilon^\kappa}.\end{aligned}$$

By the similarly argument, we obtain

$$\begin{aligned}K_2 =& \mathbb{E}\|A\int_0^t\int_{\mathbb{Z}} S_{t-\tau}(h_1(u^\varepsilon(t), z) - h_1(u^\varepsilon(\tau), z))\tilde{N}_1(d\tau, dz)\|\\ \leq& C\|u^\varepsilon(\tau) - u^\varepsilon(t)\|\\ \leq& C.\end{aligned}$$

From [13], we have

$$\mathbb{E}\|A\sigma_1\int_0^t S_{t-\tau}dW^{Q_1}\| \leq C.$$

In all,

$$\mathbb{E}\|Au^\varepsilon\|^2 \leq \frac{C}{\varepsilon^\kappa},$$

which completes the proof. ■

Lemma A.4. *Under the Hypotheses (H1) and (H2), the solution $\bar{u}$ of the averaged Eq (1.3) satisfies*

$$\begin{aligned}&\mathbb{E}\|\bar{u}(t) - \bar{u}(s)\|^2 \leq C(|t-s|^2 + |t-s|^{2(1-\kappa)}),\\ &\mathbb{E}\|A\bar{u}\|^2 \leq C,\end{aligned}$$

for any $\kappa \in (0, \frac{1}{4})$.

Proof. Notice that

$$\begin{aligned}&\int_0^t S_{t-\tau}\bar{f}(\bar{u})d\tau - \int_0^s S_{s-\tau}\bar{f}(\bar{u})d\tau\\ =& \int_s^t S_{t-\tau}\bar{f}(\bar{u})d\tau + \int_0^s (S_{t-\tau} - S_{s-\tau})\bar{f}(\bar{u})d\tau,\end{aligned}$$

with

$$\|\int_s^t S_{t-\tau}\bar{f}(\bar{u})d\tau\|^2 \leq |t-s|\int_s^t \|\bar{f}(\bar{u})\|d\tau \leq C|t-s|^2,$$

$$\mathbb{E}\|\int_0^s (S_{t-\tau}-S_{s-\tau})\bar{f}(\bar{u})d\tau\|^2 \leq C|t-s|^{2(1-\kappa)}.$$

By the same argument with Lemmas A.2 and A.3, it is easy to obtain the results of Lemma A.4. The proof is completed. ∎

Appendix B. The derivative estimates

Recall the finite dimensional approximation problem of the averaged Eq (1.3) that

$$\begin{cases} d\bar{u}_m = [\nu A_m\bar{u}_m - r\bar{u}_m - J(\psi(\bar{u}_m),\bar{u}_m) - \beta\psi_x(\bar{u}_m) + \bar{f}_m(u_m^\varepsilon, v_m^\varepsilon)]dt \\ \qquad + \sigma_1 P_m dW^{Q_1} + \int_{\mathbb{Z}} h_{1,m}(\bar{u}_m,z)\tilde{N}_1(dt,dz), \quad \text{in} \quad D, \\ \bar{u}_m = 0, \qquad \text{on} \quad EqstialD, \\ \bar{u}_m(0) = u. \end{cases} \tag{B.1}$$

Assume that $\eta^h := D_u\bar{u}_m$ with $h \in L_m^2(D)$ admits the derivative equation corresponding to Eq (B.1) as follows:

$$\begin{cases} d\eta^h = [\nu A_m\eta^h - r\eta^h - J(\psi(\bar{u}_m),\eta^h) - J(\psi(\eta^h),\bar{u}_m) - \beta\psi_x(\eta^h) + \\ \qquad \bar{f}'(\bar{u}_m)\eta^h]dt + \int_{\mathbb{Z}} h'_{1,m}(\bar{u}_m,z)\eta^h\tilde{N}_1(dt,dz), \text{in} \quad D, \\ \eta^h = 0, \qquad \text{on} \quad EqstialD, \\ \eta^h(0) = h. \end{cases} \tag{B.2}$$

Lemma B.1. *For any* $0 \leq s < t \leq T$ *and* $0 < \kappa < \frac{1}{4}$, *the solution* η^h *of (B.2) satisfies*

$$\mathbb{E}\|\eta^h\|^2 \leq C\|h\|^2, \tag{B.3}$$

$$\mathbb{E}\|\eta^h(t) - \eta^h(s)\|^2 \leq C|t-s|^{2(1-\kappa)}, \tag{B.4}$$

$$\mathbb{E}\|A\eta^h\|^2 \leq C, \tag{B.5}$$

where the operator A is defined as in Section 2.

Proof. First, from the definition of the Jacobian operator J as in Section 1, it implies that

$$\langle J(\psi(\eta^h),\bar{u}_m),\eta^h\rangle \leq \|J(\psi(\eta^h),\bar{u}_m)\|\|\eta^h\| \leq C + C\|\eta^h\|,$$

and

$$\langle J(\psi(\bar{u}_m),\eta^h),\eta^h\rangle = \langle \beta\psi_x(\eta^h),\eta^h\rangle = 0.$$

Applying the Itô formula, it follows that $\mathbb{E}\|\eta^h\|^2 \leq C\|h\|^2$. Then, using a similar argument as in [6], it is easy to prove the estimates by considering the mild formulation for η^h. Put

$$F_m(t) := -r\eta^h - J(\psi(\bar{u}_m),\eta^h) - J(\psi(\eta^h),\bar{u}_m) - \beta\psi_x(\eta^h) + \bar{f}'(\bar{u}_m)\eta^h,$$

then,

$$\eta^h(t) = S_t h + \int_0^t S_{t-\tau}F_m(\tau)d\tau + \int_0^t \int_{\mathbb{Z}} S_{t-\tau}h'_{1,m}\eta^h \tilde{N}_1(d\tau, dz).$$

We can obtain that

$$\begin{aligned}\eta^h(t) - \eta^h(s) =&(S_t - S_s)h + \int_s^t S_t F_m(\tau)d\tau + \int_0^s (S_{t-\tau} - S_{s-\tau})F_m(\tau)d\tau \\ &+ \int_s^t \int_{\mathbb{Z}} S_t h'_{1,m}\eta^h \tilde{N}_1(d\tau, dz) + \int_0^s \int_{\mathbb{Z}} (S_{t-\tau} - S_{s-\tau})h'_{1,m}\eta^h \tilde{N}_1(d\tau, dz),\end{aligned}$$

moreover,

$$\mathbb{E}\|\eta^h(t) - \eta^h(s)\|^2 \le C|t - s|^{2(1-\kappa)}. \tag{B.6}$$

Finally we also have

$$A\eta^h(t) = AS_t h + A\int_0^t S_{t-\tau}F_m(\tau)d\tau + A\int_0^t \int_{\mathbb{Z}} S_{t-\tau}h'_{1,m}\eta^h \tilde{N}_1(d\tau, dz).$$

Combining the boundness of the function in the finite-dimensional space with inequality (B.6), it can be inferred that

$$\begin{aligned}\mathbb{E}\|A\eta^h(t)\| =&\mathbb{E}\|AS_t h\| + \mathbb{E}\|A\int_0^t S_{t-\tau}F_m(\tau)d\tau\| + \mathbb{E}\|A\int_0^t \int_{\mathbb{Z}} S_{t-\tau}h'_{1,m}\eta^h \tilde{N}_1(d\tau, dz)\| \\ \le& C\|u\|_{D(A)} + \mathbb{E}\|(S_t - I)F(t)\| + \mathbb{E}\|(S_t - I)\int_{\mathbb{Z}} h'_{1,m}(\bar{u}_m, z)\nu_1(\cdot, dz))\| \\ &+ \|A\int_0^t S_{t-\tau}[F(\tau) - F(t)d\tau]\| + \mathbb{E}\|A\int_0^t \int_{\mathbb{Z}} S_{t-\tau}(h'_{1,m}(\bar{u}_m(t), z) \\ &- h'_{1,m}(\bar{u}_m(\tau), z))\eta^h \tilde{N}_1(d\tau, dz)\| \\ \le& C,\end{aligned}$$

The proof is completed. ■

Now, we introduce the second derivative of the solution $\bar{u}_m$ of Eq (B.1) with respect to the initial value u in the directions h and l, which admits

$$\begin{aligned}d\varsigma^{h,l} =& [\nu A_m\varsigma^{h,l} - r\varsigma^{h,l} - J(\psi(\bar{u}_m), \varsigma^{h,l}) - J(\psi(\varsigma^{h,l}), \bar{u}_m) - \beta\psi_x(\varsigma^{h,l}) + \\ &\bar{f}'(\bar{u}_m)\eta^h\eta^l + \bar{f}''(\bar{u}_m)\varsigma^{h,l}]dt + \int_{\mathbb{Z}} (h'_{1,m}(\bar{u}_m)\varsigma^{h,l} + h''_{1,m}(\bar{u}_m)\eta^h\eta^l)z\tilde{N}_1(dt, dz).\end{aligned} \tag{B.7}$$

Lemma B.2. *Under the Hypotheses (H1) and (H2), for Eq (B.7), it holds that*

$$\|\varsigma^{h,l}\|^2 \le C\|h\|^2\|l\|^2,$$

where $h, l \in L^2_m(D)$.

Proof. Similarly as Lemma B.1, it is easy to obtain Lemma B.2. Here, we omit it. ■

Lemma B.3. *Under the Hypotheses (H1) and (H2), for Eq (B.1), it holds that*

$$\|D_u\bar{U}\|^2 \le C, \quad \|D^2_{uu}\bar{U}\|^2 \le C, \quad \|D^3_{uuu}\bar{U}\|^2 \le C,$$

where $\bar{U} = \phi(\bar{u}_m)$.

Proof. For any $u, h, l, \tilde{l} \in L^2_m(D)$, it can be inferred from Lemmas B.1 and B.2 that

$$|D_u \bar{U} \cdot h| = |\mathbb{E}\langle \phi'(\bar{u}_m), \eta^h \rangle| \le C\|h\|,$$

and

$$\begin{aligned} |D_{uu} \bar{U} \cdot (h, k)| &= |\mathbb{E}[\phi''(\bar{u}_m \cdot (\eta^h, \eta^h)) + \phi'(\bar{u}_m) \cdot \varsigma^{h,l}]| \\ &\le C\|h\| \cdot \|l\|. \end{aligned}$$

Further, it follows that

$$|D_{uuu} \bar{U} \cdot (h, l, \tilde{l})| \le C\|h\| \cdot \|l\| \cdot \|\tilde{l}\|,$$

which completes the proof. ■