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Research article

Analysis of inclusions using s-type convex interval-valued functions via Riemann-Liouville fractional integrals

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Abstract: This paper examines a family of convex interval-valued (IVC) functions using the Riemann-Liouville (RL) integrals. Hermite-Hadamard (HH) and Hermite-Hadamard-Fejér (HHF) type inclusions are developed by employing the s-type convexity of interval-valued (IV) functions. Some inclusions for the product of s-type (IVC) functions are also established involving RL integrals. All main results are furthur refined into inclusions and inequalities for s-type IVC functions and s-type convex point-valued functions, respectively, involving the ordinary integral. In addition, several consequences of the primary results are explored demonstrating the connections between point-valued convex functions, IVC functions, and s-type IVC functions. Each key conclusion is validated numerically. The results of this paper might open the path for new avenues in modeling, optimization problems, interval differential equations, and fuzzy IV functions that involve both discrete and continuous variables simultaneously.

Keywords: convex functions; fractional integrals; interval-valued functions; integral inclusions; optimization problems; mathematical operators
Mathematics Subject Classification: 26A33, 26D07, 26E25, 28B20, 65G40

1. Introduction

Fractional calculus serves as a comprehensive extension of integer-order calculus offering enhanced capabilities that surpass those of traditional integer-order methods. Many of the greatest mathematicians in history turned their attention to fractional calculus following the ground breaking investigations of Leibniz and L'Hopital. Famous scientists such as Fourier, Euler, and Laplace studied fractional calculus in great and conducted experiments demonstrating its deep mathematical implications. The subject has diverse applications across numerous areas of science and engineering, e.g., optics [1], biological population models [2], image processing [3], electrochemistry [4], mathematics [5], physics [6], fluid mechanics [7], viscoelasticity [8, 9] and electromagnetism [10]. In a variety of mathematical and physical contexts, the fractional Fourier transform (FrFT) has been introduced and modified multiple times [1]. There are studies that investigate the diverse applications of the fractional Laplace transform in solving three distinct types of mathematical equations: those related to the study of the heat conduction equation, ODEs and electric circuits [11]. Moreover, fractional operators and fractional differential equations have been extensively applied across numerous branches of natural science. Notable terminologies and ideas have been introduced for fractional operators [12]. Convexity is a fundamental concept that plays a crucial role in addressing a broad spectrum of challenges in both pure and applied sciences. In recent years, many scholars have focused on investigating the properties and inequalities associated with convexity from both theoretical and practical perspectives.

Initially, notable attention has been directed towards two prominent inequalities related to convex maps: the HH type inequality and its weighted version HHF inequality. The HH type inequality was first presented by Hermite in Mathesis in 1883 and was later independently proved by Hadamard in 1893 [13], before being further generalized by Fejèr in 1906 [14]. The HH double inequality has a general geometric interpretation and numerous applications across a wide range of specific inequalities.

A lengthy history exists for the theory of interval analysis, beginning with Archimedes' work on calculating the perimeter of a circle. However, it remained out of focus for a long time but meaningful advancements in this area began in the 1950s. Ramon E. Moore is often considered as a pioneer of interval analysis. He published his first major textbook in 1966 on the subject aiming to compute numerical error bounds for solutions of finite-state machines. Since then, numerous researchers have devoted attention to studying interval analysis and IV functions, exploring both their mathematical foundations and practical applications. Numerous analysts have dedicated their efforts to explore interval analysis and IV functions within mathematics and their practical applications [12, 15, 16].

In the generation and analysis of fractals via s-convexity, the escape criterion plays a vital role. Escape criteria for polynomials of different orders have been developed in [17–19], involving terms such as ϑ^s and $(1-\vartheta)^s$. Solving these expressions for ϑ requires applying the binomial expansion while retaining only the linear terms $1-s\vartheta$ and $1-s(1-\vartheta)$ which led to the development of a new type of convexity known as s-type convexity introduced by Rashid et al. [20]. Motivated by this work, we introduce s-type inclusions to reduce the uncertainty arising from errors caused by truncating higher-order terms in the binomial expansion, thereby obtaining more reliable outcomes.

In [16], Xuelong Liu et al. presented a fractional HH type inclusion for IVC functions as follows:

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \supseteq \frac{\Gamma(\varpi+1)}{2(\rho_2-\rho_1)^{\varpi}}\,[J^{\varpi}_{\rho_1+}\Lambda(\rho_2)+J^{\varpi}_{\rho_2-}\Lambda(\rho_1)] \supseteq \Big(\frac{\Lambda(\rho_2)+\Lambda(\rho_1)}{2}\Big).$$

If $\underline{\Lambda}=\overline{\Lambda}=\varpi=1$. Then it reduces to HH type inequality for convex functions.

Various integral inclusions have been established by considering different types of convexity and employing various fractional operators. Abdeljawad et al. presented inclusions for p-convex IV functions using the Katugampola fractional integral in [21]. In [22], h-convexity is used to present new Jensen and HH-type inclusions by Zhao et al. In [23], Zhao et al. introduced and utilized the novel class of IV approximately h-convex functions to generalize HH-type inclusions. In [24], Macías-Díaz studied inclusions for $LR-p$-convexity. In [25], Zhao et al. used s-convex IV functions to study HH-type inclusions. Khan et al. [26] introduced and utilized $\hbar$–$\mathbb{GL}$ IV preinvex functions to generalize the HH- and HHF-type inclusions. Fahad et al. presented the class of k-harmonically IVC functions and studied applications of its inclusions in information science [27]. Fahad et al. [28] introduced the class of $GA-Cr$-convex IV functions and investigated its properties. In [29], Fahad et al. presented inequalities for $GA-Cr$-convex IV functions via IV Hadamard fractional integral operators and explored connections with information systems. Moreover, Bhardwaj et al. [30] investigated IV vector optimization problems, while Bhat et al. presented optimality conditions for IV optimization problems on Riemannian manifolds under a total order relation [31].

The goal of this paper is to introduce and study a new class of IVC functions, termed as s-type IVC functions. The HH-type inclusion, the HHF type inclusion and some inclusions for the product of functions involving the RL fractional integrals are also established using this new class. Special cases are presented that relate the main results to classically predefined ones, thereby validating their correctness. Additionally, the validity of the results is demonstrated through various examples. Overall, the findings obtained are innovative, robust, and broadly applicable.

2. Preliminaries

Let $\mathscr{H}$ be a space of closed intervals of $\mathbb{R}$ and $[\underline{\mathfrak{e}},\bar{\mathfrak{e}}],[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]\in\mathscr{H}$. Some properties of closed intervals expressed in [22] are as follows:
(1) Scalar multiplication:

$$\mathfrak{c}[\underline{\mathfrak{e}},\bar{\mathfrak{e}}]=\begin{cases}[\mathfrak{c}\cdot\underline{\mathfrak{e}},\mathfrak{c}\cdot\bar{\mathfrak{e}}] & \text{if } \mathfrak{c}>0,\\ 0 & \text{if } \mathfrak{c}=0,\\ [\mathfrak{c}\cdot\bar{\mathfrak{e}},\mathfrak{c}\cdot\underline{\mathfrak{e}}] & \text{if } \mathfrak{c}<0.\end{cases}$$

(2) Basic arithmetic operations:

$$[\underline{\mathfrak{e}},\bar{\mathfrak{e}}]+[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]=[\underline{\mathfrak{e}}+\underline{\mathfrak{f}},\bar{\mathfrak{e}}+\bar{\mathfrak{f}}],$$

$$[\underline{\mathfrak{e}},\bar{\mathfrak{e}}]-[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]=[\underline{\mathfrak{e}}-\underline{\mathfrak{f}},\bar{\mathfrak{e}}-\bar{\mathfrak{f}}],$$

$$[\underline{\mathfrak{e}},\bar{\mathfrak{e}}].[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]=[min\{\underline{\mathfrak{e}}\underline{\mathfrak{f}},\underline{\mathfrak{e}}\bar{\mathfrak{f}},\bar{\mathfrak{e}}\underline{\mathfrak{f}},\bar{\mathfrak{e}}\bar{\mathfrak{f}}\},max\{\underline{\mathfrak{e}}\underline{\mathfrak{f}},\underline{\mathfrak{e}}\bar{\mathfrak{f}},\bar{\mathfrak{e}}\underline{\mathfrak{f}},\bar{\mathfrak{e}}\bar{\mathfrak{f}}\}],$$

where $0\notin[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]$. Moreover,

$$[\underline{\mathfrak{e}},\bar{\mathfrak{e}}]\supseteq[\underline{\mathfrak{f}},\bar{\mathfrak{f}}]\iff\underline{\mathfrak{e}}\leq\underline{\mathfrak{f}},\bar{\mathfrak{e}}\geq\bar{\mathfrak{f}}.$$

Definition 2.1. *[32] Let $\varpi>0$ and $\mathfrak{R}\mathfrak{I}_{([\rho_1,\rho_2])}$ be the class of RL fractional integrable IV functions on $[\rho_1,\rho_2]$. The RL integrals of $\Lambda\in\mathfrak{R}\mathfrak{I}_{([\rho_1,\rho_2])}$ having order $\varpi>0$ are provided as:*

$$J^{\varpi}_{\rho_1+}\Lambda(\tau)=\frac{1}{\Gamma(\varpi)}\int_{\rho_1}^{\tau}(\tau-\ell)^{\varpi-1}\Lambda(\ell)d\ell,\quad\tau>\rho_1$$

and

$$J^{\varpi}_{\rho_2-}\Lambda(\tau)=\frac{1}{\Gamma(\varpi)}\int_{\tau}^{\rho_2}(\ell-\tau)^{\varpi-1}\Lambda(\ell)d\ell,\quad\tau<\rho_2$$

respectively, where $J^0_{\rho_1+}\Lambda(\tau) = J^0_{\rho_2-}\Lambda(\tau) = \Lambda(\tau)$.

Definition 2.2. *[20] Let* $\mathcal{B} \subseteq \mathbb{R}$ *be a closed interval. Then the s-type convex function* $\Lambda : \mathcal{B} \to \mathbb{R}$ *is defined as:*

$$\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) \leq (1 - s\vartheta)\Lambda(\rho_1) + (1 - s(1-\vartheta))\Lambda(\rho_2),$$

for all $\rho_1, \rho_2 \in \mathcal{B}$*, and* $\vartheta, s \in [0,1]$. *If the function is not s-type convex, then it is s-type concave and* $\leq$ *is replaced by* $\geq$ *in the above expression.*

Definition 2.3. *[33] Let* $\mathcal{B} \subseteq \mathbb{R}$ *be a closed interval, and* $\mathscr{H}^+$ *be the space of positive intervals of* $\mathbb{R}$ *that are closed. Then the IVC function* $\Lambda : \mathcal{B} \to \mathscr{H}^+$ *as*

$$\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) \supseteq \vartheta\Lambda(\rho_1) + (1-\vartheta)\Lambda(\rho_2),$$

for all $\rho_1, \rho_2 \in \mathcal{B}$*, and* $\vartheta, s \in [0,1]$. *If the function is not IV convex, then it is IV concave and* $\supseteq$ *is replaced by* $\subseteq$ *in the above expression.*

Definition 2.4. *[12] Consider an interval* $\mathcal{B} \subseteq \mathbb{R}$ *that is closed and* $\Lambda(\tau) = [\underline{\Lambda}(\tau), \overline{\Lambda}(\tau)]$ *be an IV function where* $\tau \in \mathcal{B}$. *Then* $\Lambda(\tau)$ *is Lebesgue integrable provided* $\underline{\Lambda}(\tau)$ *and* $\overline{\Lambda}(\tau)$ *are measurable as well as Lebesgue integrable over* $\mathcal{B}$*,*

$$\int_{\mathcal{B}} \Lambda(\tau)d\tau = \int_{\mathcal{B}} \underline{\Lambda}(\tau)d\tau + \int_{\mathcal{B}} \overline{\Lambda}(\tau)d\tau.$$

3. Main results

This section includes a set of new HH-type inclusions for s-type IVC functions by using the RL fractional integrals. Let $\mathcal{B} \subseteq \mathbb{R}$ be a closed interval, and $SC(\mathcal{B}, \Lambda^+)$ denotes the class of s-type IVC functions over $\mathcal{B}$, and $\Lambda, \Lambda_1, \Lambda_2$ be the functions belonging to this class. Throughout this paper Λ, Λ_1 and $\Lambda_2 \in \mathfrak{R}\mathfrak{I}_{(\mathcal{D})}$ where $\mathcal{D} = [\rho_1, \rho_2] \subseteq \mathcal{B}$ and $\rho_1 \leq \rho_2 \in \mathbb{R}$.

Building upon the definitions of IV functions and s-type convex functions, we propose a novel class of functions, designated as s-type IVC functions.

Definition 3.1. *Let* $\mathcal{B} \subseteq \mathbb{R}$ *be a closed interval, and* $\mathscr{H}^+$ *be the space of positive intervals of* $\mathbb{R}$ *that are closed. Then we define s-type IVC function* $\Lambda : \mathcal{B} \to \mathscr{H}^+$ *as*

$$\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) \supseteq (1 - s\vartheta)\Lambda(\rho_1) + (1 - s(1-\vartheta))\Lambda(\rho_2) \tag{3.1}$$

for all $\rho_1, \rho_2 \in \mathcal{B}$*, and* $\vartheta, s \in [0,1]$. *If the function is not IV s-type convex, then it is IV s-type concave, and* $\supseteq$ *is replaced by* $\subseteq$ *in the above expression.*

Example 3.1. *Consider the following assumptions in inclusion* (3.1)*:* $s = 0.98$*,* $\vartheta = 1/2$ *and* $M(x) = [-\sqrt{x} + 3, \sqrt{x} + 3]$ *where* $x \in [0, 2]$. *Furthermore, for* $\rho_1 < \rho_2 \in [0, 2]$ *assume* $\rho_2 = \rho_1 + 1/2$ *to get:*

$$\left[3 - \sqrt{\rho_1 + 1/4}, 3 + \sqrt{\rho_1 + 1/4}\right] \supseteq \left[3 - \frac{1}{2}\sqrt{\rho_1} - \frac{1}{2}\sqrt{\rho_1 + 1/2}, 3 + \frac{1}{2}\sqrt{\rho_1} + \frac{1}{2}\sqrt{\rho_1 + 1/2}\right]. \tag{3.2}$$

In Figure 1, starting from the left of the inclusion, the blue shaded region bounded by the black curves represents the first interval and the red shaded region represents the second interval of *inclusion* (3.2). *It is evident that the red shaded region is completely contained in the blue shaded region, which shows that the first interval contains the second one hence validating* (3.2) *and the ultimate inclusion* (3.1) *for s-type IVC, which is also shown numerically for a particular value of* ρ_1 *that is* $\rho_1 = 0.1$ *in* (3.2) *as follows:*

$$[2.408392, 3.591607] \supseteq [2.454587, 3.545412].$$

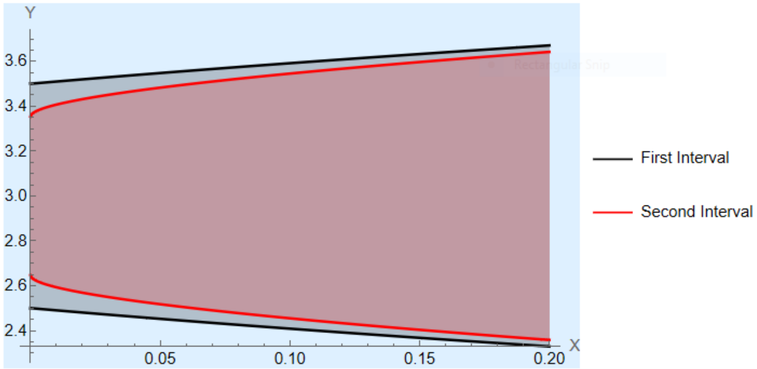

Figure 1. The graph for inclusion (3.2).

The HH-type inclusion for s-type IVC functions using the RL integrals is

Theorem 3.1. *Let* $s \in [0, 1], \varpi > 0, \rho_1, \rho_2 \in \mathcal{B}$ *such that* $\rho_2 > \rho_1$. *Then*

$$\begin{aligned}\Lambda\Big(\frac{\rho_1 + \rho_2}{2}\Big) &\supseteq (2-s)\frac{\Gamma(\varpi+1)}{2(\rho_2-\rho_1)^{\varpi}}\,[J^{\varpi}_{\rho_1+}\Lambda(\rho_2) + J^{\varpi}_{\rho_2-}\Lambda(\rho_1)]\\ &\supseteq (2-s)^2\Big(\frac{\Lambda(\rho_2)+\Lambda(\rho_1)}{2}\Big).\end{aligned}$$

Proof. Taking $\vartheta = \frac{1}{2}$ in (3.1),

$$\Lambda\left(\frac{\mathfrak{j}+\mathfrak{h}}{2}\right) \supseteq (2-s)\left(\frac{\Lambda(\mathfrak{j})+\Lambda(\mathfrak{h})}{2}\right).$$

Thus

$$2\Lambda\Big(\frac{\mathfrak{j}+\mathfrak{h}}{2}\Big) \supseteq (2-s)(\Lambda(\mathfrak{j})+\Lambda(\mathfrak{h})).$$

Take

$$\begin{aligned}\mathfrak{j} &= \vartheta\rho_1 + (1-\vartheta)\rho_2,\\ \mathfrak{h} &= (1-\vartheta)\rho_1 + \vartheta\rho_2.\end{aligned}$$

Then

$$2\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \supseteq (2-s)[\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) + \Lambda((1-\vartheta)\rho_1 + \vartheta\rho_2)]. \tag{3.3}$$

Multiply both sides of (3.3) by $\vartheta^{\varpi-1}$ and integrate

$$2\int_{[0,1]} \vartheta^{\varpi-1}\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)d\vartheta \supseteq (2-s)\int_{[0,1]} \vartheta^{\varpi-1}[\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) + \Lambda((1-\vartheta)\rho_1 + \vartheta\rho_2)]d\vartheta. \tag{3.4}$$

From (3.4)

$$\begin{aligned} 2\int_{[0,1]} \vartheta^{\varpi-1}\Lambda((\rho_1+\rho_2)/2)d\vartheta &= 2\Big[\int_{[0,1]} \underline{\Lambda}\Big(\frac{\rho_1+\rho_2}{2}\Big)\vartheta^{\varpi-1}d\vartheta, \int_{[0,1]} \overline{\Lambda}\Big(\frac{\rho_1+\rho_2}{2}\Big)\vartheta^{\varpi-1}d\vartheta\Big] \\ &= \frac{2}{\varpi}\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &(2-s)\int_{[0,1]} \vartheta^{\varpi-1}[\Lambda(\vartheta\rho_1+(1-\vartheta)\rho_2)+\Lambda(\vartheta\rho_2+(1-\vartheta)\rho_1)]d\vartheta \\ =\; &(2-s)\Big[\int_{[0,1]} \vartheta^{\varpi-1}\Lambda(\vartheta\rho_1+(1-\vartheta)\rho_2)d\vartheta + \int_{[0,1]} \vartheta^{\varpi-1}\Lambda(\vartheta\rho_2+(1-\vartheta)\rho_1)d\vartheta\Big]. \end{aligned} \tag{3.6}$$

Solving for

$$\int_{[0,1]} \vartheta^{\varpi-1}\Lambda(\vartheta\rho_1+(1-\vartheta)\rho_2)d\vartheta.$$

Take the change of variable

$$t = \vartheta\rho_1+(1-\vartheta)\rho_2, \quad dt = (\rho_2-\rho_1)\, d\vartheta,$$

with $\vartheta = 0 \Rightarrow t = \rho_2, \quad \vartheta = 1 \Rightarrow t = \rho_1$. Then

$$\begin{aligned} \int_0^1 \vartheta^{\varpi-1}\Lambda(\vartheta\rho_1+(1-\vartheta)\rho_2)\, d\vartheta &= \int_{\mathcal{D}} \left(\frac{\rho 2 - t}{\rho_2-\rho_1}\right)^{\varpi-1} \Lambda(t)\, \frac{dt}{\rho_2-\rho_1} \\ &= \frac{1}{(\rho_2-\rho_1)^{\varpi}} \int_{\mathcal{D}} (\rho_2-t)^{\varpi-1}\Lambda(t)\, dt \\ &= \frac{\Gamma(\varpi)}{(\rho_2-\rho_1)^{\varpi}} J^{\varpi}_{\rho_1+}\Lambda(\rho_2). \end{aligned} \tag{3.7}$$

Similarly, with

$$v = \vartheta\rho_2+(1-\vartheta)\rho_1, \qquad dv = (\rho_1-\rho_2)\, d\vartheta,$$

$$\int_0^1 \vartheta^{\varpi-1}\Lambda(\vartheta\rho_2+(1-\vartheta)\rho_1)\, d\vartheta = \frac{\Gamma(\varpi)}{(\rho_2-\rho_1)^{\varpi}} J^{\varpi}_{\rho_2-}\Lambda(\rho_1). \tag{3.8}$$

Inserting (3.7) and (3.8) into (3.6),

$$\begin{aligned} &(2-s)\int_{[0,1]} \vartheta^{\varpi-1}[\Lambda(\vartheta\rho_1+(1-\vartheta)\rho_2)+\Lambda(\vartheta\rho_2+(1-\vartheta)\rho_1)]d\vartheta \\ =\; &\frac{\Gamma(\varpi)}{(\rho_2-\rho_1)^{\varpi}}[J^{\varpi}_{\rho_1+}\Lambda(\rho_2)+J^{\varpi}_{\rho_2-}\Lambda(\rho_1)]. \end{aligned} \tag{3.9}$$

Using (3.5) and (3.9) in (3.4),

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \supseteq (2-s)\frac{\Gamma(\varpi+1)}{2(\rho_2-\rho_1)^{\varpi}}[J^{\varpi}_{\rho_2-}\Lambda(\rho_1)+J^{\varpi}_{\rho_1+}\Lambda(\rho_2)]. \tag{3.10}$$

Since $\Lambda \in SC(\mathcal{B}, \Lambda^{+})$

$$\Lambda[\vartheta\rho_1 + (1-\vartheta)\rho_2] \supseteq (1 - s(1-\vartheta))\Lambda(\rho_2) + (1 - s\vartheta)\Lambda(\rho_1) \tag{3.11}$$

and

$$\Lambda[\vartheta\rho_2 + (1-\vartheta)\rho_1] \supseteq (1 - s(1-\vartheta))\Lambda(\rho_1) + (1 - s\vartheta)\Lambda(\rho_2). \tag{3.12}$$

Add (3.11) and (3.12)

$$\Lambda[\vartheta\rho_1 + (1-\vartheta)\rho_2] + \Lambda[\vartheta\rho_2 + (1-\vartheta)\rho_1] \supseteq (2-s)(\Lambda(\rho_1) + \Lambda(\rho_2)). \tag{3.13}$$

Multiply both side of (3.13) by $\vartheta^{\varpi-1}$ and integrate

$$\begin{aligned}&\int_{[0,1]} \vartheta^{\varpi-1}\{\Lambda[\vartheta\rho_1 + (1-\vartheta)\rho_2] + \Lambda[\vartheta\rho_2 + (1-\vartheta)\rho_1]\}d\vartheta \\ \supseteq&(2-s)\int_{[0,1]} \vartheta^{\varpi-1}(\Lambda(\rho_1) + \Lambda(\rho_2))d\vartheta.\end{aligned} \tag{3.14}$$

Inseting (3.7) and (3.8) into (3.14)

$$\frac{\Gamma(\varpi+1)}{(\rho_2-\rho_1)^{\varpi}}[J^{\varpi}_{\rho_2-}\Lambda(\rho_1) + J^{\varpi}_{\rho_1+}\Lambda(\rho_2)] \supseteq (2-s)(\Lambda(\rho_1) + \Lambda(\rho_2)), \tag{3.15}$$

(3.10) and (3.15) give Theorem 3.1. □

Corollary 3.1. *If* $\varpi = 1$ *in* (3.1)*. Then it simplifies to the HH-type inclusion for s-type IVC function:*

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \supseteq \frac{(2-s)}{(\rho_2-\rho_1)}\int_{\mathcal{D}} \Lambda(\tau)d\tau \supseteq (2-s)^2\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}.$$

Corollary 3.2. *If* $\underline{\Lambda} = \overline{\Lambda} = \varpi = 1$ *in* (3.1)*. Then it simplifies to the HH-type inequality for the s-type convex function:*

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big) \le \frac{(2-s)}{(\rho_2-\rho_1)}\int_{\mathcal{D}} \Lambda(\tau)d\tau \le (2-s)^2\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}.$$

Remark 3.1. *From Theorem 3.1, we see that*

1) *If* $s = 1$*. Then it simplifies to the HH-type inclusion for the IV convex function [34].*
2) *If* $s = \varpi = 1$*. Then it reduces to [35, Theorem 1].*
3) *If* Λ *is a real-valued convex* ($\Re$) *function and* $s = 1$*, it further refines to [36, Theorem 2].*
4) *If* Λ *is* $\Re$ *(or concave) function and* $s = 1$*, it simplifies to the classical HH-type inequality for convex (or concave) functions [37].*

Example 3.2. *Consider the following assumptions in Theorem 3.1:*

$s = 0.98$, $\varpi = 4$, *and* $\Lambda(\tau) = [-\tau^{2/3} + 9, \tau^{2/3} + 9]$ *where* $\tau \in [0, 3]$.

Furthermore, for $\rho_1 < \rho_2 \in [0, 3]$*, assume* $\rho_2 = \rho_1 + 1/2$ *to get:*

$$\Lambda\Big(\rho_1 + \frac{1}{4}\Big) \supseteq (195.84)\ [J^{4}_{\rho_1+}\Lambda(\rho_1 + 1/2) + J^{4}_{(\rho_1+1/2)-}\Lambda(\rho_1)]$$

$$\supseteq (1.0404)\Big(\frac{\Lambda(\rho_1 + 1/2) + \Lambda(\rho_1)}{2}\Big). \tag{3.16}$$

Figure 2 reflects Theorem 3.1.

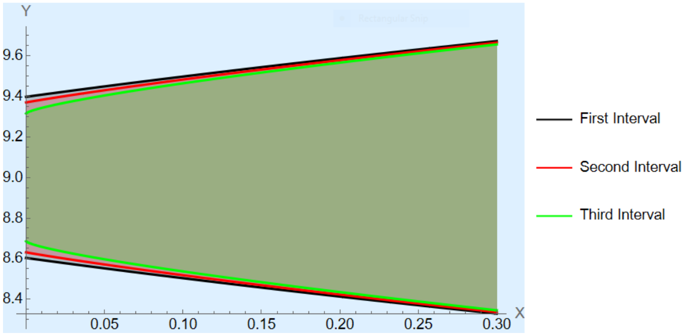

Figure 2. Graph showing inclusion (3.16).

In Figure 2, starting from the left of the inclusion, the blue-shaded region bounded by the black curves represents the first interval of inclusion (3.16)*, while the red and green shaded regions represent the second and third intervals, respectively. It is evident that the green-shaded region is completely contained within the red one, and the red-shaded region is entirely contained within the blue one. The curved boundary lines appear very close to one another because of the proximity of the corresponding interval endpoints. This nested structure shows that the first interval contains the second interval, which in turn contains the third interval, thereby validating inclusion* (3.16) *and, ultimately, the HH-type inclusion for s-type IVC in Theorem 3.1. This is also shown numerically for a particular value of* ρ_1 *that is* $\rho_1 = 0.2$ *in,* (3.16) *as below*

$$[8.41277, 9.58723] \supseteq [8.42274, 9.57726] \supseteq [8.43482, 9.56518].$$

The HHF type inclusion for s-type IVC functions using the RL integral is presented as follows:

Theorem 3.2. *Let* $s \in [0,1]$, $\varpi > 0$, $\rho_1, \rho_2 \in \mathcal{B}$ *with* $\rho_2 > \rho_1$ *and* $\Theta(\tau) = \Theta[\rho_1 + \rho_2 - \tau] \geq 0$ *for* $\tau \in \mathcal{B}$.

$$\Lambda\Big[\frac{\rho_1+\rho_2}{2}\Big][J^{\varpi}_{\rho_2-}\Theta(\rho_1) + J^{\varpi}_{\rho_1+}\Theta(\rho_2)] \supseteq (2-s)[J^{\varpi}_{\rho_2-}\Lambda\Theta(\rho_1) + J^{\varpi}_{\rho_1+}\Lambda\Theta(\rho_2)]$$
$$\supseteq (2-s)^2\Big[\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}\Big][J^{\varpi}_{\rho_2-}\Theta(\rho_1) + J^{\varpi}_{\rho_1+}\Theta(\rho_2)].$$

Proof. Since Θ is non-negative, integrable and symmetric with respect to $\Big(\frac{\rho_1+\rho_2}{2}\Big)$,

$$\Theta[\vartheta\rho_1 + (1-\vartheta)\rho_2] = \Theta[\vartheta\rho_2 + (1-\vartheta)\rho_1].$$

Multiply both sides of (3.4) by $\Theta[\vartheta\rho_2 + (1-\vartheta)\rho_1]$,

$$2\int_{[0,1]} \vartheta^{\varpi-1}\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)\Theta[\vartheta\rho_2 + (1-\vartheta)\rho_1]d\vartheta$$
$$\supseteq (2-s)\int_{[0,1]} \vartheta^{\varpi-1}[\Lambda(\vartheta\rho_1 + (1-\vartheta)\rho_2) + \Lambda((1-\vartheta)\rho_1 + \vartheta\rho_2)]\Theta[\vartheta\rho_2 + (1-\vartheta)\rho_1]d\vartheta$$

$$= (2-s)\Big[\int_{[0,1]} \vartheta^{\varpi-1}[\underline{\Lambda}(\vartheta\rho_1+(1-\vartheta)\rho_2), \overline{\Lambda}(\vartheta\rho_1+(1-\vartheta)\rho_2)]\Theta[\vartheta\rho_2+(1-\vartheta)\rho_1]d\vartheta$$

$$+\int_{[0,1]} \vartheta^{\varpi-1}[\underline{\Lambda}(\vartheta\rho_2+(1-\vartheta)\rho_1), \overline{\Lambda}(\vartheta\rho_2+(1-\vartheta)\rho_1)]\Theta[\vartheta\rho_2+(1-\vartheta)\rho_1]d\vartheta\Big]. \qquad (3.17)$$

Let $\breve{u} = \vartheta\rho_2 + (1-\vartheta)\rho_1$. Then by changing variables in (3.17), we get

$$\frac{2}{(\rho_2-\rho_1)^{\varpi}}\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\mathcal{D}}\Theta(\breve{u})(\breve{u}-\rho_1)^{\varpi-1}d\breve{u}$$
$$\supseteq \frac{(2-s)}{(\rho_2-\rho_1)^{\varpi}}\Big\{\int_{\mathcal{D}}[\underline{\Lambda}(\breve{u}),\overline{\Lambda}(\breve{u})]\Theta(\rho_2+\rho_1-\breve{u})(\rho_2-\breve{u})^{\varpi-1}d\breve{u}$$
$$+\int_{\mathcal{D}}[\underline{\Lambda}(\breve{u}),\overline{\Lambda}(\breve{u})]\Theta(\breve{u})(\breve{u}-\rho_1)^{\varpi-1}d\breve{u}\Big\}.$$

Since $\Theta(\rho_2+\rho_1-\breve{u}) = \Theta(\breve{u}) \geq 0$ and $\Lambda(\breve{u} = [\underline{\Lambda}(\breve{u}), \overline{\Lambda}(\breve{u})]$ then (3.18) becomes

$$\frac{2}{(\rho_2-\rho_1)^{\varpi}}\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\mathcal{D}}(\breve{u}-\rho_1)^{\varpi-1}\Theta(\breve{u})d\breve{u}$$
$$\supseteq \frac{(2-s)}{(\rho_2-\rho_1)^{\varpi}}\Big\{\int_{\mathcal{D}}\Lambda(\breve{u})\Theta(\breve{u})(\rho_2-\breve{u})^{\varpi-1}d\breve{u}+\int_{\mathcal{D}}\Lambda(\breve{u})\Theta(\breve{u})(\breve{u}-\rho_1)^{\varpi-1}d\breve{u}\Big\}.$$

Hence,

$$\frac{\Gamma(\varpi)}{(\rho_2-\rho_1)^{\varpi}}\Lambda\Big[\frac{\rho_1+\rho_2}{2}\Big][J^{\varpi}_{\rho_2-}\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Theta(\rho_2)] \supseteq (2-s)\frac{\Gamma(\varpi)}{(\rho_2-\rho_1)^{\varpi}}[J^{\varpi}_{\rho_2-}\Lambda\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Lambda\Theta(\rho_2)].$$

This implies that

$$\Lambda\Big[\frac{\rho_1+\rho_2}{2}\Big][J^{\varpi}_{\rho_2-}\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Theta(\rho_2)] \supseteq (2-s)[J^{\varpi}_{\rho_2-}\Lambda\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Lambda\Theta(\rho_2)]. \qquad (3.18)$$

Multiply both sides of (3.13) by $\vartheta^{\varpi-1}\Theta(\vartheta\rho_2+(1-\vartheta)\rho_1)$ and integrate

$$\int_0^1 \vartheta^{\varpi-1}\Big\{\Lambda[\vartheta\rho_1+(1-\vartheta)\rho_2]+\Lambda[\vartheta\rho_2+(1-\vartheta)\rho_1]\Big\}\Theta(\vartheta\rho_2+(1-\vartheta)\rho_1)d\vartheta$$
$$\supseteq (2-s)(\Lambda(\rho_1)+\Lambda(\rho_2))\int_0^1 \vartheta^{\varpi-1}\Theta(\vartheta\rho_2+(1-\vartheta)\rho_1)d\vartheta.$$

Taking $t = \vartheta\rho_1+(1-\vartheta)\rho_2$, $v = \vartheta\rho_2+(1-\vartheta)\rho_1$ and following similar steps as in (3.7) and (3.8), above inclusion becomes

$$(2-s)[J^{\varpi}_{\rho_2-}\Lambda\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Lambda\Theta(\rho_2)] \supseteq (2-s)^2\Big[\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}\Big][J^{\varpi}_{\rho_2-}\Theta(\rho_1)+J^{\varpi}_{\rho_1+}\Theta(\rho_2)]. \qquad (3.19)$$

(3.18) and (3.19) lead toward Theorem 3.2. □

Corollary 3.3. *If* $\varpi = 1$ *in* (3.2)*. Then it simplifies to:*

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\mathcal{D}}\Theta(\tau)d\tau \supseteq (2-s)\int_{\mathcal{D}}\Lambda(\tau)\Theta(\tau)d\tau \supseteq (2-s)^2\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}\int_{\mathcal{D}}\Theta(\tau)d\tau.$$

Corollary 3.4. *If Λ is $\Re$ and $\varpi = 1$ in (3.2). Then it reduces to:*

$$\Lambda\Big(\frac{\rho_1+\rho_2}{2}\Big)\int_{\mathcal{D}}\Theta(\tau)d\tau \le (2-s)\int_{\mathcal{D}}\Lambda(\tau)\Theta(\tau)d\tau \le (2-s)^2\frac{\Lambda(\rho_1)+\Lambda(\rho_2)}{2}\int_{\mathcal{D}}\Theta(\tau)d\tau.$$

Remark 3.2. *Theorem 3.2 leads to the conclution that*

(1) If $\Theta(\tau) = 1$, then we get Theorem 3.1.
(2) If $s = 1$, then it reduces to the HHF-type inclusion in fractional integral form for IVC functions [34].
(3) If Λ is $\Re$ function and $s = 1$, it becomes [38, Theorem 4].
(4) If Λ is $\Re$ (or concave) function and $\varpi = s = 1$, it simplifies to the classical HHF-type inequality for convex (or concave) functions [21].

Example 3.3. *Consider the following assumptions in Theorem 3.2:*
Let $s = 0.99$, $\varpi = 2$,

$$\Lambda(\tau) = [-\tau^{3/7}+7, \tau^{3/7}+7],$$

and

$$\Theta(\tau) = (\tau-\rho_1)(\rho_2-\tau),$$

for $\tau \in [0, 1.5]$. While for $\rho_1 < \rho_2 \in [0, 1.5]$, assume $\rho_2 = \rho_1 + 1/2$ to get:

$$\begin{aligned}&\Lambda\Big(\rho_1+\frac{1}{4}\Big)[J^2_{(\rho_1+1/2)-}\Theta(\rho_1) + J^2_{\rho_1+}\Theta(\rho_1+1/2)]\\ &\supseteq (1.02)[J^2_{(\rho_1+1/2)-}\Lambda\Theta(\rho_1) + J^2_{\rho_1+}\Lambda\Theta(\rho_1+1/2)]\\ &\supseteq (1.0404)\Big(\frac{\Lambda(\rho_1)+\Lambda(\rho_1+1/2)}{2}\Big)[J^2_{(\rho_1+1/2)-}\Theta(\rho_1) + J^2_{\rho_1+}\Theta(\rho_1+1/2)]. \qquad (3.20)\end{aligned}$$

Figure 3 verifies Theorem 3.2.

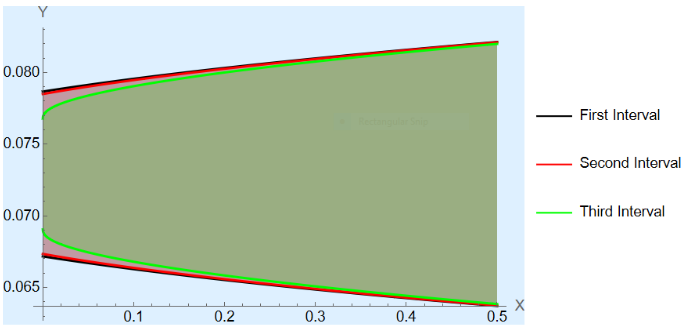

Figure 3. The graph for inclusion (3.20).

In Figure 3, starting from the left of the inclusion, the blue-shaded region bounded by the black curves represents the first interval of inclusion (3.20), while the red- and green- shaded regions represent the second and third intervals. It is evident that the green-shaded region is completely contained within the red one and the red-shaded region is entirely contained within the blue one. The curved boundary lines appear very close to one another because of the proximity of the corresponding interval endpoints. This nested structure shows that the first interval contains the second interval which *in turn contains the third interval, thereby validating inclusion* (3.20) *and, ultimately, the HHF-type inclusion for s-type IVC in Theorem 3.2. This is also confirmed numerically by taking* $\rho_1 = 0.1$ *in* (3.20) *as*

$$[0.0662742, 0.0795591] \supseteq [0.0663645, 0.0794688] \supseteq [0.0667909, 0.0790424].$$

Theorem 3.3. *Let* $s \in [0,1], \varpi > 0$, $\rho_1, \rho_2 \in \mathcal{B}$ *with* $\rho_2 > \rho_1$. *Then*

$$\begin{aligned}
&\frac{\Gamma(\varpi+1)}{2(\rho_2-\rho_1)^{\varpi}}[J^{\varpi}_{\rho_1+}\Lambda_1(\rho_2)\Lambda_2(\rho_2) + J^{\varpi}_{\rho_2-}\Lambda_1(\rho_1)\Lambda_2(\rho_1)] \\
&\supseteq \lambda_1(\rho_1,\rho_2)\Big[\frac{((1-s)^2+1)}{2} - s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big] \\
&\quad + \lambda_2(\rho_1,\rho_2)\Big[(1-s) + s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big],
\end{aligned}$$

where

$$\lambda_1(\rho_1,\rho_2) = [\Lambda_1(\rho_1)\Lambda_2(\rho_1) + \Lambda_1(\rho_2)\Lambda_2(\rho_2)], \tag{3.21}$$
$$\lambda_2(\rho_1,\rho_2) = [\Lambda_1(\rho_1)\Lambda_2(\rho_2) + \Lambda_1(\rho_2)\Lambda_2(\rho_1)]. \tag{3.22}$$

Proof. Let $\vartheta \in [0,1]$. Hence the assumptions in Theorem 3.3 imply that

$$\Lambda_1[\vartheta\rho_1 + (1-\vartheta)\rho_2] \supseteq (1-\vartheta s)\Lambda_1(\rho_1) + (1-(1-\vartheta)s)\Lambda_1(\rho_2), \tag{3.23}$$
$$\Lambda_2[\vartheta\rho_1 + (1-\vartheta)\rho_2] \supseteq (1-\vartheta s)\Lambda_2(\rho_1) + (1-(1-\vartheta)s)\Lambda_2(\rho_2). \tag{3.24}$$

Products of (3.23) and (3.24) give,

$$\begin{aligned}
&\Lambda_1[\vartheta\rho_1 + (1-\vartheta)\rho_2]\Lambda_2[\vartheta\rho_1 + (1-\vartheta)\rho_2] \\
&\supseteq (1-\vartheta s)^2\Lambda_1(\rho_1)\Lambda_2(\rho_1) + (1-(1-\vartheta)s)^2\Lambda_1(\rho_2)\Lambda_2(\rho_2) \\
&\quad + (1-\vartheta s)(1-(1-\vartheta)s)[\Lambda_1(\rho_1)\Lambda_2(\rho_2) + \Lambda_1(\rho_2)\Lambda_2(\rho_1)].
\end{aligned} \tag{3.25}$$

Analogously,

$$\begin{aligned}
&\Lambda_1[\vartheta\rho_2 + (1-\vartheta)\rho_1]\Lambda_2[\vartheta\rho_2 + (1-\vartheta)\rho_1] \\
&\supseteq (1-\vartheta s)^2\Lambda_1(\rho_2)\Lambda_2(\rho_2) + (1-(1-\vartheta)s)^2\Lambda_1(\rho_1)\Lambda_2(\rho_1) \\
&\quad + (1-\vartheta s)(1-(1-\vartheta)s)[\Lambda_1(\rho_2)\Lambda_2(\rho_1) + \Lambda_1(\rho_1)\Lambda_2(\rho_2)].
\end{aligned} \tag{3.26}$$

Add (3.25) and (3.26),

$$\begin{aligned}
&\Lambda_1[\vartheta\rho_1 + (1-\vartheta)\rho_2]\Lambda_2[\vartheta\rho_1 + (1-\vartheta)\rho_2] + \Lambda_1[\vartheta\rho_2 + (1-\vartheta)\rho_1]\Lambda_2[\vartheta\rho_2 + (1-\vartheta)\rho_1] \\
&\supseteq [(1-\vartheta s)^2 + (1-(1-\vartheta)s)^2][\Lambda_1(\rho_1)\Lambda_2(\rho_1) + \Lambda_1(\rho_1)\Lambda_2(\rho_1)] \\
&\quad + 2[(1-\vartheta s)(1-(1-\vartheta)s)].[\Lambda_1(\rho_1)\Lambda_2(\rho_2) + \Lambda_1(\rho_2)\Lambda_2(\rho_1)].
\end{aligned} \tag{3.27}$$

Multiply both sides of (3.27) by $\vartheta^{\varpi-1}$ and integrate

$$\int_{[0,1]} \vartheta^{\varpi-1}\Lambda_1[\vartheta\rho_1 + (1-\vartheta)\rho_2]\Lambda_2[\vartheta\rho_1 + (1-\vartheta)\rho_2]d\vartheta$$

$$
\begin{aligned}
&+\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1[\vartheta\rho_2+(1-\vartheta)\rho_1]\Lambda_2[\vartheta\rho_2+(1-\vartheta)\rho_1]d\vartheta\\
\supseteq\;&\lambda_1(\rho_1,\rho_2)\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)^2+(1-(1-\vartheta)s)^2]d\vartheta\\
&+2\lambda_2(\rho_1,\rho_2)\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)(1-(1-\vartheta)s)]d\vartheta.
\end{aligned}\tag{3.28}
$$

From (3.28),

$$
\begin{aligned}
&\lambda_1(\rho_1,\rho_2)\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)^2+(1-(1-\vartheta)s)^2]d\vartheta\\
&+2\lambda_2(\rho_1,\rho_2)\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)(1-(1-\vartheta)s)]\vartheta\\
=&\frac{2\lambda_1(\rho_1,\rho_2)}{\varpi}\Big[\frac{((1-s)+1)}{2}-s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big]\\
&+\frac{\lambda_2(\rho_1,\rho_2)}{\varpi}\Big[(1-s)+s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big].
\end{aligned}\tag{3.29}
$$

Inserting (3.7), (3.8), and (3.29) into (3.28) lead toward Theorem 3.3. □

Corollary 3.5. *If* $\varpi=1$ *in* (3.3). *Then it simplifies to inclusion*

$$\frac{1}{(\rho_2-\rho_1)}\int_{\mathcal{D}}\Lambda_1\Lambda_2(\tau)d\tau\supseteq\lambda_1(\rho_1,\rho_2)\Big[\frac{((1-s)^2+1)}{2}-\frac{s^2}{6}\Big]+\lambda_2(\rho_1,\rho_2)\Big[(1-s)+\frac{s^2}{6}\Big].$$

Corollary 3.6. *If* Λ *is* $\mathfrak{R}$ *and* $\varpi=1$ *in* (3.3). *Then it gives the following inequality*

$$\frac{1}{(\rho_2-\rho_1)}\int_{\mathcal{D}}\Lambda_1\Lambda_2(\tau)d\tau\le\lambda_1(\rho_1,\rho_2)\Big[\frac{((1-s)^2+1)}{2}-\frac{s^2}{6}\Big]+\lambda_2(\rho_1,\rho_2)\Big[(1-s)+\frac{s^2}{6}\Big].$$

Remark 3.3. *From Theorem 3.3, if* $s=1$*, then it reduces to [39, Theorem 3.5].*

Example 3.4. *Consider the following assumptions in Theorem 3.3:*

Let $s=0.98$, $\varpi=3$,

$$\Lambda_1(\tau)=[-\sqrt{\tau}+2,\ \sqrt{\tau}+2],$$

and

$$\Lambda_2(\tau)=[-\sqrt{\tau}+3,\ \sqrt{\tau}+3],$$

for $\tau\in[0,2]$.

While for $\rho_1<\rho_2\in[0,2]$, *assume* $\rho_2=\rho_1+1/2$ *to get:*

$$
\begin{aligned}
&24\Big[J^3_{\rho_1+}\Lambda_1(\rho_1+1/2)\Lambda_2(\rho_1+1/2)+J^3_{(\rho_1+1/2)-}\Lambda_1(\rho_1)\Lambda_2(\rho_1)\Big]\\
\supseteq\;&(0.353035)\lambda_1(\rho_1,\rho_1+1/2)+(0.157015)\lambda_2(\rho_1,\rho_1+1/2).
\end{aligned}\tag{3.30}
$$

Figure 4 validates Theorem 3.3.

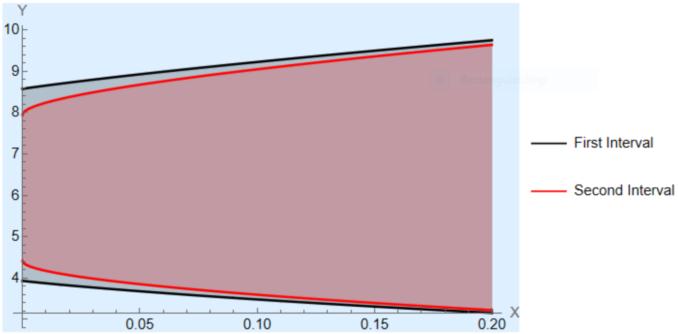

Figure 4. The graph for inclusion (3.30).

In Figure 4, starting from the left of the inclusion, the blue-shaded region bounded by the black curves represents the first interval of inclusion (3.30) *while the red shaded region represents the second interval. It is evident that the green-shaded region is completely contained within the red-region indicating that the first interval contains the second interval thereby validating Theorem 3.3 and is also confirmed numerically by taking* $\rho_1 = 0.7$ *in* (3.30) *as*

$$[2.09372, 11.8063] \supseteq [2.10969, 11.7702].$$

Theorem 3.4. *Let* $s \in [0,1]$, $\varpi > 0$, $\rho_1, \rho_2 \in \mathcal{B}$ *with* $\rho_2 > \rho_1$. *Then*

$$\begin{aligned} & 2\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big) \\ \supseteq\ & \frac{\Gamma(\varpi+1)}{2(\rho_2-\rho_1)^{\varpi}}\Big[J^{\varpi}_{\rho_1+}\Lambda_1(\rho_2)\Lambda_2(\rho_2) + J^{\varpi}_{\rho_2-}\Lambda_1(\rho_1)\Lambda_2(\rho_1)\Big] \\ & + \Big[(1-s) + s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big]\lambda_1(\rho_1,\rho_2) + \Big[\frac{(1-s)^2+1}{2} - s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big]\lambda_2(\rho_1,\rho_2), \end{aligned}$$

where $\lambda_1(\rho_1,\rho_2)$ *and* $\lambda_2(\rho_1,\rho_2)$ *are given in* (3.21) *and* (3.22), *respectively.*

Proof. Let $\varpi \in [0,1]$; then

$$\Big(\frac{\rho_1+\rho_2}{2}\Big) = \frac{\vartheta\rho_1 + (1-\vartheta)\rho_2}{2} + \frac{\vartheta\rho_2 + (1-\vartheta)\rho_1}{2}.$$

Since $\Lambda_1, \Lambda_2 \in \mathcal{SC}([\rho_1,\rho_2], \mathcal{H}^+)$. Therefore,

$$\begin{aligned} & \Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big) \\ =\ & \Lambda_1\Big[\frac{\vartheta\rho_1+(1-\vartheta)\rho_2}{2} + \frac{\vartheta\rho_2+(1-\vartheta)\rho_1}{2}\Big]\Lambda_2\Big[\frac{\vartheta\rho_1+(1-\vartheta)\rho_2}{2} + \frac{\vartheta\rho_2+(1-\vartheta)\rho_1}{2}\Big] \\ \supseteq\ & \frac{1}{4}[\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2) + \Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)][\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2) + \Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)] \\ =\ & \frac{1}{4}\Big[\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2) + \Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1) \\ & + \Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2) + \Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)\Big] \end{aligned}$$

$$
\begin{aligned}
&\supseteq \frac{1}{4}\Big[\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2)+\Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)\Big]\\
&\quad+\frac{1}{4}\Big\{[(1-\vartheta s)\Lambda_1(\rho_1)+(1-s(1-\vartheta))\Lambda_1(\rho_2)]\times[(1-\vartheta s)\Lambda_2(\rho_2)+(1-s(1-\vartheta))\Lambda_2(\rho_1)]\Big\}\\
&\quad+\frac{1}{4}\Big\{[(1-\vartheta s)\Lambda_1(\rho_2)+(1-s(1-\vartheta))\Lambda_1(\rho_1)]\times[(1-\vartheta s)\Lambda_2(\rho_1)+(1-s(1-\vartheta))\Lambda_2(\rho_2)]\Big\}\\
&= \frac{1}{4}\Big[\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2)+\Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)\Big]\\
&\quad+\frac{1}{4}\Big\{(1-\vartheta s)^2\Lambda_1(\rho_1)\Lambda_2(\rho_2)+(1-\vartheta s)(1-s(1-\vartheta))[\Lambda_1(\rho_1)\Lambda_2(\rho_1)+\Lambda_1(\rho_2)\Lambda_2(\rho_2)]\\
&\quad+(1-s(1-\vartheta))^2\Lambda_1(\rho_2)\Lambda_2(\rho_1)\Big\}\\
&\quad+\frac{1}{4}\Big\{(1-\vartheta s)^2\Lambda_1(\rho_2)\Lambda_2(\rho_1)+(1-\vartheta s)(1-s(1-\vartheta))[\Lambda_1(\rho_1)\Lambda_2(\rho_1)+\Lambda_1(\rho_2)\Lambda_2(\rho_2)]\\
&\quad+(1-s(1-\vartheta))^2\Lambda_1(\rho_1)\Lambda_2(\rho_2)\Big\}\\
&= \frac{1}{4}\Big[\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2)+\Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)\Big]\\
&\quad+\frac{1}{4}[(1-\vartheta s)^2+(1-s(1-\vartheta))^2]\lambda_2(\rho_1,\rho_2)+\frac{1}{2}[(1-\vartheta s)(1-s(1-\vartheta))]\lambda_1(\rho_1,\rho_2).
\end{aligned}
$$

Multiply by $\vartheta^{\varpi-1}$ and integrate w.r.t ϑ over (0,1),

$$
\begin{aligned}
&\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big)d\vartheta\\
\supseteq\quad &+\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2)d\vartheta\\
&+\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)d\vartheta\\
&+\lambda_2(\rho_1,\rho_2)\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)^2+(1-s(1-\vartheta))^2]d\vartheta\\
&+\lambda_1(\rho_1,\rho_2)\frac{1}{2}\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)(1-s(1-\vartheta))]d\vartheta.
\end{aligned}\tag{3.31}
$$

From (3.31),

$$
\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big)d\vartheta=\frac{1}{\varpi}\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big).\tag{3.32}
$$

Taking $t=\vartheta\rho_1+(1-\vartheta)\rho_2$, $v=\vartheta\rho_2+(1-\vartheta)\rho_1$ and following similar steps as in (3.7) and (3.8),

$$
\begin{aligned}
&\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1(\vartheta\rho_1+(1-\vartheta)\rho_2)\Lambda_2(\vartheta\rho_1+(1-\vartheta)\rho_2)d\vartheta\\
&+\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}\Lambda_1(\vartheta\rho_2+(1-\vartheta)\rho_1)\Lambda_2(\vartheta\rho_2+(1-\vartheta)\rho_1)d\vartheta\\
&+\lambda_2(\rho_1,\rho_2)\frac{1}{4}\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)^2+(1-s(1-\vartheta))^2]d\vartheta
\end{aligned}
$$

$$+\lambda_1(\rho_1,\rho_2)\frac{1}{2}\int_{[0,1]}\vartheta^{\varpi-1}[(1-\vartheta s)(1-s(1-\vartheta))]d\vartheta$$

$$= \frac{\Gamma(\varpi)}{4(\rho_2-\rho_1)^{\varpi}}\Big[J^{\varpi}_{\rho_1+}\Lambda_1(\rho_2)\Lambda_2(\rho_2)+J^{\varpi}_{\rho_2-}\Lambda_1(\rho_1)\Lambda_2(\rho_1)\Big]$$
$$+\frac{1}{2\varpi}\Big[\frac{(1-s)^2+1}{2}-s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big]\lambda_2(\rho_1,\rho_2)$$
$$+\frac{1}{2\varpi}\Big[(1-s)+s^2\Big(\frac{\varpi}{(\varpi+1)(\varpi+2)}\Big)\Big]\lambda_1(\rho_1,\rho_2). \tag{3.33}$$

(3.32) and (3.33) give Theorem 3.4. □

Corollary 3.7. *If* $\varpi = 1$ *in* (3.4). *Then it simplifies to inclusion*

$$2\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big) \supseteq \frac{1}{(\rho_2-\rho_1)}\Big[\int_{\mathcal{D}}\Lambda_1(\tau)d\tau\Big]$$
$$+\Big[(1-s)+\frac{s^2}{6}\Big]\lambda_1(\rho_1,\rho_2)+\Big[\frac{(1-s)^2+1}{2}-\frac{s^2}{6}\Big]\lambda_2(\rho_1,\rho_2).$$

Corollary 3.8. *If* Λ *is* $\mathfrak{R}$ *and* $\varpi = 1$ *in* (3.4). *Then it simplifies to an inequality*

$$2\Lambda_1\Big(\frac{\rho_1+\rho_2}{2}\Big)\Lambda_2\Big(\frac{\rho_1+\rho_2}{2}\Big) \leq \frac{1}{(\rho_2-\rho_1)}\Big[\int_{\mathcal{D}}\Lambda_1(\tau)d\tau\Big]$$
$$+\Big[(1-s)+\frac{s^2}{6}\Big]\lambda_1(\rho_1,\rho_2)+\Big[\frac{(1-s)^2+1}{2}-\frac{s^2}{6}\Big]\lambda_2(\rho_1,\rho_2).$$

Example 3.5. *Consider the following assumptions in Theorem 3.4:*

Let $s = 0.98$, $\varpi = 3$,

$$\Lambda_1(\tau) = [-\sqrt{\tau}+2,\ \sqrt{\tau}+2],$$

and

$$\Lambda_2(\tau) = [-\sqrt{\tau}+3,\ \sqrt{\tau}+3],$$

for $\tau \in [0,2]$. *While for* $\rho_1 < \rho_2 \in [0,2]$, *assume* $\rho_2 = \rho_1 + 1/2$ *to get*

$$2\Lambda_1\Big(\rho_1+\frac{1}{4}\Big)\Lambda_2\Big(\rho_1+\frac{1}{4}\Big) \supseteq (24)\Big[J^{3}_{\rho_1+}\Lambda_1\Big(\rho_1+\frac{1}{2}\Big)\Lambda_2\Big(\rho_1+\frac{1}{2}\Big)+J^{\,3}_{(\rho_1+\frac{1}{2})-}\Lambda_1(\rho_1)\Lambda_2(\rho_1)\Big]$$
$$+(0.157015)\lambda_1(\rho_1,\rho_1+1/2)+(0.353035)\lambda_2(\rho_1,\rho_1+1/2). \tag{3.34}$$

Theorem 3.4 is supported by Figure 5.

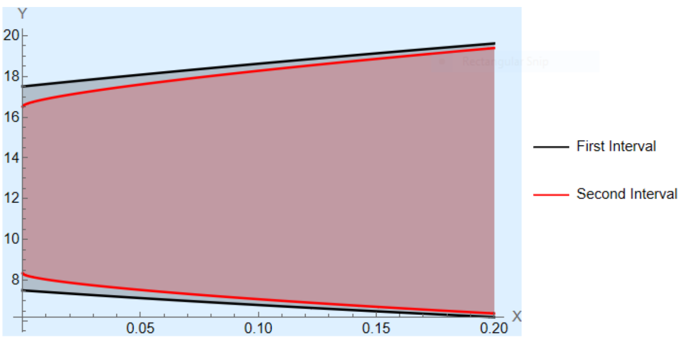

Figure 5. The graph for inclusion (3.34).

In Figure 5, starting from the left of the inclusion, the blue-shaded region bounded by the black curves represents the first interval of inclusion (3.34)*, while the red shaded region represents the second interval. It is evident that the green-shaded region is completely contained within the red-shaded region, indicating that the first interval contains the second interval. This validates Theorem 3.4 and is also shown numerically for a particular value of* ρ_1 *that is* $\rho_1 = 0.7$ *in* (3.34)*,*

$$[4.15321, 23.6468] \supseteq [4.20341, 23.5765].$$

4. Conclusions

In this article a novel class called as s-type convex interval-valued function has been introduced and used to construct several new generalized inclusions employing Riemann-Liouville fractional integrals. For each key conclusion a particular example is provided to validate the corresponding result. These findings have applications in iterative methods where expressions like ϑ^s and $(1-\vartheta)^s$ appear. Linearization of these expressions as $1 - s\vartheta$ and $1 - s(1-\vartheta)$ leads to s-type convexity, but truncating higher-order terms in the expansion introduces errors, resulting in uncertainty in the obtained results. Therefore, replacing point-valued s-type convex functions with interval-valued s-type convex functions provides more reliable bounds and better handles uncertainty in practical computations. Researchers in this discipline may investigate further inclusions, such as Hermite-Hadamard-Mercer, Ostrowski, Simpson and trapezoidal type inclusions, in the future by utilizing various generalized convexities or different fractional operators, including Atangana–Baleanu, Hadamard, Katugampola and Prabhakar fractional integrals.

Author contributions

Ammara Nosheen: Writing – original draft, validation, methodology, conceptualization; Khuram Ali Khan: Writing – review & editing, supervision, methodology, investigation; Mariam Aslam: Writing – review & editing, methodology, investigation; Atiq ur Rehman: Writing – review & editing, investigation, formal analysis, conceptualization; Tamador Alihia: Project administration, writing – review & editing, investigation, funding acquisition; Salwa El-Morsy: Writing – review & editing, validation, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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