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Research article

On quantitative weighted bounds for commutators of rough singular integral operators

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Abstract: Let Ω be a homogeneous function of degree zero, integrable, and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$, $n \geq 2$. Let T_Ω be the homogeneous convolution singular integral operator with kernel $\frac{\Omega(x)}{|x|^n}$. In this paper, we proved some quantitative two-weight estimates for the commutator, $[b, T_\Omega]$, with a BMO symbol b and the singular integral operator T_Ω under the condition $\Omega \in L^q(\mathbb{S}^{n-1})$ for $q \in (2, \infty)$.

Keywords: quantitative weighted estimate; commutator; singular integral; rough kernel of L^q type
Mathematics Subject Classification: 42B20

1. Introduction

This article will focus on studying the quantitative weighted estimates for commutators of rough singular integral operators. We will work on $\mathbb{R}^n$, $n \geq 2$. The singular integral operator with rough kernel T_Ω is defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)dy, \tag{1.1}$$

where Ω is a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$ throughout the paper.

The operator has been studied by numerous scholars since the 1950s, and was first introduced by Calderón and Zygmund [1]. For $p \in (1, \infty)$, $\Omega \in L\log L(\mathbb{S}^{n-1})$, Calderón and Zygmund [2] proved that the operator T_Ω is bounded on $L^p(\mathbb{R}^n)$. Ricci and Weiss [3] gave the same results under the condition $\Omega \in H^1(\mathbb{S}^{n-1})$, an improvement upon the Calderón-Zygmund result, where notably, the space $H^1(\mathbb{S}^{n-1})$ contains $L\log L(\mathbb{S}^{n-1})$. For other works concerning the $L^p(\mathbb{R}^n)$ boundedness for the operator T_Ω, we refer the reader to [3–5] and the references therein.

Given a linear operator T and $b \in BMO(\mathbb{R}^n)$, the commutator $[b, T]$ in the sense of Coifman-Rochberg-Weiss is defined as follows:

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

When $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ for some $\alpha \in (0, 1)$, Coifman, Rochberg, and Weiss [6] established that $b \in BMO(\mathbb{R}^n)$ is the sufficient and necessary condition for $L^p(\mathbb{R}^n)(p > 1)$ boundedness of commutator $[b, T_\Omega]$. Via the weighted estimates for T_Ω together with the relationship between A_p weights and BMO functions, Alvarez, Bagby, Kurtz, and Pérez [7] established the $L^p(\mathbb{R}^n)$ boundedness of the commutator $[b, T_\Omega]$ under the condition $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q \in (1, \infty)$. Hu [8] showed that if $\Omega \in L(\log L)^2(\mathbb{S}^{n-1})$, then the commutator $[b, T_\Omega]$ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ with the bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. For other works about the boundedness of $[b, T_\Omega]$, see [9–12], among others.

During the last two decades, there have been many significant works on the quantitative weighted bounds for singular integral operators and their commutators. Here we present a sharp, weighted estimate for some principal operators in harmonic analysis, which can be traced back to Buckley [13]. He proved that for $1 < p < \infty$ and $w \in A_p$, the Hardy-Littlewood maximal operator M has the following estimate:

$$\|Mf\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p(w)}.$$

Subsequently, Astala, Iwaniec, and Saksman proposed the famous A_2 conjecture in [14], which was solved by Petermichl and Volberg in [15]. This attracted great interest in researching the sharp weighted estimate for the Hilbert transform, Riesz transform, and general Calderón-Zygmund operators.

Later, how to find sharp quantitative weighted estimates for rough singular integral operators also received attention. In the following years, many researchers studied the quantitative weighted boundedness of T_Ω defined in (1.1) with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Hytönen, Roncal, and Tapiola first gave the following result in [16]: For $p \in (1, \infty)$ and $w \in A_p$,

$$\|T_\Omega f\|_{L^p(w)} \leq c_{n,p}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{2\max\left\{1, \frac{1}{p-1}\right\}}\|f\|_{L^p(w)}. \tag{1.2}$$

Combining estimate (1.2) and the idea in [17] originating from the conjugate method in [6, p. 621], authors in [18] proved that, for $p \in (1, \infty)$ and $w \in A_p$,

$$\|[b, T_\Omega]f\|_{L^p(w)} \leq c_{n,p}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}[w]_{A_p}^{3\max\left\{1, \frac{1}{p-1}\right\}}\|f\|_{L^p(w)}.$$

Later, Li, Pérez, Rivera-Rios, and Roncal [19] improved the bound in (1.2) for $p \in (1, \infty)$, $w \in A_p$, and $\Omega \in L^\infty(\mathbb{S}^{n-1})$:

$$\|T_\Omega f\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [w^{1-p'}]_{A_\infty}^{\frac{1}{p}}\right)\min\left\{[w^{1-p'}]_{A_\infty}, [w]_{A_\infty}\right\}\|f\|_{L^p(w)}. \tag{1.3}$$

Using the conjugate method again and combining (1.3), the authors in [9] pointed out that, for $p \in (1, \infty)$, $w \in A_p$, and $\Omega \in L^\infty(\mathbb{S}^{n-1})$,

$$\|[b, T_\Omega]f\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [w^{1-p'}]_{A_\infty}^{\frac{1}{p}}\right)\left([w^{1-p'}]_{A_\infty} + [w]_{A_\infty}\right)^2\|f\|_{L^p(w)}.$$

We next recall the quantitative $A_1 - A_\infty$ weighted estimates for some operators. In [20], Reguera and Thiele proposed that, for the Hilbert transform H, there exists a constant $c > 0$ such that for any $w \in A_1$,

$$\|Hf\|_{L^{1,\infty}(w)} \leq c[w]_{A_1}\|f\|_{L^1(w)}.$$

This is the so-called A_1 conjecture. However, it was shown in [21], by using Bellman function techniques, that the A_1 conjecture is incorrect.

Inspired by the A_1 conjecture, Lerner, Ombrosi, and Pérez [22] established the following quantitative weighted endpoint estimate for any Calderón-Zygmund operator T for $p > 1$ and $w \in A_1$:

$$\|Tf\|_{L^{1,\infty}(w)} \leq c[w]_{A_1} \log(e + [w]_{A_1})\|f\|_{L^1(w)}. \tag{1.4}$$

The key to obtain (1.4) is to prove the following two-weight L^p estimate, for $p \in (1, \infty)$, $r \in (1, \infty)$, and $w > 0$:

$$\|Tf\|_{L^p(w)} \leq c_T pp'(r')^{\frac{1}{p'}}\|f\|_{L^p(M_r w)}, \tag{1.5}$$

where $M_r w$ is the maximal function defined by $M_r w(x) := \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |w(t)|^r dt\right)^{1/r}$, and the supremum is taken over all cubes Q in $\mathbb{R}^n$. It should be noted that $(r')^{\frac{1}{p'}}$ in (1.5) is crucial to obtain the next estimate: for $p \in (1, \infty)$ and $w \in A_1$,

$$\|Tf\|_{L^p(w)} \leq cpp'[w]_{A_1}\|f\|_{L^p(w)}.$$

For $\Omega \in L^\infty(\mathbb{S}^{n-1})$, $1 < p < \infty$, and $w \in A_1$, Pérez, Rivera-Rios, and Roncal in [18] established the following quantitative weighted estimate for T_Ω:

$$\|T_\Omega f\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}\|f\|_{L^p(w)}, \tag{1.6}$$

and the following quantitative weighted estimate for the commutator $[b, T_\Omega]$ with $b \in BMO(\mathbb{R}^n)$:

$$\|[b, T_\Omega]f\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{2+\frac{1}{p}}\|f\|_{L^p(w)}. \tag{1.7}$$

Later the quantitative weighted estimates (1.6) and (1.7) were improved in [23] such that

$$\|T_\Omega f\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{\frac{1}{p'}}\|f\|_{L^p(w)},$$

and

$$\|[b, T_\Omega]f\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(p')^3 p^2[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}\|f\|_{L^p(w)}.$$

Motivated by the works mentioned above, our goal in this paper is to weaken the kernel $\Omega \in L^\infty(\mathbb{S}^{n-1})$ to $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and establish the quantitative $A_1 - A_\infty$ weighted estimates for the commutator $[b, T_\Omega]$ with rough kernel $\Omega \in L^q(\mathbb{S}^{n-1})$ and function $b \in BMO(\mathbb{R}^n)$. We will adapt the idea in [18], i.e., we decompose T_Ω into a sequence of suitable operators, whose kernels satisfy the locally L^q-Hörmander condition. Using some new refined arguments, we will get a two-weight estimate for the related commutator via sparse domination and the conjugate method.

Theorem 1.1. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $2 < q < \infty$, $b \in BMO(\mathbb{R}^n)$, and $q' < p < q$. Then for any $\gamma > \frac{q}{q-p}$ and weight $w > 0$, there exists a constant $c_n > 0$ depending only on n such that*

$$\|[b, T_\Omega]f\|_{L^p(w)} \le c_n \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} (p')^2 \left(\frac{pq-p}{q-p}\right)^{\frac{1}{q'}} \left(\left(\frac{q-p}{q}\gamma\right)'\right)^{2+\frac{1}{p'}} \|f\|_{L^p(M_\gamma(w))}. \tag{1.8}$$

Corollary 1.2. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $2 < q < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then, if $w \in A_\infty$, there exists a constant $\tau_n > 0$ depending only on n such that, for $q' < p < q/(1 + \tau_n[w]_{A_\infty})$,*

$$\|[b, T_\Omega]f\|_{L^p(w)} \le c_{n,p,q} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left(\left(\frac{q-p}{q}\left(1 + \frac{1}{\tau_n[w]_{A_\infty}}\right)\right)'\right)^{2+\frac{1}{p'}} [w]_{A_1}^{\frac{1}{p}} \|f\|_{L^p(w)}.$$

The paper is organized as follows. In Section 2, we will present some basic definitions and give a decomposition of T_Ω. In Section 3, we will present the unweighted L^p estimate and two-weight estimate for the pieces of the operator sequence. In Section 4, we will prove the main result.

2. Preliminary

2.1. Notations

In this article, c_n and C_n stand for the positive dimensional constants. C denotes the positive constants not depending on the essential variables. C, c_n, and C_n may vary at each occurrence. Given a function f, $\widehat{f}$ denotes the Fourier transform of f. Q indicates a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For $1 < p < \infty$, we denote the conjugate index of ital by $p' = \frac{p}{p-1}$. $a \sim b$ indicates that there exists an absolute constant $c > 0$ such that $\frac{1}{c}b \le a \le cb$.

2.2. A_p weight

For $1 < p < \infty$, a weight w belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ if $w^{1-p'} \in L^1_{loc}(\mathbb{R}^n)$, where $(p'-1)(p-1) = 1$, and

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\right)^{p-1} <\infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. For $w \in \cup_{p>1} A_p(\mathbb{R}^n)$, we will use the following definition of the A_∞ constant for w (see [24, 25]):

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x)dx,$$

with $w(Q) = \int_Q w(x)dx$ and χ_Q being the characteristic function of Q. For brevity, we use the following notations [16, 26]:

$$\{w\}_{A_p} := [w]_{A_p}^{\frac{1}{p}} \max\left\{[w]_{A_\infty}^{\frac{1}{p'}}, [w^{1-p'}]_{A_\infty}^{\frac{1}{p}}\right\},$$

$$(w)_{A_p} := \max\left\{[w]_{A_\infty}, [w^{1-p'}]_{A_\infty}\right\},$$

$$\{w\}_{A_{p,r;s}} := [w]_{A_p}^{\frac{1}{r}} \max\left\{[w]_{A_\infty}^{(\frac{1}{s}-\frac{1}{r})_+}, [w^{1-p'}]_{A_\infty}^{\frac{1}{r}}\right\},$$

where $(\frac{1}{r}-\frac{1}{p})_+ = \max\{\frac{1}{r}-\frac{1}{p},0\}$. Moreover, by the fact that

$$[w]_{A_\infty} \le c_n[w]_{A_p},\ [w^{1-p'}]_{A_\infty} \le c_n[w^{1-p'}]_{A_{p'}} = c_n[w]_{A_p}^{\frac{1}{p-1}},$$

we know that

$$(w)_{A_p} \le c_n\{w\}_{A_p} \le c_n'[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \quad \{w\}_{A_p,r;s} \le c_n[w]_{A_p}^{\max\{\frac{1}{s},\frac{p}{p-1}\frac{1}{r}\}}. \tag{2.1}$$

We now need some results concerning the sharp reverse Hölder inequality.

Lemma 2.1. *[16]*

(1) If $w \in A_\infty$, then there is a dimensional constant c_n such that, for any $0 < \delta < c_n/[w]_{A_\infty}$,

$$\frac{1}{|Q|}\int_Q w^{1+\delta}(x)dx \le 2\left(\frac{1}{|Q|}\int_Q w(x)dx\right)^{1+\delta},$$

and

$$\left[w^{1+\delta/2}\right]_{A_\infty} \le C_n[w]_{A_\infty}^{1+\delta/2}.$$

(2) Let $1 < p < \infty$ and $w \in A_p$. Then, by choosing c_n small enough, we have

$$\left[w^{1+\delta}\right]_{A_p} \le 4\,[w]_{A_p}^{1+\delta},$$

for every $0 < \delta \le c_n/(w)_{A_p}$. Moreover, it follows that $w^{1+\delta/2} \in A_p$ and

$$\left(w^{1+\delta/2}\right)_{A_p} \le C_n(w)_{A_p}^{1+\delta/2}, \quad \left\{w^{1+\delta/2}\right\}_{A_p} \le C_n\{w\}_{A_p}^{1+\delta/2}.$$

2.3. Sparse operator

The collection $\mathcal{S}$ of cubes is η-sparse for $0<\eta<1$, if for each fixed $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \ge \eta\,|Q|$ and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

Given a sparse family $\mathcal{S}$, $r \in (0,\infty)$, and $f \in L^r_{\text{loc}}(\mathbb{R}^n)$, we define the sparse operator $\mathcal{A}_{r,\mathcal{S}}$ by

$$\mathcal{A}_{r,\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{\frac{1}{r}}\chi_Q(x), \tag{2.2}$$

and another sparse operator $\mathcal{A}^r_{\mathcal{S}}$ by

$$\mathcal{A}^r_{\mathcal{S}}f(x) = \left\{\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q |f(y)|dy\right)^r \chi_Q(x)\right\}^{1/r}.$$

For $p \in (1,\infty)$, $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, and $b \in BMO(\mathbb{R}^n)$, we also define two sparse operators related to the commutator by

$$\mathcal{T}_{\mathcal{S},b,p}f(x) = \sum_{Q\in\mathcal{S}}|b(x)-b_Q||f|_{p,Q}\chi_Q(x) \tag{2.3}$$

and

$$\mathcal{T}^{\star}_{\mathcal{S},b,p}f(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q |(b-b_Q)f|^p\right)^{\frac{1}{p}} \chi_Q(x), \tag{2.4}$$

where $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$ and $|f|_{p,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p(x)dx\right)^{\frac{1}{p}}$.

A general dyadic grid $\mathcal{D}$ is the collection of cubes satisfying the following properties: (i) for any cube $Q \in \mathcal{D}$, its side length $\ell(Q)$ is of the form 2^k for some $k \in \mathbb{Z}$; (ii) for any cubes $Q_1, Q_2 \in \mathcal{D}$, $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$; (iii) for each $k \in \mathbb{Z}$, the cubes of side length 2^k form a partition of $\mathbb{R}^n$.

The sparse operators $\mathcal{A}^r_{\mathcal{S}}$, $\mathcal{T}_{\mathcal{S},b,p}$, and $\mathcal{T}^{\star}_{\mathcal{S},b,p}$ play an important role in getting the quantitative weighted bound, which is crucial for us.

2.4. *Young functions and general maximal functions*

We call a function Ψ a Young function if it is a continuous, convex, increasing function and $\Psi : [0,\infty) \longmapsto [0,\infty)$. Let f be a measurable function defined on a set E with finite measure in $\mathbb{R}^n$. The $\Psi-$*norm* of f over E is defined by

$$\|f\|_{\Psi(L),E} := \inf\left\{\lambda > 0 : \frac{1}{|E|}\int_E \Psi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

Let Ψ be a Young function, and we define the *Orlicz maximal operator* $M_{\Psi(L)}$ by

$$M_{\Psi(L)}f(x) = \sup_{Q\ni x}\|f\|_{\Psi(L),Q}.$$

In particular, let $\Psi(t) = t^r$ for $1 \le r < \infty$, and then $M_{\Psi(L)}$ is the maximal operator M_r defined by

$$M_r f(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f|^r dt\right)^{1/r}. \tag{2.5}$$

Let $\Psi(t) = t\log(1+t)$, and $M_{\Psi(L)}$ is the maximal operator $M_{L(\log L)}$ defined by

$$M_{L(\log L)}f(x) = \sup_{Q\ni x}\|f\|_{L(\log L),Q}.$$

Noting $\log(1+t) \le \frac{1}{\delta}t^{\delta}$ for any $0 < \delta < 1$ and all $t > 0$, we can deduce that

$$Mf(x) \le M_{L(\log L)}f(x) \le \frac{1}{\delta}M_{1+\delta}f(x), \quad x \in \mathbb{R}^n. \tag{2.6}$$

2.5. *A decomposition for rough homogeneous singular integral operator* T_Ω

Let T_Ω be the singular integral operator defined in (1.1), and one can write

$$T_\Omega f := \sum_{m\in\mathbb{Z}} T_m f := \sum_{m\in\mathbb{Z}} K^{(m)} * f, \quad K^{(m)}(x) := \frac{\Omega(x)}{|x|^n}\chi_{2^m<|x|<2^{m+1}} \;\; m \in \mathbb{Z}. \tag{2.7}$$

Lemma 2.2. *[27, Lemma 2] Let Ω be homogeneous of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$, and $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$. The following inequality holds:*

$$\left|\widehat{K^{(m)}}(\xi)\right| \le C\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \min\{|2^m\xi|^{\alpha}, |2^m\xi|^{-\alpha}\}$$

for $0 < \alpha < 1/q'$ independent of Ω and $m \in \mathbb{Z}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a radial nonnegative function with $\operatorname{supp}\varphi \subset \left\{x \in \mathbb{R}^n : |x| < \frac{1}{4}\right\}$, and $\int \varphi(x)dx = 1$. For $f \in L^p(\mathbb{R}^n)$ with $p > 1$, let $\varphi_d(x) = 2^{-nd}\varphi\left(2^{-d}x\right)$ and $S_d(f) = \varphi_d * f$. Then $S_d f$ converges in f when $d \to -\infty$ in the sense of the L^p norm.

Take a sequence of integer numbers $\{N(j)\}_{j=0}^\infty$ with

$$0 = N(0) < N(1) < N(2) < \cdots < N(j) \to \infty.$$

Then we have

$$T_m = T_m S_m + \sum_{j=0}^\infty T_m\left(S_{m-N(j+1)} - S_{m-N(j)}\right).$$

If we let, for $j \in \{0\} \cup \mathbb{N}$,

$$T_j^N f(x) = \sum_{m\in Z} T_m S_{m-N(j)} f(x) = K_j^N * f(x), \quad K_j^N(x) = \sum_{m\in\mathbb{Z}} K^{(m)} * \varphi_{m-N(j)}(x), \tag{2.8}$$

and $\widetilde{T}_j^N = T_{j+1}^N - T_j^N$, then

$$T_\Omega = T_0^N + \sum_{j=0}^\infty \left(T_{j+1}^N - T_j^N\right) = T_0^N + \sum_{j=0}^\infty \widetilde{T}_j^N. \tag{2.9}$$

Lemma 2.3. *Let Ω be a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$, and $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$. Then there are constants C and $0 < \delta < \frac{1}{q'}$ independent of ξ and $j \in \mathbb{N}$ such that for all $j \geq 0$,*

$$(1)\quad \left|\widehat{K}(\xi)\right| + \left|\widehat{K_j^N}(\xi)\right| \leq C\|\Omega\|_{L^q(\mathbb{S}^{n-1})}; \qquad (2)\quad \left|\widehat{K}(\xi) - \widehat{K_j^N}(\xi)\right| \leq C2^{-\delta N(j)}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}.$$

Proof. By a similar argument as that in [27], we obtain Lemma 2.3. For brevity, we omit the details. □

Lemma 2.3 shows that the sum in (2.9) converges strongly in the L^2-operator norm.

Lemma 2.4. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ be a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$. Then K_j^N satisfies the L^q-Hörmander condition, i.e., there is a constant $c > 0$ such that, for any $y \in \mathbb{R}^n \setminus \{0\}$ and $|y| < R$, we have*

$$\sum_{k=1}^\infty \left(2^k R\right)^{n/q'} \left(\int_{2^k R<|x|<2^{k+1}R} \left|K_j^N(x-y) - K_j^N(x)\right|^q dx\right)^{1/q} \leqslant c_{n,q} N(j).$$

Proof. By the fact that

$$\operatorname{supp}\left(K^{(m)} * \varphi_{m-N(j)}\right) \subset \left\{x \in \mathbb{R}^n : 2^{m-2} \leqslant |x| \leqslant 2^{m+2}\right\},$$

we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}(2^kR)^{n/q'}\left(\int_{2^kR<|x|<2^{k+1}R}\left|K_j^N(x-y)-K_j^N(x)\right|^q dx\right)^{1/q}\\
&\le \sum_{k=1}^{\infty}\sum_{m\in Z}(2^kR)^{n/q'}\left(\int_{2^kR<|x|<2^{k+1}R}\left|K^{(m)}*\varphi_{m-N(j)}(x-y)-K^{(m)}*\varphi_{m-N(j)}(x)\right|^q dx\right)^{1/q}\\
&\le \sum_{k=1}^{\infty}\sum_{m:2^m\sim 2^kR}(2^kR)^{n/q'}\left(\int_{2^kR<|x|<2^{k+1}R}\left|K^{(m)}*\varphi_{m-N(j)}(x-y)-K^{(m)}*\varphi_{m-N(j)}(x)\right|^q dx\right)^{1/q}\\
&= \sum_{k=1}^{N(j)}\sum_{m:2^m\sim 2^kR}(2^kR)^{n/q'}\left(\int_{2^kR<|x|<2^{k+1}R}\left|K^{(m)}*\varphi_{m-N(j)}(x-y)-K^{(m)}*\varphi_{m-N(j)}(x)\right|^q dx\right)^{1/q}\\
&\quad+\sum_{k=N(j)+1}^{\infty}\sum_{m:2^m\sim 2^kR}(2^kR)^{n/q'}\left(\int_{2^kR<|x|<2^{k+1}R}\left|K^{(m)}*\varphi_{m-N(j)}(x-y)-K^{(m)}*\varphi_{m-N(j)}(x)\right|^q dx\right)^{1/q}\\
&:= I+II.
\end{aligned}
$$

For I, by the Minkowski inequality and the fact that $\left\|K^{(m)}\right\|_{L^q(\mathbb{R}^n)} \le c_{n,q}2^{\frac{-mn}{q'}}$,

$$I \le \sum_{k=1}^{N(j)}\sum_{m:2^m\sim 2^kR}\left\|K^{(m)}\right\|_{L^q(\mathbb{R}^n)}\left(2^kR\right)^{n/q'} \le c_{n,q}N(j).$$

For II, a direct computation shows that

$$II \le \sum_{k=N(j)+1}^{\infty}\sum_{m:2^m\sim 2^kR}\left\|\varphi_{m-N(j)}(\cdot-y)-\varphi_{m-N(j)}(\cdot)\right\|_{L^1(\mathbb{R}^n)} \le c_{n,q}.$$

Combining the estimate for I and II, we complete the proof. □

Given a linear operator T, we define the grand maximal operator $\mathcal{M}_T$ by

$$\mathcal{M}_T f(x) = \sup_{Q\ni x}\operatorname*{ess\,sup}_{\xi\in Q}\left|T\left(f\chi_{\mathbb{R}^n\setminus 3Q}\right)(\xi)\right|. \tag{2.10}$$

For any fixed cube Q_0, we define the local analogy of $\mathcal{M}_{T,Q_0}$ by

$$\mathcal{M}_{T,Q_0} f(x) = \sup_{Q\ni x, Q\subset Q_0}\operatorname*{ess\,sup}_{\xi\in Q}\left|T\left(f\chi_{3Q_0\setminus 3Q}\right)(\xi)\right|.$$

By Lemma 2.4, a standard argument shows the following two lemmas.

Lemma 2.5. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ be a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$. Then T_j^N maps from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with the bound $CN(j)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}$, and is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with the bound $CN(j)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}$.*

Lemma 2.6. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ be a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$. Then*

$$\mathcal{M}_{T_j^N} f(x) \le c_n\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\left(N(j)M_{q'}f(x) + MT_j^N f(x) + N(j)Mf(x)\right).$$

Moreover, $\mathcal{M}_{T_j^N}$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{q',\infty}(\mathbb{R}^n)$ with the bound $CN(j)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}$.

Remark 2.7. The proofs of Lemmas 2.5 and 2.6 are similar to Lemmas 3 and 4 in [27] and we omit the details.

Lemma 2.8. *[28] Assume that T is bounded from $L^m(\mathbb{R}^n)$ to $L^{m,\infty}(\mathbb{R}^n)$ and $\mathcal{M}_T$ is bounded from $L^u(\mathbb{R}^n)$ to $L^{u,\infty}(\mathbb{R}^n)$ with $1 \le m \le u \le \infty$. Then, for every $f \in L^r(\mathbb{R}^n)$ with compact support, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$|Tf(x)| \le C\mathcal{A}_{r,\mathcal{S}}(|f|)(x),$$

where $C = C_{n,m,u}\left(\|T\|_{L^m(w)\to L^{m,\infty}(w)} + \|\mathcal{M}_T\|_{L^u(w)\to L^{u,\infty}(w)}\right)$.

Lemma 2.9. *[29] Let $\mathcal{A}^r_{\mathcal{S}}$ be defined as (2.2), $p \in (1,\infty)$, $r \in (0,\infty)$, and $w \in A_p$. Then for a sparse family $\mathcal{S} \subset \mathcal{D}$ with $\mathcal{D}$ as the dyadic grid,*

$$\left\|\mathcal{A}^r_{\mathcal{S}} f\right\|_{L^p(w)} \le c_{n,p}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\left(\frac{1}{r}-\frac{1}{p}\right)_+} + [w^{-\frac{1}{p-1}}]_{A_\infty}^{\frac{1}{p}}\right)\|f\|_{L^p(w)},$$

where $\left(\frac{1}{r}-\frac{1}{p}\right)_+ = \max\left\{\frac{1}{r}-\frac{1}{p}, 0\right\}$.

3. Some lemmas

3.1. Unweighted L^p estimates with good decay for $\widetilde{T}^N_j$

Recall that $\widetilde{T}^N_j = T^N_{j+1} - T^N_j$ for $j \in \{0\} \cup \mathbb{N}$. We will get the quantitative weighted estimates for $\widetilde{T}^N_j$, which are crucial for the estimate of $[b, \widetilde{T}^N_j]$ in Lemma 3.5.

Lemma 3.1. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ be a homogeneous function of degree zero and have a mean value of zero on the unit sphere $\mathbb{S}^{n-1}$. Then for any $1 < p < \infty$ and $j \in \mathbb{N}$,*

$$\left\|\widetilde{T}^N_j f\right\|_{L^p(\mathbb{R}^n)} \le c_{n,p}2^{-\delta_{p,q}N(j)}N(j+1)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},$$

with some constant $\delta_{p,q} > 0$ independent of T_Ω and j.

Proof. By Lemmas 2.3 and 2.5, we know that the operator $\widetilde{T}^N_j$ is bounded on $L^2(\mathbb{R}^n)$ with the bound $C2^{-\delta N(j)}\|\Omega\|_{L^q(S^{n-1})}$, and bounded on $L^p(\mathbb{R}^n)$ with the bound $CN(j+1)\|\Omega\|_{L^q(S^{n-1})}$ for all $1 < p < \infty$. Therefore, the Riesz-Thorin interpolation theorem implies the lemma. □

3.2. Unweighted L^p estimates with good decay for $\left[b, \widetilde{T}^N_j\right]$

We first present the sharp John-Nirenberg inequality and its application.

Lemma 3.2. *[30] [Sharp John-Nirenberg] Let $b \in BMO(\mathbb{R}^n)$. There are dimensional constants $\alpha_n = \frac{1}{2^{n+2}}$ and $\beta_n > 1$ such that*

$$\sup_Q \frac{1}{|Q|}\int_Q \exp\left(\frac{\alpha_n}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\left|b(y) - b_Q\right|\right)dy \le \beta_n. \tag{3.1}$$

Lemma 3.3. *Let $b \in BMO(\mathbb{R}^n)$ and let $\alpha_n < 1 < \beta_n$ be the dimensional constants from (3.1). Then for all $1 < p < \infty$,*

$$s \in \mathbb{R},\ |s| \le \frac{\alpha_n}{\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}} \min\{1, p-1\} \Rightarrow e^{sb} \in A_p \text{ and } \left[e^{sb}\right]_{A_p} \le \beta_n^2.$$

Proof. By a direct computation, we have

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \exp(sb)\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp\left(-\frac{sb}{p-1}\right)dx\right)^{p-1}\\ &= \left(\frac{1}{|Q|}\int_Q \exp(s(b-b_Q))\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp\left(-\frac{s(b-b_Q)}{p-1}\right)dx\right)^{p-1}.\end{aligned}$$

For the case where $p \ge 2$, Hölder's inequality and Lemma 3.2 imply

$$\left(\frac{1}{|Q|}\int_Q \exp(s(b-b_Q))\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp\left(-\frac{s(b-b_Q)}{p-1}\right)dx\right)^{p-1} \le \beta_n^2.$$

For the case where $1 < p < 2$, by a direct computation and again by Lemma 3.2,

$$\left(\frac{1}{|Q|}\int_Q \exp(s(b-b_Q))\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp\left(-\frac{s(b-b_Q)}{p-1}\right)dx\right)^{p-1} \le \beta_n^p \le \beta_n^2.$$

Thus by the definition of the constant A_p, we complete the proof. □

We also need the quantitative weighted L^p estimate for T_j^N as follow.

Lemma 3.4. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$. Then for $p \in (q', \infty)$ and $w \in A_{p/q'}$, T_j^N is bounded on $L^p(w)$ with the bound $c_{n,p}N(j)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\{w\}_{A_{p/q'},p;1}$.*

Proof. One may refer to [27] for specific proof details. In fact, combining Lemma 2.5, Lemma 2.6, and Lemma 2.8, we obtain that

$$\left|T_j^N f(x)\right| \le CN(j)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\mathcal{A}_{q',\mathcal{S}}(|f|)(x).$$

Then by Lemma 2.9, we get the lemma. □

Next we give the unweighted L^p estimate for $\left[b, \widetilde{T}_j^N\right]$ with the decay bound.

Lemma 3.5. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$, and $b \in BMO(\mathbb{R}^n)$. Then, for $q' < p < \infty$,*

$$\left\|\left[b, \widetilde{T}_j^N\right] f\right\|_{L^p(\mathbb{R}^n)} \le c_n \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\delta_{p,q,n}N(j)} N(j+1)\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Using the conjugation method, for any $\zeta > 0$,

$$\left[b, \widetilde{T}_j^N\right] f = \frac{1}{2\pi i}\int_{|z|=\zeta} \frac{e^{zb}\widetilde{T}_j^N\left(fe^{-zb}\right)}{z^2}dz.$$

Therefore, for any $\zeta > 0$,

$$\left\|\left[b, \widetilde{T}_j^N\right] f\right\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{2\pi\zeta} \sup_{|z|=\zeta} \left\|\widetilde{T}_j^N \left(f e^{-zb}\right)\right\|_{L^p(e^{p\mathrm{Re}zb})}.$$

For $|z| = \zeta > 0$, we let $w_z := e^{pRezb}$, $v_z := e^{zb}$, and $W_z := [w_z]_{A_{p/q'}}^{max\left\{1, \frac{1}{p/q'-1}\right\}}$. Take $\zeta = \alpha_n \, min\left\{\frac{1}{p}, \frac{1}{q'} - \frac{1}{p}\right\}$, and one can see that $|pRez| \leq \alpha_n min\left\{1, \frac{p}{q'} - 1\right\}$, which implies $w_z \in A_{p/q'}$ by Lemma 3.3, and so

$$[w_z]_{A_{p/q'}} = [e^{pRezb}]_{A_{p/q'}} \leq \beta_n^2. \tag{3.2}$$

Let $\varepsilon = c_n/(w_z)_{A_{p/q'}}$, and we get from Lemma 2.1 that $w_z^{1+\varepsilon} \in A_{p/q'}$ and

$$\left\{w_z^{1+\varepsilon}\right\}_{A_{p/q'},p;1} \leq c_n \left\{w_z\right\}_{A_{p/q'},p;1}^{1+\varepsilon}.$$

Thus by Lemma 3.4, we have

$$\left\|\widetilde{T}_j^N f\right\|_{L^p(w_z^{1+\varepsilon})} \leq CN(j+1)\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left\{w_z\right\}_{A_{p/q'},p;1}^{1+\varepsilon} \|f\|_{L^p(w_z^{1+\varepsilon})}. \tag{3.3}$$

Combining with Lemma 3.1 and (3.3), we apply the Stein-Weiss interpolation theorem of variable measure to get that

$$\begin{aligned}
\left\|\widetilde{T}_j^N \left(f {v_z}^{-1}\right)\right\|_{L^p(w_z)} &\leq CN(j+1)\|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\delta_{p,q,n}N(j)/(w_z)_{A_{p/q'}}} \left\{w_z\right\}_{A_{p/q'},p;1} \left\|f {v_z}^{-1}\right\|_{L^p(w_z)} \\
&\leq CN(j+1)\|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\delta_{p,a,n}N(j)/W_z} W_z \|f\|_{L^p(\mathbb{R}^n)} \\
&\leq CN(j+1)\|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\delta_{p,q,n}N(j)} \|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where the second inequality follows from (2.1), and the last inequality follows from (3.2). Thus we complete the proof. □

3.3. *Two-weight estimate for* $[b, T_j^N]$

In this subsection, we will prove the following lemma.

Lemma 3.6. *Let* $\Omega \in L^q(\mathbb{S}^{n-1})$ *with* $q > 1$ *and* $b \in BMO(\mathbb{R}^n)$. *Then, for* $1 < p < q$, $r > \frac{q}{q-p}$, *and* $w > 0$, *we have*

$$\left\|\left[b, T_j^N\right] f\right\|_{L^p(w)} \leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} N(j) \left(p'\right)^2 \left(\left(\frac{p'}{q'}\right)'\right)^{\frac{1}{q'}} \left(\left(\frac{\frac{p'}{q'} - 1}{p' - 1} r\right)'\right)^{1+\frac{1}{p'}} \|f\|_{L^p(M_r w)}.$$

3.3.1. Pointwise domination for $[b, T_j^N]$

Lemma 3.7. *Let* T *be a sublinear operator and* $\mathcal{M}_T$ *be defined in (2.10). Suppose that* T *is bounded on* $L^2(\mathbb{R}^n)$ *with bound* A. *Then for the bounded function* f *with compact support and a.e.* $x \in \mathbb{R}^n$,

$$|Tf(x)| \leq c_n \left(A|f(x)| + \mathcal{M}_T f(x)\right).$$

Proof. Let f be a bounded function with compact support. Let $x \in \mathbb{R}^n$ and $Q(x,r) \subset \mathbb{R}^n$ be a cube centered at x having side length r. It follows from Hölder's inequality and the $L^2(\mathbb{R}^n)$ boundedness of T that

$$\begin{aligned}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|Tf(y)|\mathrm{d}y &\le \frac{1}{|Q(x,r)|}\int_{Q(x,r)}\left|T(f\chi_{3Q(x,r)})(y)\right|\mathrm{d}y \\ &\quad + \frac{1}{|Q(x,r)|}\int_{Q(x,r)}|T(f\chi_{\mathbb{R}^d\backslash 3Q(x,r)})(y)|\mathrm{d}y \\ &\le \left(\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|T(f\chi_{3Q(x,r)})(y)|^2\mathrm{d}y\right)^{\frac{1}{2}} + \mathcal{M}_T f(x) \\ &\le c_n\left(A\left(\frac{1}{|3Q(x,r)|}\int_{3Q(x,r)}|f(y)|^2\mathrm{d}y\right)^{\frac{1}{2}} + \mathcal{M}_T f(x)\right).\end{aligned}$$

Let $r \to 0$ and we get the result. □

Lemma 3.8. *For any cube $Q \subset \mathbb{R}^n$, there exists 3^n dyadic lattices $\mathfrak{D}^{(i)}$ and a cube $P \in \mathfrak{D}^{(i)}$ for some i such that $3Q \subset P$ and $|P| \le 9^n|Q|$.*

Proof. This lemma is the consequence of [31, Remark 2.2] and we omit the details. □

Lemma 3.9. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $q > 1$ and let $b \in L^1_{loc}(\mathbb{R}^n)$. For every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there exist 3^n dyadic lattices $\mathfrak{D}^{(i)}$ and $\frac{1}{2\cdot 9^n}$-sparse families $\mathcal{S}_i \subset \mathfrak{D}^{(i)}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$\left|\left[b, T_j^N\right] f(x)\right| \le c_n\|\Omega\|_{L^q(\mathbb{S}^{n-1})}N(j)\sum_{i=1}^{3^n}\left(\mathcal{T}_{\mathcal{S}_i,b,q'}|f|(x) + \mathcal{T}^\star_{\mathcal{S}_i,b,q'}|f|(x)\right).$$

Proof. By Lemma 3.8, one can see that for every $Q \subset \mathbb{R}^n$, there exists 3^n dyadic lattices $\mathfrak{D}^{(1)},...,\mathfrak{D}^{(3^n)}$ and a cube $R = R_Q \in \mathfrak{D}^{(i)}$ for some i, such that $3Q \subset R_Q$ and $|R_Q| \le 9^n|Q|$.

We follow the idea in [28] and first prove a local version. Let $C_{\Omega,j} = \|\Omega\|_{L^q(\mathbb{S}^{n-1})}N(j)$, and we claim that, for a fixed cube $Q_0 \subset \mathbb{R}^n$, there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathfrak{D}(Q_0)$ such that for a.e. $x \in Q_0$,

$$\left|\left[b, T_j^N\right](f\chi_{3Q_0})(x)\right| \le c_n C_{\Omega,j}\sum_{Q\in\mathcal{F}}\left(\left|b(x) - b_{R_Q}\right||f|_{q',3Q} + \left|\left(b - b_{R_Q}\right)f\right|_{q',3Q}\right)\chi_Q(x). \tag{3.4}$$

Following the idea of recursion, it suffices to prove that there exist pairwise disjoint cubes $P_l \in \mathfrak{D}(Q_0)$ such that $\sum_l |P_l| \le \frac{1}{2}|Q_0|$ and

$$\begin{aligned}\left|\left[b, T_j^N\right](f\chi_{3Q_0})(x)\right|\chi_{Q_0} &\le c_n C_{\Omega,j}\left(\left|b(x) - b_{R_{Q_0}}\right||f|_{q',3Q_0} + \left|\left(b - b_{R_{Q_0}}\right)f\right|_{q',3Q_0}\right) \\ &\quad + \sum_l\left|\left[b, T_j^N\right](f\chi_{3P_l})(x)\right|\chi_{P_l}.\end{aligned}$$

In fact, we obtain (3.4) by establishing a $\frac{1}{2}$-sparse family $\mathcal{F} = \{P_l^k\}, l \in \mathbb{Z}_+$, which is the union of the cubes obtained from the iterative process. Next, for arbitrary pairwise disjoint cubes $P_l \in \mathfrak{D}(Q_0)$, $\left|\left[b, T_j^N\right](f\chi_{3Q_0})\right|\chi_{Q_0}$ can be dominated by the sum of three parts:

$$\left|\left[b, T_j^N\right](f\chi_{3Q_0})\right|\chi_{Q_0\backslash\cup_l P_l}, \sum_l\left|\left[b, T_j^N\right](f\chi_{3Q_0\backslash 3P_l})\right|\chi_{P_l}, \sum_l\left|\left[b, T_j^N\right](f\chi_{3P_l})\right|\chi_{P_l}.$$

It remains to show that for a.e. $x \in Q_0$,

$$\begin{aligned} &\left|\left[b, T_j^N\right](f\chi_{3Q_0})\right|\chi_{Q_0\setminus\cup_l P_l} + \sum_l \left|\left[b, T_j^N\right](f\chi_{3Q_0\setminus 3P_l})\right|\chi_{P_l} \\ &\le c_n C_{\Omega,j}\left(\left|b(x) - b_{R_{Q_0}}\right| |f|_{q',3Q_0} + \left|\left(b - b_{R_{Q_0}}\right) f\right|_{q',3Q_0}\right). \end{aligned} \tag{3.5}$$

By the fact that $[b, T]f = [b - c, T]f$ for any $c \in \mathbb{R}$, one has

$$\begin{aligned} &\left|\left[b, T_j^N\right](f\chi_{3Q_0})\right|\chi_{Q_0\setminus\cup_l P_l} + \sum_l \left|\left[b, T_j^N\right](f\chi_{3Q_0\setminus 3P_l})\right|\chi_{P_l} \\ &\le \left|b - b_{R_{Q_0}}\right|\left(\left|T_j^N(f\chi_{3Q_0})\right|\chi_{Q_0\setminus\cup_l P_l} + \sum_l \left|T_j^N(f\chi_{3Q_0\setminus 3P_l})\right|\chi_{P_l}\right) \\ &\quad + \left|T_j^N\left(\left(b - b_{R_{Q_0}}\right) f\chi_{3Q_0}\right)\right|\chi_{Q_0\setminus\cup_l P_l} + \sum_l \left| T_j^N\left(\left(b - b_{R_{Q_0}}\right) f\chi_{3Q_0\setminus 3P_l}\right)\right|\chi_{P_l}. \end{aligned} \tag{3.6}$$

Set $E = E_1 \cup E_2$, where

$$E_i = \left\{x \in Q_0 : |f_i| > \alpha_n |f_i|_{q',3Q_0}\right\} \cup \left\{x \in Q_0 : \mathcal{M}_{T_j^N}(f_i\chi_{3Q_0}) > \alpha_n C_{\Omega,j}|f_i|_{q',3Q_0}\right\},$$

for $i = 1, 2$, where we take $f_1 = f$ and $f_2 = \left(b - b_{R_{Q_0}}\right) f$. By Lemma 2.6, one can choose α_n large enough so that $|E| \le \frac{1}{2^{n+2}}|Q_0|$. Applying Calderón-Zygmund decomposition to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$, we obtain a class of pairwise disjoint cubes $P_l \in \mathfrak{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_l| \le |P_l \cap E| \le \frac{1}{2}|P_l|$$

and $|E \setminus \cup_l P_l| = 0$. It follows that $\sum_l |P_l| \le \frac{1}{2}|Q_0|$ and by the fact that $P_l \cap E^c \neq \emptyset$,

$$\operatorname*{ess\,sup}_{\xi\in P_l} \left|T_j^N(f_i\chi_{3Q_0\setminus 3P_l})(\xi)\right| \le c_n\|\Omega\|_{L^q(\mathbb{S}^{n-1})}N(j)|f_i|_{q',3Q_0}.$$

Also, by Lemma 2.3 and the Plancherel theorem, we have that $\left\|T_j^N\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C\|\Omega\|_{L^q(\mathbb{S}^{n-1})}$. One can see that by applying Lemma 3.7 and $|E \setminus \cup_l P_l| = 0$, for a.e. $x \in Q_0 \setminus \cup_l P_l$,

$$\begin{aligned} \left|T_j^N(f_i\chi_{3Q_0})\right| &\le c_n\left(\|\Omega\|_{L^q(\mathbb{S}^{n-1})}|f_i(x)| + \mathcal{M}_{T_j^N}(f_i\chi_{3Q_0})(x)\right) \\ &\le c_n\|\Omega\|_{L^q(\mathbb{S}^{n-1})}N(j)|f_i|_{q',3Q_0}. \end{aligned}$$

Combining the estimates above together with (3.6) proves (3.5), and then (3.4) is also proved.

The remaining is similar to [28] and we omit the details. We take a cube Q_0 containing supp(f), and then give a partition of $\mathbb{R}^n$ by a union of cubes produced by Q_0. Applying (3.4) to each cube of the union and by Lemma 3.8, the proof is complete. □

3.3.2. Proof of Lemma 3.6

We need the folowing two lemmas.

Lemma 3.10. *[18, Lemma 3.8] Let $w \in A_\infty$. Let $\mathfrak{D}$ be a dyadic lattice and $\mathcal{S} \subset \mathfrak{D}$ be an η-sparse family. Let Ψ be a Young function. Given a measurable function f on $\mathbb{R}^n$, define*

$$\mathcal{B}_{\mathcal{S}} f(x) := \sum_{Q \in \mathcal{S}} \|f\|_{\Psi(L),Q} \chi_Q(x).$$

Then we have

$$\|\mathcal{B}_{\mathcal{S}} f\|_{L^1(w)} \leq \frac{4}{\eta}[w]_{A_\infty} \|M_{\Psi(L)} f\|_{L^1(w)}.$$

Lemma 3.11. *[32, Lemma 2.9] Let $1 < p, r < \infty$ and let M be the Hardy-Littlewood maximal operator. Then for any $w > 0$,*

$$\|Mf\|_{L^p((M_r w)^{1-p})} \leq c p' (r')^{\frac{1}{p}} \|f\|_{L^p(w^{1-p})}.$$

We now prove Lemma 3.6. By duality, it suffices to prove that

$$\left\| \frac{\left[b, T_j^N\right] f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} N(j) \, (p')^2 \left(\left(\frac{p'}{q'} \right)' \right)^{\frac{1}{q'}} (s')^{1+\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}, \tag{3.7}$$

where we set $s = \frac{\frac{p'}{q'}-1}{p'-1} r$. Again by the duality argument, we can write

$$\left\| \frac{\left[b, T_j^N\right] f}{M_r w} \right\|_{L^{p'}(M_r w)} = \sup_{\|h\|_{L^p(M_r w)}=1} \left| \int_{\mathbb{R}^n} \left[b, T_j^N\right] f(x) h(x) dx \right|.$$

We follow the idea in [18], which is called the Rubio de Francia algorithm (see [33, 34] for details). For any $h \in L^p(M_r w)$ such that $\|h\|_{L^p(M_r w)} = 1$, let

$$R(h) := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k h}{\|S\|^k_{L^p(M_r w)}}, \quad S(h) := \frac{M\left(h\,(M_r w)^{\frac{1}{p}}\right)}{(M_r w)^{\frac{1}{p}}},$$

which has the following properties:

$$\begin{aligned} &(a)\ \ 0 \leq h \leq R(h), \quad (b) \|Rh\|_{L^p(M_r w)} \leq 2\|h\|_{L^p(M_r w)}, \\ &(c)\ \ R(h)(M_r w)^{\frac{1}{p}} \in A_1 \ with \left[R(h)\,(M_r w)^{\frac{1}{p}}\right]_{A_1} \leq c p'. \end{aligned}$$

One can also see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$. By property (a) and Lemma 3.9, we have

$$\int_{\mathbb{R}^n} \left| \left[b, T_j^N\right] f(x) \right| h(x) dx \leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} N(j) \int_{\mathbb{R}^n} \sum_{i=1}^{3^n} \left(\mathcal{T}_{\mathcal{S}_i, b, q'} |f|(x) + \mathcal{T}^{\star}_{\mathcal{S}_i, b, q'} |f|(x) \right) Rh(x) dx.$$

Using the John-Nirenberg inequality, (2.6), Lemma 2.1, and property (c) above, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}_{\mathcal{S}_i,b,q'}|f|(x)Rh(x)dx &\le \sum_{Q\in\mathcal{S}_i} |f|_{q',Q}\int_Q \left|b(x)-b_Q\right| Rh(x)dx\\ &\le 2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{Q\in\mathcal{S}_i}|Q||f|_{q',Q}\|Rh\|_{L\log L,Q}\\ &\le 2s'_{Rh}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{Q\in\mathcal{S}_i}\left(\frac{1}{|Q|}\int_Q Rh(x)^{s_{Rh}}dx\right)^{\frac{1}{s_{Rh}}}|Q||f|_{q',Q}\\ &\le c_n[Rh]_{A_\infty}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{Q\in\mathcal{S}_i}Rh(Q)|f|_{q',Q}\\ &\le c_n p'\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{Q\in\mathcal{S}_i}Rh(Q)|f|_{q',Q},\end{aligned}$$

where we take $s_{Rh} := 1 + \frac{1}{c_n[Rh]_{A_\infty}}$. We apply Lemma 3.10 with $\Psi(t) = t^{q'}$, and we have

$$\sum_{Q\in\mathcal{S}_i}Rh(Q)|f|_{q',Q} \le 8[Rh]_{A_\infty}\left\|M_{q'}f\right\|_{L^1(Rh)} \le c_n p'\left\|M_{q'}f\right\|_{L^1(Rh)}.$$

By property (b) above with $\|h\|_{L^p(M_r w)} = 1$, we have

$$\left\|M_{q'}f\right\|_{L^1(Rh)} \le \left(\int_{\mathbb{R}^n}\left|M_{q'}f\right|^{p'}(M_r w)^{1-p'}dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n}(Rh)^p M_r w dx\right)^{\frac{1}{p}} \le 2\left\|\frac{M_{q'}f}{M_r w}\right\|_{L^{p'}(M_r w)}.$$

We note that $q' < p' < \infty$, $1 < s < r$, and $\frac{s}{r}(p'-1) = \frac{p'}{q'} - 1$. By Lemma 3.11, one has that

$$\begin{aligned}\left\|\frac{M_{q'}f}{M_r w}\right\|_{L^{p'}(M_r w)} &= \left(\int_{\mathbb{R}^n}\left(M(|f|)^{q'}\right)^{\frac{p'}{q'}}(M_r w)^{1-p'}dx\right)^{\frac{1}{p'}}\\ &= \left(\int_{\mathbb{R}^n}\left(M(|f|)^{q'}\right)^{\frac{p'}{q'}}\left(M_s\left(w^{\frac{r}{s}}\right)\right)^{\frac{s}{r}(1-p')}dx\right)^{\frac{1}{p'}}\\ &\le c_n\left(\left(\frac{p'}{q'}\right)'(s')^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}\left\||f|^{q'}\right\|_{L^{\frac{p'}{q'}}\left(w^{\left(1-\frac{p'}{q'}\right)\frac{r}{s}}\right)}\\ &= c_n\left(\left(\frac{p'}{q'}\right)'\right)^{\frac{1}{q'}}\left(\left(\frac{\frac{p'}{q'}-1}{p'-1}r\right)'\right)^{\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)},\end{aligned} \tag{3.8}$$

and then

$$\int_{\mathbb{R}^n}\mathcal{T}_{\mathcal{S}_i,b,q'}|f|(x)Rh(x)dx \le c_n\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left(\left(\frac{p'}{q'}\right)'\right)^{\frac{1}{q'}}(p')^2(s')^{\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}. \tag{3.9}$$

On the other hand, the John-Nirenberg inequality tells us that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}^{\star}_{\mathcal{S}_i,b,q'}|f|(x)Rh(x)dx &\le \sum_{Q\in\mathcal{S}_i}\left(\frac{1}{|Q|}\int_Q \left|(b-b_Q)f(y)\right|^{q'}dy\right)^{\frac{1}{q'}}Rh(Q)\\ &\le \sum_{Q\in\mathcal{S}_i}\left(\left\||b-b_Q|^{q'}\right\|_{\exp L^{1/q'},Q}\cdot\left\||f|^{q'}\right\|_{L(\log L)^{q'},Q}\right)^{\frac{1}{q'}}Rh(Q)\\ &\le c_n\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{Q\in\mathcal{S}_i}\||f|^{q'}\|^{\frac{1}{q'}}_{L(\log L)^{q'},Q}Rh(Q).\end{aligned}$$

By Lemma 3.10 with $\Psi(t)=t\log^{q'}(1+t)$, (2.6), and property (b), we can deduce that

$$\begin{aligned}\sum_{Q\in\mathcal{S}_i}\left\||f|^{q'}\right\|^{\frac{1}{q'}}_{L(\log L)^{q'},Q}Rh(Q) &\le 8[Rh]_{A_\infty}\left\|M_{L(\log L)^{q'}}(|f|^{q'})^{1/q'}\right\|_{L^1(Rh)}\\ &\le c_n\frac{p'}{\delta}\left\|M_{q'(1+\delta q')}f\right\|_{L^1(Rh)}\le 2c_n\frac{p'}{\delta}\left\|\frac{M_{q'(1+\delta q')}f}{M_r w}\right\|_{L^{p'}(M_r w)},\end{aligned}$$

for any $0<\delta<1$. A similar argument as the estimate of (3.8) implies

$$\frac{p'}{\delta}\left\|\frac{M_{q'(1+\delta q')}f}{M_r w}\right\|_{L^{p'}(M_r w)}\le\frac{p'}{\delta}\left(\frac{p'}{q'(1+q'\delta)}\right)^{\frac{1}{q'(1+q'\delta)}}\left(\left(\frac{\frac{p'}{q'(1+q'\delta)}-1}{p'-1}r\right)'\right)^{\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}. \tag{3.10}$$

Recall the notion that $s=\frac{\frac{p'}{q'}-1}{p'-1}r>1$, and set $\Delta=\frac{(2s')'}{s}$. Then $0<\Delta<1$ and $\frac{1}{1-\Delta}=2s'-1$. Now take $\delta=\min\{\frac{(1-\Delta)(p'+q')}{(1-\Delta)p'+\Delta q'}\frac{1}{q'},\ 1\}$, and one has

$$\frac{p'}{q'(1+q'\delta)}=\max\{\Delta(\frac{p'}{q'}-1)+1,\frac{p'}{q'(1+q')}\},$$

and so

$$\left(\left(\frac{\frac{p'}{q'(1+q'\delta)}-1}{p'-1}r\right)'\right)^{\frac{1}{p'}}=\left(\left(\frac{\frac{p'}{q'(1+q'\delta)}-1}{\frac{p'}{q'}-1}s\right)'\right)^{\frac{1}{p'}}\le((\Delta s)')^{\frac{1}{p'}}=(2s')^{\frac{1}{p'}}, \tag{3.11}$$

$$\frac{1}{\delta}=\max\left\{\frac{(1-\Delta)p'+\Delta q'}{(1-\Delta)(p'+q')}q',\ 1\right\}\le\max\{\frac{q'}{1-\Delta},1\}\le 2q's'. \tag{3.12}$$

Combining the inequalities (3.10)–(3.12), we have

$$\int_{\mathbb{R}^n}\mathcal{T}^{\star}_{\mathcal{S}_i,b,q'}|f|(x)Rh(x)dx\le c_n\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}p'q'\left(\frac{p'}{q'}\right)^{\frac{1}{q'}}(s')^{1+\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}. \tag{3.13}$$

Now the desired inequality (3.7) follows from (3.9) and (3.13).

4. Proof of the main results

Lemma 4.1. *Let* $N(j) = 2^j - 1$ *for* $j \in \{0\} \cup \mathbb{N}$*. Then, for any* $0 < \delta < 1$*,*

$$\sum_{j=0}^{\infty} N(j+1) 2^{-N(j)\delta} < \frac{16}{\delta}.$$

Proof. In fact, note that $2^{-x} < 2x^{-2}$ for $x \geq 1$, and a simple computation shows

$$\sum_{j=0}^{\infty} N(j+1) 2^{-N(j)\delta} < \sum_{j=0}^{\infty} 2^{j+1} 2^{-(2^j-1)\delta} < 4 \sum_{j=0}^{\infty} 2^j 2^{-2^j\delta} \leq \frac{16}{\delta}.$$

□

4.1. Proof of Theorem 1.1

Let $\Omega \in L^q(\mathbb{S}^{n-1})$ and $b \in BMO(\mathbb{R}^n)$. For $q' < p < q$, by using Lemmas 3.5 and 3.6 and applying the interpolation theorem with a change of measures in [35], we have, for $0 < \theta < 1$ and any weight function $w > 0$,

$$\left\| \left[b, \widetilde{T}_j^N \right] f \right\|_{L^p(w)} \leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} (p')^{2\theta} \left(\frac{p'}{q'} \right)^{\frac{\theta}{q'}} \left(\left(\frac{\frac{p'}{q'} - 1}{p' - 1} r \right)' \right)^{\theta + \frac{\theta}{p'}} \|f\|_{L^p(M_{r/\theta} w)}$$
$$\times N(j+1) 2^{-\delta_{p,q,n} N(j)(1-\theta)}.$$

By the triangle inequality and Lemma 4.1 with $N(j) = 2^j - 1$, we have

$$\begin{aligned} \|[b, T_\Omega] f\|_{L^p(w)} &\leq \left\| \left[b, T_0^N \right] f \right\|_{L^p(w)} + \sum_{j=0}^{\infty} \left\| \left[b, \widetilde{T}_j^N \right] f \right\|_{L^p(w)} \\ &\leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{(p')^2}{1-\theta} \left(\frac{p'}{q'} \right)^{\frac{1}{q'}} \left(\left(\frac{\frac{p'}{q'} - 1}{p' - 1} r \right)' \right)^{1 + \frac{1}{p'}} \|f\|_{L^p(M_{r/\theta} w)}. \end{aligned}$$

Changing the indicator by letting $r = \theta\gamma$, and taking $\theta = \frac{(2\eta')'}{\eta}$ and $\eta = \frac{q-p}{q}\gamma > 1$, then the term on the right-hand side of the last inequality above is equal to

$$\begin{aligned} & c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{1}{1-\theta} (p')^2 \left(\frac{pq-p}{q-p} \right)^{\frac{1}{q'}} \left(\left(\frac{q-p}{q} \theta\gamma \right)' \right)^{1+\frac{1}{p'}} \|f\|_{L^p(M_\gamma(w))} \\ & \leq c_n \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} 2\eta' (p')^2 \left(\frac{pq-p}{q-p} \right)^{\frac{1}{q'}} (2\eta')^{1+\frac{1}{p'}} \|f\|_{L^p(M_\gamma(w))}, \end{aligned}$$

which completes the proof.

4.2. Proof of Corollary 1.2

By Lemma 2.1, if we take $\gamma = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$, then $M_\gamma(w)(x) \leq c_n M(w)(x)$ for $w \in A_\infty$. Moreover we have $M(w)(x) \leq c_n [w]_{A_1} w(x)$ for $w \in A_1$. Therefore, Theorem 1.1 implies Corollary 1.2.

5. Conclusions

In this article, the two-weight estimate was studied for the commutator $[b, T_\Omega]$ with $b \in BMO(\mathbb{R}^n)$ and the singular integral operator T_Ω with the rough kernel Ω. We extended the previous works by weakening the kernel $\Omega \in L^\infty(\mathbb{S}^{n-1})$ to $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$, and established the quantitative $A_1 - A_\infty$ weighted estimates for the commutator $[b, T_\Omega]$ with the rough kernel $\Omega \in L^q(\mathbb{S}^{n-1})$ $(q > 1)$ and the function $b \in BMO(\mathbb{R}^n)$.

Author contributions

Xiangxing Tao: Writing—original draft preparation, writing—review and editing, funding acquisition; Peize Lv: Writing—original draft preparation, writing—review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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