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Research article

A novel mathematical programming method for interval-valued Pythagorean fuzzy MCGDM with incomplete weight information

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Abstract: The interval-valued Pythagorean fuzzy set possesses a strong capability to characterize uncertainty and fuzziness, and has been widely applied to multi-criteria group decision-making. However, extant studies seldom consider the fuzzy truth degrees in pairwise comparisons of alternatives and often overlook incomplete information regarding criteria weights. Therefore, this paper investigated interval-valued Pythagorean fuzzy multi-criteria group decision-making, incorporating both interval-valued Pythagorean fuzzy truth degrees for pairwise comparisons and incomplete information on criterion weights. First, recognizing that decision-makers may have different weights under different criteria, their weights with respect to each criterion were determined based on the relative closeness of each alternative to the positive ideal solution and the negative ideal solution under that criterion. To derive the criteria weights, this paper defined the interval-valued Pythagorean fuzzy positive ideal solution and the interval-valued Pythagorean fuzzy negative ideal solution, and established the interval-valued Pythagorean fuzzy group consistency index and inconsistency index. By minimizing the group inconsistency index, a bi-objective interval-valued Pythagorean fuzzy programming model was constructed and skillfully transformed into a linear programming model to compute the criteria weights. Subsequently, the relative closeness degree of each alternative for each decision-maker was calculated and used to generate individual rankings of the alternatives. To obtain a collective ranking, a multi-objective allocation model was established and then converted into a single-objective programming model for the solution. Finally, a wireless network selection example was provided to demonstrate the effectiveness of the proposed method.

Keywords: interval-valued Pythagorean fuzzy set; multi-criteria group decision-making; incomplete information
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1. Introduction

Decision-making is a complex cognitive process through which individuals make choices in various situations. This process entails collecting relevant information, assessing potential alternatives, forming judgments, and reaching conclusions. Since real-world decision-making often involves multiple criteria, an integrated approach known as multi-criteria decision-making (MCDM) has been developed to tackle such challenges. In modern contexts, decision-making problems typically encompass a range of considerations—economic, social, human, ecological, and more—making it essential to involve multiple participants in the process. As a result, scenarios involving a group of decision-makers are referred to as multi-criteria group decision-making (MCGDM). However, given the inherent complexity of these problems and the often imprecise knowledge of the decision-makers, the information used in decisions is frequently characterized by uncertainty. Traditional MCGDM methods, which rely on precise data, often prove inadequate or impractical in addressing such uncertain decision-making environments. Therefore, scholars proposed various fuzzy information, such as the fuzzy set (FS) [1], intuitionistic fuzzy set (IFS) [2], interval-valued IFS (IVIFS) [3], Pythagorean fuzzy set (PFS) [4, 5], interval-valued Pythagorean fuzzy set (IVPFS) [6], and so on. These different types of fuzzy information can capture the uncertainty and fuzziness of decision-makers (DMs).

Pythagorean fuzzy sets and their extensions have become a research hotspot due to their enhanced ability to flexibly express the uncertain relationship between membership and non-membership degrees. Asif et al. [7] integrated the Hamacher aggregation operator with a PFS to propose interactive aggregation operators, such as Pythagorean fuzzy Hamacher interactive weighted averaging and ordered weighted averaging operators. Palanikumar et al. [8] further introduced the redefined square root interval-valued normal Pythagorean fuzzy set, constructing operators like redefined square root interval-valued normal Pythagorean fuzzy weighted averaging and weighted geometric operators. This work refines the algebraic operational rules in an interval-valued fuzzy environment. Tahir et al. [9] integrated PFS, soft set, and hypersoft set theories to propose frameworks for Pythagorean soft sets and Pythagorean hypersoft sets, overcoming the limitations of traditional soft sets in uncertainty expression. Their application in areas like technology selection and cloud configuration demonstrates the framework's advantage in handling multi-criteria fuzzy information.

Furthermore, Razak et al. [10] combined the interval-valued Pythagorean neutrosophic set with the comprehensive distance-based ranking method, innovatively incorporating 5-point and 7-point linguistic scales. This addresses the previous lack of linguistic variables in the interval-valued Pythagorean neutrosophic set, providing a decision-making tool better aligned with human subjective judgment for e-commerce strategy selection and IT supplier evaluation. Kamari et al. [11] proposed a Pythagorean neutrosophic TOPSIS-VIKOR integrated framework, which optimizes distance measurement through a flexible indeterminacy quantifier. This enhances score differentiation and ranking stability in digital supplier selection for manufacturing SMEs. Collectively, these studies enrich the theoretical system of decision-making methods under uncertainty.

In group decision-making and large-scale group decision-making, determining weights, conducting cluster analysis, and the consensus-reaching process are key research areas. Jiang et al. [12] determined weights based on the Shapley value function. This method quantified the interconnections and uncertainty among decision-makers through a trust propagation model, and generated objective weights by calculating each decision-maker's marginal contribution within different coalitions, thereby comprehensively reflecting their influence in large-scale group decision-making. Liu et al. [13] did not directly calculate explicit weights. Instead, they integrated expert knowledge and experience organically into the retrieval and adaptation process of case-based reasoning. Experts' judgments directly influenced the similarity calculation between cases, achieving an indirect and dynamic form of "weighting" that avoided the subjective parameter setting. The best-worst method (BWM), valued for its fewer pairwise comparisons and high reliability, has been widely extended. Wan et al. [14] proposed an intuitionistic fuzzy BWM based on additive consistency for intuitionistic fuzzy preference relations. They used a goal programming model to derive optimal weights and a 0-1 integer programming model to aggregate group rankings, significantly improving group decision-making consistency. Their subsequent research extended the BWM to a hesitant multiplicative environment, proposing a hesitant multiplicative BWM that derives weights via a mixed goal programming model [15]. Chen et al. [16] innovated an interval-valued intuitionistic fuzzy group BWM, which determines global criteria weights by solving a group weight assignment model once, substantially reducing the computational load in evaluating emergency medical waste disposal modes. Luo et al. [17] combined trapezoidal intuitionistic fuzzy numbers with BWM constraints to propose the TrIF-BWMC-HDEA framework. This ensures the prioritization of key criteria while supporting heterogeneous data like crisp values and intervals, providing an effective tool for rescue route assessment.

Addressing the complexity of large-scale group decision-making, Wan et al. [18], focusing on the role of trust relationships in social network large-scale group decision-making, designed an improved grey clustering algorithm that fuses preference similarity and trust relationships. They proposed a two-stage consensus-reaching process incorporating self-adjustment and subgroup binding force. By defining weights for decision-makers and subgroups, the consensus process better aligns with real-world scenarios. Another study by Wan et al. [19] focused on ranking consensus, proposing a dual-strategy consensus-reaching process based on probabilistic linguistic term sets. It designs differentiated adjustment strategies for decision-makers with different similarity characteristics, achieving efficient ranking consensus by optimizing decision matrices. These studies provide new approaches to tackling the scale and consensus challenges in large-scale group decision-making.

These decision-making methods have been widely applied across fields such as healthcare, manufacturing, e-commerce, and insurance. Chen et al. [16] integrated the group BWM, regret theory, and the multi-attributive border approximation area comparison method to build a heterogeneous decision-making framework. This effectively handles criteria weight determination, regret-rejoice matrix construction, and optimal alternative selection in evaluating emergency medical waste disposal modes. Wan et al. [20] applied a dual-similarity-based consensus mechanism to quality function deployment to handle heterogeneous linguistic preference relations, improving group decision-making quality in product design. Another study by Wan et al. [21] proposed a probabilistic linguistic group bi-matrix game model, opening a new path for the intersection of game theory and group decision-making. These application studies not only validate the effectiveness of theoretical methods but also demonstrate the core value of decision theory in solving practical problems.

While existing methods can effectively address certain MCDM or MCGDM problems involving interval-valued Pythagorean fuzzy (IVPF) information, they face significant and interconnected limitations when confronted with realistic scenarios characterized by incomplete decision information, which encompasses not only incomplete weight information but also, more fundamentally, incomplete or missing pairwise comparison preferences provided by experts.

(1) Current research predominantly focuses on single decision-maker settings or assumes that DMs can provide complete evaluation matrices. However, in practice, it is often more feasible and less cognitively demanding for experts to express partial pairwise preferences between alternatives rather than fully rating all against all criteria. As decision complexity increases, synthesizing such incomplete yet rich preference information within an MCGDM framework becomes crucial for credible outcomes. Thus, constructing a dedicated IVPF MCGDM method that can operate directly on incomplete preference judgments presents a critical yet underexplored task.

(2) The reliable derivation of both DM and criteria weights is critical for MCGDM. A prevalent limitation is that most methods require these weights as pre-defined inputs, even when dealing with incomplete preferences. In scenarios where experts only provide partial pairwise comparisons, the very data needed to objectively infer these weights are inherently sparse. Therefore, a method that can simultaneously and objectively derive both weight sets directly from the incomplete preference judgments themselves is essential to ensure fairness and reduce arbitrariness.

(3) The linear programming technique for multidimensional analysis of preference (LINMAP) [22] is a well-established method precisely for MCDM with incomplete preference information, as it constructs an optimal solution consistent with given pairwise judgments. Although extended to various fuzzy environments [23, 24], its application within the IVPF setting, particularly for MCGDM with incomplete preferences, remains unexplored. Furthermore, most existing LINMAP-based models rely solely on the distance from the positive ideal solution (PIS), neglecting the negative ideal solution (NIS). This omission can be particularly detrimental when preferences are scarce, as the contrastive information from the NIS is vital for stabilizing the solution and enhancing discriminative power in data-sparse conditions.

Hence, the study introduces an innovative IVPF programming model designed to address MCGDM problems. First, the weights of decision-makers are determined objectively using a technique for order preference by similarity to ideal solution (TOPSIS)-based approach that considers the relative proximity of alternatives to the PIS and NIS. Within the LINMAP framework, specific indices are then defined to measure the levels of agreement and disagreement within the group. These indices are then used to establish an IVPF model for determining weights of criteria by minimizing group inconsistency. The model is solved by converting it into an eight-objective program, which is subsequently converted into a linear program for computational tractability. Next, individual alternatives are ranked based on their calculated closeness coefficients. An aggregation model is then constructed to synthesize these individual rankings into a final group decision. Finally, the effectiveness of the proposed method is demonstrated through a case study on network selection.

The main contributions of this paper can be summarized as follows:

(1) This research advances preference modeling by employing IVPFSs to characterize DMs' fuzzy truth degrees in pairwise alternative comparisons. As an extension of Pythagorean fuzzy sets,

IVPFSs provide a more powerful and flexible framework that allows DMs to fully express their opinions while better capturing the inherent uncertainty and fuzziness in their judgments, leading to a more nuanced and realistic representation of preferences.

(2) A core methodological contribution is the development of a novel, objective framework for determining both DMs' and criteria weights. First, DMs' weights are calculated by defining relative closeness degrees inspired by the TOPSIS method, incorporating distances to both PIS and NIS to ensure reliability and avoid subjective randomness. Second, a bi-objective IVPF programming model is constructed to determine criteria weights. This model uniquely minimizes group inconsistency by considering both PIS and NIS, offering a more reliable and effective approach compared to methods that overlook criteria weights or rely on aggregation operators. The model is then transformed into a solvable linear program, enhancing the logical rigor and credibility of the derived weights.

(3) To obtain a final collective ranking, this paper constructs a multi-objective assignment model designed to aggregate individual alternative rankings. This approach moves beyond simply aggregating original evaluation matrices, thereby effectively avoiding the information loss and potential biases inherent in the direct aggregation of fuzzy evaluations.

The subsequent sections of this paper are organized as follows. In Section 2, an overview of the IFS, IVIFS, PFS, IVPFS, and certain operational laws of the IVPFN are presented. In Section 3, the discussion pertains to IVPF MCGDM problems incorporating IVPF truth degrees. Moving to Section 4, a novel IVPF mathematical programming model is introduced, specifically developed to tackle MCGDM problems within an IVPF framework. Section 5 includes an illustrative example and conducts an in-depth comparative analysis against the IVPF-ELECTRE, IVPF-TOPSIS, and IVPF-TODIM methods. Section 6 demonstrates the validity and practicality of the proposed method within a decision support system. Finally, Section 7 summarizes the primary content encapsulated in this paper.

2. Preliminaries

This section serves as the theoretical foundation, presenting the essential concepts of IVPFSs and IVPFNs. Specifically, it covers their mathematical definitions, fundamental operational rules, methods for comparison and ranking, and the calculation of distance measures between them. The entire analytical framework and methodological development of this study are built upon these core constructs.

2.1. Interval-valued Pythagorean fuzzy sets

Definition 2.1. [6] Let X be a universe of discourse. A PFS P of X is given by

$$P = \{< x, u_P(x), v_P(x) > \ |x \in X\}, \tag{1}$$

where $u_P: X \to [0,1]$ and $v_P: X \to [0,1]$ refer to the membership function and the non-membership function, respectively, of the element $x \in X$ to P. They must satisfy the condition that $0 \leq u_P^2(x) + v_P^2(x) \leq 1$. Additionally, $\pi_P(x) = \sqrt{1 - u_P^2(x) - v_P^2(x)}$ is referred to as the indeterminacy function for x to P.

For simplicity, we can refer to $P = (u_P, v_P)$ as a PFN [5].

Definition 2.2. [6] Let X be a *universe* of discourse. An IVPFS α of X is given by

$$\alpha = \{< x, [u_\alpha^L(x), u_\alpha^U(x)], [v_\alpha^L(x), v_\alpha^U(x)] > | x \in X\}. \quad (2)$$

Element α has two corresponding intervals, namely the membership $[u_\alpha^L(x), u_\alpha^U(x)] \subseteq [0,1]$ and non-membership $[v_\alpha^L(x), v_\alpha^U(x)] \subseteq [0,1]$, which fulfill the condition $(u_\alpha^U(x))^2 + (v_\alpha^U(x))^2 \leq 1$. Moreover, element $x \in X$ is associated with the indeterminacy $[\pi_\alpha^L(x), \pi_\alpha^U(x)]$, where $\pi_\alpha^L(x) = \sqrt{1 - (u_\alpha^U(x))^2 - (v_\alpha^U(x))^2}$ and $\pi_\alpha^U(x) = \sqrt{1 - (u_\alpha^L(x))^2 - (v_\alpha^L(x))^2}$.

An IVPFN is represented by a pair $\alpha = ([u_\alpha^L(x), u_\alpha^U(x)], [v_\alpha^L(x), v_\alpha^U(x)])$, which is denoted as $\alpha = ([u_\alpha^L, u_\alpha^U], [v_\alpha^L, v_\alpha^U])$ [25], where $[u_\alpha^L, u_\alpha^U] \subseteq [0,1]$, $[v_\alpha^L, v_\alpha^U] \subseteq [0,1]$, and $(u_\alpha^U)^2 + (v_\alpha^U)^2 \leq 1$.

Definition 2.3. [6] Let $\alpha = ([u_\alpha^L, u_\alpha^U], [v_\alpha^L, v_\alpha^U])$, $\alpha_1 = ([u_{\alpha_1}^L, u_{\alpha_1}^U], [v_{\alpha_1}^L, v_{\alpha_1}^U])$, and $\alpha_2 = ([u_{\alpha_2}^L, u_{\alpha_2}^U], [v_{\alpha_2}^L, v_{\alpha_2}^U])$ be three IVPFNs, $\lambda > 0$. Then,

(1) $\alpha^c = ([v_\alpha^L, v_\alpha^U], [u_\alpha^L, u_\alpha^U])$,

(2) $\alpha_1 \oplus \alpha_2 = ([\sqrt{(u_{\alpha_1}^L)^2 + (u_{\alpha_2}^L)^2 - (u_{\alpha_1}^L)^2(u_{\alpha_2}^L)^2}, \sqrt{(u_{\alpha_1}^U)^2 + (u_{\alpha_2}^U)^2 - (u_{\alpha_1}^U)^2(u_{\alpha_2}^U)^2}], [v_{\alpha_1}^L v_{\alpha_2}^L, v_{\alpha_1}^U v_{\alpha_2}^U])$,

(3) $\alpha_1 \otimes \alpha_2 = ([u_{\alpha_1}^L u_{\alpha_2}^L, u_{\alpha_1}^U u_{\alpha_2}^U], [\sqrt{(v_{\alpha_1}^L)^2 + (v_{\alpha_2}^L)^2 - (v_{\alpha_1}^L)^2(v_{\alpha_2}^L)^2}, \sqrt{(v_{\alpha_1}^U)^2 + (v_{\alpha_2}^U)^2 - (v_{\alpha_1}^U)^2(v_{\alpha_2}^U)^2}])$,

(4) $\lambda\alpha = ([\sqrt{1 - (1 - (u_\alpha^L)^2)^\lambda}, \sqrt{1 - (1 - (u_\alpha^U)^2)^\lambda}], [(v_\alpha^L)^\lambda, (v_\alpha^U)^\lambda])$,

(5) $\alpha^\lambda = ([(u_\alpha^L)^\lambda, (u_\alpha^U)^\lambda], [\sqrt{1 - (1 - (v_\alpha^L)^2)^\lambda}, \sqrt{1 - (1 - (v_\alpha^U)^2)^\lambda}])$.

Theorem 2.1. [6] Let $\alpha = ([u_\alpha^L, u_\alpha^U], [v_\alpha^L, v_\alpha^U])$ and $\alpha_i = ([u_{\alpha_i}^L, u_{\alpha_i}^U], [v_{\alpha_i}^L, v_{\alpha_i}^U])(i = 1,2)$ be three IVPFNs, $\lambda, \lambda_1, \lambda_2 > 0$. Then,

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$,

(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$,

(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$,

(4) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$,

(5) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$,

(6) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1 + \lambda_2}$.

Definition 2.4. [6] Let $\alpha_i = ([u_{\alpha_i}^L, u_{\alpha_i}^U], [v_{\alpha_i}^L, v_{\alpha_i}^U])(i = 1,2)$ be two IVPFNs, a nature quasi-ordering on IVPFNs is defined as $\alpha_1 \succ \alpha_2$ if and only if $u_{\alpha_1}^L \geq u_{\alpha_2}^L$, $u_{\alpha_1}^U \geq u_{\alpha_2}^U$, $v_{\alpha_1}^L \leq v_{\alpha_2}^L$, and $v_{\alpha_1}^U \leq v_{\alpha_2}^U$, where " $\succ$ " means "bigger than or indifferent to".

2.2. Distance of IVPFNs

This paper introduces a novel method for evaluating the relative positioning between a pair of IVPFNs, based on the distance metric used for interval-valued intuitionistic fuzzy values [3].

Definition 2.5. Let $\alpha_i = ([u_{\alpha_i}^L, u_{\alpha_i}^U], [v_{\alpha_i}^L, v_{\alpha_i}^U])(i = 1,2)$ be two IVPFNs, $q_1 \geq 1$. The Minkowski distance between α_1 and α_2 is defined as follows:

$$\begin{aligned} d_{q_1}(\alpha_1, \alpha_2) = \frac{1}{4}(&|(u_{\alpha_1}^L)^2 - (u_{\alpha_2}^L)^2|^{q_1} + |(u_{\alpha_1}^U)^2 - (u_{\alpha_2}^U)^2|^{q_1} + |(v_{\alpha_1}^L)^2 - (v_{\alpha_2}^L)^2|^{q_1} \\ &+ |(v_{\alpha_1}^U)^2 - (v_{\alpha_2}^U)^2|^{q_1} + |(\pi_{\alpha_1}^L)^2 - (\pi_{\alpha_2}^L)^2|^{q_1} + |(\pi_{\alpha_1}^U)^2 - (\pi_{\alpha_2}^U)^2|^{q_1})^{\frac{1}{q_1}}. \end{aligned} \quad (3)$$

The indeterminacy of element $\alpha_i(i=1,2)$ is denoted by $\bar{\pi}_{\alpha_i}=[\pi^L_{\alpha_i},\pi^U_{\alpha_i}]=[\sqrt{1-(u^U_{\alpha_i})^2-(v^U_{\alpha_i})^2},\sqrt{1-(u^L_{\alpha_i})^2-(v^L_{\alpha_i})^2}](i=1,2)$.

It can be easily verified that $d_{q_1}(\alpha_1,\alpha_2)$ satisfies the axioms of distance:

(1) Non-negativity: $d_{q_1}(\alpha_1,\alpha_2)\geq 0$,

(2) Symmetry: $d_{q_1}(\alpha_1,\alpha_2)=d_{q_1}(\alpha_2,\alpha_1)$,

(3) Triangle inequality: If $\alpha_1\succ\alpha_2\succ\alpha_3$, then $d_{q_1}(\alpha_1,\alpha_3)\geq max\{d_{q_1}(\alpha_1,\alpha_2),d_{q_1}(\alpha_2,\alpha_3)\}$.

When $q_1=1$ is present, Eq (3) becomes degenerated to the Hamming distance within the range of [26] as shown below:

$$d_1(\alpha_1,\alpha_2)=\frac{1}{4}(|(u^L_{\alpha_1})^2-(u^L_{\alpha_2})^2|+|(u^U_{\alpha_1})^2-(u^U_{\alpha_2})^2|+|(v^L_{\alpha_1})^2-(v^L_{\alpha_2})^2|$$
$$+|(v^U_{\alpha_1})^2-(v^U_{\alpha_2})^2|+|(\pi^L_{\alpha_1})^2-(\pi^L_{\alpha_2})^2|+|(\pi^U_{\alpha_1})^2-(\pi^U_{\alpha_2})^2|).$$

When $q_1=2$ is present, Eq (3) can be converted into the Euclidean distance as follows:

$$d_2(\alpha_1,\alpha_2)=\frac{1}{4}(|(u^L_{\alpha_1})^2-(u^L_{\alpha_2})^2|^2+|(u^U_{\alpha_1})^2-(u^U_{\alpha_2})^2|^2+|(v^L_{\alpha_1})^2-(v^L_{\alpha_2})^2|^2$$
$$+|(v^U_{\alpha_1})^2-(v^U_{\alpha_2})^2|^2+|(\pi^L_{\alpha_1})^2-(\pi^L_{\alpha_2})^2|^2+|(\pi^U_{\alpha_1})^2-(\pi^U_{\alpha_2})^2|^2)^{\frac{1}{2}}.$$

When $q_1\to+\infty$ is present, Eq (3) can be converted into the Chebyshev distance as shown below:

$$d_{+\infty}(\alpha_1,\alpha_2)=max\{\frac{1}{4}(|(u^L_{\alpha_1})^2-(u^L_{\alpha_2})^2|,|(u^U_{\alpha_1})^2-(u^U_{\alpha_2})^2|,|(v^L_{\alpha_1})^2-(v^L_{\alpha_2})^2|,$$
$$|(v^U_{\alpha_1})^2-(v^U_{\alpha_2})^2|,|(\pi^L_{\alpha_1})^2-(\pi^L_{\alpha_2})^2|,|(\pi^U_{\alpha_1})^2-(\pi^U_{\alpha_2})^2|)\}.$$

2.3. Distance of IVPFSs

The distance between two IVPFSs is a fundamental concept for measuring their similarity or dissimilarity. Since an IVPFS is composed of IVPFNs, the overall distance between two IVPFSs is typically defined based on the distances between their corresponding IVPFNs. The distance between two IVPFSs can be defined with respect to the distance between two IVPFNs as follows.

Definition 2.6. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a universe of discourse. Let A and B be two IVPFSs in X, $q_2\geq 1$, and then the Minkowski distance between A and B is defined as

$$d_{q_2}(A,B)=[\sum_{j=1}^{n}\frac{1}{4n}(|(u^L_A(x_j))^2-(u^L_B(x_j))^2|^{q_2}+|(u^U_A(x_j))^2-(u^U_B(x_j))^2|^{q_2}$$
$$+|(v^L_A(x_j))^2-(v^L_B(x_j))^2|^{q_2}+|(v^U_A(x_j))^2-(v^U_B(x_j))^2|^{q_2} \quad (4)$$
$$+|(\pi^L_A(x_j))^2-(\pi^L_B(x_j))^2|^{q_2}+|(\pi^U_A(x_j))^2-(\pi^U_B(x_j))^2|^{q_2}]^{\frac{1}{q_2}}.$$

Definition 2.7. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse. Let A and B be two IVPFSs in X. A is included in B (denoted by $A \subseteq B$) if and only if for every $x_j \in X$, the following inequalities hold simultaneously:

$$u_A^L(x_j) \le u_B^L(x_j), \qquad u_A^U(x_j) \le u_B^U(x_j), \qquad v_A^L(x_j) \ge v_B^L(x_j), \qquad v_A^U(x_j) \ge v_B^U(x_j).$$

Theorem 2.2. Let A, B, and C be three IVPFSs in X, $q_2 \ge 1$, then the following axiom holds:
(1) Non-negativity: $d_{q_2}(A, B) \ge 0$,
(2) Symmetry: $d_{q_2}(A, B) = d_{q_2}(B, A)$,
(3) Triangle inequality: If $A \subseteq B \subseteq C$, then $d_{q_2}(A, C) \ge max\{ d_{q_2}(A, B), d_{q_2}(B, C)\}$.
Proof. It follows directly from Definition 2.6 that (1) and (2) are satisfied.
(3) Assume $A \subseteq B \subseteq C$, by Definition 2.7, we have for all $x_j \in X$:

$$u_A^L(x_j) \le u_B^L(x_j) \le u_C^L(x_j), \quad u_A^U(x_j) \le u_B^U(x_j) \le u_C^U(x_j),$$

$$v_A^L(x_j) \ge v_B^L(x_j) \ge v_C^L(x_j), \quad v_A^U(x_j) \ge v_B^U(x_j) \ge v_C^U(x_j),$$

and consequently for the hesitancy degrees:

$$\pi_A^L(x_j) \ge \pi_B^L(x_j) \ge \pi_C^L(x_j), \quad \pi_A^U(x_j) \ge \pi_B^U(x_j) \ge \pi_C^U(x_j).$$

Since all squared differences are non-negative, and using the monotonicity of the function $f(t) = t^{q_2}$ for $t \ge 0$ and $q_2 \ge 1$, we have

$$|(u_A^L(x_j))^2 - (u_C^L(x_j))^2|^{q_2} \ge |(u_A^L(x_j))^2 - (u_B^L(x_j))^2|^{q_2},$$

$$|(u_A^L(x_j))^2 - (u_C^L(x_j))^2|^{q_2} \ge |(u_B^L(x_j))^2 - (u_C^L(x_j))^2|^{q_2},$$

$$|(u_A^U(x_j))^2 - (u_C^U(x_j))^2|^{q_2} \ge |(u_A^U(x_j))^2 - (u_B^U(x_j))^2|^{q_2},$$

$$|(u_A^U(x_j))^2 - (u_C^U(x_j))^2|^{q_2} \ge |(u_B^U(x_j))^2 - (u_C^U(x_j))^2|^{q_2},$$

$$|(v_A^L(x_j))^2 - (v_C^L(x_j))^2|^{q_2} \ge |(v_A^L(x_j))^2 - (v_B^L(x_j))^2|^{q_2},$$

$$|(v_A^L(x_j))^2 - (v_C^L(x_j))^2|^{q_2} \ge |(v_B^L(x_j))^2 - (v_C^L(x_j))^2|^{q_2},$$

$$|(v_A^U(x_j))^2 - (v_C^U(x_j))^2|^{q_2} \ge |(v_A^U(x_j))^2 - (v_B^U(x_j))^2|^{q_2},$$

$$|(v_A^U(x_j))^2 - (v_C^U(x_j))^2|^{q_2} \ge |(v_B^U(x_j))^2 - (v_C^U(x_j))^2|^{q_2},$$

$$|(\pi_A^L(x_j))^2 - (\pi_C^L(x_j))^2|^{q_2} \geq |(\pi_A^L(x_j))^2 - (\pi_B^L(x_j))^2|^{q_2},$$

$$|(\pi_A^L(x_j))^2 - (\pi_C^L(x_j))^2|^{q_2} \geq |(\pi_B^L(x_j))^2 - (\pi_C^L(x_j))^2|^{q_2},$$

$$|(\pi_A^U(x_j))^2 - (\pi_C^U(x_j))^2|^{q_2} \geq |(\pi_A^U(x_j))^2 - (\pi_B^U(x_j))^2|^{q_2},$$

$$|(\pi_A^U(x_j))^2 - (\pi_C^U(x_j))^2|^{q_2} \geq |(\pi_B^U(x_j))^2 - (\pi_C^U(x_j))^2|^{q_2}.$$

Summing over all $j = 1,2,\ldots,n$ and multiplying by $\frac{1}{4n}$, we obtain:

$$\sum_{j=1}^{n} \frac{1}{4n} (|(u_A^L(x_j))^2 - (u_C^L(x_j))^2|^{q_2} + \ldots + |(\pi_A^U(x_j))^2 - (\pi_C^U(x_j))^2|^{q_2})$$

$$\geq \sum_{j=1}^{n} \frac{1}{4n} (|(u_A^L(x_j))^2 - (u_B^L(x_j))^2|^{q_2} + \ldots + |(\pi_A^U(x_j))^2 - (\pi_B^U(x_j))^2|^{q_2})$$

and likewise for the pair B, C. Taking the $\frac{1}{q_2}$-th power preserves the inequalities, hence:

$$d_{q_2}(A,C) \geq d_{q_2}(A,B) \text{ and } d_{q_2}(A,C) \geq d_{q_2}(B,C).$$

Therefore, $d_{q_2}(A,C) \geq max\{d_{q_2}(A,B), d_{q_2}(B,C)\}$.

Definition 2.8. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse. Let A and B be two IVPFSs in X, $q_2 \geq 1$. A weighted Minkowski distance between A and B is defined as

$$\begin{aligned} \bar{d}_{q_2}(A,B) = [\sum_{j=1}^{n} \frac{\omega_j}{4} (&|(u_A^L(x_j))^2 - (u_B^L(x_j))^2|^{q_2} + |(u_A^U(x_j))^2 - (u_B^U(x_j))^2|^{q_2} \\ &+ |(v_A^L(x_j))^2 - (v_B^L(x_j))^2|^{q_2} + |(v_A^U(x_j))^2 - (v_B^U(x_j))^2|^{q_2} \\ &+ |(\pi_A^L(x_j))^2 - (\pi_B^L(x_j))^2|^{q_2} + |(\pi_A^U(x_j))^2 - (\pi_B^U(x_j))^2|^{q_2}]^{\frac{1}{q_2}}. \end{aligned} \tag{5}$$

Here, weight ω_j of x_j satisfies conditions $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \geq 0$ $(j = 1,2,\cdots,n)$.

The exponent q_2 in the (weighted) Minkowski distance is a norm-order parameter that determines "how differences are aggregated". When considering $q_2 = 1$, $q_2 = 2$, and $q_2 \to +\infty$, the respective (weighted) distances are referred to as the (weighted) Hamming distance, (weighted) Euclidean distance, and (weighted) Chebyshev distance. The smaller the value of q_2, the more the distance focuses on the accumulation of multiple small differences; the larger the value of q_2, the more sensitive the distance becomes to a single prominent difference. In the decision-making process, the exponent q_2 in the (weighted) Minkowski distance reflects the DM's value-risk orientation: $q_2 = 1$ represents full compensation between dimensions; $q_2 = 2$ offers balanced compensation; while $q_2 \to +\infty$ indicates strict non-compensatory logic where any critical deficiency dominates.

When addressing practical decision-making problems, the determination of q_2 must integrate the characteristics of the problem, the data structure, and the decision logic. Typically, the following approach can be adopted: First, analyze the decision attributes, if the attributes are mutually substitutable, a smaller q_2 should be selected; if there are critical constraint indicators, a larger q_2 or even $+\infty$ should be chosen. Second, conduct sensitivity analysis to test the stability of alternative rankings under typical values of q_2: if the rankings remain consistent, the choice of q_2 offers greater flexibility; if the rankings fluctuate significantly, decision preferences must be further clarified to determine the value of q_2 that best aligns with the decision-maker's intent.

3. IVPF MCGDM with IVPF truth degrees

This section delineates the interval-valued Pythagorean fuzzy multi-criteria group decision-making problem under investigation. First, an overview of the MCGDM within the IVPF context is presented, along with the associated data normalization procedures. Subsequently, the framework of incomplete criteria weight information and the preference relations expressed via IVPF truth degrees are formally introduced.

3.1. Description of problems and normalization methods

For the sake of convenience, let $L=\{1,2,\ldots,l\}$, $M=\{1,2,\ldots,m\}$, and $N=\{1,2,\ldots,n\}$. Suppose a group of DMs, denoted as E_k $(k\in L)$, are responsible for evaluating and ranking a set of alternatives A_i $(i\in M)$ based on criteria C_j $(j\in N)$.

Suppose we have a set of alternatives, denoted by $A=\{A_1,A_2,\ldots,A_m\}$, and a set of criteria, denoted by $C=\{C_1,C_2,\ldots,C_n\}$. Let $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be a vector of criteria weights that satisfies conditions $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j\geq 0$ $(j\in N)$. Let IVPFN $z_{ij}^k=([u_{ij}^{kL},u_{ij}^{kU}],[v_{ij}^{kL},v_{ij}^{kU}])$ $(i\in M,\ j\in N,\ k\in L)$ denote the rating given by DM E_k to alternative A_i on criterion C_j. Thus, we can formulate the MCGDM problem using IVPF decision matrices $\boldsymbol{Z}^k=(z_{ij}^k)_{m\times n}(k\in L)$.

To eliminate the effect of different dimensions on decision-making results, the criteria values z_{ij}^k $(i\in M,\ j\in N,\ k\in L)$ should be normalized into $s_{ij}^k=([\bar{u}_{ij}^{kL},\bar{u}_{ij}^{kU}],[\bar{v}_{ij}^{kL},\bar{v}_{ij}^{kU}])$ as follows:

$$s_{ij}^k=([\bar{u}_{ij}^{kL},\bar{u}_{ij}^{kU}],[\bar{v}_{ij}^{kL},\bar{v}_{ij}^{kU}])=\begin{cases}([u_{ij}^{kL},u_{ij}^{kU}],[v_{ij}^{kL},v_{ij}^{kU}]), & \text{if } C_j\in C^b\\ ([v_{ij}^{kL},v_{ij}^{kU}],[u_{ij}^{kL},u_{ij}^{kU}]), & \text{if } C_j\in C^c\end{cases}\ (i\in M,\ j\in N,\ k\in L). \quad (6)$$

In this context, C^b denotes a collection of advantageous criteria, while C^c stands for a collection of disadvantageous criteria.

Hence, the IVPF decision matrices $\boldsymbol{Z}^k=(z_{ij}^k)_{m\times n}$ $(k\in L)$ are transformed into the normalized IVPF decision matrices $\boldsymbol{Z'}^k=(s_{ij}^k)_{m\times n}$ $(k\in L)$.

3.2. Incomplete criteria weight information structure

Criteria weights constitute a critical and non-negligible element in decision-making. While DMs can express preferences over these weights based on their knowledge and experience, individual differences often result in incomplete information regarding the weight assignments. Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ denote a vector of criteria weights, where the weight of criterion C_j is represented by ω_j and satisfies conditions $\sum_{j=1}^{n} \omega_j = 1$ and$\omega_j \geq 0$ $(j \in N)$. The vector $\boldsymbol{\omega}$ is incompletely known in this paper and needs to be determined. Consider $\Lambda_0 = \{\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T | \sum_{j=1}^{n} \omega_j = 1,\ \omega_j \geq \varepsilon$ for $j = 1,2, \ldots, n\}$, where $\varepsilon > 0$ is a positive number that is sufficiently small. The limitations $\omega_j \geq \varepsilon$ $(j = 1,2, \cdots, n)$ can guarantee that every magnitude of Λ_0 is not lesser than a sufficiently minuscule positive quantity ε, as it might materialize in the LINMAP technique [27]. Li [28] officially and scrupulously determined weight information framework employing the ensuing quintessential relations amidst attribute weights.

3.3. Preference relations with IVPF truth degrees

In multi-criteria decision-making and fuzzy set theory, in order to handle the uncertainty and fuzziness of DMs' interval-valued Pythagorean fuzzy preference information, a tool is often needed to extract “sufficiently certain” or “sufficiently significant” preference relations. This tool is the cut set, which filters out those preference pairs with sufficiently high membership degrees and sufficiently low non-membership degrees by setting thresholds, thereby simplifying the decision analysis process while retaining key information. The cut set not only helps focus on the judgments that DMs are relatively certain about, but can also be used to evaluate the adequacy and reliability of decision information, providing a basis for subsequent weight determination and alternative ranking.

To formalize the preference structure, we first define the concept of truth degrees in the context of preference relations.

Definition 3.1. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives. For any decision-maker E_k, a truth degree of a preference relation $A_g \succ_k A_h$ is a numerical value $\alpha_k(g, h) \in [0,1]$ that quantifies the degree to which E_k prefers alternative A_g over A_h. A value closer to 1 indicates stronger preference, while a value closer to 0 indicates weaker or negligible preference.

In many practical situations, however, a single numerical value may not capture the inherent uncertainty in human judgments. To model such uncertainty in both membership and non-membership assessments, we extend the notion of truth degrees to IVPF truth degrees.

Definition 3.2. Let A_g and A_h be two alternatives, and let $E_k$be a decision-maker. An IVPF truth degree for the preference $A_g \succ_k A_h$ is expressed as:

$$\alpha_k(g, h) = ([u_{(g,h)}^{kL}, u_{(g,h)}^{kU}], [v_{(g,h)}^{kL}, v_{(g,h)}^{kU}]),$$

where $[u_{(g,h)}^{kL}, u_{(g,h)}^{kU}] \subseteq [0,1]$ is the interval membership degree, indicating the extent to which A_g

is preferred to A_h; $[v_{(g,h)}^{kL}, v_{(g,h)}^{kU}] \subseteq [0,1]$ is the interval non-membership degree, indicating the extent to which A_g is not preferred to A_h; and the following condition holds:

$$(u_{(g,h)}^{kU})^2 + (v_{(g,h)}^{kU})^2 \leq 1.$$

The hesitation degree corresponding to an IVPF truth degree is given by $\pi_{(g,h)} = [\pi_{(g,h)}^{kL}, \pi_{(g,h)}^{kU}]$, where

$$\pi_{(g,h)}^{kL} = \sqrt{1 - (u_{(g,h)}^{kU})^2 - (v_{(g,h)}^{kU})^2},\ \pi_{(g,h)}^{kU} = \sqrt{1 - (u_{(g,h)}^{kL})^2 - (v_{(g,h)}^{kL})^2}.$$

Based on the IVPF truth degree structure, we now introduce the IVPFS representation of a decision-maker's preference relations. Assuming DM E_k utilizes an IVPFS of $\Omega_k = \{<(g,h), \alpha_k(g,h) > | A_g \succ_k A_h\}$ with an associated IVPF truth degree $\alpha_k(g,h)$ $(g,h \in M)$ to denote preference relations between alternatives, we can observe that (g,h) represents the preference of DM E_k for alternative A_g over A_h (represented as $A_g \succ_k A_h$) with an IVPF truth degree $\alpha_k(g,h) = ([u_{(g,h)}^{kL}, u_{(g,h)}^{kU}], [v_{(g,h)}^{kL}, v_{(g,h)}^{kU}])$. To formally characterize a cut set within the IVPFS framework, we introduce the following definition.

Definition 3.3. Let $[a_0, b_0] \subseteq [0,1]$ and $[c_0, d_0] \subseteq [0,1]$ satisfy $b_0^2 + d_0^2 \leq 1$. The $([a_0, b_0], [c_0, d_0])$-cut set of Ω_k is defined as:

$$\Omega_k^{([a_0,b_0],[c_0,d_0])} = \{(g,h) | u_{(g,h)}^{kL} \geq a_0, u_{(g,h)}^{kU} \geq b_0,\ v_{(g,h)}^{kL} \leq c_0, v_{(g,h)}^{kU} \leq d_0\ (g,h \in M)\}.$$

In particular, when $a_0 = 0$, $b_0 = 0$, $c_0 = 1$, and $d_0 = 1$, the cut set reduces to the support set, denoted by $\Omega_k^{(0,1)}$, which includes all possible preference pairs. Its cardinality $|\Omega_k^{(0,1)}|$ reflects the amount of valid preference information provided by that DM. In group decision-making, a larger total cardinality $\sum_{k=1}^{l} \left| \Omega_k^{(0,1)} \right|$ indicates richer available preference information, often leading to a more precise and reliable criterion weight vector $\boldsymbol{\omega}$.

4. A novel IVPF mathematical programming method for IVPF MCGDM problems

This section proposes a novel LINMAP methods to solve IVPF MCGDM problems with IVPF truth degree. First, the DMs' weights on each criterion are determined based on the relative closeness degree. Then, the process of LINMAP is adhered to for establishing group consistency and group inconsistency utilizing IVPF PIS and IVPF NIS, correspondingly. Based on the group consistency and group inconsistency, an innovative bi-objective IVPF mathematical programming model is constructed to ascertain criteria weights. Subsequently, the individual relative proximity degrees of alternatives are computed and the individual hierarchy matrix for each DM is generated. Furthermore, according to the aim to minimize the variance between each DM's individual alternative order and the collective order, this paper formulated a multi-objective assignment model to derive a collective ranking matrix for determining the ranking of alternatives.

4.1. Determining DM's weight under each criterion based on the relative closeness degree

It is significant and should not be ignored to determine DMs' weights during the process of MCGDM. In fact, because each DM involved in decision-making may have different preferences for the alternatives under different criteria, it is reasonable to think each DM's weight under different criteria is not the same in real life.

For a criterion C_j, suppose that the PIS and NIS of alternatives given by DMs are $\boldsymbol{r}_j^+ = (r_j^{1+}, r_j^{2+}, \ldots, r_j^{l+})^T$ and $\boldsymbol{r}_j^- = (r_j^{1-}, r_j^{2-}, \ldots, r_j^{l-})^T$, respectively, where r_j^{k+} and r_j^{k-} are the best rating and the worst rating by $E_k (k \in L)$. Namely, one has

$$r_j^{k+} = ([u_j^{kL+}, u_j^{kU+}], [v_j^{kL+}, v_j^{kU+}]),\ r_j^{k-} = ([u_j^{kL-}, u_j^{kU-}], [v_j^{kL-}, v_j^{kU-}]), \tag{7}$$

where $u_j^{kL+} = \max_{i \in M}\{u_{ij}^{kL}\}$, $u_j^{kU+} = \max_{i \in M}\{u_{ij}^{kU}\}$, $v_j^{kL+} = \min_{i \in M}\{v_{ij}^{kL}\}$, $v_j^{kU+} = \min_{i \in M}\{v_{ij}^{kU}\}$ and $u_j^{kL-} = \min_{i \in M}\{u_{ij}^{kL}\}$, $u_j^{kU-} = \min_{i \in M}\{u_{ij}^{kU}\}$, $v_j^{kL-} = \max_{i \in M}\{v_{ij}^{kL}\}$, $v_j^{kU-} = \max_{i \in M}\{v_{ij}^{kU}\}$.

For $E_k (k \in L)$, the smaller $d_{q_1}(s_{ij}^k, r_j^{k+})$, the better A_i. Meanwhile, the bigger $d_{q_1}(s_{ij}^k, r_j^{k-})$, the better A_i. Thus, the relative closeness degree of alternative A_i on criterion C_j given by DM E_k can be defined as follows:

$$R_{ij}^k = \frac{d_{q_1}(s_{ij}^k, r_j^{k-})}{d_{q_1}(s_{ij}^k, r_j^{k+}) + d_{q_1}(s_{ij}^k, r_j^{k-})}, \tag{8}$$

where $d_{q_1}(s_{ij}^k, r_j^{k-})$ and $d_{q_1}(s_{ij}^k, r_j^{k+})$ are the Minkowski distance between normalized IVPF evaluation s_{ij}^k and the worst rating r_j^{k-} and the Minkowski distance between normalized IVPF evaluation s_{ij}^k and the best rating r_j^{k+}, respectively. Concretely,

$$d_{q_1}(s_{ij}^k, r_j^{k-}) = \frac{1}{4}(|(u_{ij}^{kL})^2 - (u_j^{kL-})^2|^{q_1} + |(u_{ij}^{kU})^2 - (u_j^{kU-})^2|^{q_1} + |(v_{ij}^{kL})^2 - (v_j^{kL-})^2|^{q_1}$$
$$+ |(v_{ij}^{kU})^2 - (v_j^{kU-})^2|^{q_1} + |(\pi_{ij}^{kL})^2 - (\pi_j^{kL-})^2|^{q_1} + |(\pi_{ij}^{kU})^2 - (\pi_j^{kU-})^2|^{q_1})^{\frac{1}{q_1}},$$

$$d_{q_1}(s_{ij}^k, r_j^{k+}) = \frac{1}{4}(|(u_{ij}^{kL})^2 - (u_j^{kL+})^2|^{q_1} + |(u_{ij}^{kU})^2 - (u_j^{kU+})^2|^{q_1} + |(v_{ij}^{kL})^2 - (v_j^{kL+})^2|^{q_1}$$
$$+ |(v_{ij}^{kU})^2 - (v_j^{kU+})^2|^{q_1} + |(\pi_{ij}^{kL})^2 - (\pi_j^{kL+})^2|^{q_1} + |(\pi_{ij}^{kU})^2 - (\pi_j^{kU+})^2|^{q_1})^{\frac{1}{q_1}}.$$

It is apparent that $0 \leq R_{ij}^k \leq 1$. Especially, if $d_{q_1}(s_{ij}^k, r_j^{k-}) = 0$, then $R_{ij}^k = 0$; if $d_{q_1}(s_{ij}^k, r_j^{k+}) = 0$, then $R_{ij}^k = 1$. Moreover, the bigger the value of R_{ij}^k, the better alternative A_i on criterion C_j for DM E_k.

If we consider an alternative, the weight of a DM E_k under criterion C_j is directly proportional to the sum of the relative closeness degrees of the alternative on criterion C_j given by DM E_k. In other words, as the sum of the relative closeness degree increases, the weight of DM E_k under criterion C_j also increases. Hence, we can define the weight of DM E_k under criterion C_j as w_j^k, given by:

$$w_j^k = c_j^k / \sum_{k=1}^{l} c_j^k,\ c_j^k = \sum_{i=1}^{m} R_{ij}^k = \sum_{i=1}^{m} \frac{d_{q_1}(s_{ij}^k, r_j^{k-})}{d_{q_1}(s_{ij}^k, r_j^{k+}) + d_{q_1}(s_{ij}^k, r_j^{k-})}. \quad (9)$$

Apparently, the DMs' weights satisfy the conditions that $0 \leq w_j^k \leq 1$ $(j = 1,2,\cdots,n;\ k = 1,2,\cdots,l)$ and $\sum_{k=1}^{l} w_j^k = 1$ $(j \in N)$. The steps of determination of DMs' weights can be summarized as follows:

Step 1. Normalize the original IVPF evaluations z_{ij}^k into s_{ij}^k by Eq (6);

Step 2. Derive the PIS and NIS of alternatives by Eq (7);

Step 3. Calculate the relative closeness degree of alternative A_i on criterion C_j given by DM E_k by Eq (8);

Step 4. Calculate the weight of DM E_k under criterion C_j by Eq (9).

Accordingly, the DM's weight vector under criterion C_j can be derived as $\boldsymbol{w}_j = (w_j^1, w_j^2, \ldots, w_j^l)^T (j \in N)$.

4.2. IVPF group consistency and inconsistency

IVPF-based group consistency and inconsistency are key concepts for quantifying the convergence and divergence of collective preferences under uncertainty. Group consistency measures the extent to which individual IVPF judgments converge toward a coherent group preference structure, typically evaluated using aggregated similarity indices or distance-based metrics. In contrast, group inconsistency reflects the degree of disagreement, ambiguity, or conflict among decision-makers, often manifested in the dispersion of IVPF evaluations or deviations from a common reference. By explicitly modeling both consistency and inconsistency within the IVPF framework, this approach enables a more nuanced and flexible representation of group decision dynamics. Suppose that the IVPF PIS and IVPF NIS of alternatives under criteria are $\boldsymbol{s}^+ = (s_1^+, s_2^+, \ldots, s_n^+)^T$ and $\boldsymbol{s}^- = (s_1^-, s_2^-, \ldots, s_n^-)^T$, respectively, where s_j^+ and s_j^- are the best rating and the worst rating on the criterion C_j $(j \in N)$. Namely, one has

$$s_j^+ = ([u_j^{L+}, u_j^{U+}], [v_j^{L+}, v_j^{U+}]),\ s_j^- = ([u_j^{L-}, u_j^{U-}], [v_j^{L-}, v_j^{U-}]), \tag{10}$$

where $u_j^{L+} = \max_{i\in M,k\in L}\{u_{ij}^{kL}\}$, $u_j^{U+} = \max_{i\in M,k\in L}\{u_{ij}^{kU}\}$, $v_j^{L+} = \min_{i\in M,k\in L}\{v_{ij}^{kL}\}$, $v_j^{U+} = \min_{i\in M,k\in L}\{v_{ij}^{kU}\}$ and $u_j^{L-} = \min_{i\in M,k\in L}\{u_{ij}^{kL}\}$, $u_j^{U-} = \min_{i\in M,k\in L}\{u_{ij}^{kU}\}$, $v_j^{L+} = \max_{i\in M,k\in L}\{v_{ij}^{kL}\}$, $v_j^{U+} = \max_{i\in M,k\in L}\{v_{ij}^{kU}\}$.

Considering the weight vector of DM under criterion $C_j\ (j \in N)$ as $\boldsymbol{w}_j = (w_j^1, w_j^2, \ldots, w_j^l)^T (j \in N)$, Eq (5) can be used to compute the weighted Minkowski distance between $\boldsymbol{s}_i^k = (s_{i1}^k, s_{i2}^k, \ldots, s_{in}^k)^T$ and $\boldsymbol{s}^+$, $\boldsymbol{s}^-$ in the following way:

$$\begin{aligned} T_i^{k+} = \sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [&|(u_{ij}^{kL})^2 - (u_j^{L+})^2|^{q_2} + |(u_{ij}^{kU})^2 - (u_j^{U+})^2|^{q_2} + |(v_{ij}^{kL})^2 - (v_j^{L+})^2|^{q_2} \\ &+ |(v_{ij}^{kU})^2 - (v_j^{U+})^2|^{q_2} + |(\pi_{ij}^{kL})^2 - (\pi_j^{L+})^2|^{q_2} + |(\pi_{ij}^{kU})^2 - (\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}} \end{aligned} \tag{11}$$

$$\begin{aligned} T_i^{k-} = \sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [&|(u_{ij}^{kL})^2 - (u_j^{L-})^2|^{q_2} + |(u_{ij}^{kU})^2 - (u_j^{U-})^2|^{q_2} + |(v_{ij}^{kL})^2 - (v_j^{L-})^2|^{q_2} \\ &+ |(v_{ij}^{kU})^2 - (v_j^{U-})^2|^{q_2} + |(\pi_{ij}^{kL})^2 - (\pi_j^{L-})^2|^{q_2} + |(\pi_{ij}^{kU})^2 - (\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}, \end{aligned} \tag{12}$$

where $(\pi_j^{L+})^2 = 1 - (u_j^{U+})^2 - (v_j^{U+})^2$, $(\pi_j^{U+})^2 = 1 - (u_j^{L+})^2 - (v_j^{L+})^2$, $(\pi_j^{L-})^2 = 1 - (u_j^{U-})^2 - (v_j^{U-})^2$, $(\pi_j^{U-})^2 = 1 - (u_j^{L-})^2 - (v_j^{L-})^2 (j \in N)$. It is necessary to determine the criteria weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Assuming that the criteria weights are fully specified, the weighted Minkowski distances T_g^{k+}, T_g^{k-}, T_h^{k+}, and T_h^{k-} for DM E_k can be derived with respect to an ordered pair (g, h). In the case where $T_g^{k+} < T_h^{k+}$ holds for every $(g, h) \in \Omega_k^{(0,1)}$, then A_g is superior to A_h , which is consistent with the subjective preference of DM E_k. Conversely, In the case where $T_g^{k+} \geq T_h^{k+}$, then A_g is superior to A_h, which contradicts the subjective preference provided by DM E_k. Similarly, In the case where $T_g^{k-} > T_h^{k-}$ holds for every $(g, h) \in \Omega_k^{(0,1)}$, then A_g is superior to A_h, which is consistent with the subjective preference of DM E_k. In the case where $T_g^{k-} \leq T_h^{k-}$, then A_g is inferior to A_h, which contradicts the subjective preference provided by DM E_k.

Please note that the MCGDM problems under consideration involve incomplete criteria weights which must be determined. To integrate this incomplete information on criteria weights into the decision-making process, a bi-objective IVPF programming model is created to obtain them. According to the IVPF PIS approach for evaluating alternatives against criteria, when considering each $(g,h) \in \Omega_k^{(0,1)}$, if condition $T_g^{k+} < T_h^{k+}$ is satisfied, then A_g is superior to A_h. Therefore, the ranking of alternatives A_g and A_h based on T_g^{k+} and T_h^{k+} aligns with the subjective preference expressed by DM E_k. On the other hand, if condition $T_g^{k+} \geq T_h^{k+}$ holds true, then the ranking of alternatives A_g and A_h based on T_g^{k+} and T_h^{k+} will not match the subjective preference of DM E_k.

According to IVPF NIS $\boldsymbol{s}^- = (s_1^-, s_2^-, \ldots, s_n^-)^T$, given any $(g,h) \in \Omega_k^{(0,1)}$, if $T_g^{k-} > T_h^{k-}$ is satisfied, then A_g is preferred over A_h. Thus, the ranking of alternatives A_g and A_h based on criteria T_g^{k-} and T_h^{k-} corresponds to the subjective preference expressed by DM E_k. On the other hand, if $T_g^{k-} \leq T_h^{k-}$ holds true, then the ranking of alternatives A_g and A_h based on T_g^{k-} and T_h^{k-} does not align with the subjective preference of DM E_k.

The statement indicates that the criteria weight vector ω results in the ranking order of A_g and A_h.

Definition 4.1. An index $(T_h^{k+} - T_g^{k+})^+$ is defined for each $(g,h) \in \Omega_k^{(0,1)}$ as follows:

$$(T_h^{k+} - T_g^{k+})^+ = \begin{cases} \alpha_k(g,h)(T_h^{k+} - T_g^{k+}) & \text{if } T_g^{k+} < T_h^{k+}, \\ 0 & \text{if } T_g^{k+} \geq T_h^{k+}. \end{cases} \tag{13}$$

Evidently, index $(T_h^{k+} - T_g^{k+})^+$ quantifies the extent of agreement between the objective ranking sequence and the subjective preferences conveyed by DM E_k using the IVPF PIS. Provided that condition $T_g^{k+} < T_h^{k+}$ is met, the ranking of alternatives A_g and A_h based on T_g^{k+} and T_h^{k+} corresponds to the preferences of DM E_k. Thus, $(T_h^{k+} - T_g^{k+})^+$ is defined as $\alpha_k(g,h)(T_h^{k+} - T_g^{k+})$. Conversely, if $T_g^{k+} \geq T_h^{k+}$ holds true, the ranking of alternatives A_g and A_h based on T_g^{k+} and T_h^{k+} does not align with the preferences expressed by DM E_k. In this case, $(T_h^{k+} - T_g^{k+})^+$ is defined as 0. Hence, the degree of consistency can be expressed as $(T_h^{k+} - T_g^{k+})^+ = \alpha_k(g,h)\, max\{0, T_h^{k+} - T_g^{k+}\}$.

To make readers understand this definition better, we provide the following example.

Example 1. Let $(2,4)$ be a preference relation that is given by DM E_k for alternative A_2 over A_4 (represented as $A_2 \succ_k A_4$) with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k+} = 0.4$ and $T_4^{k+} = 0.6$ be the distance between the weighted Minkowski distance between the alternative A_2 and the PIS and the distance between the weighted Minkowski distance between the alternative A_4 and the PIS, respectively. Since the preference relation is $(2,4)$ and $T_2^{k+} < T_4^{k+}$, then the consistency index $(T_2^{k+} - T_4^{k+})^+$ is calculated as follows:

$$(T_2^{k+} - T_4^{k+})^+ = ([0.5,0.6],[0.2,0.3]) \times (0.6 - 0.4) = ([0.23,0.29],[0.72,0.79]).$$

Example 2. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k+} = 0.6$ and $T_4^{k+} = 0.4$. Since the preference relation is $(2,4)$ and $T_2^{k+} \geq T_4^{k+}$, then the consistency index $(T_2^{k+} - T_4^{k+})^+ = 0$.

Thus, the definition of group consistency using the IVPF PIS is as follows:

$$K^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} (T_h^{k+} - T_g^{k+})^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_h^{k+} - T_g^{k+}\}. \tag{14}$$

Definition 4.2. An index $(T_h^{k+} - T_g^{k+})^-$ is defined for each $(g,h) \in \Omega_k^{(0,1)}$ as follows:

$$(T_h^{k+} - T_g^{k+})^- = \begin{cases} \alpha_k(g,h)(T_g^{k+} - T_h^{k+}) & \text{if } T_g^{k+} \geq T_h^{k+}, \\ 0 & \text{if } T_g^{k+} < T_h^{k+}. \end{cases} \tag{15}$$

Clearly, index $(g,h) \in \tilde{\Omega}_k^{(0,1)}$ quantifies the level of inconsistency. In a similar way, the degree of inconsistency can be expressed as $(T_h^{k+} - T_g^{k+})^- = \alpha_k(g,h)\, max\{0, T_g^{k+} - T_h^{k+}\}$.

Example 3. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k+} = 0.6$ and $T_4^{k+} = 0.4$. Since the preference relation is $(2,4)$ and $T_2^{k+} \geq T_4^{k+}$, then the inconsistency index $(T_2^{k+} - T_4^{k+})^-$ is calculated as follows:

$$(T_2^{k+} - T_4^{k+})^- = ([0.5,0.6],[0.2,0.3]) \times (0.6 - 0.4) = ([0.23,0.29],[0.72,0.79]).$$

Example 4. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k+} = 0.4$ and $T_4^{k+} = 0.6$. Since the preference relation is $(2,4)$ and $T_2^{k+} < T_4^{k+}$, then the inconsistency index $(T_2^{k+} - T_4^{k+})^- = 0$.

Thus, the definition of group inconsistency using the IVPF PIS is as follows:

$$J^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} (T_h^{k+} - T_g^{k+})^- = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_g^{k+} - T_h^{k+}\}. \tag{16}$$

Definition 4.3. An index $(T_g^{k-} - T_h^{k-})^+$ is defined for each $(g,h) \in \Omega_k^{(0,1)}$ as follows:

$$(T_g^{k-} - T_h^{k-})^+ = \begin{cases} \alpha_k(g,h)(T_g^{k-} - T_h^{k-}) & \text{if } T_g^{k-} > T_h^{k-}, \\ 0 & \text{if } T_g^{k-} \leq T_h^{k-}. \end{cases} \tag{17}$$

$(T_g^{k-} - T_h^{k-})^+ = \alpha_k(g,h)\, max\{0, T_g^{k-} - T_h^{k-}\}$ can be expressed as the measurement of how well the subjective preference aligns with the objective ranking based on the IVPF NIS, with $(T_h^{k+} - T_g^{k+})^+$ being the metric used to quantify this consistency.

Example 5. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k-} = 0.6$ and $T_4^{k-} = 0.4$. Since the preference relation is $(2,4)$ and $T_2^{k-} \geq T_4^{k-}$, then the consistency index $(T_2^{k-} - T_4^{k-})^+$ is calculated as follows:

$$(T_2^{k-} - T_4^{k-})^+ = ([0.5,0.6],[0.2,0.3]) \times (0.6 - 0.4) = ([0.23,0.29],[0.72,0.79]).$$

Example 6. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k-} = 0.4$ and $T_4^{k-} = 0.6$. Since the preference relation is $(2,4)$ and $T_2^{k-} < T_4^{k-}$, then the consistency index $(T_2^{k-} - T_4^{k-})^+ = 0$.

Therefore, the definition of group consistency, which relies on the IVPF NIS, can be expressed as:

$$K^- = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} (T_g^{k-} - T_h^{k-})^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_g^{k-} - T_h^{k-}\}. \quad (18)$$

Definition 4.4. An index $(T_g^{k-} - T_h^{k-})^-$ is defined for each $(g,h) \in \Omega_k^{(0,1)}$ as follows:

$$(T_g^{k-} - T_h^{k-})^- = \begin{cases} \alpha_k(g,h)(T_h^{k-} - T_g^{k-}) & \text{if } T_g^{k-} < T_h^{k-}, \\ 0 & \text{if } T_g^{k-} \geq T_h^{k-}. \end{cases} \quad (19)$$

Example 7. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k-} = 0.4$ and $T_4^{k-} = 0.6$. Since the preference relation is $(2,4)$ and $T_2^{k-} < T_4^{k-}$, then the inconsistency index $(T_2^{k-} - T_4^{k-})^+$ is calculated as follows:

$$(T_2^{k-} - T_4^{k-})^- = ([0.5,0.6],[0.2,0.3]) \times (0.6 - 0.4) = ([0.23,0.29],[0.72,0.79]).$$

Example 8. Let $(2,4)$ be a preference relation with truth degree $\alpha_k(g,h) = ([0.5,0.6],[0.2,0.3])$. Let $T_2^{k-} = 0.6$ and $T_4^{k-} = 0.4$. Since the preference relation is $(2,4)$ and $T_2^{k-} \geq T_4^{k-}$, then the inconsistency index $(T_2^{k-} - T_4^{k-})^- = 0$.

Thus, the definition of group inconsistency, which is based on the IVPF NIS, can be stated as follows:

$$J^- = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} (T_g^{k-} - T_h^{k-})^- = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_h^{k-} - T_g^{k-}\}. \quad (20)$$

It can be observed that the truth degree significantly influences the consistency index. In actual decision-making environments, it is often difficult for DMs to provide precise preferences between two alternatives. Many existing studies do not account for the truth degree. In contrast to these approaches, this paper employs IVPFNs to represent DMs' truth degrees, which not only captures the uncertainty and fuzziness in DMs' judgments but also better reflects their actual cognitive processes when comparing alternatives.

4.3. IVPF programming model

In group decision-making scenarios, reducing inter-group disparities generally contributes to more consistent and logically sound outcomes. To determine the weight vector $\boldsymbol{\omega}$ for the evaluation criteria, a dual-objective mathematical programming model based on an IVPFS is constructed. This model aims to minimize group inconsistency by incorporating the IVPF PIS and the IVPF NIS, which serve as benchmarks for optimal and anti-ideal performance, respectively. The formulation of this model is presented mathematically as follows:

$$\begin{array}{l} \min\{J^+\} \\ \min\{J^-\} \\ \text{s.t.} \begin{cases} K^+ - J^+ \geq \rho, \\ K^- - J^- \geq \sigma, \\ \boldsymbol{\omega} \in \Lambda. \end{cases} \end{array} \tag{21}$$

Here, ρ and σ represent two IVPF thresholds predetermined by DMs, defined as $\rho = ([u_\rho^L, u_\rho^U], [v_\rho^L, v_\rho^U])$ and $\sigma = ([u_\sigma^L, u_\sigma^U], [v_\sigma^L, v_\sigma^U])$. These thresholds denote the tolerance lower bounds for "best consistency" and "worst consistency" of the decision group, respectively. Typically, they can be set as small positive IVPF numbers. If decision-makers require stricter consistency, the lower bound of membership degrees in ρ and σ can be appropriately increased, or the upper bound of non-membership degrees can be decreased, ensuring $K^+ - J^+ \geq \rho$ and $K^- - J^- \geq \sigma$, which enforces that the degree of consistency must be significantly higher than that of inconsistency.

The programming model in Eq (21) is an IVPF programming model because its objective functions and constraint conditions are expressed using IVPFNs. To solve it, the model needs to be transformed.

First, the objective functions need to be transformed. Since $J^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_g^{k+} - T_h^{k+}\}$ and $J^- = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \alpha_k(g,h)\, max\{0, T_h^{k-} - T_g^{k-}\}$ are the sum of piecewise functions, let $\eta_{gh}^{k+} = max\{0, T_g^{k+} - T_h^{k+}\}$ and $\eta_{hg}^{k-} = max\{0, T_h^{k-} - T_g^{k-}\}$ for each $(g,h) \in \Omega_k^{(0,1)}$, where $\eta_{gh}^{k+} \geq 0$, $\eta_{hg}^{k-} \geq 0$, $\eta_{gh}^{k+} \geq T_g^{k+} - T_h^{k+}$, and $\eta_{hg}^{k-} \geq T_h^{k-} - T_g^{k-}$ hold true, respectively. Let $\vartheta_{hg}^{k+} = T_h^{k+} - T_g^{k+}$ and $\vartheta_{gh}^{k-} = T_g^{k-} - T_h^{k-}$. Then, the objective functions can be transformed into $\sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} [\alpha_k(g,h)\eta_{gh}^{k+}]$ and $\sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} [\alpha_k(g,h)\eta_{hg}^{k-}]$ with the constraint conditions $\vartheta_{hg}^{k+} + \eta_{gh}^{k+} \geq 0$ $((g,h) \in \Omega_k^{(0,1)}, k \in L)$, $\vartheta_{gh}^{k-} + \eta_{hg}^{k-} \geq 0$ $((g,h) \in \Omega_k^{(0,1)}, k \in L)$, and $\eta_{gh}^{k+} \geq 0, \eta_{hg}^{k-} \geq 0$ $((g,h) \in \Omega_k^{(0,1)}, k \in L)$.

Then, the constraint conditions need to be transformed. From Eqs (14) and (16), we can deduce that $K^+ - J^+ = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} [\alpha_k(g,h)(T_h^{k+} - T_g^{k+})]$, and similarly, from Eqs (18) and (20), we obtain $K^{-}-J^{-}=\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}[\alpha_k(g,h)(T_g^{k-}-T_h^{k-})]$.

Therefore, Eq (21) can be reformulated into the following programming model:

$$\min\{\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}[\alpha_k(g,h)\eta_{gh}^{k+}]\}$$

$$\min\{\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}[\alpha_k(g,h)\eta_{hg}^{k-}]\}$$

$$\text{s.t.}\begin{cases}\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}[\alpha_k(g,h)\vartheta_{hg}^{k+}]\geq\rho\,,\\ \sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}[\alpha_k(g,h)\vartheta_{gh}^{k-}]\geq\sigma,\\ \vartheta_{hg}^{k+}+\eta_{gh}^{k+}\geq 0\quad\left((g,h)\in\Omega_k^{(0,1)},k\in L\right),\\ \vartheta_{gh}^{k-}+\eta_{hg}^{k-}\geq 0\quad\left((g,h)\in\Omega_k^{(0,1)},k\in L\right),\\ \eta_{gh}^{k+}\geq 0,\eta_{hg}^{k-}\geq 0\quad\left((g,h)\in\Omega_k^{(0,1)},k\in L\right),\\ \boldsymbol{\omega}\in\Lambda.\end{cases}\tag{22}$$

4.4. Solving the PF programming model

Although Section 4.3 transforms the original programming model with piecewise objective functions, the model in Eq (22) remains a bi-objective IVPF model that requires further transformation. Since each objective function is represented by an IVPF containing four elements, we convert the bi-objective IVPF model into an eight-objective crisp programming model using Definitions 2.1 and 2.2, and Eq (22). The transformation focuses on minimizing the membership degrees and maximizing the non-membership degrees. The resulting eight-objective crisp programming model is presented below.

$$\begin{aligned}&min\left\{\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kL})^2)^{\eta_{gh}^{k+}}}\right\}\\&min\left\{\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kU})^2)^{\eta_{gh}^{k+}}}\right\}\end{aligned}\tag{23}$$

$$max\left\{\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kL})^{\eta_{gh}^{k+}}\right\}$$

$$max\left\{\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kU})^{\eta_{gh}^{k+}}\right\}$$

$$min\left\{\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kL})^2)^{\eta_{hg}^{k-}}}\right\}$$

$$min\left\{\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kU})^2)^{\eta_{hg}^{k-}}}\right\}$$

$$max\left\{\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kL})^{\eta_{hg}^{k-}}\right\}$$

$$max\left\{\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kU})^{\eta_{hg}^{k-}}\right\}$$

$$
\text{s.t.}\begin{cases}
\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kL})^2)^{\vartheta_{hg}^{k+}}} \ge u_{\rho}^{L}\\
\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kU})^2)^{\vartheta_{hg}^{k+}}} \ge u_{\rho}^{U}\\
\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kL})^{\vartheta_{hg}^{k+}} \le v_{\rho}^{L}\\
\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kU})^{\vartheta_{hg}^{k+}}) \le v_{\rho}^{U}\\
\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kL})^2)^{\vartheta_{gh}^{k-}}} \ge u_{\sigma}^{L}\\
\sqrt{1-\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(1-(u_{(g,h)}^{kU})^2)^{\vartheta_{gh}^{k-}}} \ge u_{\sigma}^{U}\\
\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kL})^{\vartheta_{gh}^{k-}} \le v_{\sigma}^{L}\\
\prod_{k=1}^{l}\prod_{(g,h)\in\Omega_k^{(0,1)}}(v_{(g,h)}^{kU})^{\vartheta_{gh}^{k-}} \le v_{\sigma}^{U}\\
\vartheta_{hg}^{k+}+\eta_{gh}^{k+}\ge 0 \quad ((g,h)\in\Omega_k^{(0,1)}, k\in L)\\
\vartheta_{gh}^{k-}+\eta_{hg}^{k-}\ge 0 \quad ((g,h)\in\Omega_k^{(0,1)}, k\in L)\\
\eta_{gh}^{k+}\ge 0, \eta_{hg}^{k-}\ge 0 \quad ((g,h)\in\Omega_k^{(0,1)}, k\in L)\\
\boldsymbol{\omega}\in \Lambda.
\end{cases}
$$

The core of the proposed method lies in transforming the group consensus problem under incomplete IVPF preferences into a solvable mathematical programming model without losing the essential information. This transformation is not a simple discretization but rather an equivalent mathematical reformulation based on the axiomatic definition of IVPF distance. The distance measure between two IVPFNs is essentially a function of the boundary parameters of their membership and non-membership intervals. Therefore, the programming model aimed at minimizing group inconsistency within the LINMAP framework can be equivalently reformulated as a coordinated optimization of these underlying interval-bound parameters, adhering to the dimension-preserving principle. This equivalence ensures that the model remains firmly grounded in the IVPF context throughout, as its inputs, constraints, and interpretations are all derived from and belong to that framework. This reformulation is a necessary step to render the model solvable via linear programming while fully preserving the uncertainty structure captured by IVPFNs. By the logarithmic function, Eq (23) is converted into

$$\begin{aligned}
&max\{ z_1 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{gh}^{k+} \, log(1 - (u_{(g,h)}^{kL})^2)\},\\
&max\{ z_2 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{gh}^{k+} \, log(1 - (u_{(g,h)}^{kU})^2)\},\\
&max\{ z_3 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{gh}^{k+} \, log(v_{(g,h)}^{kL})\},\\
&max\{ z_4 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{gh}^{k+} \, log(v_{(g,h)}^{kU})\},\\
&max\{ z_5 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{hg}^{k-} \, log(1 - (u_{(g,h)}^{kL})^2)\},\\
&max\{ z_6 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{hg}^{k-} \, log(1 - (u_{(g,h)}^{kU})^2)\},\\
&max\{ z_7 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{hg}^{k-} \, log(v_{(g,h)}^{kL})\},\\
&max\{ z_8 = \sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \eta_{hg}^{k-} \, log(v_{(g,h)}^{kU})\}.
\end{aligned} \tag{24}$$

$$
s.t.\begin{cases}
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{hg}^{k+}\,log(1-(u_{(g,h)}^{kL})^2)\le log(1-(u_\rho^L)^2)\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{hg}^{k+}\,log(1-(u_{(g,h)}^{kU})^2)\le log(1-(u_\rho^U)^2)\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{hg}^{k+}\,log(v_{(g,h)}^{kL})\le log\,v_\rho^L\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{hg}^{k+}\,log(v_{(g,h)}^{kU})\le log\,v_\rho^U\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{gh}^{k-}\,log(1-(u_{(g,h)}^{kL})^2)\le log(1-(u_\sigma^L)^2)\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{gh}^{k-}\,log(1-(u_{(g,h)}^{kU})^2)\le log(1-(u_\sigma^U)^2)\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{gh}^{k-}\,log(v_{(g,h)}^{kL})\le log\,v_\sigma^L\\
\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\vartheta_{gh}^{k-}\,log(v_{(g,h)}^{kU})\le log\,v_\sigma^U\\
\vartheta_{hg}^{k+}+\eta_{gh}^{k+}\ge 0\quad((g,h)\in\Omega_k^{(0,1)},k\in L)\\
\vartheta_{gh}^{k-}+\eta_{hg}^{k-}\ge 0\quad((g,h)\in\Omega_k^{(0,1)},k\in L)\\
\eta_{gh}^{k+}\ge 0,\eta_{hg}^{k-}\ge 0\quad((g,h)\in\Omega_k^{(0,1)},k\in L)\\
\boldsymbol{\omega}\in\Lambda.
\end{cases}
$$

By utilizing the linear weighted summation method and substituting Eqs (11) and (12) into Eq (24), a linear programming model can be formulated as shown below:

$$max\{z=z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8\}$$

subject to:

$$
\begin{aligned}
&\textstyle\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{hj}^{kL})^2-\\
&(v_j^{L+})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L+})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}-\\
&\textstyle\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L+})^2|^{q_2}+\\
&|(v_{gj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L+})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}\}\,log(1-
\end{aligned}
\qquad (25)
$$

$$(u_{(g,h)}^{kL})^2) \leq log(1 - (u_\rho^L)^2),$$

$$\sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \{\sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [|(u_{hj}^{kL})^2 - (u_j^{L+})^2|^{q_2} + |(u_{hj}^{kU})^2 - (u_j^{U+})^2|^{q_2} + |(v_{hj}^{kL})^2$$

$$- (v_j^{L+})^2|^{q_2} + |(v_{hj}^{kU})^2 - (v_j^{U+})^2|^{q_2} + |(\pi_{hj}^{kL})^2 - (\pi_j^{L+})^2|^{q_2} + |(\pi_{hj}^{kU})^2$$

$$- (\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}$$

$$- \sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [|(u_{gj}^{kL})^2 - (u_j^{L+})^2|^{q_2} + |(u_{gj}^{kU})^2 - (u_j^{U+})^2|^{q_2} + |(v_{gj}^{kL})^2$$

$$- (v_j^{L+})^2|^{q_2} + |(v_{gj}^{kU})^2 - (v_j^{U+})^2|^{q_2} + |(\pi_{gj}^{kL})^2 - (\pi_j^{L+})^2|^{q_2} + |(\pi_{gj}^{kU})^2$$

$$- \left(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}\right\} log(1 - \left(u_{(g,h)}^{kU})^2\right) \leq log(1 - \left(u_\rho^U)^2\right),$$

$$\sum_{k=1}^{l} \sum_{(g,h)\in\Omega_k^{(0,1)}} \{\sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [|(u_{hj}^{kL})^2 - (u_j^{L+})^2|^{q_2} + |(u_{hj}^{kU})^2 - (u_j^{U+})^2|^{q_2} + |(v_{hj}^{kL})^2$$

$$- (v_j^{L+})^2|^{q_2} + |(v_{hj}^{kU})^2 - (v_j^{U+})^2|^{q_2} + |(\pi_{hj}^{kL})^2 - (\pi_j^{L+})^2|^{q_2} + |(\pi_{hj}^{kU})^2$$

$$- (\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}$$

$$- \sum_{j=1}^{n} \frac{\omega_j}{4} \cdot w_j^k [|(u_{gj}^{kL})^2 - (u_j^{L+})^2|^{q_2} + |(u_{gj}^{kU})^2 - (u_j^{U+})^2|^{q_2} + |(v_{gj}^{kL})^2$$

$$- (v_j^{L+})^2|^{q_2} + |(v_{gj}^{kU})^2 - (v_j^{U+})^2|^{q_2} + |(\pi_{gj}^{kL})^2$$

$$- \left(\pi_j^{L+})^2|^{q_2} + \left| \left(\pi_{gj}^{kU})^2 - \left(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}\right\} log(v_{(g,h)}^{kL}\right) \leq log(v_\rho^L\right),$$

$$\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{hj}^{kL})^2$$

$$-(v_j^{L+})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L+})^2|^{q_2}+|(\pi_{hj}^{kU})^2$$

$$-(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}$$

$$-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{gj}^{kL})^2$$

$$-(v_j^{L+})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{gj}^{kL})^2$$

$$-\left(\pi_j^{L+})^2|^{q_2}+\left|\left(\pi_{gj}^{kU})^2-\left(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}\right\}log(v_{(g,h)}^{kU}\right)\le log(v_\rho^U\right),$$

$$\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}\}log(1-(u_{(g,h)}^{kL})^2)\le log(1-(u_\sigma^L)^2),$$

$$\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}\}log(1-(u_{(g,h)}^{kU})^2)\le log(1-(u_\sigma^U)^2),$$

$$\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L-})^2|^{q_2}+$$

$$|(v_{hj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}\}\, log(\, v_{(g,h)}^{kL}) \leq log(\, v_\sigma^{L}),$$

$$\sum_{k=1}^{l}\sum_{(g,h)\in\Omega_k^{(0,1)}}\{\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}\}\, log(\, v_{(g,h)}^{kU}) \leq log(\, v_\sigma^{U}),$$

$$\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L+})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L+})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L+})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U+})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L+})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U+})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L+})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U+})^2|^{q_2}]^{\frac{1}{q_2}}+\eta_{gh}^{k+}\geq 0 \quad ((g,h)\in \Omega_k^{(0,1)}, k\in L),$$

$$\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{gj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{gj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{gj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{gj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{gj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{gj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}-\sum_{j=1}^{n}\frac{\omega_j}{4}\cdot w_j^k[|(u_{hj}^{kL})^2-(u_j^{L-})^2|^{q_2}+|(u_{hj}^{kU})^2-(u_j^{U-})^2|^{q_2}+|(v_{hj}^{kL})^2-(v_j^{L-})^2|^{q_2}+|(v_{hj}^{kU})^2-(v_j^{U-})^2|^{q_2}+|(\pi_{hj}^{kL})^2-(\pi_j^{L-})^2|^{q_2}+|(\pi_{hj}^{kU})^2-(\pi_j^{U-})^2|^{q_2}]^{\frac{1}{q_2}}+\eta_{hg}^{k-}\geq 0 \quad \left((g,h)\in \Omega_k^{(0,1)}, k\in L\right),$$

$$\eta_{gh}^{k+}\geq 0, \eta_{hg}^{k-}\geq 0 \ ((g,h)\in \Omega_k^{(0,1)}, k\in L), \boldsymbol{\omega}\in \Lambda.$$

It is evident that in Eq (25), the objective function is linear in the decision variables, and all constraints are linear equalities or inequalities in these variables. Therefore, Eq (25) constitutes a linear programming (LP) problem. The process for determining the criterion weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is embedded within the structure and solution mechanism of the linear programming model (Eq (25)), rather than being conducted separately outside the model. The objective function and constraints of this linear programming formulation fully define the mathematical relationships and optimization criteria that the weight vector must satisfy. Therefore, by directly solving this linear programming model using the simplex method, its optimal solution yields the desired criterion weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. This indicates that weight determination and the model solution are achieved simultaneously, eliminating the need for additional independent weight calculation steps after solving the model. Subsequently, T_i^{k+} and T_i^{k-} can be derived using Eqs (11) and (12), respectively.

Remark 1. In Eq (25), there are $(8\sum_{k=1}^{l}|\Omega_k^{(0,1)}| + n)$ variables to be determined, including n weights ω_j $(j \in N)$ and $8\sum_{k=1}^{l}|\Omega_k^{(0,1)}|$ variables $u_{(g,h)}^{kL}$, $u_{(g,h)}^{kU}$, $v_{(g,h)}^{kL}$, and $v_{(g,h)}^{kU}$ $((g,h) \in \Omega_k^{(0,1)}, k \in L)$. There exist $8\sum_{k=1}^{l}|\Omega_k^{(0,1)}| + 8$ inequalities at least. Generally, a higher value of $\sum_{k=1}^{l}|\Omega_k^{(0,1)}|$ (i.e., the larger the number of pairwise comparisons of alternatives) means more pairwise comparisons, leading to a more precise and reliable weight vector derived from Eq (25). Due to the fact that Eq (25) is a linear programming model, it can be easily solved by Wan's algorithm [29]. Since the number of decision variables in the linear programming model is $(8\sum_{k=1}^{l}|\Omega_k^{(0,1)}| + n)$, it is easy to see that the model complexity of the model in Eq (25) is calculated as $O\left(max\left\{(\sum_{k=1}^{l}|\Omega_k^{(0,1)}|)^{3.5}, n^{3.5}\right\}L^{3.5}\right)$, where L is the number of bits in the input. Therefore, according to the complexity, the developed method has a low complexity. Since there are some mature software programs, such as LINGO and MATLAB, the processing time for solving this type of linear program is very little.

4.5. Derive the ranking by a collective ranking matrix

DM E_k considers alternative A_i to be superior when the value of T_i^{k+} is smaller. Likewise, alternative A_i is viewed as superior if the value of T_i^{k-} is greater, from the perspective of DM E_k. Thus, the relative closeness degree of alternative A_i for DM E_k is calculated as follows:

$$R_i^k = T_i^{k-}/(T_i^{k-} + T_i^{k+})(i \in M, k \in L). \tag{26}$$

It is evident that $0 \le R_i^k \le 1$ holds true, and if $T_i^{k-} = 0$, then $R_i^k = 0$ follows. Similarly, $T_i^{k+} = 0$ implies $R_i^k = 1$. Moreover, DM E_k considers alternative A_i to be superior when the value of R_i^k is greater. According to R_i^k, the individual ranking of alternatives for each DM can be gained.

The individual ranking matrix $\boldsymbol{X}^k = (x_{io}^k)_{m\times m}$ is generated for DM E_k, where

$$x_{io}^k = \begin{cases} 1, & \text{if } E_k \text{ ranks } A_i \text{ in the } o-\text{th position,} \\ 0, & \text{otherwise.} \end{cases}$$

Denote

$$x_{io}=\begin{cases}1, & \text{if the decision group ranks } A_i \text{ in the } o-\text{th position},\\ 0, & \text{otherwise.}\end{cases} \tag{27}$$

To derive the collective ranking matrix $\boldsymbol{X}=(x_{io})_{m\times m}$, an assignment model is constructed. This model aims to minimize the deviation between each DM's individual ranking of alternatives and the overall group ranking, as formulated below:

$$\begin{gathered} min \ \sum_{i=1}^{m}\sum_{o=1}^{m}|x_{io}^{k}-x_{io}| \quad (k\in L) \\ \text{s.t.}\begin{cases}\sum_{i=1}^{m}x_{io}=1 \quad (o\in M),\\ \sum_{o=1}^{m}x_{io}=1 \quad (i\in M),\\ x_{io}=0 \text{ or } 1 \quad (i,o\in M).\end{cases}\end{gathered} \tag{28}$$

The constraints $\sum_{i=1}^{m}x_{io}=1 \ \ (o\in M)$ ensure that each alternative is exclusively ranked within a solitary position, and the constraints $\sum_{o=1}^{m}x_{io}=1 \ \ (i\in M)$ affirm that each position solely accommodates a single alternative.

The multi-objective assignment model in Eq (28) can be transformed into a single-objective model as follows:

$$\begin{gathered} min \ \sum_{k=1}^{l}\sum_{i=1}^{m}\sum_{o=1}^{m}|x_{io}^{k}-x_{io}| \\ \text{s.t.}\begin{cases}\sum_{i=1}^{m}x_{io}=1 \quad (o\in M),\\ \sum_{o=1}^{m}x_{io}=1 \quad (i\in M),\\ x_{io}=0 \text{ or } 1 \quad (i,o\in M).\end{cases}\end{gathered} \tag{29}$$

The ranking of alternatives can be determined by solving the assignment model in Eq (29) using the Hungarian algorithm. The optimal solution yields the final priority order.

4.6. IVPF programming model for IVPF MCGDM problems

Based on the above analysis, we have developed a mathematical programming approach capable of addressing practical IVPF MCGDM problems with IVPF information. Inspired by [30], the algorithmic description of the proposed method is presented in Algorithm 1 as follows:

Algorithm 1. The proposed decision method.

Input: Alternative set A, criteria set C, and criteria weight Λ.

Output: The final ranking of alternatives.

Step 1: Supply the IVPF ordered pairs representing the subjective preference relations among alternatives by $\Omega_k = \{< (g,h), \alpha_k(g,h) > | A_g \succ_k A_h\}$ with IVPF truth degrees $\alpha_k(g,h)$ $(g,h \in M)$.

Step 2: Elicit $\boldsymbol{Z}^k$ $(k \in L)$ and transform into $\boldsymbol{Z}'^k$ $(k \in L)$ by Eq (6).

Step 3: Obtain PIS $\boldsymbol{r}^+$ and NIS $\boldsymbol{r}^-$ under a criterion C_j according to Eq (7).

Step 4: Derive DM's weight vector $\boldsymbol{w}_j = (w_j^1, w_j^2, \dots, w_j^l)^T (j \in N)$ under each criterion by Eq (9).

Step 5: By utilizing Eq (10), IVPF PIS $\boldsymbol{s}^+$ and IVPF NIS $\boldsymbol{s}^-$ can be derived.

Step 6: Construct an IVPF programming model (i.e., Eq (23)) and transform it into a single-objective model (i.e., Eq (25)).

Step 7: Solve Eq (25) to obtain $\boldsymbol{\omega}$.

Step 8: Calculate R_i^k $(i \in M, k \in L)$ of alternative A_i for DM E_k using Eq (26).

Step 9: Construct $\boldsymbol{X}^k = (x_{io}^k)_{m \times m}$ for the alternatives as per DM E_k by arranging them in descending order of R_i^k.

Step 10: Utilize Eq (28) to compute the collective ranking matrix $\boldsymbol{X} = (x_{io})_{m \times m}$, which is then employed to establish the sequence of alternatives according to their rankings.

The complete application of the proposed method is visually illustrated in Figure 1.

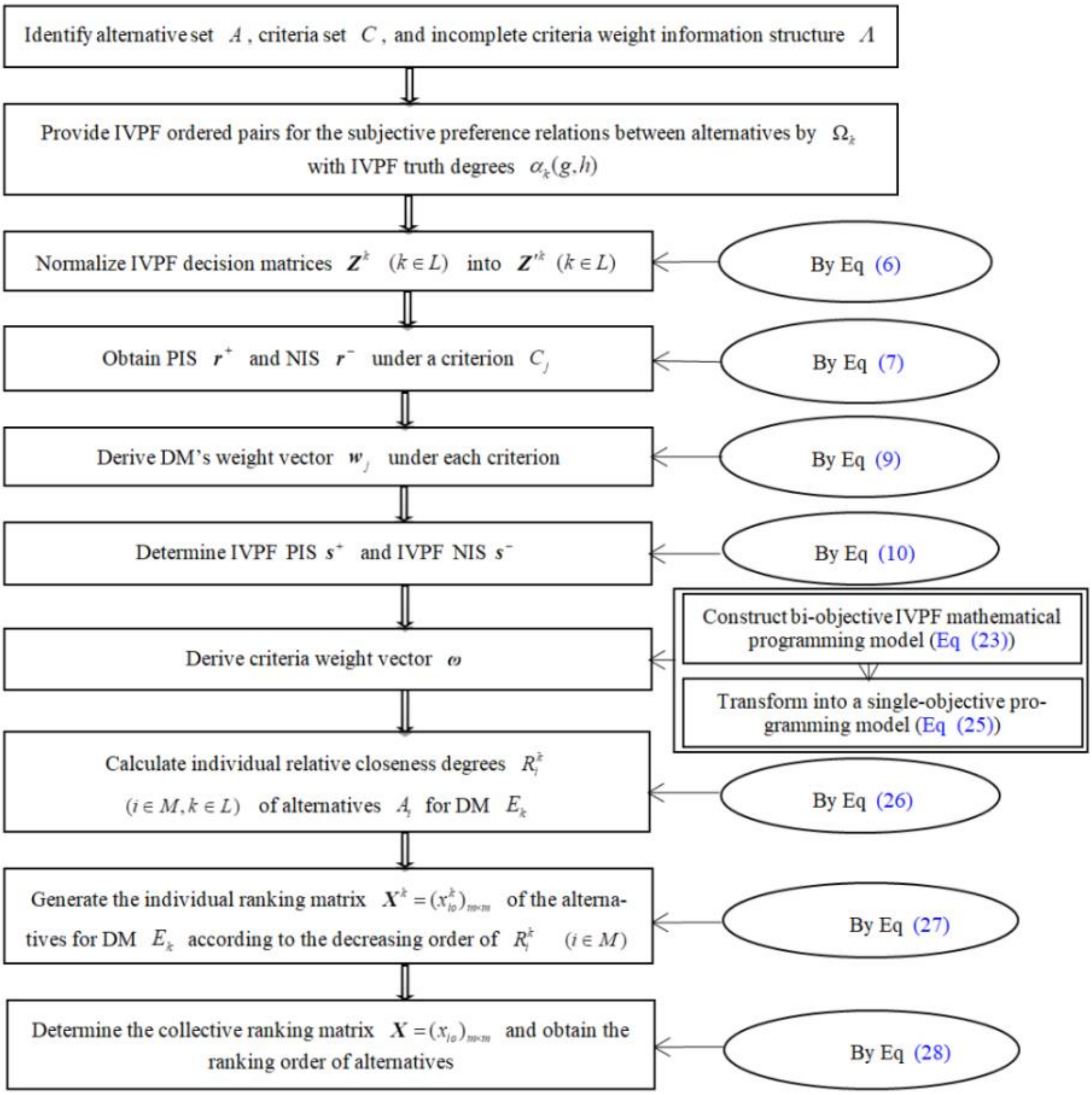

Figure 1. Decision-making process of an IVPF mathematical programming method.

5. A wireless network example and comparative analysis

To illustrate the viability and dependability of the proposed method, a wireless network selection example is presented within this section. Furthermore, the effectiveness of the suggested methodology is evaluated via a comparative examination against the IVPF-ELECTRE [26], IVPF-TOPSIS [31], and IVPF-TODIM [32].

5.1. A wireless network example

The rapid advancement of wireless communication technologies has greatly facilitated daily life, offering users a diverse range of access services within heterogeneous network environments. Consequently, heterogeneous wireless networks have emerged as a dominant trend for future communication systems, attracting considerable research interest. Among the key enabling technologies, network selection stands out as a critical research focus. To illustrate this problem, consider a scenario where a company needs to select the most suitable wireless network from among the following five alternatives: UMTS A_1, GSM A_2, WIMAX A_3, WLAN A_4, and WMAN A_5. To pick the best one from all alternatives, the evaluation expert team consisting of project manager E_1, CEO E_2, and technical personnel E_3 will evaluate the five alternatives from four criteria: signal strength C_1, network security C_2, network speed C_3, and coverage C_4.

Table 1 presents the individual decision matrices, which contain the linguistic ratings provided by each decision-maker for all alternatives with respect to every criterion.

Table 1. Individual linguistic term decision matrices.

DM	Alternative	Criterion			
		C_1	C_2	C_3	C_4
E_1	A_1	EG	VG	F	G
	A_2	P	G	F	G
	A_3	F	G	EG	F
	A_4	EG	F	G	VG
	A_5	VP	F	F	G
E_2	A_1	F	EG	VG	EG
	A_2	VG	VP	P	F
	A_3	F	F	G	G
	A_4	G	P	F	VP
	A_5	F	F	G	EG
E_3	A_1	EG	G	VG	F
	A_2	G	P	VP	G
	A_3	VG	G	VG	F
	A_4	F	F	G	F
	A_5	EG	G	F	P

To reflect the uniform expression of DMs, the mapping between linguistic terms and IVPFNs [26] is illustrated in Table 2.

Table 2. Relation between linguistic terms and IVPFNs.

Linguistic term	Abbreviation	IVPFN
Extremely good	EG	([0.8,0.9],[0.1,0.2])
Very good	VG	([0.7,0.8],[0.2,0.3])
Good	G	([0.6,0.7],[0.3,0.4])
Fair	F	([0.5,0.6],[0.4,0.5])
Poor	P	([0.3,0.4],[0.6,0.7])
Very poor	VP	([0.2,0.3],[0.7,0.8])
Extremely poor	EP	([0.1,0.2],[0.8,0.9])

Based on their experience and expertise, the three DMs have provided their IVPF preference relations among the alternatives, as shown below:

$$\Omega_1 = \{< (1,2), \alpha_1(1,2) >, < (3,2), \alpha_1(3,2) >, < (3,5), \alpha_1(3,5) > \},$$

$$\Omega_2 = \{< (1,2), \alpha_2(1,2) >, < (1,3), \alpha_2(1,3) >, < (1,4), \alpha_2(1,4) > \},$$

$$\Omega_3 = \{< (1,3), \alpha_3(1,3) >, < (3,4), \alpha_3(3,4) >, < (5,2), \alpha_3(5,2) > \},$$

where the associated IVPF truth degrees are given by:

$\alpha_1(1,2) = ([0.7,0.8],[0.2,0.3])$, $\alpha_1(3,2) = ([0.6,0.7],[0.3,0.4])$, $\alpha_1(3,5) = ([0.5,0.6],[0.4,0.5])$,

$\alpha_2(1,2) = ([0.7,0.8],[0.2,0.3])$, $\alpha_2(1,3) = ([0.6,0.7],[0.3,0.4])$, $\alpha_2(1,4) = ([0.8,0.9],[0.1,0.2])$,

$\alpha_3(1,3) = ([0.5,0.6],[0.4,0.5])$, $\alpha_3(3,4) = ([0.7,0.8],[0.2,0.3])$, $\alpha_3(5,2) = ([0.5,0.6],[0.4,0.5])$.

After discussion and negotiation, the expert group provides the following preference information structure Λ of criteria importance:

$$\Lambda = \{\omega | \omega \in \Lambda_0, \omega_3 \geq \omega_2,\ \omega_1 \geq 2\omega_4,\ 0.1 \leq \omega_2 - \omega_4 \leq 0.2,\ 0.1 \leq \omega_4 \leq 0.15\}.$$

Because each of criteria $C_1 - C_4$ is a benefit, Table 3 displays the normalized individual IVPF decision matrices $\boldsymbol{Z}'^k$ $(k = 1,2,3)$.

Table 3. Normalized individual IVPF decision matrices.

DM	Alternative	Criterion			
		C_1	C_2	C_3	C_4
E_1	A_1	([0.8,0.9],[0.1,0.2])	([0.7,0.8],[0.2,0.3])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])
	A_2	([0.3,0.4],[0.6,0.7])	([0.6,0.7],[0.3,0.4])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])
	A_3	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])	([0.8,0.9],[0.1,0.2])	([0.5,0.6],[0.4,0.5])
	A_4	([0.8,0.9],[0.1,0.2])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])	([0.7,0.8],[0.2,0.3])
	A_5	([0.2,0.3],[0.7,0.8])	([0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])
E_2	A_1	([0.5,0.6],[0.4,0.5])	([0.8,0.9],[0.1,0.2])	([0.7,0.8],[0.2,0.3])	([0.8,0.9],[0.1,0.2])
	A_2	([0.7,0.8],[0.2,0.3])	([0.2,0.3],[0.7,0.8])	([0.3,0.4],[0.6,0.7])	([0.5,0.6],[0.4,0.5])
	A_3	([0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])	([0.6,0.7],[0.3,0.4])
	A_4	([0.6,0.7],[0.3,0.4])	([0.3,0.4],[0.6,0.7])	([0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.7,0.8])
	A_5	([0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])	([0.8,0.9],[0.1,0.2])
E_3	A_1	([0.8,0.9],[0.1,0.2])	([0.6,0.7],[0.3,0.4])	([0.7,0.8],[0.2,0.3])	([0.5,0.6],[0.4,0.5])
	A_2	([0.6,0.7],[0.3,0.4])	([0.3,0.4],[0.6,0.7])	([0.2,0.3],[0.7,0.8])	([0.6,0.7],[0.3,0.4])
	A_3	([0.7,0.8],[0.2,0.3])	([0.6,0.7],[0.3,0.4])	([0.7,0.8],[0.2,0.3])	([0.5,0.6],[0.4,0.5])
	A_4	([0.5,0.6],[0.4,0.5])	([0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.3,0.4])	([0.5,0.6],[0.4,0.5])
	A_5	([0.8,0.9],[0.1,0.2])	([0.6,0.7],[0.3,0.4])	([0.5,0.6],[0.4,0.5])	([0.3,0.4],[0.6,0.7])

PIS r_j^+ and NIS r_j^- under the criterion C_j are obtained by Eq (7) as:

$$r_1^+ = (r_1^{1+}, r_1^{2+}, r_1^{3+})^T = (([0.8,0.9],[0.1,0.2]),([0.7,0.8],[0.2,0.3]),([0.8,0.9],[0.1,0.2]))^T,$$

$$r_1^- = (r_1^{1-}, r_1^{2-}, r_1^{3-})^T = (([0.2,0.3],[0.7,0.8]),([0.5,0.6],[0.4,0.5]),([0.5,0.6],[0.4,0.5]))^T,$$

$$r_2^+ = (r_2^{1+}, r_2^{2+}, r_2^{3+})^T = (([0.7,0.8],[0.2,0.3]),([0.8,0.9],[0.1,0.2]),([0.6,0.7],[0.3,0.4]))^T,$$

$r_2^- = (r_2^{1-}, r_2^{2-}, r_2^{3-})^T = (([0.5,0.6],[0.4,0.5]),([0.2,0.3],[0.7,0.8]),([0.3,0.4],[0.6,0.7]))^T,$

$r_3^+ = (r_3^{1+}, r_3^{2+}, r_3^{3+})^T = (([0.8,0.9],[0.1,0.2]),([0.7,0.8],[0.2,0.3]),([0.7,0.8],[0.2,0.3]))^T,$

$r_3^- = (r_3^{1-}, r_3^{2-}, r_3^{3-})^T = (([0.5,0.6],[0.4,0.5]),([0.3,0.4],[0.6,0.7]),([0.2,0.3],[0.7,0.8]))^T,$

$r_4^+ = (r_4^{1+}, r_4^{2+}, r_4^{3+})^T = (([0.7,0.8],[0.2,0.3]),([0.8,0.9],[0.1,0.2]),([0.6,0.7],[0.3,0.4]))^T,$

$r_4^- = (r_4^{1-}, r_4^{2-}, r_4^{3-})^T = (([0.5,0.6],[0.4,0.5]),([0.2,0.3],[0.7,0.8]),([0.3,0.4],[0.6,0.7]))^T.$

To simplify the analysis, $q_1 = 2$ can be substituted into Eq (29). Then, the distance matrices of decision information of these three DMs and the PIS can be displayed in Tables 4–6, respectively. The distance matrices of decision information of these three DMs and the NIS can be displayed in Tables 7–9, respectively.

Table 4. Distance matrix of decision information of DM E_1 and the PIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0	0	0.1831	0.0610
A_2	0.2658	0.0610	0.1831	0.0610
A_3	0.1831	0.0610	0	0.1131
A_4	0	0.1131	0.1327	0
A_5	0.3059	0.1131	0.1831	0.0610

Table 5. Distance matrix of decision information of DM E_2 and the PIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0.1131	0	0	0
A_2	0	0.3059	0.2040	0.1831
A_3	0.1131	0.1831	0.0610	0.1327
A_4	0.0610	0.2658	0.1131	0.3059
A_5	0.1131	0.1831	0.0610	0

Table 6. Distance matrix of decision information of DM E_3 and the PIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0	0	0	0.0532
A_2	0.1327	0.1507	0.2512	0
A_3	0.0723	0	0	0.0532
A_4	0.1831	0.0532	0.0610	0.0532
A_5	0	0	0.1131	0.1507

Table 7. Distance matrix of decision information of DM E_1 and the NIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0.3059	0.1131	0	0.0532
A_2	0.0610	0.0532	0	0.0532
A_3	0.1595	0.0532	0.1831	0
A_4	0.3059	0	0.0532	0.1131
A_5	0	0	0	0.0532

Table 8. Distance matrix of decision information of DM E_2 and the NIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0	0.3059	0.2040	0.3059
A_2	0.1131	0	0	0.1595
A_3	0	0.1595	0.1507	0.2040
A_4	0.0532	0.0610	0.1020	0
A_5	0	0.1595	0.1507	0.3059

Table 9. Distance matrix of decision information of DM E_3 and the NIS.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0.1831	0.1507	0.2512	0.1020
A_2	0.0532	0	0	0.1507
A_3	0.1131	0.1507	0.2512	0.1020
A_4	0	0.1020	0.2040	0.1020
A_5	0.1831	0.1507	0.1595	0

Using Eq (9), the weight of DM E_1 under criterion C_1 can be calculated as $w_1^1 = \frac{2.6523}{2.6523+1.4655+2.8962} = 0.3781$.

Likewise, the weight vector $\boldsymbol{w}_j = (w_j^1, w_j^2, \dots, w_j^l)^T (j \in N)$ for each DM under each criterion can be derived as:

$$\boldsymbol{w}_1 = (w_1^1, w_1^2, w_1^3)^T = (0.3781, 0.2090, 0.4129)^T,$$

$$\boldsymbol{w}_2 = (w_2^1, w_2^2, w_2^3)^T = (0.2506, 0.2748, 0.4746)^T,$$

$$\boldsymbol{w}_3 = (w_3^1, w_3^2, w_3^3)^T = (0.1706, 0.3844, 0.4450)^T,$$

$$\boldsymbol{w}_4 = (w_4^1, w_4^2, w_4^3)^T = (0.2839, 0.3639, 0.3522)^T.$$

By Eq (10), IVPF PIS $\boldsymbol{s}^+$ and IVPF NIS $\boldsymbol{s}^-$ are determined as

$$s^+ = (([0.8,0.9],[0.1,0.2]),([0.8,0.9],[0.1,0.2]),([0.8,0.9],[0.1,0.2]),([0.8,0.9],[0.1,0.2]))^T,$$

$$s^- = (([0.2,0.3],[0.7,0.8]),([0.2,0.3],[0.7,0.8]),([0.2,0.3],[0.7,0.8]),([0.2,0.3],[0.7,0.8]))^T.$$

Let $\rho = ([0.0001,0.0002],[0.9997,0.9998])$ and $\sigma = ([0.0001,0.0002],[0.9997,0.9998])$. To facilitate simplicity, let $q_2 = 2$ be substituted into Eq (29), and then the vector of criteria weights $\boldsymbol{\omega}$ can be obtained as:

$$\omega_1 = 0.2952,\ \omega_2 = 0.2408,\ \omega_3 = 0.3412,\ \omega_4 = 0.1228.$$

Based on the model solution, the obtained optimal criterion weight vector is $\boldsymbol{\omega} = (0.2952,0.2408,0.3412,0.1228)^T$. The determination of these weights originates from minimizing the group inconsistency index, i.e., seeking the weight configuration that best reconciles the conflicts among the evaluations provided by the three experts.

Criterion C_3 has the highest weight of 0.3412. This prominent weight indicates that, regarding this criterion, the experts' evaluations of different alternatives exhibit relatively large divergence or the highest degree of uncertainty. Reviewing the original linguistic decision matrix (Table 1), for example, for alternative A_2, the three experts' evaluations on C_3 are "F", "P", and "VP", whose corresponding IVPF representations already demonstrate significant variability. Assigning a higher weight to C_3 implies that, during consensus formation, greater importance must be placed on the evaluative differences concerning this criterion. By weighting it more heavily, the model reconciles the judgmental conflicts among different experts in this area, thereby ensuring that the final ranking result better reflects the group's compromise and agreement on this key point of divergence.

Criterion C_4 has the lowest weight of 0.1228. This lower weight suggests that the experts' evaluations on this criterion are relatively consistent or exhibit higher certainty. For example, for alternative A_3, the three experts' evaluations on C_4 are "F", "G", and "F". Although variations exist, their overall fluctuation is less disruptive to the overall consensus compared to other criteria. Consequently, the model automatically reduces its weight, indicating that this criterion contributes relatively less to distinguishing and forming the final consensus ranking.

Criteria C_1 and C_2 demonstrate their balanced role in consensus formation. For instance, on C_1, the evaluations received by alternative A_1 are "EG", "F", and "EG", showing a certain degree of fuzziness and inconsistency, though not to the extent observed for C_3. Therefore, their weights reflect an intermediate role for these criteria in the group decision-making process: they cannot be ignored, yet they are not the primary sources of disagreement.

This illustrates that the allocation of weights is not a "black-box" output but rather a quantified feedback and semantic mapping of the inherent consensus difficulty and informational certainty embedded within the experts' original linguistic evaluations. A higher weight signifies that the corresponding criterion requires greater emphasis and reconciliation during the consensus process, whereas a lower weight indicates that the experts' opinions on that criterion are relatively convergent, naturally diminishing its influence on the final ranking.

Then, the relative closeness degree matrix of these three DMs can be obtained in Tables 10–12.

Table 10. Relative closeness degree matrix of DM E_1.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	1	1	0	0.4655
A_2	0.1868	0.4655	0	0.4655
A_3	0.4655	0.4655	1	0
A_4	1	0	0.2861	1
A_5	0	0	0	0.4655
Sum	2.6523	1.9310	1.2861	2.3965

Table 11. Relative closeness degree matrix of DM E_2.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	0	1	1	1
A_2	1	0	0	0.4655
A_3	0	0.4655	0.7118	0.6059
A_4	0.4655	0.1868	0.4741	0
A_5	0	0.4655	0.7118	1
Sum	1.4655	2.1178	2.8977	3.0714

Table 12. Relative closeness degree matrix of DM E_3.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	1	1	1	0.6574
A_2	0.2861	0	0	1
A_3	0.6101	1	1	0.6574
A_4	0	0.6574	0.7697	0.6574
A_5	1	1	0.5850	0
Sum	2.8962	3.6573	3.3546	2.9722

Thus, after obtaining distances between $\boldsymbol{r}_i^k$ $(i = 1,2,3,4,5; k = 1,2,3)$ and $\boldsymbol{s}^+$, $\boldsymbol{s}^-$ as T_i^{k+}, T_i^{k-}, respectively, the relative closeness degrees can be calculated. The results are presented in Table 13.

Table 13. Weighted Euclidean distances T_i^{k+}, T_i^{k-} and relative closeness degrees R_i^k.

DM	Alternative	T_i^{k+}	T_i^{k-}	R_i^k
	A_1	0.0196	0.0657	0.7698
	A_2	0.0530	0.0355	0.4014
E_1	A_3	0.0348	0.0535	0.6056
	A_4	0.0213	0.0644	0.7516
	A_5	0.0605	0.0260	0.3008
	A_1	0.0208	0.0767	0.7869
	A_2	0.0677	0.0306	0.3114
E_2	A_3	0.0467	0.0563	0.5462
	A_4	0.0635	0.0375	0.3717
	A_5	0.0408	0.0608	0.5984
	A_1	0.0341	0.1056	0.7562
	A_2	0.0987	0.0407	0.2917
E_3	A_3	0.0429	0.0990	0.6978
	A_4	0.0713	0.0755	0.5144
	A_5	0.0545	0.0875	0.6163

The individual ranking of alternatives is obtained by sorting them in descending order of their relative closeness degrees, denoted as $A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$ for DM E_1, $A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$ for DM E_2, and $A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$ for DM E_3, respectively. Therefore, the individual ranking matrices $\mathbf{X}^k (k = 1,2,3)$ are generated for DM $E_k (k = 1,2,3)$ as follows:

$$\mathbf{X}^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{X}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{X}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By Eq (29), the assignment model can be fabricated:

$$min \sum_{k=1}^{3}\sum_{i=1}^{5}\sum_{o=1}^{5} |x_{io}^k - x_{io}|$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{5} x_{io} = 1 \quad (o = 1,2,3,4,5), \\ \sum_{o=1}^{5} x_{io} = 1 \quad (i = 1,2,3,4,5), \\ x_{io} = 0 \text{ or } 1 \quad (i, o = 1,2,3,4,5). \end{cases} \tag{30}$$

By employing LINGO 11.0 for the resolution of Eq (30), the resultant matrix of collective ranking is acquired:

$$\mathbf{X}=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&0\end{pmatrix}.$$

Thus, the overall order may be established as $A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$, with alternative A_1 identified as the optimal choice. The method proposed in this paper integrates the determination of criterion weights into a linear programming model (Eq (25)). The optimal solution of this model directly contains the criterion weight vector, thereby avoiding potential distortions that may arise in methods such as fuzzy information aggregation. Furthermore, after obtaining the rankings of alternatives from each DM, an assignment model is used to aggregate the DMs' ranking results, yielding the final ranking of alternatives. In summary, the method proposed in this paper is both reliable and reasonable.

5.2. Sensitivity analyses

The distance measure between IVPFSs is critical not only for determining DMs' weights but also for assessing group consistency and inconsistency. In particular, the weighted Minkowski distance metric incorporates a parameter q_2. Different values of q_2 lead to different distance calculations, which consequently alter the final decision outcomes. It is therefore essential to investigate how variations in q_2 affect the decision results.

First, we take the values of parameter q_2 by 1, 2, 10, 100, and 392. According to the MCGDM proposed in this paper, the DM's weights, criteria weights, and the collective ranking orders obtained by different values of parameters are listed in Table 14.

Remark 2. When $q_2 \geq 393$, the distances T_i^{k+} and T_i^{k-} are close to zero in Eq (26). Consequently, the relative closeness degree R_i^k assigned by DM E_k to alternative A_i holds no significance.

Then, based on the weights of DMs and weights of criteria derived in Section 5.1, we take the values of parameter q_2 as 1, 2, 10, 100, and 392, respectively, and further obtain the final decision results in Table 15.

Table 14. Collective ranking orders with different values of q_2.

q_2	DM's weight vector $\boldsymbol{w}_j$ under a criterion	Criteria weight vector $\boldsymbol{\omega}$	Collective ranking order
	$\boldsymbol{w}_1 = (0.3778,0.2082,0.4140)^T$ $\boldsymbol{w}_2 = (0.2503,0.2750,0.4747)^T$ $\boldsymbol{w}_3 = (0.1727, 0.3789,0.4484)^T$ $\boldsymbol{w}_4 = (0.2845,0.3646,0.3509)^T$	$\boldsymbol{\omega} = (0.2488,0.1040,0.5271,0.1201)^T$	$A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$
	$\boldsymbol{w}_1 = (0.3781,0.2090,0.4129)^T$ $\boldsymbol{w}_2 = (0.2506,0.2748,0.4746)^T$ $\boldsymbol{w}_3 = (0.1706,0.3844,0.4450)^T$ $\boldsymbol{w}_4 = (0.2839,0.3639,0.3522)^T$	$\boldsymbol{\omega} = (0.2952,0.2408,0.3412,0.1228)^T$	$A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$
0	$\boldsymbol{w}_1 = (0.3776,0.2084,0.4140)^T$ $\boldsymbol{w}_2 = (0.2505,0.2750,0.4745)^T$ $\boldsymbol{w}_3 = (0.1718, 0.3820,0.4462)^T$ $\boldsymbol{w}_4 = (0.2844,0.3645,0.3511)^T$	$\boldsymbol{\omega} = (0.3592,0.3197,0.1371,0.1840)^T$	$A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$
00	$\boldsymbol{w}_1 = (0.3774,0.2084,0.4142)^T$ $\boldsymbol{w}_2 = (0.2507,0.2750,0.4743)^T$ $\boldsymbol{w}_3 = (0.1725, 0.3800,0.4475)^T$ $\boldsymbol{w}_4 = (0.2848,0.3647,0.3505)^T$	$\boldsymbol{\omega} = (0.1171,0.4074,0.1335,0.3420)^T$	$A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$
92	$\boldsymbol{w}_1 = (0.4227,0.1593, 0.4180)^T$ $\boldsymbol{w}_2 = (0.1405,0.2974,0.5621)^T$ $\boldsymbol{w}_3 = (0.1392,0.3951,0.4657)^T$ $\boldsymbol{w}_4 = (0.1240,0.3800,0.4960)^T$	$\boldsymbol{\omega} = (0.2029,0.2277,0.1880,0.3814)^T$	$A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$

Table 15. Collective ranking orders with different values of q_2 by the equal weights of criteria and equal weights of DMs.

q_2	Collective ranking order
1	$A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_2$
2	$A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_2$
10	$A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_2$
100	$A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$
392	$A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$

According to Tables 14 and 15, we can obtain the following three conclusions.

(1) The optimal alternative is always A_1, which is not changed when the parameter q_2 takes different values. Thus, the proposed method is reliable.

(2) In Table 14, it can be found that the weights information and the suboptimal alternative are greatly affected by the values of parameter q_2. If q_2 = 1 or q_2 = 2, then the suboptimal solution is A_5. However, if q_2 = 10 or q_2 = 100, then A_5 is the worst alternative. Although the different values of parameter q_2 will also cause huge changes in weight information and ranking of alternatives, the optimal alternative is still A_1. Therefore, the proposed method is reliable.

(3) As shown in Table 15, the ranking of alternatives changes only slightly across different values of q_2. This stability can be attributed to the relatively stable weight assignments derived from

the process. Since these weights are closely dependent on the distances between IVPFSs, which themselves vary with q_2, any variation in q_2 will first affect the weight calculations and, consequently, the final ranking results. The observed minimal change in ranking indicates that the derived weights are robust to the chosen distance parameter within the tested range. However, the decision results in Table 15 are obtained based on the weight information obtained in Section 5.1 and the weight information will not change with the change of values of parameter q_2. Thus, even if the values of parameter q_2 change, the alternatives ranking is not changed greatly.

Therefore, Tables 14 and 15 reveal that the optimal alternative obtained by the proposed method is always A_1, which demonstrates that the proposed method is robust and reliable.

5.3. Comparative analyses

To validate the effectiveness and advantages of our proposed method, this section conducts a comparative analysis with several established IVPF-MCGDM methods. Specifically, we select the IVPF-ELECTRE [26], IVPF-TOPSIS [31], and IVPF-TODIM [32] methods for comparison, as they are all designed to address similar decision-making problems under interval-valued Pythagorean fuzzy information. The comparative results are then analyzed, followed by a robustness assessment using Spearman's rank correlation test.

5.3.1. Comparative analysis with IVPF-ELECTRE

Peng et al. [26] first defined the score and accuracy functions for IVPFNs, along with their operational rules and distance measures. Building upon these foundations, they then introduced an IVPF point-weighted average operator by integrating the IVPF point operator. Based on this operator, they subsequently developed an IVPF-ELECTRE method to solve MCGDM problems.

Although method [26] can be employed to resolve the MCGDM quandaries, the relative importance $\lambda_k = ([\mu_{\lambda_\kappa}^-, \mu_{\lambda_\kappa}^+], [\upsilon_{\lambda_\kappa}^-, \upsilon_{\lambda_\kappa}^+], [\pi_{\lambda_\kappa}^-, \pi_{\lambda_\kappa}^+])(k = 1,2,3)$ of DM $E_k\,(k = 1,2,3)$ should be given in advance, as shown in Table 16.

Table 16. The relative importance of DMs.

	E_1	E_2	E_3
Linguistic term	Medium	Important	Very important
IVPFN	([0.5,0.6],[0.4,0.5])	([0.7,0.8],[0.2,0.3])	([0.8,0.9],[0.1,0.2])

In approach [26], the weight of DM E_k is computed using the following equation:

$$\xi_k = \frac{\mu_{\lambda_k}^- + \mu_{\lambda_k}^+ + (\pi_{\lambda_k}^- + \pi_{\lambda_k}^+)\frac{\mu_{\lambda_k}^- + \mu_{\lambda_k}^+}{\mu_{\lambda_k}^- + \mu_{\lambda_k}^+ + \upsilon_{\lambda_k}^- + \upsilon_{\lambda_k}^+}}{\sum_{k=1}^{l} \mu_{\lambda_k}^- + \mu_{\lambda_k}^+ + (\pi_{\lambda_k}^- + \pi_{\lambda_k}^+)\frac{\mu_{\lambda_k}^- + \mu_{\lambda_k}^+}{\mu_{\lambda_k}^- + \mu_{\lambda_k}^+ + \upsilon_{\lambda_k}^- + \upsilon_{\lambda_k}^+}}.$$

Then, DMs' weights can be obtained as

$$\xi_1 = 0.\,2743,\ \ \xi_2 = 0.\,3534,\ \ \xi_3 = 0.\,3723.$$

Furthermore, objective criteria weight ω_j^k of the DM is calculated based on the formula

$$\omega_j^k = \frac{\sum_{i=1}^m \sum_{i'=1}^m d(p_{ij}^k, p_{i'j}^k)}{\sqrt{\sum_{j=1}^n (\sum_{i=1}^m \sum_{i'=1}^m d(p_{ij}^k, p_{i'j}^k))^2}}.$$

Thus, it is possible to derive the objective criteria weights of DMs as

$$\omega^{(1)} = (0.4904,0.1538,0.2308,0.1250),$$

$$\omega^{(2)} = (0.1231,0.3269,0.1962,0.3538),$$

$$\omega^{(3)} = (0.2923,0.1897,0.3487,0.1693).$$

The final integrated weight for each criterion is calculated by aggregating the subjective criterion weights provided by individual DMs, weighted by their respective DM importance weights. This integration is performed using the following equation:

$$\omega_j = \sum_{k=1}^3 \omega_j^k \xi_k.$$

Therefore, the weights of the criteria are acquired as $\omega_1 = 0.29$, $\omega_2 = 0.23$, $\omega_3 = 0.26$, $\omega_4 = 0.22$.

The individual IVPF decision matrices provided by the DM are aggregated into a collective IVPF decision matrix using the interval-valued Pythagorean fuzzy weighted average operator, which is defined as follows:

$$p_{ij} = \text{IVPFWA}(p_{ij}^1, p_{ij}^2, \cdots, p_{ij}^l) = \left(\left[\sum_{k=1}^l \xi_k u_{ij}^{k-}, \sum_{k=1}^l \xi_k u_{ij}^{k+}\right], \left[\sum_{k=1}^l \xi_k v_{ij}^{k-}, \sum_{k=1}^l \xi_k v_{ij}^{k+}\right]\right).$$

The collective IVPF decision matrix is computed and listed in Table 17.

Table 17. Collective IVPF decision matrix.

Alternative	Criterion			
	C_1	C_2	C_3	C_4
A_1	([0.69,0.79], [0.21,0.31])	([0.70,0.80], [0.20,0.30])	([0.65,0.75], [0.25,0.35])	([0.63,0.73], [0.27,0.37])
A_2	([0.55,0.65], [0.35,0.45])	([0.35,0.45], [0.55,0.65])	([0.32,0.42], [0.58,0.68])	([0.56,0.66], [0.34,0.44])
A_3	([0.57,0.67], [0.33,0.43])	([0.56,0.66], [0.34,0.44])	([0.69,0.79], [0.21,0.31])	([0.54,0.64], [0.36,0.46])
A_4	([0.62,0.72], [0.28,0.38])	([0.43,0.53], [0.47,0.57])	([0.56,0.66], [0.34,0.44])	([0.45,0.55], [0.45,0.55])
A_5	([0.53,0.63], [0.37,0.47])	([0.54,0.64], [0.36,0.46])	([0.54,0.64], [0.36,0.46])	([0.56,0.66], [0.34,0.44])

After identifying the concordance and discordance sets, the concordance and discordance Boolean matrices are constructed. Subsequently, the outranking matrix is derived as follows:

$$O = \begin{pmatrix} - & 1 & 0 & 1 & 1 \\ 0 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 1 \\ 0 & 1 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}.$$

Consequently, A_1 is considered better than A_2, A_4, A_5, and as a result, alternative A_1 emerges as the optimal choice. Figure 2 illustrates the decision-making result.

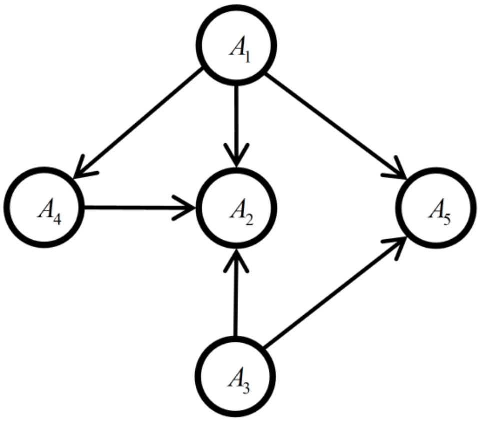

Figure 2. Decision result by IVPF-ELECTRE.

Comparing the decision result obtained by Peng et al.'s method [26] with that obtained by the proposed method, the following conclusions can be obtained.

(1) Peng et al.'s method directly provided the weighs of DMs in advance and it is difficult to avoid the subjectivity and randomness. Different from Peng et al.'s method, this paper determines the weights of DMs by the relative closeness degree, which is objective and credible. Compared with Peng et al.'s method, the weights of DMs obtained by the proposed method in this paper are more reasonable and reliable, which create the condition to obtain the correct decision result.

(2) Although Peng et al.'s method [26] can yield the optimal alternative, it only provides the partial orders of the remaining alternatives. The decision results obtained by Peng et al.'s method cannot distinguish the rankings of all alternatives. According to Figure 2, alternative A_1 is not compared with alternative A_3. The optimal alternative obtained by Peng et al.'s method is also not reliable. Thus, compared with Peng et al.'s method, the proposed method in this paper not only can obtain rankings of all alternatives but also can provide the reliable optimal alternatives.

Therefore, compared with the approach introduced by Peng et al.'s method, our method offers several benefits and has a broader range of potential applications.

5.3.2. Comparative analysis with IVPF-TOPSIS

Yu et al. [31] conducted supplier performance assessments by synthesizing the distance and similarity among various options. Following this, a protracted IVPF-TOPSIS plan was presented to tackle the MCGDM predicament in the sustainable evaluation of suppliers. Following the method in Yu et al. based on the data shown in Table 3, the assemblage amalgamated decision matrix is calculated by taking into account the degree of optimism λ_k and the relative weight η_k of DM E_k in [31]. The resulting matrix is derived from Table 3 and presented below.

$$R=\begin{pmatrix}(0.7727,\ 0.2340) & (0.7710,\ 0.2324) & (0.7155,\ 0.2860) & (0.7324,\ 0.2756)\\ (0.6566, 0.3507) & (0.4219, 0.6046) & (0.3801, 0.6374) & (0.6126, 0.3887)\\ (0.6417, 0.3618) & (0.6126, 0.3887) & (0.7486, 0.2547) & (0.5931, 0.4084)\\ (0.6911, 0.3179) & (0.4834, 0.5232) & (0.6126, 0.3887) & (0.5403, 0.4868)\\ (0.6827, 0.3349) & (0.5903, 0.4104) & (0.5931, 0.4084) & (0.7035, 0.3176)\end{pmatrix},$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3)=(0.8,0.5,0.3)$, $\eta=(\eta_1,\eta_2,\eta_3)=(0.2,0.4,0.4)$.

Next, the divergence vector d and the normalized criteria weight vector w are acquired through Eqs (16) and (17) in [31].

$$d=(1.8037, 1.8905, 1.8423, 1.8780),$$

$$w=(0.2433, 0.2550, 0.2485, 0.2533).$$

Afterward, the weighted distance vectors are computed using given equations.

$$d^{+}=(0.0130, 0.3538, 0.1629, 0.2650, 0.1770)^{T},$$

$$d^{-}=(0.3761, 0.0359, 0.2303, 0.1290, 0.2182)^{T}.$$

Then the grey relational grade vectors are obtained as follows by Eqs (22) and (23) in [31].

$$\gamma^{+}=(0.4890, 0.3128, 0.3917, 0.3404, 0.3814)^{T},$$

$$\gamma^{-}=(0.2999, 0.4720, 0.3741, 0.4163, 0.3618)^{T}.$$

According to Eqs (25) and (26) in [31], the derived values of distance and grey relational grade, seamlessly integrated in a vector, are procured in the following manner.

$$I^{+}=(1.0000, 0.3675, 0.7068, 0.5196, 0.6800)^{T},$$

$$I^{-}=(0.3360, 1.0000, 0.6266, 0.8154, 0.6334)^{T},$$

where $\alpha=0.5,\beta=0.5$.

$CC=(0, 0.9170, 0.4128, 0.6787, 0.4368)^{T}$ is the calculated distance vector from each alternative to the optimized ideal reference point $G=(1, 0.3360)$. Ultimately, the obtained ranking result is $A_1\succ A_3\succ A_5\succ A_4\succ A_2$.

TOPSIS and similar distance-based methods calculate the distance from each alternative to the PIS and NIS as a fixed geometric measure. The final ranking is based on a predefined relative closeness formula. The PIS and NIS are given or constructed from the decision matrix, a process that does not inherently optimize or calibrate these distances against the decision-maker's implicit preferences. In contrast, our method is fundamentally designed for scenarios where weight information is incomplete or unavailable. The simultaneous use of both the PIS and NIS in our optimization objective is precisely what enables the model to extract and uniquely determine criteria weights from simple pairwise comparisons. According to the solving process and ranking result that is presented above, we can derive the following three conclusions.

(1) The ranking result obtained by Yu et al.'s method [31] is consistent with the result obtained by the proposed method, which demonstrates the effectiveness and correctness of the proposed method.

(2) Yu et al.'s method involves too many parameters and those parameters need to be determined in advance, which may cause the decision result to be unreliable and difficult to implement in actual situations. Thus, the proposed method can more effectively solve IVPFN MCGDM problem.

(3) Yu et al.'s method transforms IVPFNs into PFNs, which may cause the loss of decision information. Different from Yu et al.'s method, the proposed method does not transform IVPFNs but utilizes the distances between IVPFNs to determine the weight information and consistency index as well as the inconsistency index. Thus, the proposed method can preserve more original decision information than Yu et al.'s method.

In addition, the method described in this paper is more germane than the approach proposed by Yu et al. since it is capable of solving MCGDM problems with IVPF true degrees. Involving IVPF true degrees makes the proposed approach more practical and effective for decision-making in real-world scenarios.

5.3.3. Comparative analysis with IVPF-TODIM

Biswas and Sarkar [32] introduced some new point operator-based similarity measures (POSMs) for IVPFSs, aiming to decrease the uncertainty level of elements in the universe of discourse associated with IVPFSs. These newly defined POSMs are subsequently employed to calculate the relative dominance measure of each alternative in the IVPF-TODIM framework. Finally, the generalized mean aggregation operator is utilized to select the best alternative.

Although the method in [32] can solve the group decision-making problem of IVPFSs, the weights of the criteria are pre-given, precise numbers. In the process of calculating the group value after combining the solution set using the generalized mean operator, the weights of the experts are considered equal.

Using the method in [32] to solve the normalized data (Table 3) in this paper, assuming that the weights of the criteria are $w = (0.4, 0.2, 0.15, 0.25)^T$, the relative weights of the criteria are:

$$w_{1r} = 1,\ w_{2r} = 0.5,\ w_{3r} = 0.375,\ w_{4r} = 0.625.$$

Consider the POSM $S^3_{F^{0.4}_{0.1}}(A, B)$ when assessing the degree of dominance of alternative A_i over each alternative A_l in relation to criteria C_j for the first expert E_1. For $\theta = 2.5$, the dominance degree matrices corresponding to the criteria $C_j (j = 1,2,3,4)$ for E_1 are shown in Table 18. Similarly, we calculate the dominance degree matrices for other decision matrices.

Dominance degree matrices $\delta^{(1)}$ for each alternative $A_i (i = 1,2,3,4,5)$ of the expert E_1 are shown in Table 19. In the same way, overall dominance degree matrices for other experts can be determined.

Table 18. Dominance degree matrix of alternatives for E_1.

Dominance degree matrix	A_1	A_1	A_3	A_4	A_5
$\phi_1^{(1)}$					
A_1	0	0.4556	0.5353	0	0.4051
A_2	-0.4556	0	-0.5627	-0.4556	0.5943
A_3	-0.5353	0.5627	0	-0.5353	0.52
A_4	0	0.4556	0.5353	0	0.4051
A_5	-0.4051	-0.5943	-0.52	-0.4051	0
$\phi_2^{(1)}$					
A_1	0	0.4255	0.4255	0.4025	0.4025
A_2	-0.8511	0	0	0.4252	0.4252
A_3	-0.8511	0	0	0.4252	0.4252
A_4	-0.805	-0.8503	-0.8503	0	0
A_5	-0.805	-0.8503	-0.8503	0	0
$\phi_3^{(1)}$					
A_1	0	0	-0.8741	-0.9818	0
A_2	0	0	-0.8741	-0.9818	0
A_3	0.3278	0.3278	0	0.3485	0.3278
A_4	0.3682	0.3682	-0.9293	0	0.3682
A_5	0	0	-0.8741	-0.9818	0
$\phi_4^{(1)}$					
A_1	0	0	0.4753	-0.7612	0
A_2	0	0	0.4753	-0.7612	0
A_3	-0.7605	-0.7605	0	-0.7201	-0.7605
A_4	0.4758	0.4758	0.45	0	0.4758
A_5	0	0	0.4753	-0.7612	0

Table 19. Overall dominance degree matrix of alternatives for E_1.

Overall dominance degree matrix $\delta^{(1)}$	A_1	A_1	A_3	A_4	A_5
A_1	0	0.8811	0.562	-1.3406	0.8076
A_2	-1.3067	0	-0.9615	-1.7735	1.0194
A_3	-1.8191	0.13	0	-0.4817	0.5124
A_4	0.0389	0.4492	-0.7943	0	1.2491
A_5	-1.2101	-1.4446	-1.7691	-2.1482	0

Subsequently, the overall values of each expert's assessment for each alternative are presented in Table 20. Finally, for each alternative, the overall values from all experts can be aggregated to obtain ζ_i and then the alternatives can be ranked based on it.

Table 20. The overall alternative's values of the DMs.

Alternative's overall value	A_1	A_1	A_3	A_4	A_5
$\zeta_i^{(1)}$	0.9957	0.4724	0.6538	1	0
$\zeta_i^{(2)}$	1	0	0.4341	0.0318	0.643
$\zeta_i^{(3)}$	1	0.0641	0.811	0	0.3471
Final overall value ζ_i	0.9986	0.3278	0.6681	0.6934	0.4681

According to ζ_i, it is easy to see that the ranking result is $A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$, which is consistent with the result by the method in this article. In comparison with the method in [32], the advantages of this paper are outlined as the following.

(i) Although the method in [32] can address MCGDM problems, it simply presets the weights of all experts to be equal. In contrast, our approach determines the DMs' weights objectively based on the original decision information.

(ii) In the method in [32], the criterion weights are predetermined. In contrast, this paper establishes a bi-objective IVPF mathematical programming model to derive these weights. This model-based approach yields more objective and credible weight assignments.

5.3.4. Rank-correlation analysis based on Spearman's rank-correlation coefficient

To deeply compare these ranking orders by the IVPF-ELECTRE method [26], IVPF-TOPSIS method [31], IVPF-TODIM method [32], and the method proposed in this paper, Spearman's rank-correlation test [33] is considered to estimate whether there is statistical significance of the ranking difference among them. During the process of Spearman's test, a rank-correlation coefficient r_s and a test statistic Z are defined to determine the similarity of the rankings between two sets of rankings $\{x^\kappa\}$ and $\{y^\kappa\}$, where

$$r_s = 1 - 6\sum_{\kappa=1}^{K} \frac{(d^\kappa)^2}{K(K^2-1)},\ Z = r_s\sqrt{K-1},\ d^\kappa = x^\kappa - y^\kappa\ (\kappa = 1,2,\ldots,K).$$

The closer r_s is to ± 1, the stronger the relationship between $\{x^\kappa\}$ and $\{y^\kappa\}$. Especially, when the rank-correlation coefficient r_s varies to $+1$, it denotes a perfect positive relationship between $\{x^\kappa\}$ and $\{y^\kappa\}$. If the relative measure r_s varies to -1, it implies a perfect negative relationship between $\{x^\kappa\}$ and $\{y^\kappa\}$. In addition, the test statistic Z is utilized to compare with a pre-determined level of significance α value. Usually set at $\alpha = 0.05$, the critical Z value is 1.645, i.e., $Z_{0.05} = 1.645$. When Z exceeds 1.645, it can be derived that $\{x^\kappa\}$ and $\{y^\kappa\}$ are similar. Otherwise, we know that there is no evidence of a positive relationship between $\{x^\kappa\}$ and $\{y^\kappa\}$.

When solving the above investment example, there are three sets of preference rankings obtained by the proposed method and the methods in [31, 32], denoted by A, B, and C, respectively.To compare these ranking orders, the rank-correlation coefficients and the test statistics are calculated in Table 21.

Table 21. Comparison of alternative rankings of IVPF-TOPSIS [31], IVPF-TODIM [32], and the proposed method.

Alternative	Ranking			Ranking difference	
	This paper (A)	IVPF-TOPSIS (B)	IVPF-TODIM (C)	A-B	A-C
A_1	1	1	1	0	0
A_2	5	5	5	0	0
A_3	3	2	2	1	1
A_4	4	4	4	0	0
A_5	2	3	3	-1	-1
Spearman's rank-correlation coefficient r_s				0.9	0.9
Test value Z				1.8	1.8

From Table 21, it can be summarized that the ranking (A) in this paper is positively correlated with the rankings (B) and (C) because the test value $Z = 1.8$ is bigger than 1.645.

As the preceding discussion indicates, the ranking of alternatives obtained by IVPF-ELECTRE [26] is not a complete linear order but a partial order. It reveals only that Alternative A_1 is superior to Alternatives A_2, A_4, and A_5, and that Alternative A_3 is superior to Alternatives A_2 and A_5. In contrast, both IVPF-TOPSIS [31] and IVPF-TODIM [32] yield an identical ranking of $A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$. Our proposed IVPF-LINMAP method, however, produces a different order of $A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$. The distinction is that the two sequences differ solely in the transposition of the second and third positions. In our result, Alternative A_5 is ranked second and Alternative A_3 third. This discrepancy can be attributed to the following reasons:

(1) Alternative A_5 received consistently outstanding ratings on several key criteria. For instance, Expert E_1 assigned an "EG" (Extremely Good) rating on Criterion C_2, while Expert E_2 gave "EG" on C_1 and "VG" (Very Good) on C_3. In contrast, evaluations for Alternative A_3 were highly polarized. On Criterion C_1, ratings varied from "VG" (E_2) to "F" (Fair, E_1) to "EG" (E_3). More critically, on Criterion C_2, Expert E_2 assigned a "VP" (Very Poor) rating—a potential veto that signals a severe drawback. In group decision-making, such an extreme negative evaluation often carries significant weight, as it may point to an unacceptable risk. The presence of this"VP" rating inherently elevates the perceived risk associated with Alternative A_3.

(2) The proposed method is fundamentally different because it simultaneously incorporates both the PIS and NIS within its optimization framework. This dual consideration enables the model to actively penalize extreme negative evaluations (like the "VP" rating). Consequently, the robust and consensus-driven profile of Alternative A_5 is rewarded with a higher ranking.

When weight information is incomplete, our method derives weights implicitly through its optimization process, which is guided by DMs' preference judgments. This process naturally favors alternatives that exhibit high expert consensus and no critical weaknesses (such as Alternative A_5), while applying greater scrutiny to those carrying extreme negative feedback (such as Alternative A_3). This behavior aligns closely with practical decision-making principles of risk aversion and consensus-seeking, thereby making the preferential ranking of Alternative A_5 over Alternative A_3 not only methodologically sound but also intuitively more reasonable.

Several decision-making methods exist in the IVPF environment. We have selected three of them for comparison with the proposed method in this paper, as each of these four approaches has its own strengths and limitations. IVPF-ELECTRE [26] effectively models veto scenarios through non-compensatory concordance/discordance relations, making it robust for high-risk decisions but suffers from complex parameter calibration and often yields an incomplete partial order rather than a definitive ranking. IVPF-TOPSIS [31] is valued for its intuitive logic and computational efficiency, providing a complete ranking as a common benchmark. Its major limitations are insensitivity to critical flaws and a heavy, subjective dependence on pre-defined weights. IVPF-TODIM [32] incorporates prospect theory to better reflect human psychological biases under uncertainty. However, its complex calculations and the subjective setting of a key behavioral parameter introduce complexity and arbitrariness. IVPF-LINMAP in this paper uniquely derives weights objectively from incomplete preferences via mathematical programming, simultaneously optimizing toward the ideal and away from the anti-ideal solution. This promotes consensus-seeking, robust alternatives but at the cost of higher computational load, sensitivity to input preference quality, and reduced interpretability due to its "black-box" optimization core.

It is worth mentioning that Kamari et al. [11] proposed a new distance metric called the flexible indeterminacy quantifier, which addresses the shortcomings of traditional distance metrics in handling uncertainty. The flexible indeterminacy quantifier features three key characteristics: adaptive weighting, self-regulating exponents, and dynamic norm selection, enabling it to better capture differences in uncertain environments and improve the stability and discriminative power of rankings. Furthermore, the paper extends TOPSIS and VIKOR into PNTOPSIS and PNVIKOR, respectively, and integrates them into the Pythagorean neutrosophic set framework, thereby enhancing the applicability of these methods in fuzzy and uncertain contexts.

6. System verification

Furthermore, we have translated this case into a practical implementation within the decision-making system. The procedure is detailed in the following steps.

Step 1. The interface, titled "Parameter Settings", is designed to initialize the fundamental structure of the decision-making process. This stage allows the user to define the key participants and framework before any evaluations begin. In this phase, the system is configured for 3 experts to provide consolidated judgments, 5 alternatives, and 4 defined criteria: signal strength, network security, network speed, and coverage. The interface of Step 1 is depicted in Figure 3.

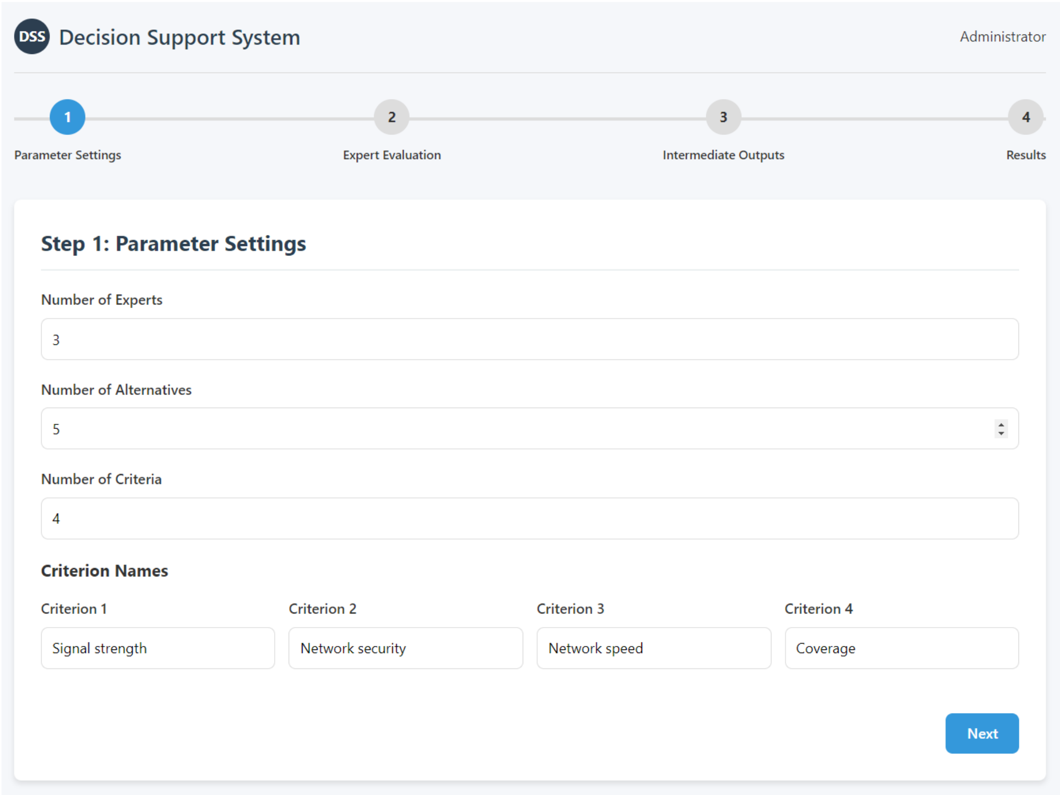

Figure 3. Configure the parameters of the decision-making system.

Step 2. In this step, the experts' evaluations are input through the interface. The three predefined experts provide their qualitative assessments for each alternative across all four criteria. The data is presented in a consolidated table where the rows list the five alternatives and the columns are dedicated to the criteria: signal strength, network security, network speed, and coverage. Each expert has a dedicated sub-column under each criterion to select their judgment. The evaluations use a linguistic scale, which includes terms such as "Extremely Good," "Very Good," "Good," "Fair," "Poor," and "Very Poor," allowing experts to express nuanced preferences. Figure 4 shows the interface where experts input their decision data.

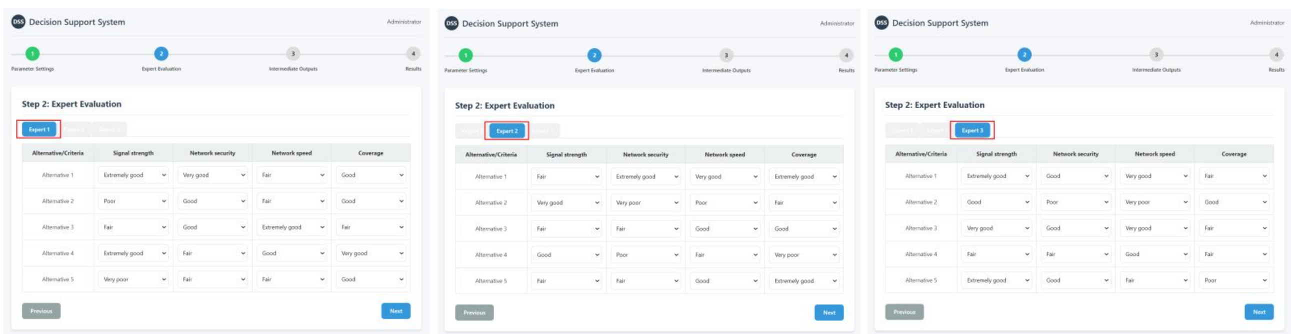

Figure 4. Input the evaluation data from experts for each alternative under all criteria.

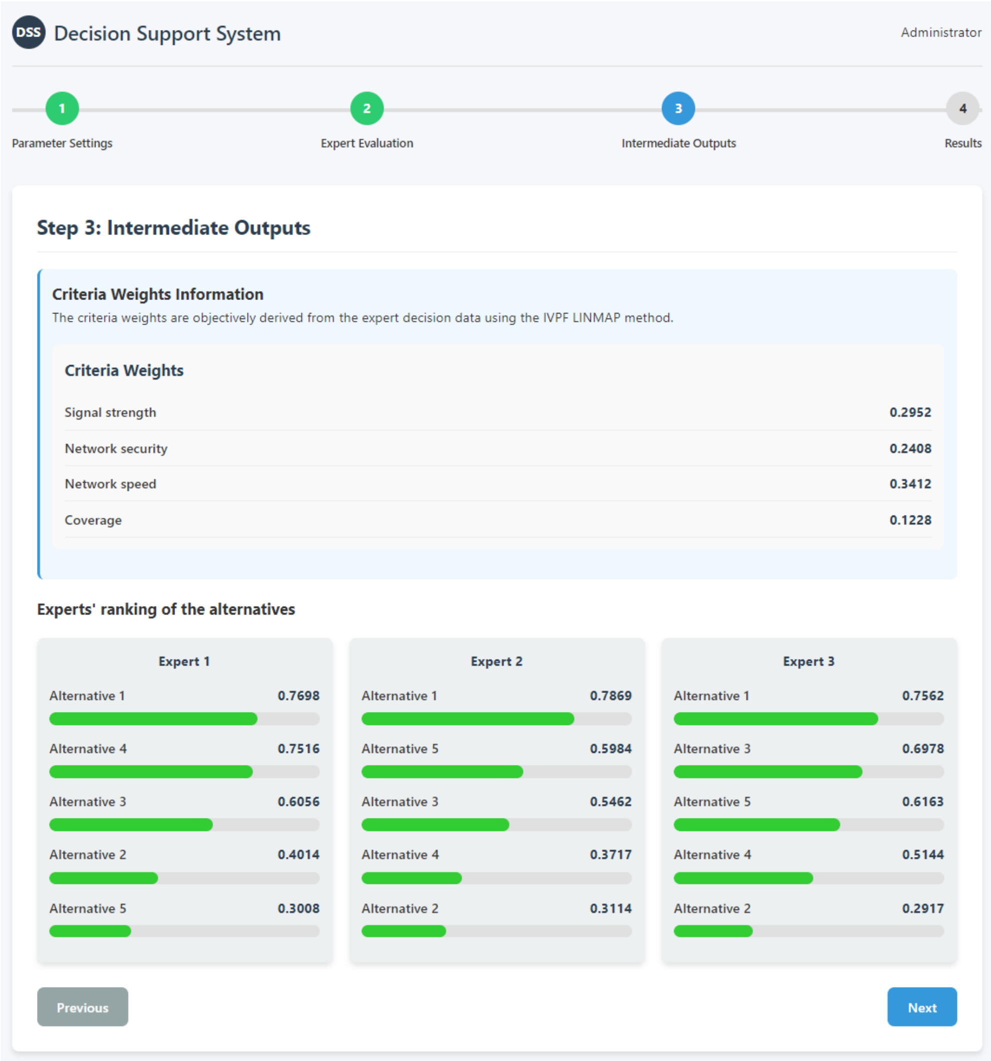

Figure 5. The ranking results of the alternatives by each expert.

Step 3. This step presents the key intermediate results calculated by the system using the method proposed in this paper. The process begins by objectively deriving the weights of the criteria from the raw expert evaluations. This is accomplished using the IVPF LINMAP method, which eliminates subjective bias in determining the importance of each criterion. Subsequently, based on these objectively determined criterion weights, the system processes each expert's original evaluations to generate an individual ranking of the alternatives for each expert. The experts' ranking of the alternatives is displayed in a comparative table. These intermediate results output by the system are shown in Figure 5.

Step 4. This step presents the conclusive outcome of the group decision-making process. The core of this phase is the assignment model for group decision-making, which is specifically designed to integrate the individual rankings from all experts into a single, collective ranking. The final ranking of the alternatives, as determined by the system, is $A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$, which is presented in Figure 6.

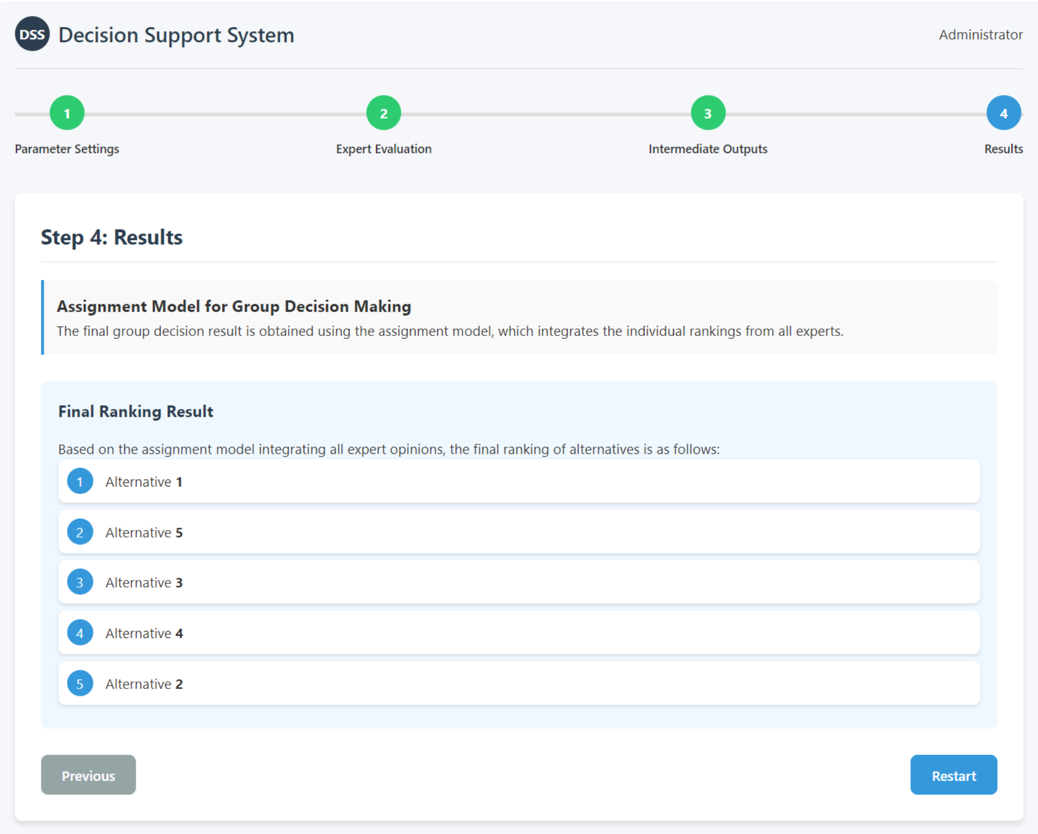

Figure 6. Final aggregated ranking via an assignment model.

7. Conclusions

This paper introduces an IVPF mathematical programming approach to tackle MCGDM issues encompassing incomplete IVPF information. First, this paper determines the weighs of DMs based on the relative closeness degree. Then, based on the IVPF group consistency and inconsistency, the IVPF programming model is constructed to obtain the weights of criteria. Finally, a multi-objective assignment model is constructed and solved by the Hungarian algorithm to generate the overall ranking matrix for the group of decision-makers and obtain the optimal alternative. Additionally, the case study and comparative analysis are conducted to underscore the advantages of the approach delineated in the study.

The main contributions and advantages of this paper can be summarized as follows:

(1) A criterion-specific weighting scheme is introduced for DMs, leveraging relative closeness degrees to quantify and integrate their individual expertise and preferences. This mechanism ensures objective weight determination while closely mirroring real-world, differentiated expert contributions in complex decision-making scenarios.

(2) Novel group consensus and inconsistency indices are defined, explicitly incorporating both the positive and negative ideal IVPF solutions. This dual-reference-point framework enables a more nuanced assessment of group alignment, effectively mitigating the risk of suboptimal alternative selection when alternatives deviate significantly from both reference points.

(3) A dual-objective mathematical programming model is established within the IVPF

framework, designed to jointly minimize the group inconsistency indices associated with the IVPF positive and negative ideal solutions. The model is transformed into a linear programming problem, facilitating the efficient and unbiased determination of optimal criteria weights.

(4) The method synthesizes the criterion-specific DM weights and the derived criteria weights to calculate the comprehensive relative closeness of each alternative. Finally, a multi-objective assignment model, solved efficiently via the Hungarian algorithm, aggregates individual preference structures into a robust and reliable group ranking, thereby significantly enhancing the stability and credibility of the decision-making process.

Despite the theoretical contributions of the proposed mathematical programming model for interval-valued Pythagorean fuzzy MCGDM, it is essential to acknowledge its inherent limitations to provide a balanced perspective and guide subsequent research. The primary challenges pertain to computational scalability in complex scenarios and the need for broader methodological enhancements.

(1) A key practical limitation of the model is its growing computational demand in large-scale settings. The number of variables and constraints increases polynomially with the number of decision-makers, alternatives, and criteria, particularly when comprehensive pairwise comparisons are employed. This combinatorial explosion can render the exact optimization model computationally intensive for real-time or very-large-scale problems. Therefore, the current framework is most applicable to strategic, moderate-scale decision problems where analytical precision is paramount and the problem size remains manageable.

(2) The transformation of linguistic, fuzzy evaluations into a crisp linear programming model, while mathematically rigorous, may create a disconnect between the numerical output and the original preference semantics used by decision-makers. A key area for improvement to ensure practical adoption is enhancing the solution's post-hoc interpretability; this involves translating optimization results back into intuitive, fuzzy-linguistic terms.

To address the aforementioned limitations and enhance the practical value of this framework, the following research directions have been systematically outlined, with the aim of advancing the field from theoretical validation to engineering implementation:

(1) Development of Scalable Solution Algorithms. Future work will prioritize creating efficient heuristic and metaheuristic algorithms to obtain high-quality approximate solutions for large-scale instances. Additionally, exploring model decomposition techniques to break down the problem into smaller, solvable units will be crucial for enhancing computational feasibility.

(2) Using statistical methods such as the Wilcoxon signed-rank test, the ranking results of the proposed model will be systematically compared with established methods like IVPF-TOPSIS and IVPF-TODIM. This work will rigorously validate the model's consistency and clarify its practical advantages and differences.

(3) The method proposed in this paper can be naturally extended to broader research directions, such as heterogeneous MCGDM problems, the integration of social network analysis among decision-makers, and the study of the consensus-reaching process in group decision-making.

Author Contributions

Conceptualization: Zhen Jin, Xiaofang Deng; Formal Analysis: Zhen Jin, Xiaofang Deng, Gaili Xu; Methodology: Xiaofang Deng; Resources: Gaili Xu; Software: Xiaofang Deng, Gaili Xu; Writing–Original Draft Preparation: Zhen Jin; Writing–Review and Editing: Zhen Jin, Xiaofang Deng.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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Appendix

To facilitate clear discussion and presentation in this paper, and to ensure uniformity and clarity in the use of terminology, a systematic compilation of the common abbreviations involved in this study is provided. Table 22 lists the abbreviations of relevant terms along with their corresponding full forms, enabling readers to accurately understand and refer to them.

Table 22. List of abbreviations.

Abbreviation	Full term
MCDM	Multi-criteria decision-making
MCGDM	Multi-criteria group decision-making
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
IVIFS	Interval-valued intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
IVPF	Interval-valued Pythagorean fuzzy
IVPFS	Interval-valued Pythagorean fuzzy set
IVPFN	Interval-valued Pythagorean fuzzy number
PIS	Positive ideal solution
NIS	Negative ideal solution