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Research article

Fractional inclusions of superquadraticity via multiplicative calculus

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Abstract: In this paper, we introduced a novel class of superquadraticity, termed multiplicatively (superquadratic interval-valued functions) superquadratic $\mathcal{IVF}$s and investigated their unique properties using interval order relations and (Riemann-Liouville) $\mathcal{R}.\mathcal{L}$-fractional integral operators. By employing the framework of multiplicative calculus, we established new fractional integral inequalities specifically of Hermite-Hadamard ($\mathcal{H}.\mathcal{H}$) type, for these superquadratic $\mathcal{IVF}$. Furthermore, we extended our analysis to derive fractional inequalities for the product and quotient of multiplicatively superquadratic and subquadratic $\mathcal{IVF}$ functions within the same calculus setting. By setting $\ell = 1$, the results naturally reduced to their corresponding integer-order forms for multiplicatively superquadratic $\mathcal{IVF}$. To validate our theoretical findings, we present numerical computations and graphical illustrations based on several illustrative examples, showcasing the practical utility and robustness of the results. In addition, we explored potential applications of these inequalities, particularly in the context of linear combinations of special means. This provides a fresh perspective on superquadratic $\mathcal{IVF}s$ and expands the scope of multiplicative convex analysis. The results presented in this work are entirely new within the framework of fractional multiplicative calculus and have not been previously reported in the literature. We believe that this study will pave the way for future research, offering a deeper understanding of convexity phenomena and powerful tools for mathematical modeling involving $\mathcal{IVF}s$.

Keywords: superquadratic function; multiplicatively superquadratic $\mathcal{IVF}$; multiplicative calculus; $\mathcal{R}.\mathcal{L}$-fractional integral operators; $\mathcal{H}.\mathcal{H}$'s type inequality; interval calculus
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1. Introduction

Convexity is an intrinsic concept in mathematics and engineering because it supports a wide range of theoretical and practical applications. Convex sets and convex functions help simplify complex systems and make optimization problems more manageable due to their well-structured nature and unique global minima. Beyond mathematics, convexity plays an essential role in fields such as control theory, where it contributes to system stability and economics, and explains consumer preferences and market behavior. Its significance extends to real-world engineering practices, where convex structures routinely appear in system design and performance optimization. Convexity also has an elegant presence in integral inequalities, where convex functions serve as key components in estimating bounds. One of the most notable results are the inequalities of $\mathcal{H}.\mathcal{H}$ type, which provides bounds for the mean functional output of a convex function on a specified interval. This deep connection between convexity and integral inequalities makes the area a vibrant and influential part of modern mathematical research [1–3].

Fractional calculus has become a powerful and widely used mathematical tool, attracting growing interest from physicists, engineers, and computer scientists [4]. By extending classical calculus to derivatives and integrals of arbitrary orders, it provides an effective framework for modeling systems with memory and hereditary effects, which explains its rising popularity. Beyond modeling, Fractional-order calculus has also shaped the development of the theory of inequalities through fractional order inequalities. A major breakthrough in this area was achieved by Sarikaya et al. in 2013 [5], who introduced a fractional version of the inequalities of $\mathcal{H}.\mathcal{H}$ type using $\mathcal{R}.\mathcal{L}$ fractional integrals. This generalization broadened the scope of convex analysis and opened new directions for studying the behavior of convex functions under fractional integration. Following the groundbreaking presentation of the inequalities of $\mathcal{H}.\mathcal{H}$ type via $\mathcal{R}.\mathcal{L}$ fractional integrals, there has been significant progress in expanding inequalities of $\mathcal{H}.\mathcal{H}$ type across various definitions of fractional integrals. Since the inequalities of $\mathcal{H}.\mathcal{H}$ type were first introduced in the context of $\mathcal{R}.\mathcal{L}$ fractional operators, numerous extensions of inequalities of $\mathcal{H}.\mathcal{H}$ type utilizing a broad class of fractional operators have been thoroughly studied. The Sarikaya fractional operators [6], k-fractional operators [7], ψ-$\mathcal{R}.\mathcal{L}$ fractional operators [8], generalized proportional fractional operators [9], generalized $\mathcal{R}.\mathcal{L}$ operators [10], $(k-p)$ fractional operators [11], Katugampola fractional operators [12] and conformable fractional operators [13] are notable examples. In addition to these operator-based generalizations, new classes of non-integer order inequalities have also been established, including those of the Hadamard-Mercer type [14], Bullen-type [15], Euler-Maclaurin-type [16], Simpson-type [17] and Ostrowski-type inequalities [18]. Studies like [19, 20] and the references cited therein, offer readers who want a more thorough understanding of the most recent advancements and thorough summaries in this constantly changing field.

The idea of superquadraticity was initially introduced by Abramovich et al. [21]. Superquadraticity provides tighter and better bounds than general convexity, which greatly improves the theory of integral inequalities. Such an improvement is highly desired in applied mathematics applications, where better approximation translates into higher modelling accuracy, and in optimization problems, where the quality of the boundary estimate determines the optimal solution. Superquadraticity thus enriches the theoretical foundations of inequality structures and broadens their scope of application. It offers a robust analytical framework that facilitates the formulation and utilization of inequalities across practical and mathematical issues. Later, Abramovich et al. [22] provided a formal definition of superquadraticity along with key theoretical insights that enhanced comprehension of this type of function. Building upon this foundation, Li and Chen [23] advanced the notion by examining the fractional perspective of the inequalities of $\mathcal{H}.\mathcal{H}$ type using $\mathcal{R}.\mathcal{L}$ fractional integrals, marking a significant extension of superquadraticity into the realm of fractional calculus. Further developments were made by Alomari et al. [24], who initiated the notion of h-superquadratic functionn, investigated their fundamental features and established a new variant within the broader family. Krnić et al. [25] advanced the theory by formulating the concept of logarithmically superquadratic functions, which serves as a natural logarithmic enhancement of the traditional superquadratic framework. In a significant contribution to analysis over interval-valued structures and generalized function frameworks, Khan and Butt [26] proposed a new class of function known cr-order superquadratic function along with their fractional analogs, highlighting their utility through various applications. Butt and Khan [27] were the first to develop the classical and fractional forms inequalities of $\mathcal{H}.\mathcal{H}$ and Fejér type within the framework of h-superquadratic functions. Extending this line of research, Khan et al. [28] proposed the (P, m)-superquadratic function, a more general form that incorporates examples, key properties, integral inequalities, and practical applications, thereby enriching the functional landscape of superquadraticity. For further exploration of these ideas via multiplicative calculus, fuzzy calculus and fractional calculus the reader is referred to the works in [29–32].

Recent years have seen growing interest in multiplicative calculus, largely sparked by the impactful work of Ali et al. [33]. Their effort unleashed a flood of attention around the application of multiplicative calculus, with a focus on the area of integral inequalities in particular. Unlike traditional calculus, multiplicative calculus is founded on another type of foundation structure and is therefore extremely effective in handling issues related to growth procedures and ratio-based systems. Such a structure has served well to examine inequalities of other classes of functions. For this purpose, researchers have found several integer order multiplicative inequalities for other variants of functions. Worth mentioning is the substantial advancement in establishing inequalities for multiplicative preinvex P-convex functions [34] and multiplicative harmonically convex functions [35], which apply classical convexity principles within a multiplicative framework. These results not only enrich the theoretical framework of multiplicative calculus but also provide valuable tools for application to optimization and mathematical analysis.

In recent years, there has been growing scholarly interest in the study of integer-order inequalities within the framework of multiplicative calculus. Numerous results have been established for various classes of functions under this setting. Yet, fractional versions, especially those employing multiplicative fractional operators, remain relatively underexplored. A significant advancement occurred in 2020 when Budak and Özçelik [36] introduced a novel approach for deriving new inequalities of $\mathcal{H}.\mathcal{H}$ type using multiplicative $\mathcal{R}.\mathcal{L}$ fractional operators. This pioneering work marked a milestone in the field, generating substantial interest within the mathematical community due to its innovative methodology and its potential for broader applications. Building on this foundation, Fu et al. [37] investigated a family of operators known as multiplicative tempered fractional integrals, thereby extending the scope of inequalities of $\mathcal{H}.\mathcal{H}$ type to multiplicatively convex function and enriching the theoretical framework of fractional order multiplicative inequalities. Further developments were made by Peng and Du [38], who introduced differentiable multiplicative (s,m)-preinvex and m-preinvex functions. Within the setting of multiplicative tempered fractional integrals, they established new inequalities of $\mathcal{H}.\mathcal{H}$ type, broadening the applicability of this theory to more generalized convexity frameworks. Collectively, these works represent important contributions to the systematic advancement of inequality theory in the context of fractional multiplicative calculus.

In 2022, Peng et al. [39] introduced a new class of operators called multiplicative fractional integrals with exponential kernels, a significant breakthrough in multiplicative fractional calculus. Merad et al. [40] made another significant addition by deriving symmetric Maclaurin-type inequalities for functions whose multiplicative derivatives are both bounded and convex. Their work was conducted within the framework of multiplicative fractional calculus and stands out for its symmetry-based approach and theoretical rigor. The works of Lakhdari and Saleh [41] and Saleh et al. [42] provide important advances in multiplicative fractional inequality theory. In [41], the authors extend the $\mathcal{H}.\mathcal{H}$ inequality using new fractional operators in G-calculus. Their results generalize several classical inequalities and deepen the study of multiplicative convexity. Moreover, [42] develops $\mathcal{H}.\mathcal{H}$ type inequalities based on the Katugampola fractional multiplicative integral. Together, these studies offer unified and powerful tools for research in multiplicative fractional calculus. To gain a broader perspective on the current state of research in this dynamic area, readers are directed to [43,44], offering detailed discussions on extensions, applications and ongoing developments in multiplicative-fractional calculus.

Making decisions entails deciding on the best course of action when faced with paradoxes, which are encountered across real-life situations. It is also essential to explore a broad spectrum of disciplines, such as management science, optimization theory, and operations research. When making a choice, several aspects are considered, including risk and future uncertainty. Decision-making involves varying degrees of confidence, forming a scale that extends from complete certainty to absolute doubt. There are many different contexts in which decisions can be made, such as conflicting demands, decisions with confidence, decisions with ambiguity, decisions with risk and more. There are further classifications for decisions made in the face of uncertainty. Two of them make optimistic choices, while one makes depressing ones. While a pessimistic decision-maker prioritizes the most secure outcome under uncertainty, an optimistic one selects the most favorable outcome despite the presence of ambiguity. Real numbers are commonly used to express deterministic parameters in mathematical models and they can provide a traditional explanation for some issues.

In numerous uncertain real-world contexts, particularly within engineering and decision-making domains, such as operations research and management science, it is challenging to treat parameters as fixed real values. By nature, models exhibit certain inexact or imperfect characteristics, requiring decision-makers to exercise judgment in the presence of ambiguity. Operations research and management experts typically use stochastic or fuzzy methodologies to deal with imprecise or uncertain parameters. Imprecise parameters are treated as random variables with known probability distributions when the stochastic approach is used. Alternatively, fuzzy approaches address uncertainty by employing fuzzy sets with corresponding membership functions or representing values as fuzzy numbers. In practice, there are scenarios where integrating both methods provides a more robust solution to imprecise results. These approaches raise the issue of which probability distributions or membership functions to use. This is truly a difficult task for a decision maker in an uncertain setting. In response to this challenge, many researchers have turned to interval representations to model inaccurate or uncertain parameters. In cases when decisions must be made, the order in which the intervals are arranged determines the best option. Over the past few decades, researchers have presented order relations in the context of intervals in a number of mathematical approaches. Reformulating interval-oriented optimization issues was the primary goal of these methods. Although their main goal was to improve solution techniques, they did not necessarily go into great depth into the pertinent interval ordering idea. Once they accomplished their goal, the researchers cut off the discussion.

Moore [45] was the pioneer in using interval analysis in 1969 to automatically evaluate computational errors, a breakthrough that improved accuracy and attracted attention from the academic community. Interval numbers are used as variables in interval analysis and interval operations are preferred over number operations because intervals may be stated as unknown quantities. Moreover convex functions and integral inequalities have been connected by scholars in the setting of interval-valued analysis, leading to some intriguing discoveries. The generalization of classical integral inequalities to fuzzy-valued functions and $\mathcal{IVF}$s has been the focus of many researchers, including Chalco-Cano et al. [46], Flores-Franulič et al. [47], Costa et al. [48], and others.

The interval-inclusion connection was used by Zhao et al. [49] to explicitly design an interval h-convex function and verify the pertinent integral inequalities. Using the Kulisch-Miranker order, Khan et al. [50] created an h-convex $\mathcal{IVF}$ in 2021 and deduced several inequalities for such convex functions. According to this line of research, Jensen's inequalities were derived by Zhang et al. [51] using fuzzy set-valued functions. One can refer to [52–55] for detailed studies on convex $\mathcal{IVF}$s, including their various forms, inequality results, and fractional generalizations through multiple fractional integral operators.

Extending the literature on multiplicative calculus, fractional calculus, interval calculus, superquadraticity, and convexity, we identify a significant gap: Fractional forms of inequalities for superquadratic $\mathcal{IVF}$ have not been studied from the perspective of multiplicative calculus. Accordingly, we investigate the properties of multiplicative superquadratic $\mathcal{IVF}$ and formulates corresponding fractional-order inequalities of $\mathcal{H}.\mathcal{H}$-type.

The paper is structured as follows:

In Section 1, we review the essential background and foundational concepts related to convex and superquadratic functions, as well as fractional and multiplicative calculi and their associated inequalities. In Section 2, we present key formulas and fundamental results concerning convexity, superquadraticity and both types of calculus.In Section 3, we introduce new fractional integral inequalities within the framework of multiplicative calculus, employing multiplicatively superquadratic functions. To demonstrate the applicability of these results, relevant examples and graphical illustrations are provided in Section 4. In Section 5, application is provided to strengthen our results. Finally, in Section 6, we offer a concise summary and discuss potential directions for future research based on the findings.

2. Preliminaries

In this section, we begin by reviewing several fundamental definitions, properties, and concepts related to multiplicative calculus.

Definition 2.1. *[44] Let the function $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ be a positive then the multiplicative derivative $\mathcal{F}^*$ is defined as given below:*

$$\frac{d^*\mathcal{F}}{dt} = \mathcal{F}^*(t) = \lim_{h\to 0}\left(\frac{\mathcal{F}(t+h)}{\mathcal{F}(t)}\right)^{\frac{1}{h}}.$$

Remark 2.2. *The relationship between the multiplicative derivative and the ordinary (classical) derivative of a positive differentiable function is given by the following expression:*

$$\mathcal{F}^* = \exp\{(\ln\mathcal{F}(t))'\} = \exp\left\{\frac{\mathcal{F}'(t)}{\mathcal{F}(t)}\right\}.$$

The major properties of the derivative with respect to multiplicative calculus are as follows:

Proposition 2.3. *[44] Let the functions $\mathcal{F}$ and φ are multiplicatively differentiable then*

(1) $(c\mathcal{F})^*(t) = \mathcal{F}^*(t), \quad c \in \mathcal{R}$.

(2) $(\mathcal{F}\varphi)^*(t) = \mathcal{F}^*(t)\varphi^*(t)$.

(3) $(\mathcal{F} + \varphi)^*(t) = \mathcal{F}^*(t)^{\frac{\mathcal{F}(t)}{\mathcal{F}(t)+\varphi(t)}}\varphi^*(t)^{\frac{\varphi(t)}{\mathcal{F}(t)+\varphi(t)}}$.

(4) $\left(\frac{\mathcal{F}}{\varphi}\right)^*(t) = \frac{\mathcal{F}^*(t)}{\varphi^*(t)}$.

(5) $(\mathcal{F}^{\varphi})^*(t) = \mathcal{F}^*(t)^{\varphi(t)}\mathcal{F}(t)^{\varphi'(t)}$.

The notation $\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt}$ is used for the multiplicative integral, sometimes called the *integral. Its correspondence with the standard Riemann integral is given by the following relation, as documented in [44]:

Proposition 2.4. *The Riemann integrability of the function $\mathcal{F}$ on the interval $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$ implies its multiplicative integrability over the same interval.*

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt} = \exp\left\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\ln(\mathcal{F}(t))dt\right\}.$$

In addition, Bashirov et al. [44] established that a function which is multiplicatively integrable possesses the following features and implications:

Proposition 2.5. *The Riemann integrability of $\mathcal{F}$ on $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$ ensures that its multiplicative integrability on* $[\mathfrak{a}\circ, \mathfrak{b}\circ]$.

1. $\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\mathcal{F}(t))^p)^{dt} = \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\mathcal{F}(t))^{dt}\right)^p, \quad p > 0.$

2. $\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t)\varphi(t))^{dt} = \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt}\cdot\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\varphi(t))^{dt}.$

3. $\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\frac{\mathcal{F}(t)}{\varphi(t)}\right)^{dt} = \frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\varphi(t))^{dt}}.$

4. $\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt} = \int_{\mathfrak{a}_\circ}^{c}(\mathcal{F}(t))^{dt}\cdot\int_{c}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt}, \quad \mathfrak{a}_\circ \le c \le \mathfrak{b}_\circ.$

5. $\int_{\mathfrak{a}_\circ}^{\mathfrak{a}_\circ}(\mathcal{F}(t))^{dt} = 1, \quad \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(t))^{dt} = \left(\int_{\mathfrak{b}_\circ}^{\mathfrak{a}_\circ}(\mathcal{F}(t))^{dt}\right)^{-1}.$

Theorem 2.6. *[44] Let the functions $\mathcal{F}$ and φ be *differentiable and differentiable respectively, and the function $\mathcal{F}^{\varphi}$ be a *integrable, so that*

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\mathcal{F}^*(\mathfrak{t})^{\varphi(\mathfrak{t})}\right)^{dt}=\frac{\mathcal{F}(\mathfrak{b}_\circ)^{\varphi(\mathfrak{b}_\circ)}}{\mathcal{F}(\mathfrak{a}_\circ)^{\varphi(\mathfrak{a}_\circ)}}\times\frac{1}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\mathcal{F}(\mathfrak{t})^{\varphi'(\mathfrak{t})}\right)^{dt}}.$$

Lemma 2.7. *[22] Let $\mathcal{F}:[\mathfrak{a}_\circ,\mathfrak{b}_\circ]\to\mathcal{R}$ and $\varphi:[\mathfrak{a}_\circ,\mathfrak{b}_\circ]\to\mathcal{R}$ be *differentiable and differentiable, respectively, and $\mathcal{F}^{\varphi}$ is *integrable, so that*

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\mathcal{F}^*(\mathrm{h}(\mathfrak{t}))^{\mathrm{h}'(\mathfrak{t})\varphi(\mathfrak{t})}\right)^{dt}=\frac{\mathcal{F}(\mathfrak{b}_\circ)^{\varphi(\mathfrak{b}_\circ)}}{\mathcal{F}(\mathfrak{a}_\circ)^{\varphi(\mathfrak{a}_\circ)}}\times\frac{1}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\mathcal{F}(\mathrm{h}(\mathfrak{t}))^{\varphi'(\mathfrak{t})}\right)^{dt}}.$$

Next, we elaborate the basic mathematics pertaining to interval order relations, which is useful for constructing the results associated with $\mathcal{IVF}$s.

Let $\mathbb{K}_c$ be the space of bounded and closed intervals of $\mathcal{R}$, and any $\mathfrak{a}_\circ\in\mathbb{K}_c$ may be defined using the following definition:

$$\mathfrak{a}_\circ=[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]=\{\mathfrak{t}:\underline{\mathfrak{a}_\circ}\leq\mathfrak{t}\leq\overline{\mathfrak{a}_\circ},\mathfrak{t}\in\mathcal{R}\}$$

with width $(\overline{\mathfrak{a}_\circ}-\underline{\mathfrak{a}_\circ})$. If $(\overline{\mathfrak{a}_\circ}-\underline{\mathfrak{a}_\circ})=0$, then $\mathfrak{a}_\circ$ is called degenerate. The upper and lower bounds of $\mathfrak{a}_\circ$ are denoted by $\overline{\mathfrak{a}_\circ}$ and $\underline{\mathfrak{a}_\circ}\in\mathcal{R}$, respectively. With no gape, each $\mathfrak{t}\in\mathcal{R}$ may be identified as an interval $[\mathfrak{t},\mathfrak{t}]$. It is deemed positive if $(\overline{\mathfrak{a}_\circ}-\underline{\mathfrak{a}_\circ})>0$. It is represented by $\mathbb{K}_c^+$ and provides the space of positive intervals.

$$\mathbb{K}_c^+=\{[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]:[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]\in\mathbb{K}_c\wedge\underline{\mathfrak{a}_\circ}\geq 0\}.\tag{2.1}$$

For each $\beta\in\mathcal{R}$, we attain

$$\beta\mathfrak{a}_\circ=\beta[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]=\begin{cases}[\beta\underline{\mathfrak{a}_\circ},\beta\overline{\mathfrak{a}_\circ}], & \text{if } \beta\geq 0\\ \left[\beta\overline{\mathfrak{a}_\circ},\beta\underline{\mathfrak{a}_\circ}\right], & \text{if } \beta<0.\end{cases}$$

The definitions of fundamental arithmetic operations on interval numbers $[\mathfrak{a}_\circ],[\mathfrak{b}_\circ]\in\mathbb{K}_c$ are given in the following, where $[\mathfrak{a}_\circ]=[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]$ and $[\mathfrak{b}_\circ]=[\underline{\mathfrak{b}_\circ},\overline{\mathfrak{b}_\circ}]$:

- **Addition:** $[\mathfrak{a}_\circ]+[\mathfrak{b}_\circ]=[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]+[\underline{\mathfrak{b}_\circ},\overline{\mathfrak{b}_\circ}]=[\underline{\mathfrak{a}_\circ}+\underline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}+\overline{\mathfrak{b}_\circ}]$
- **Subtraction:** $[\mathfrak{a}_\circ]-[\mathfrak{b}_\circ]=[\underline{\mathfrak{a}_\circ}-\underline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}-\overline{\mathfrak{b}_\circ}]$.
- **Multiplication:** $[\mathfrak{a}_\circ]\times[\mathfrak{b}_\circ]=[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]\times[\underline{\mathfrak{b}_\circ},\overline{\mathfrak{b}_\circ}]=\left[\min(\underline{\mathfrak{a}_\circ}\,\underline{\mathfrak{b}_\circ},\underline{\mathfrak{a}_\circ}\,\overline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}\,\underline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}\,\overline{\mathfrak{b}_\circ}),\max(\underline{\mathfrak{a}_\circ}\,\underline{\mathfrak{b}_\circ},\underline{\mathfrak{a}_\circ}\,\overline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}\,\underline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}\,\overline{\mathfrak{b}_\circ})\right]$.
- **Division:** $\frac{[\mathfrak{a}_\circ]}{[\mathfrak{b}_\circ]}=[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]\times\left[\frac{1}{\overline{\mathfrak{b}_\circ}},\frac{1}{\underline{\mathfrak{b}_\circ}}\right]$, provided $0\notin[\underline{\mathfrak{b}_\circ},\overline{\mathfrak{b}_\circ}]$.

Remark 2.8. *[51] We can claim that $[\underline{\mathfrak{a}_\circ},\overline{\mathfrak{a}_\circ}]\leq_I[\underline{\mathfrak{b}_\circ},\overline{\mathfrak{b}_\circ}]\Leftrightarrow\underline{\mathfrak{a}_\circ}\leq\underline{\mathfrak{b}_\circ},\overline{\mathfrak{a}_\circ}\leq\overline{\mathfrak{b}_\circ}$, partial ordering is the name given to such a relationship.*

Remark 2.9. *[56] The lenght between* $[\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]$, *and* $[\underline{\mathfrak{b}_\circ}, \overline{\mathfrak{b}_\circ}]$ *defined as follows*

$$d_{H_s}([\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}], [\underline{\mathfrak{b}_\circ}, \overline{\mathfrak{b}_\circ}]) = max\{\left|\underline{\mathfrak{a}_\circ} - \underline{\mathfrak{b}_\circ}\right|, \left|\overline{\mathfrak{a}_\circ} - \overline{\mathfrak{b}_\circ}\right|\}. \tag{2.2}$$

This distance is known as the Hausdorff-Pompeiu distance. It is evident that a complete metric space is formed by the pair $(\mathbb{K}_c, d_{H_s})$.

Definition 2.10. *[57] Let* $[\mathfrak{a}_\circ] = [\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]$ *and* n *be any positive integer, then*

$$[\mathfrak{a}_\circ]^{\mathrm{n}} = [\underline{\mathfrak{a}_\circ}^{\mathrm{n}}, \overline{\mathfrak{a}_\circ}^{\mathrm{n}}], \quad for \;\; \underline{\mathfrak{a}_\circ} \geq 0.$$

Definition 2.11. *[57] The nth root of* $[\mathfrak{a}_\circ] = [\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]$, *where* $\mathrm{n} \in Z^+$, *is defined as*

$$[\mathfrak{a}_\circ]^{\frac{1}{\mathrm{n}}} = [\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]^{\frac{1}{\mathrm{n}}} = \sqrt[n]{[\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]} = [\sqrt[n]{\underline{\mathfrak{a}_\circ}}, \sqrt[n]{\overline{\mathfrak{a}_\circ}}], \quad for \;\; \underline{\mathfrak{a}_\circ} \geq 0.$$

Definition 2.12. *[57] The one that follows is a definition of the modulus of* $[\mathfrak{a}_\circ]$*:*

$$\begin{aligned} \|[\mathfrak{a}_\circ]\| = \|[\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]\| &= [\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}], \quad if \;\; \underline{\mathfrak{a}_\circ} \geq 0 \\ &= [|\overline{\mathfrak{a}_\circ}|, |\underline{\mathfrak{a}_\circ}|], \quad if \;\; \overline{\mathfrak{a}_\circ} \leq 0. \end{aligned}$$

Logarithmic, exponential, cosine, sine, and other basic useful functions have interval representations introduced by [57], such that

$$\begin{aligned} &exp([\mathfrak{a}_\circ]) = exp([\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]) = [exp(\underline{\mathfrak{a}_\circ}), exp(\overline{\mathfrak{a}_\circ})] \\ &log([\mathfrak{a}_\circ]) = log([\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]) = [log(\underline{\mathfrak{a}_\circ}), log(\overline{\mathfrak{a}_\circ})], \quad Provided \;\; \underline{\mathfrak{a}_\circ} > 0. \\ &Sin([\mathfrak{a}_\circ]) = Sin([\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}]) = [min\{Sin(\underline{\mathfrak{a}_\circ}), Sin(\overline{\mathfrak{a}_\circ})\}, max\{Sin(\underline{\mathfrak{a}_\circ}), Sin(\overline{\mathfrak{a}_\circ})\}] = [\underline{\mathfrak{b}}, \overline{\mathfrak{b}}]. \end{aligned}$$

Here $\underline{\mathfrak{b}} = min\{Sin(\underline{\mathfrak{a}_\circ}), Sin(\overline{\mathfrak{a}_\circ})\}$ and, $\overline{\mathfrak{b}} = max\{Sin(\underline{\mathfrak{a}_\circ}), Sin(\overline{\mathfrak{a}_\circ})\}$. A Similar case holds for the function $Cos([\underline{\mathfrak{a}_\circ}, \overline{\mathfrak{a}_\circ}])$.

Theorem 2.13. *[52] Let* $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \to \mathcal{R}_I$ *is an* $\mathcal{IVF}$ *given as* $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. *Then,* $\mathcal{F}$ *is interval Riemann integrable on* $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$ *if and only if* $\underline{\mathcal{F}}$ *and* $\overline{\mathcal{F}}$ *are Riemann integrable on* $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$, *i.e.,*

$$(IR) \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \mathcal{F}(\mathrm{t}) d\mathrm{t} = \left[(R) \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \underline{\mathcal{F}}(\mathrm{t}) d\mathrm{t}, (R) \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \overline{\mathcal{F}}(\mathrm{t}) d\mathrm{t}\right]. \tag{2.3}$$

Let us consider the notion of superquadraticity.

Definition 2.14. *A function* $\mathcal{F} : [0, \infty) \longrightarrow \mathcal{R}$ *is superquadratic, if*

$$\mathcal{F}(\mathrm{t}_o) \geq \mathcal{F}(\mathrm{t}) + \mathfrak{C}_\mathrm{t}(\mathrm{t}_o - \mathrm{t}) + \mathcal{F}(|\mathrm{t}_o - \mathrm{t}|), \tag{2.4}$$

holds for each $\mathrm{t}_o \geq 0$, $\mathfrak{C}_\mathrm{t} \in \mathcal{R}$, *where* $\mathrm{t} \geq 0$ *and* $\mathfrak{C}_\mathrm{t}$ *is a constant.*

Remark 2.15. $\mathcal{F}$ *is subquadratic provided, the inequality* (2.4) *is flipped.*

More specifically, any arbitrarily selected superquadratic function satisfies the three additional conditions stated in Lemma 2.16:

Lemma 2.16. *If the function $\mathcal{F}$ is superquadratic, then*

- $\mathcal{F}(0) \leq 0$.
- $\mathcal{F}(0) = \mathcal{F}'(0) = 0$ *implies* $\mathcal{F}'(\mathfrak{t}) = \mathfrak{C}_{\mathfrak{t}}$, *provided* $\mathcal{F}$ *is differentiable at* $\mathfrak{t} > 0$.
- $\mathcal{F}$ *is convex provided* $\mathcal{F}(0) = \mathcal{F}'(0) = 0$ *and* $\mathcal{F}$ *is positive on* $(0, \infty)$.

Theorem 2.17. *[22] If the function $\mathcal{F}$ is superquadratic, then*

$$\sum_{i=1}^{\mathfrak{n}} {}_i\mathcal{F}(\mathfrak{t}_i) \geq \mathcal{F}(\bar{\mathfrak{t}}) + \sum_{i=1}^{\mathfrak{n}} {}_i\mathcal{F}(|\mathfrak{t}_i - \bar{\mathfrak{t}}|), \tag{2.5}$$

holds for each ${}_i \in (0, 1)$ *and* $\mathfrak{t}_i \geq 0$, *where* $\bar{\mathfrak{t}} = \sum_{i=1}^{\mathfrak{n}} {}_i\mathfrak{t}_i$, *and* $\sum_{i=1}^{\mathfrak{n}} {}_i = 1$.

In their work, Banić et al. [31] contributed to superquadratic theory by deriving the following inequalities of $\mathcal{H}.\mathcal{H}$ type:

Theorem 2.18. *If the function $\mathcal{F}$ is superquadratic on* $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$, *then*

$$\begin{aligned} &\mathcal{F}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big) + \frac{1}{\mathfrak{b}_\circ - \mathfrak{a}_\circ} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \mathcal{F}\Big(\Big|\mathfrak{t} - \frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big|\Big) d\mathfrak{t} \leq \frac{1}{\mathfrak{b}_\circ - \mathfrak{a}_\circ} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \mathcal{F}(\mathfrak{t}) d\mathfrak{t} \\ &\leq \frac{\mathcal{F}(\mathfrak{a}_\circ) + \mathcal{F}(\mathfrak{b}_\circ)}{2} - \frac{1}{(\mathfrak{b}_\circ - \mathfrak{a}_\circ)^2} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} [(\mathfrak{b}_\circ - \mathfrak{t})\mathcal{F}(\mathfrak{t} - \mathfrak{a}_\circ) + (\mathfrak{t} - \mathfrak{a}_\circ)\mathcal{F}(\mathfrak{b}_\circ - \mathfrak{t})] d\mathfrak{t}. \end{aligned} \tag{2.6}$$

The notion of the supporting line for superquadraticity is introduced in Definition 2.14. Below, we present an additional related definition.

Definition 2.19. *[21] If the function $\mathcal{F}$ is superquadratic, then*

$$\begin{aligned} &\mathcal{F}((1-\beta)\mathfrak{t}_1 + \beta\mathfrak{t}_2) \leq (1-\beta)\mathcal{F}(\mathfrak{t}_1) + \beta\mathcal{F}(\mathfrak{t}_2) - \beta\mathcal{F}((1-\beta)|\mathfrak{t}_1 - \mathfrak{t}_2|) \\ &- (1-\beta)\mathcal{F}(\beta|\mathfrak{t}_1 - \mathfrak{t}_2|), \end{aligned} \tag{2.7}$$

holds for each $\beta \in (0, 1)$ *and* $\mathfrak{t}_1, \mathfrak{t}_2 \geq 0$.

Definition 2.20. *[25] A function* $\mathcal{F} : I \subseteq \mathcal{R} \to (0, \infty)$ *is said to be multiplicatively superquadratic function on I if*

$$\mathcal{F}((1-\beta)\mathfrak{t}_1 + \beta\mathfrak{t}_2) \leq \frac{[\mathcal{F}(\mathfrak{t}_1)]^{(1-\beta)}[\mathcal{F}(\mathfrak{t}_2)]^{\beta}}{[\mathcal{F}((1-\beta)|\mathfrak{t}_1 - \mathfrak{t}_2|)]^{\beta}[\mathcal{F}(\beta|\mathfrak{t}_1 - \mathfrak{t}_2|)]^{(1-\beta)}}, \tag{2.8}$$

holds $\forall$ $\mathfrak{t}_1, \mathfrak{t}_2 \geq 0$ *and* $0 < \beta < 1$.

The notion of superquadratic $\mathcal{IVF}$s was first established in [58]. In the following, we list the major results of the said notion:

Definition 2.21. *A function* $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \to \mathcal{R}_I^+$, *is said to be superquadratic* $\mathcal{IVF}$ *on* $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$, *if*

$$\begin{aligned} &\mathcal{F}((1-\beta)\mathfrak{t}_2 + \beta\mathfrak{t}_1) \supseteq (1-\beta)\mathcal{F}(\mathfrak{t}_2) + \beta\mathcal{F}(\mathfrak{t}_1) - \beta\mathcal{F}((1-\beta)|\mathfrak{t}_2 - \mathfrak{t}_1|) \\ &- (1-\beta)\mathcal{F}(\beta|\mathfrak{t}_2 - \mathfrak{t}_1|), \end{aligned} \tag{2.9}$$

holds $\forall\mathfrak{t}_1, \mathfrak{t}_2 \in [\mathfrak{a}_\circ, \mathfrak{b}_\circ]$ *and for each* $\beta \in [0, 1]$.

Remark 2.22. *The function $\mathcal{F}$ is called subquadratic $\mathcal{IVF}$ on* J *if the relation "$\supseteq$" in* (3.1) *is reversed.*

Remark 2.23. *The function $\mathcal{F}$ is called affine $\mathcal{IVF}$ on* J *when the symbol "$\supseteq$" in* (3.1) *is substituted by "=".*

Remark 2.24. *A non-negative scalar multiple of a superquadratic $\mathcal{IVF}$ on* J *is also superquadratic $\mathcal{IVF}$ on* J.

Remark 2.25. *Let $\mathcal{F}(\mathrm{t})$ and $\varphi(\mathrm{t})$ be superquadratic $\mathcal{IVF}$s on* J. *Then the function defined by* $\max \mathcal{F}(\mathrm{t}), \varphi(\mathrm{t})$ *is also superquadratic $\mathcal{IVF}$s on* J *for every* $\mathrm{t} \in \mathrm{J}$.

Theorem 2.26. *If $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \longrightarrow \mathcal{R}_I^+$ is a superquadratic $\mathcal{IVF}$ given by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$, then for all* $\mathrm{t}_i, \beta_i \geq 0,\ \ 1 \leq i \leq n$, *we have*

$$\mathcal{F}\Big(\sum_{i=1}^{\mathfrak{n}} \beta_i(\mathrm{t}_i)\Big) \supseteq \sum_{i=1}^{\mathfrak{n}} \beta_i \mathcal{F}(\mathrm{t}_i) - \sum_{i=1}^{\mathfrak{n}} \beta_i \mathcal{F}\Big(|\mathrm{t}_i - \sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{t}_i|\Big), \tag{2.10}$$

where $\sum_{i=1}^{\mathfrak{n}} \beta_i = 1$.

Theorem 2.27. *Suppose that $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \longrightarrow \mathcal{R}_I^+$ is superquadratic $\mathcal{IVF}$ on $[\mathfrak{a}_\circ, \mathfrak{b}_\circ]$ provided by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$, then*

$$\begin{aligned}
&\mathcal{F}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big) + \frac{1}{\mathfrak{b}_\circ - \mathfrak{a}_\circ} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \mathcal{F}\Big(\Big|\mathrm{t} - \frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big|\Big) d\mathrm{t} \supseteq \frac{1}{\mathfrak{b}_\circ - \mathfrak{a}_\circ} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \mathcal{F}(\mathrm{t}) d\mathrm{t} \\
&\supseteq \frac{\mathcal{F}(\mathfrak{a}_\circ) + \mathcal{F}(\mathfrak{b}_\circ)}{2} - \frac{1}{(\mathfrak{b}_\circ - \mathfrak{a}_\circ)^2} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} [(\mathfrak{b}_\circ - \mathrm{t})\mathcal{F}(\mathrm{t} - \mathfrak{a}_\circ) + (\mathrm{t} - \mathfrak{a}_\circ)\mathcal{F}(\mathfrak{b}_\circ - \mathrm{t})] d\mathrm{t}.
\end{aligned} \tag{2.11}$$

Next, we define the $\mathcal{R}.\mathcal{L}$ fractional operators, which form the foundation for the analysis that follows.

Definition 2.28. *The $\mathcal{R}.\mathcal{L}$ fractional operators of order $\ell \geq 0$ with $\mathfrak{a}_\circ \geq 0$ are denoted by $\mathrm{J}^{\ell}_{\mathfrak{b}_\circ^-}\mathcal{F}(\mathrm{t})$ and $\mathrm{J}^{\ell}_{\mathfrak{a}_\circ^+}\mathcal{F}(\mathrm{t})$, respectively, and given by*

$$\mathrm{J}^{\ell}_{\mathfrak{a}_\circ^+}\mathcal{F}(\mathrm{t}) = \frac{1}{\Gamma(\ell)} \int_{\mathfrak{a}_\circ}^{\mathrm{t}} (\mathrm{t} - \mathrm{t}_o)^{\ell-1} \mathcal{F}(\mathrm{t}_o) d\mathrm{t}_o, \quad (\mathrm{t} > \mathfrak{a}_\circ),$$

and

$$\mathrm{J}^{\ell}_{\mathfrak{b}_\circ^-}\mathcal{F}(\mathrm{t}) = \frac{1}{\Gamma(\ell)} \int_{\mathrm{t}}^{\mathfrak{b}_\circ} (\mathrm{t}_o - \mathrm{t})^{\ell-1} \mathcal{F}(\mathrm{t}_o) d\mathrm{t}_o, \quad (\mathrm{t} < \mathfrak{b}_\circ).$$

Here $\Gamma(\ell) = \int_0^\infty {\mathrm{t}_o}^{\ell-1} e^{-\mathrm{t}_o} d\mathrm{t}_o$.

A significant extension of the classical $\mathcal{R}.\mathcal{L}$ fractional operators, referred to as the multiplicative $\mathcal{R}.\mathcal{L}$ fractional operators, is presented in [43], representing an important advancement in the development of multiplicative fractional calculus.

Definition 2.29. *The multiplicative left-sided $\mathcal{R}.\mathcal{L}$ fractional operators* ${}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}\mathcal{F}(\mathrm{t})$ *of order* $\ell \in C$, *Re*$(\ell) > 0$ *and* $\mathrm{t} > \mathfrak{a}_\circ$, *is defined by*

$$\begin{aligned} {}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}\mathcal{F}(\mathrm{t}) &= exp\{\mathrm{J}_{\mathfrak{a}_\circ^+}^{\ell}(ln \circ \mathcal{F})(\mathrm{t})\} \\ &= exp\left\{\frac{1}{\Gamma(\ell)}\int_{\mathfrak{a}_\circ}^{\mathrm{t}}(\mathrm{t}-\mathrm{t}_o)^{\ell-1}(ln \circ \mathcal{F})(\mathrm{t}_o)\mathrm{d}\mathrm{t}_o\right\}, \end{aligned} \tag{2.12}$$

and the right-sided one ${}_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}\mathcal{F}(\mathrm{t})$, *where* $\mathrm{t} < \mathfrak{b}_\circ$ *is defined by*

$$\begin{aligned} {}_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}\mathcal{F}(\mathrm{t}) &= exp\{\mathrm{J}_{\mathfrak{b}_\circ^-}^{\ell}(ln \circ \mathcal{F})(\mathrm{t})\} \\ &= exp\left\{\frac{1}{\Gamma(\ell)}\int_{\mathrm{t}}^{\mathfrak{b}_\circ}(\mathrm{t}_o-\mathrm{t})^{\ell-1}(ln \circ \mathcal{F})(\mathrm{t}_o)\mathrm{d}\mathrm{t}_o\right\}, \end{aligned} \tag{2.13}$$

with $\Gamma(\ell+1) = \ell\Gamma(\ell)$ *and* $\Gamma(1) = 1$.

3. Multiplicatively superquadratic $\mathcal{IVF}$s and its integral inequalities via multiplicative calculus

In light of the analytical framework, we now present the following definition. Throughout the article, the notation G(.,.) represents the geometric mean.

Definition 3.1. *A function* $\mathcal{F} : \mathrm{J} \subset [0,\infty) \to \mathcal{R}_I^+$, *is said to be multiplicatively superquadratic* $\mathcal{IVF}$ *on* J, *if the following inequality*

$$\frac{[\mathcal{F}(\mathrm{t}_1)]^{(1-\beta)}[\mathcal{F}(\mathrm{t}_2)]^{\beta}}{[\mathcal{F}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\mathcal{F}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}} \subseteq \mathcal{F}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2), \tag{3.1}$$

holds for any $\mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$ *and* $\beta \in [0,1]$.

Remark 3.2. *Reversing the inequality in* (3.1) *characterizes the function* $\mathcal{F}$ *as a multiplicatively subquadratic* $\mathcal{IVF}$ *on* J.

Remark 3.3. *The function* $\mathcal{F}$ *is called a multiplicatively affine* $\mathcal{IVF}$ *on* J *when the inequality in* (3.1) *holds with equality.*

Remark 3.4. *A non-negative scalar multiple of a multiplicatively superquadratic* $\mathcal{IVF}$ *on* J *is also multiplicatively superquadratic* $\mathcal{IVF}$ *on* J.

Remark 3.5. *Let* $\mathcal{F}(\mathrm{t})$ *and* $\varphi(\mathrm{t})$ *be multiplicatively superquadratic* $\mathcal{IVF}$*s on* J. *Then, for every* $\mathrm{t} \in \mathrm{J}$, *the function* $\max\{\mathcal{F}(\mathrm{t}), \varphi(\mathrm{t})\}$ *is also multiplicatively superquadratic* $\mathcal{IVF}$ *on* J.

Theorem 3.6. *Let* $\mathcal{F} : \mathrm{J} \to \mathcal{R}_I^+$ *be an* $\mathcal{IVF}$ *on* J *such that*

$$\mathcal{F}(\mathrm{t}) = [\underline{\mathcal{F}}(\mathrm{t}), \overline{\mathcal{F}}(\mathrm{t})], \tag{3.2}$$

$\forall\ \mathrm{t} \in \mathrm{J}$. *Then* $\mathcal{F}$ *is multiplicatively superquadratic* $\mathcal{IVF}$ *on* J *provided* $\underline{\mathcal{F}}(\mathrm{t})$ *is multiplicatively superquadratic and* $\overline{\mathcal{F}}(\mathrm{t})$ *is multiplicatively subquadratic.*

Proof. Let $\mathcal{F}$ be a multiplicatively superquadratic $\mathcal{IVF}$ on J. Then $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in$ J and $\beta \in [0,1]$, so we have

$$\mathcal{F}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2) \supseteq \frac{[\mathcal{F}(\mathrm{t}_1)]^{(1-\beta)}[\mathcal{F}(\mathrm{t}_2)]^{\beta}}{[\mathcal{F}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\mathcal{F}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}. \tag{3.3}$$

Accordingly, using (3.2), we can rewrite the left-hand side and right-hand side of (3.3) as follows:

$$\begin{aligned}&\frac{[\mathcal{F}(\mathrm{t}_1)]^{(1-\beta)}[\mathcal{F}(\mathrm{t}_2)]^{\beta}}{[\mathcal{F}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\mathcal{F}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}\\ &=\left[\frac{[\underline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\underline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\underline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\underline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}, \frac{[\overline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\overline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\overline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\overline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}\right],\end{aligned} \tag{3.4}$$

and

$$\mathcal{F}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2) = [\underline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2), \overline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2)], \tag{3.5}$$

respectively. Thus, it implies that

$$\begin{aligned}&\left[\underline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2), \overline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2)\right]\\ &\supseteq\left[\frac{[\underline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\underline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\underline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\underline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}, \frac{[\overline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\overline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\overline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\overline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}}\right].\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\frac{[\underline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\underline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\underline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\underline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}} \geq \underline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2), \tag{3.7}$$

and

$$\frac{[\overline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\overline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\overline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\overline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}} \leq \overline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2). \tag{3.8}$$

From the inequalities (3.7) and (3.8) that $\underline{\mathcal{F}}$ is multiplicatively superquadratic and $\overline{\mathcal{F}}$ is multiplicatively subquadratic.
Conversely: Let $\underline{\mathcal{F}}$ be a multiplicatively superquadratic function and $\overline{\mathcal{F}}$ a multiplicatively subquadratic one. Then, by Definition 2.20, the following inequality holds:

$$\frac{[\underline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\underline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\underline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\underline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}} \geq \underline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2), \tag{3.9}$$

and

$$\frac{[\overline{\mathcal{F}}(\mathrm{t}_1)]^{(1-\beta)}[\overline{\mathcal{F}}(\mathrm{t}_2)]^{\beta}}{[\overline{\mathcal{F}}((1-\beta)|\mathrm{t}_1-\mathrm{t}_2|)]^{\beta}[\overline{\mathcal{F}}(\beta|\mathrm{t}_1-\mathrm{t}_2|)]^{(1-\beta)}} \leq \overline{\mathcal{F}}((1-\beta)\mathrm{t}_1+\beta\mathrm{t}_2). \tag{3.10}$$

From (3.9) and (3.10), we can have

$$\left[\frac{[\underline{\mathcal{F}}(t_1)]^{(1-\beta)}[\underline{\mathcal{F}}(t_2)]^{\beta}}{[\underline{\mathcal{F}}((1-\beta)|t_1-t_2|)]^{\beta}[\underline{\mathcal{F}}(\beta|t_1-t_2|)]^{(1-\beta)}}, \frac{[\overline{\mathcal{F}}(t_1)]^{(1-\beta)}[\overline{\mathcal{F}}(t_2)]^{\beta}}{[\overline{\mathcal{F}}((1-\beta)|t_1-t_2|)]^{\beta}[\overline{\mathcal{F}}(\beta|t_1-t_2|)]^{(1-\beta)}}\right]$$
$$\subseteq \left[\underline{\mathcal{F}}((1-\beta)t_1+\beta t_2), \overline{\mathcal{F}}((1-\beta)t_1+\beta t_2)\right]. \tag{3.11}$$

This implies that

$$\frac{[\mathcal{F}(t_1)]^{(1-\beta)}[\mathcal{F}(t_2)]^{\beta}}{[\mathcal{F}((1-\beta)|t_1-t_2|)]^{\beta}[\mathcal{F}(\beta|t_1-t_2|)]^{(1-\beta)}} \subseteq \mathcal{F}((1-\beta)t_1+\beta t_2).$$

Hence, $\mathcal{F}$ is multiplicatively superquadratic $\mathcal{IVF}$ on J. □

Remark 3.7. *When $\underline{\mathcal{F}}(t) = \overline{\mathcal{F}}(t)$, the notion of a multiplicatively superquadratic $\mathcal{IVF}$ coincides with that of the classical multiplicatively superquadratic function.*

Proposition 3.8. *Let $\mathcal{F} : J \to \mathcal{R}_I^+$ be an $\mathcal{IVF}$ on J such that*

$$\mathcal{F}(t) = [\underline{\mathcal{F}}(t), \overline{\mathcal{F}}(t)]. \tag{3.12}$$

Where $\underline{\mathcal{F}}, \overline{\mathcal{F}} : J \to \mathcal{R}^+$. If $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are *integrable on J, then $\mathcal{F}$ is *integrable on J and

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\mathcal{F}(t))^{dt} = exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\mathcal{F}(t))dt\Big\}. \tag{3.13}$$

Proof. Let $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are *integrable on J, then by Proposition 2.4, $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are (RI)-integrable on J, and

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\underline{\mathcal{F}}(t))^{dt} = exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\underline{\mathcal{F}}(t))dt\Big\}, \tag{3.14}$$

and

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\overline{\mathcal{F}}(t))^{dt} = exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\overline{\mathcal{F}}(t))dt\Big\}. \tag{3.15}$$

Thus, by Theorem 2.13, $\mathcal{F}$ is interval (RI)-integrable on J. Again by Proposition 2.4, $\mathcal{F}$ is interval *integrable on J, and from (3.14) and (3.15), we attain

$$\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\underline{\mathcal{F}}(t))^{dt}, \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\overline{\mathcal{F}}(t))^{dt}\Big] = \Big[exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\underline{\mathcal{F}}(t))dt\Big\}, exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\overline{\mathcal{F}}(t))dt\Big\}\Big],$$

or

$$\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\mathcal{F}(t))^{dt} = exp\Big\{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} ln(\mathcal{F}(t))dt\Big\}.$$

□

Theorem 3.9. *Let $\mathcal{F} : \mathrm{J} \to \mathcal{R}_I^+$ be a multiplicatively superquadratic $\mathcal{IVF}$ on* J*, such that*

$$\mathcal{F}(\mathrm{t}) = [\underline{\mathcal{F}}(\mathrm{t}), \overline{\mathcal{F}}(\mathrm{t})], \tag{3.16}$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}} : \mathrm{J} \to \mathcal{R}^+$. If $\mathcal{F}$ is interval *integrable on J*, then*

$$\begin{aligned}
&\mathcal{F}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big)\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big(\Big(\mathcal{F}(|\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2} - \mathrm{t}|)\Big)^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{dt}}\Big)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\supseteq \Big({}_{\mathfrak{a}_\circ}\mathrm{J}_*^\ell(\mathcal{F})(\mathfrak{b}_\circ) \cdot_* \mathrm{J}_{\mathfrak{b}_\circ}^\ell(\mathcal{F})(\mathfrak{a}_\circ)\Big)^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\mathcal{F}(|\mathfrak{b}_\circ - \mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)^\ell} \cdot (\mathcal{F}(|\mathrm{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathrm{t})(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{dt}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}},
\end{aligned}$$

holds, $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ be a multiplicatively superquadratic $\mathcal{IVF}$ on J, then

$$\mathcal{F}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) = \Big[\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), \overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big]. \tag{3.17}$$

Taking Log on both sides of (3.17),

$$ln\mathcal{F}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) = ln\Big[\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), \overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big] = \Big[ln\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), ln\overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big]. \tag{3.18}$$

Consider $ln\underline{\mathcal{F}}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})$ from (3.18), and it is to be noted that $\underline{\mathcal{F}}(\mathrm{t})$ is multiplicatively superquadratic.

$$\begin{aligned}
&\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}) = \ln\underline{\mathcal{F}}\Big(\frac{\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ + \beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ}{2}\Big) \\
&\leq \frac{1}{2}\ln\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\mathcal{F}}(\beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ) \\
&- \frac{1}{2}\ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big) - \frac{1}{2}\ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big).
\end{aligned}$$

Multiplying both sides of the inequality by $\beta^{\ell-1}$ and then integrating the resulting expression with respect to β over the interval $[0, 1]$, we obtain:

$$\begin{aligned}
&\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\mathrm{d}\beta \leq \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ)\mathrm{d}\beta \\
&+ \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta)\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ)\mathrm{d}\beta - \int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big)\mathrm{d}\beta.
\end{aligned}$$

After simple calculations and then with a change of variables, we obtain

$$\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}) \leq \frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ - \mathfrak{a}_\circ)^\ell}\Big[J_{\mathfrak{a}_\circ^+}^\ell \ln\underline{\mathcal{F}}(\mathfrak{b}_\circ) + J_{\mathfrak{b}_\circ^-}^\ell \ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big]$$

$$-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)d\mathfrak{t}.$$

This implies

$$exp\Big\{\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big\}\leq exp\Big\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big(J^\ell_{\mathfrak{a}_\circ+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big)$$
$$-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)d\mathfrak{t}\Big\}.$$

Thus,

$$\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\leq\frac{\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}}{\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\big(\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\big)^{d\mathfrak{t}}\Big)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}},$$

or

$$\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Bigg(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\big(\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\big)^{d\mathfrak{t}}\Bigg)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\leq\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \quad (3.19)$$

As $\overline{\mathcal{F}}$ is multiplicatively subquadratic and moving in the same fashion, we yield

$$\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Bigg(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\big(\overline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\big)^{d\mathfrak{t}}\Bigg)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\geq\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \quad (3.20)$$

From (3.19) and (3.20), we attain

$$\Bigg[\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Bigg(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\big(\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\big)^{d\mathfrak{t}}\Bigg)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},$$
$$\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Bigg(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\big(\overline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\big)^{d\mathfrak{t}}\Bigg)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Bigg]$$
$$\supseteq\Bigg[\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},$$
$$\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Bigg]. \quad (3.21)$$

To prove the second part, we consider the RHS of inequality (3.19), therefore we have:

$$\big[{}_{\mathfrak{a}_\circ}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}=exp\Big\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[J^\ell_{\mathfrak{a}_\circ^+}(ln\circ\underline{\mathcal{F}})(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ^-}(ln\circ\underline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]\Big\}$$
$$=exp\Big\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln(\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ-(1-\beta)\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln(\underline{\mathcal{F}}(\beta\mathfrak{b}_\circ-(1-\beta)\mathfrak{a}_\circ)d\beta\Big]\Big\}$$

$$\leq exp\Big\{\frac{\ell}{2}\Big[\frac{\ln(\underline{\mathcal{F}}(\mathfrak{a}_\circ)\underline{\mathcal{F}}(\mathfrak{b}_\circ)}{\ell} - \frac{2}{(\mathfrak{b}_\circ - \mathfrak{a}_\circ)^{\ell+1}}\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \ln(\underline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}d\mathfrak{t}$$

$$+ \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \ln(\underline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}d\mathfrak{t}\Big)\Big]\Big\}$$

$$= \frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ), \underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}. \tag{3.22}$$

Similarly by considering the right term of (3.19), we have

$$\Big[{}_{\mathfrak{a}_\circ}J_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^{\ell}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$

$$\geq \frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ), \overline{\mathcal{F}}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}. \tag{3.23}$$

From (3.22) and (3.23), we achieve

$$\Big[\big[{}_{\mathfrak{a}_\circ}J_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^{\ell}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}, \big[{}_{\mathfrak{a}_\circ}J_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^{\ell}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Big]$$

$$\supseteq \Big[\frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ), \underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}$$

$$\frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ)\overline{\mathcal{F}}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}\Big]. \tag{3.24}$$

Merging (3.21) and (3.24), we attain the required result.

□

Remark 3.10. *If we set $\ell = 1$ in Theorem 3.9, we attain*

$$\mathcal{F}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big)\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big(\mathcal{F}(|\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2} - \mathfrak{t}|)\Big)^{d\mathfrak{t}}\Big)^{\frac{1}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)}} \supseteq \Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(\mathfrak{t}))^{d\mathfrak{t}}\Big)^{\frac{1}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}$$

$$\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\mathcal{F}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)} \cdot (\mathcal{F}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}\Big)^{d\mathfrak{t}}\Big]^{\frac{1}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}}.$$

Theorem 3.11. *Let $\mathcal{F}, \varphi : \mathrm{J} \to \mathcal{R}_I^+$ be a multiplicatively superquadratic $\mathcal{IVF}s$ on J, such that*

$$\mathcal{F}(\mathfrak{t}) = [\underline{\mathcal{F}}(\mathfrak{t}), \overline{\mathcal{F}}(\mathfrak{t})]$$
$$\varphi(\mathfrak{t}) = [\underline{\varphi}(\mathfrak{t}), \overline{\varphi}(\mathfrak{t})],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}}, \underline{\varphi}, \overline{\varphi} : \mathrm{J} \to \mathcal{R}^{+}$*. If* $\mathcal{F}$ *and* φ *are interval* **integrable on* J*, then*

$$\mathcal{F}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\varphi(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\left(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\left(\mathcal{F}\varphi(|\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2} - \mathfrak{t}|)\right)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{dt}\right)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\supseteq \left[{}_{\mathfrak{a}_\circ}\mathrm{J}_{*}^{\ell}(\mathcal{F})(\mathfrak{b}_\circ) \cdot_{*} \mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\mathcal{F})(\mathfrak{a}_\circ) \cdot {}_{\mathfrak{a}_\circ}\mathrm{J}_{*}^{\ell}(\varphi)(\mathfrak{b}_\circ) \cdot_{*} \mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\varphi)(\mathfrak{a}_\circ)\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}} \cdot (\mathcal{F}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{dt}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}$$
$$\times \frac{G(\varphi(\mathfrak{a}_\circ), \varphi(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\varphi(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}} \cdot (\varphi(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{dt}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}},$$

holds, $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ and φ be multiplicatively superquadratic $\mathcal{IVF}$s on J, then

$$\begin{aligned}
&ln\Big(\mathcal{F}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) \cdot \varphi\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big) = ln\mathcal{F}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) + ln\varphi\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\\
&= ln\Big[\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), \overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big] + ln\Big[\underline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), \overline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big]\\
&= \Big[ln\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), ln\overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big] + \Big[ln\underline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), ln\overline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big]\\
&= \Big[ln\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) + ln\underline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big), ln\overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big) + ln\overline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big]\\
&= \Big[ln\Big(\underline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\underline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big), ln\Big(\overline{\mathcal{F}}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\overline{\varphi}\Big(\frac{\mathfrak{b}_\circ + \mathfrak{a}_\circ}{2}\Big)\Big)\Big].
\end{aligned} \tag{3.25}$$

Consider $ln(\underline{\mathcal{F}}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}) \cdot \underline{\varphi}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}))$ from (3.25), and it is to be noted that $\underline{\mathcal{F}}(\mathfrak{t})$ and $\underline{\varphi}(\mathfrak{t})$ are multiplicatively superquadratic.

$$\begin{aligned}
&\ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\Big) = \ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}) + \ln\underline{\varphi}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\\
&= \ln\underline{\mathcal{F}}\Big(\frac{\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ + \beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ}{2}\Big) + \ln\underline{\varphi}\Big(\frac{\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ + \beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ}{2}\Big)\\
&\leq \frac{1}{2}\ln\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\mathcal{F}}(\beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ) - \frac{1}{2}\ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big)\\
&- \frac{1}{2}\ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big) + \frac{1}{2}\ln\underline{\varphi}(\beta\mathfrak{a}_\circ + (1-\beta)\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\varphi}(\beta\mathfrak{b}_\circ + (1-\beta)\mathfrak{a}_\circ)\\
&- \frac{1}{2}\ln\underline{\varphi}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big) - \frac{1}{2}\ln\underline{\varphi}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|}{2}\Big).
\end{aligned} \tag{3.26}$$

Multiplying (3.26) both sides with $\beta^{\ell-1}$, then integrating the resulting inequality with respect to β over $[0, 1]$, we have

$$
\begin{aligned}
&\int_0^1 \beta^{\ell-1} \ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big)\mathrm{d}\beta \\
&\leq \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)\mathrm{d}\beta+\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)\mathrm{d}\beta \\
&-\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}\left(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\right)\mathrm{d}\beta+\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)\mathrm{d}\beta \\
&+\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)\mathrm{d}\beta-\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}\left(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\right)\mathrm{d}\beta.
\end{aligned}
$$

After simple calculations and with change of variables, we obtain

$$
\begin{aligned}
&\ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big) \\
&\leq \frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)+J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big] \\
&-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|)\mathrm{dt}+\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|)\mathrm{dt}\Big].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&exp\Big\{\ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big)\Big\} \\
&\leq exp\left\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)+J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big]\right. \\
&\left.-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}\ln(\underline{\mathcal{F}}\underline{\varphi})(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|)\mathrm{dt}\right\}.
\end{aligned}
$$

This follows as

$$
\begin{aligned}
&\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\left(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\Big(\underline{\mathcal{F}}\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|)\Big)^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}\right)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\leq \Big[{}_{\mathfrak{a}_\circ}\mathrm{J}^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}^\ell_{\mathfrak{b}_\circ}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}^\ell_*(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}^\ell_{\mathfrak{b}_\circ}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}.
\end{aligned}
\tag{3.27}
$$

As $\overline{\mathcal{F}}$ and $\overline{\varphi}$ are multiplicatively subquadratic and moving in the same fashion, we yield

$$
\begin{aligned}
&\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\left(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left(\Big(\overline{\mathcal{F}}\overline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|)\Big)^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}\right)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\geq \Big[{}_{\mathfrak{a}_\circ}\mathrm{J}^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}^\ell_{\mathfrak{b}_\circ}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}^\ell_*(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}^\ell_{\mathfrak{b}_\circ}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}.
\end{aligned}
\tag{3.28}
$$

From (3.27) and (3.28), we attain

$$\Big[\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big(\big(\underline{\mathcal{F}}\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)\big)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{d}\mathfrak{t}}\Big)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},$$
$$\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big(\big(\overline{\mathcal{F}}\overline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)\big)^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{d}\mathfrak{t}}\Big)^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Big]$$
$$\supseteq\Big[\Big[{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},$$
$$\Big[{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Big]. \tag{3.29}$$

To prove the second part, we consider the RHS of (3.27), from which we obtain:

$$\Big[{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$=exp\Big\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J_{\mathfrak{a}_\circ^+}^{\ell}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J_{\mathfrak{b}_\circ^-}^{\ell}\ln\mathcal{F}(\mathfrak{a}_\circ)+J_{\mathfrak{a}_\circ^+}^{\ell}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J_{\mathfrak{b}_\circ^-}^{\ell}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big]\Big\}$$
$$=exp\Big\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)\mathrm{d}\beta+\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)\mathrm{d}\beta$$
$$+\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)\mathrm{d}\beta+\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)\mathrm{d}\beta\Big]\Big\}$$
$$\leq exp\Big\{\frac{1}{2}\Big(\ln(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ))+\ln(\varphi(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))\Big)$$
$$-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\ln\Big((\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)\mathrm{d}\mathfrak{t}\Big]$$
$$-\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\ln\Big((\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)\mathrm{d}\mathfrak{t}\Big]\Big\}.$$

Thus, it follows as

$$\Big[{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}} \tag{3.30}$$
$$\leq\frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}\cdot(\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{d}\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}$$
$$\times\frac{G(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))}{\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\Big((\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}\cdot(\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)^{\mathrm{d}\mathfrak{t}}\Big]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}.$$

Similarly, by considering the right term of (3.28), we have

$$\Big[{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\mathfrak{a}_\circ}\mathrm{J}_*^{\ell}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_*\mathrm{J}_{\mathfrak{b}_\circ}^{\ell}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$

$$\geq \frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ), \overline{\mathcal{F}}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\overline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\overline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}} \times \frac{G(\overline{\varphi}(\mathfrak{a}_\circ), \overline{\varphi}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\overline{\varphi}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\overline{\varphi}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}. \tag{3.31}$$

From (3.30) and (3.31), we achieve an interval

$$\begin{aligned}
&\Bigg[\left[{}_{\mathfrak{a}_\circ}J_*^\ell(\underline{\mathcal{F}})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^\ell(\underline{\mathcal{F}})(\mathfrak{a}_\circ) \cdot {}_{\mathfrak{a}_\circ}J_*^\ell(\underline{\varphi})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^\ell(\underline{\varphi})(\mathfrak{a}_\circ)\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}, \\
&\left[{}_{\mathfrak{a}_\circ}J_*^\ell(\overline{\mathcal{F}})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^\ell(\overline{\mathcal{F}})(\mathfrak{a}_\circ) \cdot {}_{\mathfrak{a}_\circ}J_*^\ell(\overline{\varphi})(\mathfrak{b}_\circ) \cdot_* J_{\mathfrak{b}_\circ}^\ell(\overline{\varphi})(\mathfrak{a}_\circ)\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Bigg] \\
&\supseteq \Bigg[\frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ), \underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\underline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\underline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}} \\
&\times \frac{G(\underline{\varphi}(\mathfrak{a}_\circ), \underline{\varphi}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\underline{\varphi}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\underline{\varphi}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}, \\
&\frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ), \overline{\mathcal{F}}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\overline{\mathcal{F}}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\overline{\mathcal{F}}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}} \\
&\times \frac{G(\overline{\varphi}(\mathfrak{a}_\circ), \overline{\varphi}(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\overline{\varphi}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^\ell} \cdot (\overline{\varphi}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}\Bigg].
\end{aligned} \tag{3.32}$$

Merging (3.29) and (3.32), we attain the required result.

□

Remark 3.12. *If we set $\ell = 1$ in Theorem 3.11, we attain*

$$\begin{aligned}
&\mathcal{F}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big)\varphi\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big)\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \Big(\mathcal{F}\varphi\big(\big|\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2} - \mathfrak{t}\big|\big)\Big)^{d\mathfrak{t}}\Big)^{\frac{1}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}} \\
&\supseteq \Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\mathcal{F}(\mathfrak{t}))^{d\mathfrak{t}} \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} (\varphi(\mathfrak{t}))^{d\mathfrak{t}}\Big)^{\frac{1}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}} \\
&\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ)) \cdot G(\varphi(\mathfrak{a}_\circ), \varphi(\mathfrak{b}_\circ))}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ} \left((\mathcal{F}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}(\mathcal{F}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\varphi(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}(\varphi(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}\right)^{d\mathfrak{t}}\right]^{\frac{1}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}}.
\end{aligned}$$

Theorem 3.13. *Let $\mathcal{F}, \varphi : \mathrm{J} \to \mathcal{R}_I^+$ be a multiplicatively superquadratic $\mathcal{IVF}$ and multiplicatively subquadratic $\mathcal{IVF}$, respectively, on* J*, such that*

$$\mathcal{F}(\mathfrak{t}) = [\underline{\mathcal{F}}(\mathfrak{t}), \overline{\mathcal{F}}(\mathfrak{t})]$$

$$\varphi(\mathrm{t}) = [\underline{\varphi}(\mathrm{t}), \overline{\varphi}(\mathrm{t})],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}}, \underline{\varphi}, \overline{\varphi} : \mathrm{J} \to \mathcal{R}^{+}$. *If* $\mathcal{F}$ *and* φ *are interval* **integrable on* J*, then*

$$\frac{\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\varphi(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\varphi(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\supseteq \left[\frac{{}_{\mathfrak{a}_\circ}J_{*}^{\ell}(\mathcal{F})(\mathfrak{b}_\circ)\cdot_{*} J_{\mathfrak{b}_\circ}^{\ell}(\mathcal{F})(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J_{*}^{\ell}(\varphi)(\mathfrak{b}_\circ)\cdot_{*} J_{\mathfrak{b}_\circ}^{\ell}(\varphi)(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(b-a)^{\ell}}}$$
$$\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))}{G(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ))}\cdot\left[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\varphi(|\mathfrak{b}_\circ-\mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell}}\cdot(\varphi(|\mathrm{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathrm{t})(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{b}_\circ-\mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell}}\cdot(\mathcal{F}(|\mathrm{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathrm{t})(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},$$

holds, $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ and φ be multiplicatively superquadratic $\mathcal{IVF}$ and multiplicatively subquadartic $\mathcal{IVF}$, respectively, on J, then

$$ln\left(\frac{\mathcal{F}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}{\varphi(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}\right) = ln\mathcal{F}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right) - ln\varphi\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)$$
$$= ln\left[\underline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right),\overline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)\right] - ln\left[\underline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right),\overline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)\right]$$
$$= \left[ln\underline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right), ln\overline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)\right] - \left[ln\underline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right), ln\overline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)\right]$$
$$= \left[ln\underline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right) + ln\underline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right), -ln\overline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right) - ln\overline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)\right]$$
$$= \left[ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}\right), ln\left(\frac{\overline{\mathcal{F}}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}{\overline{\varphi}(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2})}\right)\right]. \tag{3.33}$$

Consider $ln\left(\frac{\underline{\mathcal{F}}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)}{\underline{\varphi}\left(\frac{\mathfrak{b}_\circ+\mathfrak{a}_\circ}{2}\right)}\right)$ from (3.33), and $\underline{\mathcal{F}}(\mathrm{t})$ and $\underline{\varphi}(\mathrm{t})$ are multiplicatively superquadratic and multiplicatively subquadratic, respectively.

$$\ln\left(\frac{\underline{\mathcal{F}}\left(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\right)}{\underline{\varphi}\left(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\right)}\right) = \ln\underline{\mathcal{F}}\left(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\right) - \ln\underline{\varphi}\left(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\right)$$
$$= \ln\underline{\mathcal{F}}\left(\frac{(1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ}{2}\right) - \ln\underline{\varphi}\left(\frac{(1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ}{2}\right)$$
$$\leq \frac{1}{2}\ln\underline{\mathcal{F}}\Big((1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ\Big) + \frac{1}{2}\ln\underline{\mathcal{F}}\Big((1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ\Big) - \ln\underline{\mathcal{F}}\left(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\right)$$
$$-\frac{1}{2}\ln\underline{\varphi}\Big((1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ\Big) - \frac{1}{2}\ln\varphi\Big((1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ\Big) + \ln\underline{\varphi}\left(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\right). \tag{3.34}$$

Multiplying $\beta^{\ell-1}$ on both sides of (3.34) and integrating the inequality with respect to β over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 \beta^{\ell-1} \ln\Big(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Big)d\beta \\
&\le \frac{1}{2}\int_0^1 \beta^{\ell-1} \ln\underline{\mathcal{F}}\Big((1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ\Big)d\beta + \frac{1}{2}\int_0^1 \beta^{\ell-1} \ln\underline{\mathcal{F}}\Big((1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ\Big)d\beta \\
&- \int_0^1 \beta^{\ell-1} \ln\underline{\mathcal{F}}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\Big)d\beta - \frac{1}{2}\int_0^1 \beta^{\ell-1} \ln\underline{\varphi}\Big((1-\beta)\mathfrak{a}_\circ+\beta\mathfrak{b}_\circ\Big)d\beta \\
&- \frac{1}{2}\int_0^1 \beta^{\ell-1} \ln\underline{\varphi}\Big((1-\beta)\mathfrak{b}_\circ+\beta\mathfrak{a}_\circ\Big)d\beta + \int_0^1 \beta^{\ell-1} \ln\underline{\varphi}\Big(\frac{|(1-2\beta)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|}{2}\Big)d\beta.
\end{aligned}$$

After simple calculations and with change of variables, we obtain

$$\begin{aligned}
\ln\Big(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Big) &\le \frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ) + J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ) - J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\varphi}(\mathfrak{b}_\circ) - J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big] \\
&+ \frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^\ell \ln\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)d\mathfrak{t} - \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^\ell \ln\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)d\mathfrak{t}\Big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
exp\Bigg\{\ln\Bigg(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg)\Bigg\} &\le exp\Bigg\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[(J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ) + J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)) - (J^\ell_{\mathfrak{a}_\circ^+}\ln\underline{\varphi}(\mathfrak{b}_\circ) + J^\ell_{\mathfrak{b}_\circ^-}\ln\underline{\varphi}(\mathfrak{a}_\circ))\Big] \\
&+ \frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg(\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^\ell \ln\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|)d\mathfrak{t} - \int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathfrak{t}-\mathfrak{a}_\circ)^\ell \ln\underline{\mathcal{F}}\Big(\Big|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}\Big|\Big)d\mathfrak{t}\Bigg)\Bigg\}.
\end{aligned}$$

It follows as

$$\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}\Bigg]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \le \Bigg[\frac{{}_{\mathfrak{a}_\circ}J^\ell_*\underline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}\underline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^\ell_*\underline{\varphi}(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}\underline{\varphi}(\mathfrak{a}_\circ)}\Bigg]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \tag{3.35}$$

As $\overline{\mathcal{F}}$ and $\overline{\varphi}$ are multiplicatively subquadratic and multiplicatively superquadratic respectively, and moving in the same fashion, we yield

$$\frac{\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\overline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\overline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}\Bigg]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \ge \Bigg[\frac{{}_{\mathfrak{a}_\circ}J^\ell_*\overline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}\overline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^\ell_*\overline{\varphi}(\mathfrak{b}_\circ)\cdot_* J^\ell_{\mathfrak{b}_\circ}\overline{\varphi}(\mathfrak{a}_\circ)}\Bigg]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \tag{3.36}$$

From (3.35) and (3.36), we attain

$$\begin{aligned}
\Bigg[&\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\underline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\underline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}\Bigg]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}, \\
&\frac{\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\overline{\mathcal{F}}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}((\overline{\varphi}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}})^{d\mathfrak{t}}}\Bigg]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Bigg]
\end{aligned}$$

$$\supseteq\left[\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\right]. \tag{3.37}$$

To prove the second part, we consider the RHS of (3.35), from which we derive:

$$\begin{aligned}
&\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\mathcal{F}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\mathcal{F}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\varphi(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\varphi(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&= exp\left\{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J^{\ell}_{\mathfrak{a}_\circ^{+}}\ln\mathcal{F}(\mathfrak{b}_\circ)+J^{\ell}_{\mathfrak{b}_\circ^{-}}\ln\mathcal{F}(\mathfrak{a}_\circ)-J^{\ell}_{\mathfrak{a}_\circ^{+}}\ln\varphi(\mathfrak{b}_\circ)-J^{\ell}_{\mathfrak{b}_\circ^{-}}\ln\varphi(\mathfrak{a}_\circ)\Big]\right\}\\
&= exp\left\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln\mathcal{F}(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln\mathcal{F}(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)d\beta\right.\\
&\left.-\int_0^1\beta^{\ell-1}\ln\varphi(\beta\mathfrak{a}_\circ+(1-\beta)\mathfrak{b}_\circ)d\beta-\int_0^1\beta^{\ell-1}\ln\varphi(\beta\mathfrak{b}_\circ+(1-\beta)\mathfrak{a}_\circ)d\beta\Big]\right\}\\
&\le exp\left\{\frac{1}{2}\Big(\ln\left(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ)\right)-\ln\left(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ)\right)\Big)\right.\\
&+\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\ln\Big((\varphi(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\varphi(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)d\mathfrak{t}\\
&\left.-\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\ln\Big((\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\Big)d\mathfrak{t}\Big]\right\}.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&\le\frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ))}{G(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}.
\end{aligned}\tag{3.38}$$

Similarly, considering the right term of (3.36), we have

$$\begin{aligned}
&\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&\ge\frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ))}{G(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}.
\end{aligned}\tag{3.39}$$

From (3.38) and (3.39), we achieve an interval

$$\begin{aligned}
&\left[\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\underline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\underline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},\left[\frac{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\mathcal{F}}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\mathcal{F}}(\mathfrak{a}_\circ)}{{}_{\mathfrak{a}_\circ}J^{\ell}_{*}\overline{\varphi}(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\mathfrak{b}_\circ}\overline{\varphi}(\mathfrak{a}_\circ)}\right]^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\right]\\
&\supseteq\Bigg[\frac{G(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ))}{G(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},
\end{aligned}$$

$$\frac{G(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ))}{G(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ))}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell}}(\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})(\mathfrak{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{d\mathfrak{t}}}\Bigg]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}\Bigg]. \tag{3.40}$$

Merging (3.37) and (3.40), we attain the required result.

Remark 3.14. *If we set* $\ell = 1$ *in Theorem 3.13, we attain*

$$\frac{\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\varphi(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\varphi(|\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}-\mathfrak{t}|))^{d\mathfrak{t}}}\Bigg]^{\frac{1}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}} \supseteq \left(\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\mathcal{F}(\mathfrak{t}))^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}(\varphi(\mathfrak{t}))^{d\mathfrak{t}}}\right)^{\frac{1}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}$$

$$\supseteq \frac{G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))}{G(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ))}\Bigg[\frac{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\varphi(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\cdot(\varphi(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}\right)^{d\mathfrak{t}}}{\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\cdot(\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}\right)^{d\mathfrak{t}}}\Bigg]^{\frac{1}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}.$$

□

Theorem 3.15. *Let* $\mathcal{F}: \mathrm{J} \to \mathcal{R}_I^+$ *be a multiplicatively superquadratic* $\mathcal{IVF}$ *on* J*, such that*

$$\mathcal{F}(\mathfrak{t}) = [\underline{\mathcal{F}}(\mathfrak{t}),\overline{\mathcal{F}}(\mathfrak{t})],$$

where $\underline{\mathcal{F}},\overline{\mathcal{F}}: \mathrm{J} \to \mathcal{R}^+$*. If* $\mathcal{F}$ *is interval* **integrable on* J*, then*

$$\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\left(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}\right)^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$

$$\supseteq \left[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\mathcal{F})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathcal{F})(\mathfrak{a}_\circ)\right]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$

$$\supseteq G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))\left[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}\right]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},$$

holds, $\forall\ \mathfrak{a}_\circ,\mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ be a multiplicatively superquadratic $\mathcal{IVF}$ on J, then

$$ln\mathcal{F}(\mathfrak{t}) = \Big[ln(\underline{\mathcal{F}}(\mathfrak{t})), ln(\overline{\mathcal{F}}(\mathfrak{t}))\Big].$$

Since $\underline{\mathcal{F}}$ be a positive and multiplicative superquadratic function, we have

$$\ln\underline{\mathcal{F}}\left(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\right) = \ln\underline{\mathcal{F}}\left(\frac{1}{2}\left(\left(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ\right)+\left(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ\right)\right)\right)$$
$$\leq \frac{1}{2}\ln\underline{\mathcal{F}}\left(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ\right)+\frac{1}{2}\ln\underline{\mathcal{F}}\left(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ\right)-\ln\underline{\mathcal{F}}\left(\frac{1}{2}\left|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)\right|\right). \tag{3.41}$$

Multiplying with $\beta^{\ell-1}$ on both sides of (3.41) and integrating the resulting inequality with respect to β over $[0, 1]$, we get

$$\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})d\beta \leq \frac{1}{2}\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta$$

$$+\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta-\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|)d\beta.$$

By changing of variables and after simple calculations, we get

$$\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\leq\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big[J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big]$$
$$-\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t}.$$

Thus, it follows that

$$exp\Big\{\ln\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big\}\leq exp\Big\{\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big(J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big)$$
$$-\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t}\Big\}.$$

Thus, we have

$$\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{d\mathfrak{t}}\Big]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\leq\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \tag{3.42}$$

Similarly, considering $ln\overline{\mathcal{F}}(\mathfrak{t})$, where $\overline{\mathcal{F}}$ is multiplicatively subquadratic, and moving in the same fashion, we get

$$\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\overline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{d\mathfrak{t}}\Big]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\leq\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}. \tag{3.43}$$

From (3.42) and (3.43), we have

$$\Big[\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{d\mathfrak{t}}\Big]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},$$
$$\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\overline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{d\mathfrak{t}}\Big]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Big]$$
$$\supseteq\Big[\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},$$
$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Big]. \tag{3.44}$$

To establish the second part, we consider the right term of (3.42). Accordingly, we have:

$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$

$$= exp\Big\{\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big(J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^\ell_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big)\Big\}$$

$$= exp\Big\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta\Big]\Big\}$$

$$\leq exp\Big\{\ln\sqrt{\underline{\mathcal{F}}(\mathfrak{a}_\circ)\underline{\mathcal{F}}(\mathfrak{b}_\circ)}-\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\ln\big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\big)d\mathfrak{t}\Big)\Big\}$$

$$= G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \quad (3.45)$$

Similarly, considering the right term of (3.43), we obtain

$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$

$$\geq G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \quad (3.46)$$

From (3.42) and (3.46), we obtain

$$\Big[\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^\ell_*(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\Big]$$

$$\supseteq\Big[G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},$$

$$G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}\Big]. \quad (3.47)$$

Combining (3.44) and (3.47), we obtain the required result. □

Remark 3.16. *If we set $\ell = 1$ in Theorem 3.15, we attain*

$$\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\mathcal{F}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)\Big)^{d\mathfrak{t}}\Big)^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}\supseteq\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathcal{F}(\mathfrak{t}))^{d\mathfrak{t}}\Big)^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}$$

$$\supseteq G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\Big)^{d\mathfrak{t}}\Big]^{\frac{-2}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}.$$

Theorem 3.17. *Let $\mathcal{F},\varphi : \mathrm{J}\to\mathcal{R}^+_I$ be a multiplicatively superquadratic $\mathcal{IVF}$s on J, such that*

$$\mathcal{F}(\mathfrak{t}) = [\underline{\mathcal{F}}(\mathfrak{t}),\overline{\mathcal{F}}(\mathfrak{t})]$$
$$\varphi(\mathfrak{t}) = [\underline{\varphi}(\mathfrak{t}),\overline{\varphi}(\mathfrak{t})].$$

Where $\underline{\mathcal{F}},\overline{\mathcal{F}},\underline{\varphi},\overline{\varphi} : \mathrm{J}\to\mathcal{R}^+$. If $\mathcal{F}$ and φ are interval *integrable on J, then

$$\Big[\mathcal{F}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\varphi\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\mathcal{F}\varphi(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big)^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$

$$\supseteq \Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\mathcal{F})(\mathfrak{b}_\circ) \cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathcal{F})(\mathfrak{a}_\circ) \cdot_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}} J_*^{\ell}(\varphi)(\mathfrak{b}_\circ) \cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\varphi)(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}} \Big]$$
$$\supseteq \Big[G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))\cdot\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}$$
$$\times G(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ))\cdot\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\varphi(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\varphi(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big]^{\frac{-\ell\cdot2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}\Big],$$

holds, $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ and φ be multiplicatively superquadratic $\mathcal{IVF}$s on J, then

$$\begin{aligned} ln\Big(\underline{\mathcal{F}}(\mathfrak{t})\underline{\varphi}(\mathfrak{t})\Big) &= ln\underline{\mathcal{F}}(\mathfrak{t}) + ln\underline{\varphi}(\mathfrak{t}) = ln\Big[\underline{\mathcal{F}}(\mathfrak{t}),\overline{\mathcal{F}}(\mathfrak{t})\Big] + ln\Big[\underline{\varphi}(\mathfrak{t}),\overline{\varphi}(\mathfrak{t})\Big] \\ &= \Big[ln\underline{\mathcal{F}}(\mathfrak{t}), ln\overline{\mathcal{F}}(\mathfrak{t})\Big] + \Big[ln\underline{\varphi}(\mathfrak{t}), ln\overline{\varphi}(\mathfrak{t})\Big] = \Big[ln\underline{\mathcal{F}}(\mathfrak{t}) + ln\underline{\varphi}(\mathfrak{t}), ln\overline{\mathcal{F}}(\mathfrak{t}) + ln\overline{\varphi}(\mathfrak{t})\Big] \\ &= \Big[ln\Big(\underline{\mathcal{F}}(\mathfrak{t})\underline{\varphi}(\mathfrak{t})\Big), ln\Big(\overline{\mathcal{F}}(\mathfrak{t})\overline{\varphi}(\mathfrak{t})\Big)\Big]. \end{aligned} \tag{3.48}$$

Next, consider

$$\begin{aligned} &\ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big) = \ln\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big) + \ln\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big) \\ &= \ln\underline{\mathcal{F}}\Big(\frac{1}{2}\Big((\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + (\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)\Big)\Big) + \ln\underline{\varphi}\Big(\frac{1}{2}\Big((\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + (\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)\Big)\Big) \\ &\le \frac{1}{2}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ) - \ln\underline{\mathcal{F}}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|) \\ &+ \frac{1}{2}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ) - \ln\underline{\varphi}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|). \end{aligned} \tag{3.49}$$

Multiplying with $\beta^{\ell-1}$ on both sides of (3.49) and integrating the resulting inequality with respect to β over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 \beta^{\ell-1}\ln\Big(\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})\Big)d\beta \\ &\le \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta + \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta - \int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|)d\beta \\ &+ \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta + \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta - \int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|)d\beta. \end{aligned}$$

By changing of variables and after simple calculations, we get

$$\begin{aligned} &\ln\Big(\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big) \\ &\le \frac{\Gamma(\ell+1)\cdot2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ) + J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ) + J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\varphi}(\mathfrak{b}_\circ) + J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big] \\ &- \frac{\ell\cdot2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t} + \int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t}\Big]. \end{aligned}$$

Thus, it follows that

$$
\begin{aligned}
&exp\Big\{\ln\Big(\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big)\Big\}\\
&\leq exp\Big\{\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big]\\
&-\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t}+\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}\ln\underline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathfrak{t}\Big)\Big\},
\end{aligned}
$$

or

$$
\begin{aligned}
&\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\underline{\mathcal{F}}\underline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big)^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&\leq\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}.
\end{aligned}
\tag{3.50}
$$

Similarly,

$$
\begin{aligned}
&\overline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\overline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\overline{\mathcal{F}}\overline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big)^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&\geq\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}.
\end{aligned}
\tag{3.51}
$$

Combining (3.50) and (3.51), we get the interval

$$
\begin{aligned}
&\Big[\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\underline{\mathcal{F}}\underline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big)^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},\\
&\overline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\overline{\varphi}\Big(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}\Big)\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\overline{\mathcal{F}}\overline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Big)^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Big]\\
&\supseteq\Big[\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},\\
&\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Big].
\end{aligned}
\tag{3.52}
$$

To prove the second part, we consider the RHS of (3.50). Accordingly, we have:

$$
\begin{aligned}
&\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_{*}J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\\
&=exp\Big\{\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big]\Big\}\\
&=exp\Big\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta\\
&+\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta\Big]\Big\}
\end{aligned}
$$

$$\leq exp\Bigg\{\frac{\ell}{2}\Bigg[\frac{\ln(\underline{\mathcal{F}}(\mathfrak{a}_\circ)\underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\ell}-\frac{2^{\ell+1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\ln\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)d\mathfrak{t}\Big]$$
$$+\frac{\ln(\underline{\varphi}(\mathfrak{a}_\circ)\underline{\varphi}(\mathfrak{b}_\circ))}{\ell}-\frac{2^{\ell+1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\ln\Big((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)d\mathfrak{t}\Big]\Bigg]\Bigg\}.$$

It follows that

$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\leq G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}$$
$$\times G\Big(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \tag{3.53}$$

Similarly,

$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\geq G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}$$
$$\times G\Big(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\overline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \tag{3.54}$$

From (3.53) and (3.54), we get the interval

$$\Bigg[\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},$$
$$\Big[{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)\cdot{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^{\ell}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)\Big]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Bigg]$$
$$\supseteq\Bigg[G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}$$
$$\times G\Big(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},$$
$$G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\overline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}$$
$$\times G\Big(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ)\Big)\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)^{d\mathfrak{t}}\Bigg]^{\frac{-\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}\Bigg]. \tag{3.55}$$

Combining (3.52) and (3.55), we get the required result. □

Remark 3.18. *If we set $\ell = 1$ in Theorem 3.17, we attain*

$$\begin{aligned}
&\mathcal{F}(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2})\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big(\mathcal{F}\varphi(|\mathfrak{t} - \frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}|)\Big)^{\mathrm{dt}}\Big)^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}} \\
&\supseteq \Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathcal{F}(\mathfrak{t}))^{\mathrm{dt}}\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\varphi(\mathfrak{t}))^{\mathrm{dt}}\Big)^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}} \\
&\supseteq G(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ))\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\mathcal{F}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\mathcal{F}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\Big)^{\mathrm{dt}}\Big]^{\frac{-2}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}} \\
&\times G(\varphi(\mathfrak{a}_\circ), \varphi(\mathfrak{b}_\circ))\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\varphi(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\varphi(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\Big)^{\mathrm{dt}}\Big]^{\frac{-2}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}.
\end{aligned}$$

Theorem 3.19. *Let $\mathcal{F}, \varphi : \mathrm{J} \to \mathcal{R}_I^+$ be a multiplicatively superquadratic $\mathcal{IVF}$ and multiplicatively subquadratic $\mathcal{IVF}$, respectively, on* J*, such that*

$$\begin{aligned}
\mathcal{F}(\mathfrak{t}) &= [\underline{\mathcal{F}}(\mathfrak{t}), \overline{\mathcal{F}}(\mathfrak{t})] \\
\varphi(\mathfrak{t}) &= [\underline{\varphi}(\mathfrak{t}), \overline{\varphi}(\mathfrak{t})],
\end{aligned}$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}}, \underline{\varphi}, \overline{\varphi} : \mathrm{J} \to \mathcal{R}^+$. If $\mathcal{F}$ and φ are interval *integrable on J*, then*

$$\begin{aligned}
&\frac{\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\varphi(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\mathcal{F}(|\mathfrak{t} - \frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{\mathrm{dt}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\varphi(|\mathfrak{t} - \frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{\mathrm{dt}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\supseteq \left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\mathcal{F})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathcal{F})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\varphi)(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\varphi)(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}} \\
&\supseteq \frac{G\big(\mathcal{F}(\mathfrak{a}_\circ), \mathcal{F}(\mathfrak{b}_\circ)\big)}{G(\varphi(\mathfrak{a}_\circ), \varphi(\mathfrak{b}_\circ))}\cdot\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\varphi(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\varphi(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{\mathrm{dt}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}((\mathcal{F}(|\mathfrak{t} - \mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\mathcal{F}(|\mathfrak{b}_\circ - \mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}})^{\mathrm{dt}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},
\end{aligned}$$

holds, $\forall\ \mathfrak{a}_\circ, \mathfrak{b}_\circ \in \mathrm{J}$.

Proof. Let $\mathcal{F}$ and φ be multiplicatively superquadratic and multiplicatively subquadratic $\mathcal{IVF}$ respectively on J, and $\underline{\mathcal{F}}$ and $\underline{\varphi}$ be multiplicative superquadratic and multiplicative subquadratic functions, respectively, so we have

$$\begin{aligned}
&\ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\right) = \ln\underline{\mathcal{F}}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big) - \ln\underline{\varphi}\Big(\frac{\mathfrak{a}_\circ + \mathfrak{b}_\circ}{2}\Big) \\
&= \ln\underline{\mathcal{F}}\Big(\frac{1}{2}\Big((\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + (\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)\Big)\Big) - \ln\underline{\varphi}\Big(\frac{1}{2}\Big((\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + (\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ)\Big)\Big) \\
&\leq \frac{1}{2}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) + \frac{1}{2}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ) - \ln\underline{\mathcal{F}}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|) \\
&- \frac{1}{2}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ + \frac{2-\beta}{2}\mathfrak{b}_\circ) - \frac{1}{2}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ + \frac{2-\beta}{2}\mathfrak{a}_\circ) + \ln\underline{\varphi}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ - \mathfrak{b}_\circ)|). \qquad (3.56)
\end{aligned}$$

Multiplying with $\beta^{\ell-1}$ on both sides of (3.56) and integrating the resulting inequality with respect to β over [0, 1], we get

$$\begin{aligned}
&\int_0^1 \beta^{\ell-1} \ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\right)d\beta\\
&\leq \frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta+\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta\\
&-\int_0^1 \beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|)d\beta-\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta\\
&-\frac{1}{2}\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta+\int_0^1 \beta^{\ell-1}\ln\underline{\varphi}(\frac{1}{2}|(\beta-1)(\mathfrak{a}_\circ-\mathfrak{b}_\circ)|)d\beta.
\end{aligned}$$

After simple calculations and with change of variables, we obtain

$$\begin{aligned}
&\ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\right)\leq\frac{\ell\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg[\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\mathcal{F}}(\mathrm{t})d\mathrm{t}+\int_{\mathfrak{a}_\circ}^{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\mathcal{F}}(\mathrm{t})d\mathrm{t}\Big)\\
&-\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\varphi}(\mathrm{t})d\mathrm{t}+\int_{\mathfrak{a}_\circ}^{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}\ln\underline{\varphi}(\mathrm{t})d\mathrm{t}\Big)\Bigg]\\
&-\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}-\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\varphi}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}\Bigg).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\right)\\
&\leq\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg[J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)-J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)-J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Bigg]\\
&+\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\varphi}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}-\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}\Bigg].
\end{aligned}$$

It follows that

$$\begin{aligned}
&exp\left\{\ln\left(\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\right)\right\}\leq exp\left\{\frac{\Gamma(\ell+1)\cdot 2^{\ell-1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Bigg[\Big(J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)\Big)\right.\\
&-\Big(J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^+}\ln\underline{\varphi}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^-}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big)\Bigg]+\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\varphi}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}\\
&\left.-\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}\ln\underline{\mathcal{F}}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|)d\mathrm{t}\Big)\right\}.
\end{aligned}$$

Thus, we have

$$\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\mathcal{F}}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}}\Big)^{d\mathrm{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\Big((\underline{\varphi}(|\mathrm{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}}\Big)^{d\mathrm{t}}}\right]^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$

$$\leq \left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}. \tag{3.57}$$

Similarly,

$$\frac{\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}\right]^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$\geq \left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}. \tag{3.58}$$

Combining (3.57) and (3.58), we get the interval

$$\Bigg[\frac{\underline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\underline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}\right]^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},$$
$$\frac{\overline{\mathcal{F}}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\overline{\varphi}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\mathcal{F}}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{\mathrm{dt}}}\right]^{\frac{\ell\cdot 2^{\ell}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Bigg]$$
$$\supseteq \Bigg[\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}},$$
$$\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\Bigg]. \tag{3.59}$$

Next, considering the RHS of (3.59),

$$\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J^{\ell}_{*}(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_{*} J^{\ell}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}$$
$$= exp\left\{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}\Big[J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^{+}}\ln\underline{\mathcal{F}}(\mathfrak{b}_\circ)+J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^{-}}\ln\underline{\mathcal{F}}(\mathfrak{a}_\circ)-J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^{+}}\ln\underline{\varphi}(\mathfrak{b}_\circ)-J^{\ell}_{(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})^{-}}\ln\underline{\varphi}(\mathfrak{a}_\circ)\Big]\right\}$$
$$= exp\Bigg\{\frac{\ell}{2}\Big[\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta+\int_0^1\beta^{\ell-1}\ln\underline{\mathcal{F}}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta$$
$$-\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{a}_\circ+\frac{2-\beta}{2}\mathfrak{b}_\circ)d\beta-\int_0^1\beta^{\ell-1}\ln\underline{\varphi}(\frac{\beta}{2}\mathfrak{b}_\circ+\frac{2-\beta}{2}\mathfrak{a}_\circ)d\beta\Big]\Bigg\}$$
$$\leq exp\Bigg\{\frac{\ell}{2}\Bigg[\frac{\ln(\underline{\mathcal{F}}(\mathfrak{a}_\circ)\underline{\mathcal{F}}(\mathfrak{b}_\circ))}{\ell}-\frac{2^{\ell+1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\ln\Big((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^{\ell}}\cdot(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)\mathrm{dt}\Big)$$

$$-\frac{\ln(\underline{\varphi}(\mathfrak{a}_\circ)\underline{\varphi}(\mathfrak{b}_\circ))}{\ell}+\frac{2^{\ell+1}}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}\Big(\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\ln\Big((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}\cdot(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\Big)d\mathfrak{t}\Big)\Big]\Big\}$$

$$=\frac{G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)}{G(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \tag{3.60}$$

Similarly,

$$\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\geq\frac{G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)}{G(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}. \tag{3.61}$$

Combining (3.60) and (3.61), we get the interval

$$\left[\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\underline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\underline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\underline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}},\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\overline{\mathcal{F}})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\mathcal{F}})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\overline{\varphi})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\overline{\varphi})(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}\right]$$
$$\supseteq\left[\frac{G\Big(\underline{\mathcal{F}}(\mathfrak{a}_\circ),\underline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)}{G(\underline{\varphi}(\mathfrak{a}_\circ),\underline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\underline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\underline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}},\right.$$
$$\left.\frac{G\Big(\overline{\mathcal{F}}(\mathfrak{a}_\circ),\overline{\mathcal{F}}(\mathfrak{b}_\circ)\Big)}{G(\overline{\varphi}(\mathfrak{a}_\circ),\overline{\varphi}(\mathfrak{b}_\circ))}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\overline{\varphi}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\overline{\varphi}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\underline{\mathcal{F}}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})^\ell}(\overline{\mathcal{F}}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathfrak{t})^{\ell-1}}\right)^{d\mathfrak{t}}}\right]^{\frac{\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}\right]. \tag{3.62}$$

Merging the results (3.59) and (3.62), we get the required result. □

Remark 3.20. *If we set $\ell = 1$ in Theorem 3.19, we attain*

$$\frac{\mathcal{F}(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}{\varphi(\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2})}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathcal{F}(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\varphi(|\mathfrak{t}-\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}|))^{d\mathfrak{t}}}\right]^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}\supseteq\left(\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{f}(\mathfrak{t}))^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}(\mathfrak{g}(\mathfrak{t}))^{d\mathfrak{t}}}\right)^{\frac{2}{\mathfrak{b}_\circ-\mathfrak{a}_\circ}}$$
$$\supseteq\frac{G(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ))}{G(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ))}\left[\frac{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\varphi(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\varphi(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\right)^{d\mathfrak{t}}}{\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathfrak{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathfrak{t})}(\mathcal{F}(|\mathfrak{b}_\circ-\mathfrak{t}|))^{(\mathfrak{t}-\mathfrak{a}_\circ)}\right)^{d\mathfrak{t}}}\right]^{\frac{2}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^2}}. \tag{3.63}$$

Corollary 3.21. *Let $\mathcal{F},\varphi:\mathrm{J}\to\mathcal{R}_I^+$ be a multiplicatively superquadratic $\mathcal{IVF}$ and multiplicatively convex $\mathcal{IVF}$, respectively, on J, such that*

$$\mathcal{F}(\mathfrak{t})=[\underline{\mathcal{F}}(\mathfrak{t}),\overline{\mathcal{F}}(\mathfrak{t})]$$
$$\varphi(\mathfrak{t})=[\underline{\varphi}(\mathfrak{t}),\overline{\varphi}(\mathfrak{t})],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}}, \underline{\varphi}, \overline{\varphi} : \mathrm{J} \to \mathcal{R}^+$. *If* $\mathcal{F}$ *and* φ *are interval* **integrable on* J*, then*

$$\left[\frac{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\mathcal{F})(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\mathcal{F})(\mathfrak{a}_\circ)}{{}_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}J_*^\ell(\varphi)(\mathfrak{b}_\circ)\cdot_* J^\ell_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}(\varphi)(\mathfrak{a}_\circ)}\right]^{\frac{2^{\ell-1}\cdot\Gamma(\ell+1)}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^\ell}}$$
$$\leq \frac{G\Big(\mathcal{F}(\mathfrak{a}_\circ),\mathcal{F}(\mathfrak{b}_\circ)\Big)}{G(\varphi(\mathfrak{a}_\circ),\varphi(\mathfrak{b}_\circ))}\Big[\int_{\frac{\mathfrak{a}_\circ+\mathfrak{b}_\circ}{2}}^{\mathfrak{b}_\circ}\left((\mathcal{F}(|\mathrm{t}-\mathfrak{a}_\circ|))^{(\mathfrak{b}_\circ-\mathrm{t})^\ell}\cdot(\mathcal{F}(|\mathfrak{b}_\circ-\mathrm{t}|))^{(\mathrm{t}-\mathfrak{a}_\circ)(\mathfrak{b}_\circ-\mathrm{t})^{\ell-1}}\right)^{d\mathrm{t}}\Big]^{\frac{-\ell\cdot 2^\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}.$$

holds.

4. Graphical and simulation analysis

To verify the efficacy of the generated results, we carry out a thorough numerical study in this part. We evaluate the suggested method's precision and effectiveness in approximating integrals of multiplicatively superquadratic $\mathcal{IVF}$s with a battery of numerical tests. Our objectives are to assess the effectiveness of $\mathcal{H}.\mathcal{H}$'s type inequalities in a variety of circumstances and provide tangible numerical data to support the theoretical conclusions. First, we consider examples to explain how the functions are multiplicative superquadratic $\mathcal{IVF}$s.

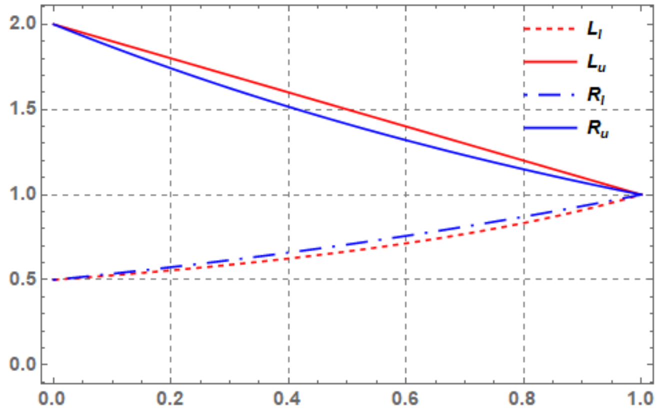
Example 4.1. *Let us examine the multiplicatively* $\mathcal{IVF}$*,* $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \subseteq \mathcal{R}^+ \to \mathcal{R}_I^+$*, given by*

$$\mathcal{F}(\mathrm{t}) = \left[exp\{\mathrm{t}^{2.5}\}, exp\{\sqrt{\mathrm{t}}\}\right], \quad \forall\ \mathrm{t} \in [\mathfrak{a}_\circ, \mathfrak{b}_\circ]. \tag{4.1}$$

Since the endpoint functions in (4.1) *are defined as* $\underline{\mathcal{F}}(\mathrm{t}) = \exp\left(\mathrm{t}^{2.5}\right)$ *and* $\overline{\mathcal{F}}(\mathrm{t}) = \exp\left(\sqrt{\mathrm{t}}\right)$*, the subsequent graphs and table constructed based on the values of* L_{l}*,* L_{u}*,* R_{l}*, and* R_{u} *verify that* $\mathcal{F}(\mathrm{t})$ *is a multiplicatively superquadratic* $\mathcal{IVF}$ *for all* $\in [0, 1]$*. Here,* L_{l} *and* L_{u} *denote the lower and upper bounds of the interval on the left term of* (3.6)*, while* R_{l} *and* R_{u} *represent the corresponding bounds on the right term. It is important to note from Figure 1(b) that the dotted and solid blue curves lie between the corresponding dotted and solid red curves. This indicates that the left term of inclusion* (3.6) *is a superset of its right term. Likewise, Figure 1(a) shows that the values associated with* R_{l} *and* R_{u} *remain between the values corresponding to* L_{l}*, and* L_{u}*, verifying that* $[L_{\mathrm{l}}, L_{\mathrm{u}}] \supseteq [R_{\mathrm{l}}, R_{\mathrm{u}}]$*. Figure 1(a),(b) confirm the validity of Theorem 3.6.*

	$L_{\mathfrak{l}}$	$L_{\mathfrak{u}}$	$R_{\mathfrak{l}}$	$R_{\mathfrak{u}}$
0.1	0.52	1.9	0.53	1.86
0.2	0.55	1.8	0.57	1.74
0.3	0.58	1.7	0.61	1.62
0.4	0.62	1.6	0.65	1.51
0.5	0.66	1.5	0.70	1.41
0.6	0.71	1.4	0.75	1.31
0.7	0.76	1.3	0.81	1.23
0.8	0.83	1.2	0.87	1.14
0.9	0.90	1.1	0.93	1.07
1.0	1.00	1.0	1.00	1.00

(a) Numerical illustration of Theorem 3.6.



(b) Graphical illustration of Theorem 3.6.

Figure 1. Numerical and graphical illustration of $\mathcal{F}(\mathfrak{t}) = \left[exp\{\mathfrak{t}^{2.5}\}, exp\{\sqrt{\mathfrak{t}}\}\right]$ as multiplicative superquadratic $\mathcal{IVF}$s via Theorem 3.6 for $\mathfrak{a}_\circ = 0, \mathfrak{b}_\circ = 1$ and $\in [0,1]$.

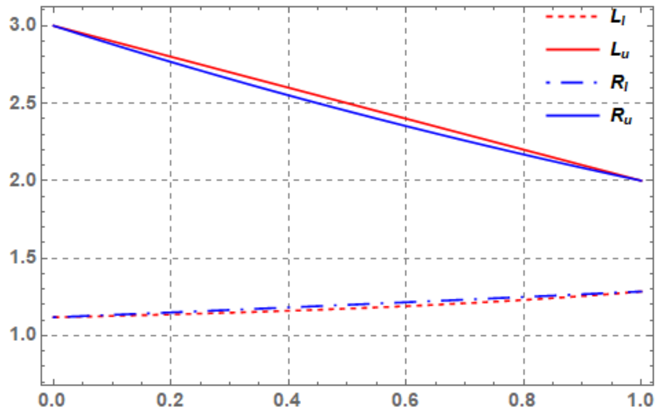
Example 4.2. *Let us examine the* $\mathcal{IVF}$, $\mathcal{F} : [\mathfrak{a}_\circ, \mathfrak{b}_\circ] \subseteq [2,\infty) \to \mathcal{R}_I^+$, *given by*

$$\mathcal{F}(\mathfrak{t}) = \left[exp\{\mathfrak{t}^3\}, exp\{\mathfrak{t}\}\right], \quad \forall\, \mathfrak{t} \in [\mathfrak{a}_\circ, \mathfrak{b}_\circ]. \tag{4.2}$$

Since the endpoint functions in (4.2) *are given by* $\underline{\mathcal{F}}(\mathfrak{t}) = \exp\left(\mathfrak{t}^3\right)$ *and* $\overline{\mathcal{F}}(\mathfrak{t}) = \exp(\mathfrak{t})$, *the subsequent graphs and table constructed based on the values of* $L_{\mathfrak{l}}$, $L_{\mathfrak{u}}$, $R_{\mathfrak{l}}$, *and* $R_{\mathfrak{u}}$ *demonstrate that* $\mathcal{F}(\mathfrak{t})$ *is a multiplicatively superquadratic* $\mathcal{IVF}$ *for all* $\in [0,1]$. *Here,* $L_{\mathfrak{l}}$ *and* $L_{\mathfrak{u}}$ *denote the lower and upper bounds of the interval on the LHS of* (3.6), *while* $R_{\mathfrak{l}}$ *and* $R_{\mathfrak{u}}$ *correspond to the respective bounds on the RHS. The numerical and graphical representations are provided below (see Figure 2):*

	$L_{\mathfrak{l}}$	$L_{\mathfrak{u}}$	$R_{\mathfrak{l}}$	$R_{\mathfrak{u}}$
0.1	1.12	2.9	1.13	2.88
0.2	1.13	2.8	1.14	2.76
0.3	1.14	2.7	1.16	2.65
0.4	1.15	2.6	1.18	2.55
0.5	1.17	2.5	1.19	2.44
0.6	1.18	2.4	1.21	2.35
0.7	1.20	2.3	1.23	2.25
0.8	1.22	2.2	1.24	2.16
0.9	1.25	2.1	1.26	2.08
1.0	1.28	2.0	1.28	2.00

(a) Numerical illustration of Theorem 3.6.



(b) Graphical illustration of Theorem 3.6.

Figure 2. Numerical and graphical illustration of $\mathcal{F}(\mathfrak{t}) = [exp\{\mathfrak{t}^3\}, exp\{\mathfrak{t}\}]$ via Theorem 3.6 for $\mathfrak{a}_\circ = 0, \mathfrak{b}_\circ = 1$ and $\in [0,1]$.

The following example provides the authenticity of Theorem 3.9.

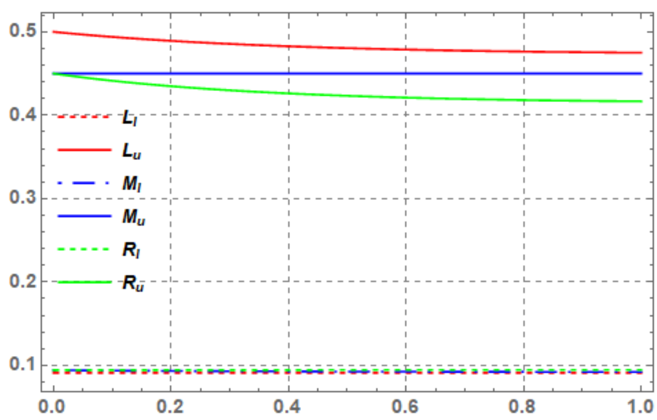
Example 4.3. *Let us examine the same multiplicatively superquadratic $\mathcal{IVF}$ as shown by Example 4.1, given by*

$$\mathcal{F}(\mathfrak{t}) = \left[exp\{\mathfrak{t}^{2.5}\}, exp\{\sqrt{\mathfrak{t}}\}\right], \quad \forall\, \mathfrak{t} \in [\mathfrak{a}_\circ, \mathfrak{b}_\circ]. \tag{4.3}$$

Since the endpoint functions in (4.3) *are given by* $\underline{\mathcal{F}}(\mathfrak{t}) = \frac{1}{\mathfrak{t}}$ *and* $\overline{\mathcal{F}}(\mathfrak{t}) = \mathfrak{t}$, *the corresponding graphs constructed using the values of* $L_{\mathfrak{l}}$, $L_{\mathfrak{u}}$, $M_{\mathfrak{l}}$, $M_{\mathfrak{u}}$, $R_{\mathfrak{l}}$, *and* $R_{\mathfrak{u}}$ *confirm the validity of Theorem 3.9. Here,* $L_{\mathfrak{l}}$ *and* $L_{\mathfrak{u}}$ *represent the lower and upper bounds of the interval associated with the left term in the statement of Theorem 3.9;* $M_{\mathfrak{l}}$ *and* $M_{\mathfrak{u}}$ *denote the bounds for the middle term; and* $R_{\mathfrak{l}}$ *and* $R_{\mathfrak{u}}$ *correspond to the bounds of the RHS term in the same theorem. It is important to note from Figure 3(b) that the dotted and solid green curves lie between the corresponding dotted and solid blue curves, while the blue curves lie between red curves. This indicates that the left term of inclusion of Theorem 3.9 is a superset of its middle term, and the middle term is the superset of its right term. Likewise, Figure 3(a) shows that the values associated with* $R_{\mathfrak{l}}$ *and* $R_{\mathfrak{u}}$ *remain between the values corresponding to* $M_{\mathfrak{l}}$, *and* $M_{\mathfrak{u}}$, *while* $M_{\mathfrak{l}}$, *and* $M_{\mathfrak{u}}$ *lie between* $L_{\mathfrak{l}}$, *and* $L_{\mathfrak{u}}$ *verifying that* $[L_{\mathfrak{l}}, L_{\mathfrak{u}}] \supseteq [M_{\mathfrak{l}}, M_{\mathfrak{u}}] \supseteq [R_{\mathfrak{l}}, R_{\mathfrak{u}}]$. *Thus Figure 3(a),(b) confirm the validity of Theorem 3.9.*

	$L_{\mathfrak{l}}$	$L_{\mathfrak{u}}$	$M_{\mathfrak{l}}$	$M_{\mathfrak{u}}$	$R_{\mathfrak{l}}$	$R_{\mathfrak{u}}$
0.1	0.12	0.54	0.09	0.54	0.09	0.45
0.2	0.12	0.53	0.09	0.53	0.09	0.45
0.3	0.12	0.53	0.09	0.53	0.09	0.45
0.4	0.12	0.53	0.09	0.53	0.09	0.45
0.5	0.12	0.53	0.09	0.53	0.09	0.45
0.6	0.12	0.52	0.09	0.52	0.09	0.45
0.7	0.12	0.52	0.09	0.52	0.09	0.45
0.8	0.12	0.52	0.09	0.52	0.09	0.45
0.9	0.12	0.52	0.09	0.52	0.09	0.45
1.0	0.12	0.52	0.09	0.52	0.09	0.45

(a) Numerical illustration of Theorem 3.9.



(b) Graphical illustration of Theorem 3.9.

Figure 3. Numerical and Graphical illustration of $\left[exp\{\mathfrak{t}^{2.5}\}, exp\{\sqrt{\mathfrak{t}}\}\right]$ via Theorem 3.9 for $\mathfrak{a}_\circ = 0, \mathfrak{b}_\circ = 1$ and $\ell \in [0, 1]$.

5. Applications

In this section, we demonstrate the practical significance of the theoretical findings by applying them to a range of mathematical constructs. In particular, we focus on classical and generalized special means, examining the related inequalities.

Now we describe some formulas of special means and then take into account the multiplicatively superquadrati $\mathcal{IVF}$ for Theorem 3.9, demonstrating that how the linear combination of special means are associated with each other:

Let $\mathfrak{a}_o < \mathfrak{b}_o$ and $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{R}$, considering the following special means.

(1) Arithmetic mean:

$$\mathrm{A}(\mathfrak{a}_o, \mathfrak{b}_o) = \frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}. \tag{5.1}$$

(2) $\mathfrak{p}$-logarithmic mean:

$$\mathrm{L}_{\mathfrak{p}}(\mathfrak{a}_o, \mathfrak{b}_o) = \left[\frac{1}{\mathfrak{p}+1} \cdot \frac{\mathfrak{b}_o{}^{(\mathfrak{p}+1)} - \mathfrak{a}_o{}^{(\mathfrak{p}+1)}}{(\mathfrak{b}_o - \mathfrak{a}_o)}\right]^{\frac{1}{\mathfrak{p}}}, \tag{5.2}$$

where $\mathfrak{p} \in \mathrm{Z}\backslash\{-1, 0\}$, $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{R}$, and $\mathfrak{a}_o \neq \mathfrak{b}_o$.
(3) Geometric mean:

$$\mathrm{G}(\mathfrak{a}_o, \mathfrak{b}_o) = \sqrt{\mathfrak{a}_o \mathfrak{b}_o}. \tag{5.3}$$

Proposition 5.1. *The inequality*

$$\begin{aligned}
&\left[exp\left\{A^{\mathfrak{p}}(\mathfrak{a}_\circ, \mathfrak{b}_\circ) + \frac{2^{1+\mathfrak{p}}A^{1+\mathfrak{p}}(-\mathfrak{a}_\circ, \mathfrak{b}_\circ)\Gamma(1+\ell)\Gamma(1+\mathfrak{p})}{\Gamma(1+\ell+\mathfrak{p})}\right\}, exp\left\{A(\mathfrak{a}_\circ, \mathfrak{b}_\circ) + \frac{2^2A^2(-\mathfrak{a}_\circ, \mathfrak{b}_\circ)\Gamma(1+\ell)\Gamma(2)}{\Gamma(2+\ell)}\right\}\right] \\
&\supseteq \left[\left({}_{\mathfrak{a}_\circ}\mathrm{J}^{\ell}_{*}(\mathfrak{b}_\circ)^{\mathfrak{p}} \cdot_* \mathrm{J}^{\ell}_{\mathfrak{b}_\circ}(\mathfrak{a}_\circ)^{\mathfrak{p}}\right)^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}, \left({}_{\mathfrak{a}_\circ}\mathrm{J}^{\ell}_{*}(\mathfrak{b}_\circ) \cdot_* \mathrm{J}^{\ell}_{\mathfrak{b}_\circ}(\mathfrak{a}_\circ)\right)^{\frac{\Gamma(\ell+1)}{2(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell}}}\right] \\
&\supseteq \left[\frac{G(\mathfrak{a}_\circ^{\mathfrak{p}}, \mathfrak{b}_\circ^{\mathfrak{p}})}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((|\mathfrak{b}_\circ - \mathrm{t}|)^{\mathfrak{p}(\mathrm{t}-\mathfrak{a}_\circ)^{\ell}} \cdot (|\mathrm{t} - \mathfrak{a}_\circ|)^{\mathfrak{p}(\mathfrak{b}_\circ-\mathrm{t})(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}},\right. \\
&\left.\frac{G(\mathfrak{a}_\circ, \mathfrak{b}_\circ)}{\left[\int_{\mathfrak{a}_\circ}^{\mathfrak{b}_\circ}\left((|\mathfrak{b}_\circ - \mathrm{t}|)^{(\mathrm{t}-\mathfrak{a}_\circ)^{\ell}} \cdot (|\mathrm{t} - \mathfrak{a}_\circ|)^{(\mathfrak{b}_\circ-\mathrm{t})(\mathrm{t}-\mathfrak{a}_\circ)^{\ell-1}}\right)^{\mathrm{dt}}\right]^{\frac{\ell}{(\mathfrak{b}_\circ-\mathfrak{a}_\circ)^{\ell+1}}}}\right],
\end{aligned}$$

holds for all $[\mathfrak{a}_\circ, \mathfrak{b}_\circ] \subset [0, 1]$.

Proof. Theorem 3.6 can be utilized to demonstrate the outcome for a multiplicatively superquadratic function $\mathcal{F}(\mathrm{t}) = [exp\{\mathrm{t}^{\mathfrak{p}}\}, exp\{\mathrm{t}\}]$ for $\mathfrak{p} \geq 2$. □

6. Conclusions

In this study, we introduced and systematically explored a novel class of functions, namely multiplicatively superquadratic $\mathcal{IVF}$, within the framework of multiplicative calculus. By employing interval order relations and the $\mathcal{R}.\mathcal{L}$ fractional integral operators, we successfully established new $\mathcal{H}.\mathcal{H}$-type inequalities of fractional order for these functions. The theoretical development was further extended to encompass fractional integral inequalities for the product and quotient of multiplicatively superquadratic and subquadratic $\mathcal{IVF}$s, highlighting the structural richness of this function class under multiplicative operations.

Our findings exhibit a seamless reduction to integer-order results when the fractional parameter is set to $\alpha = 1$, thereby demonstrating the generality and robustness of the proposed approach. The validity and applicability of the results have been substantiated through a series of well-chosen examples, supported by numerical and graphical illustrations. Moreover, the applications presented in terms of linear combinations of special means not only validate the theoretical constructs but also emphasize their practical significance in the broader context of convexity and mean value analysis.

This work contributes a new direction to the growing field of fractional multiplicative calculus by unifying interval analysis, superquadraticity, and fractional integration in a coherent and original framework. The theoretical advancements introduced here are expected to inspire further investigations into related classes of functions, integral inequalities, and their applications across mathematical modeling, optimization, and information theory. In the future researchers may extend these ideas to stochastic settings, multidimensional interval-valued functions, and generalized fractional operators, thereby broadening the horizons of multiplicative convex analysis and its interdisciplinary applications.

Author contributions

G.J: Conceptualization, investigation, supervision; D.K: Formal analysis, writing-original draft & editing, methodology; S.I.B: Software, writing-review, supervision; Y.S: Formal analysis, supervision, funding acquisition; D.K: Data curation, validation, visualization, and literature review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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