Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1954–1967.
DOI:10.3934/math.2026081
Received: 01 December 2025
Revised: 06 January 2026
Accepted: 13 January 2026
Published: 21 January 2026

Research article

Gauss sums over some subsets

Xiaoying Liu and Zhefeng Xu*

Research Center for Number Theory and Its Applications, Northwest University, Xi'an 710127, China

* **Correspondence:** Email: zfxu@nwu.edu.cn.

Abstract: Let $q \geq 3$ be an integer, χ a non-principal character modulo q, and $A, B, H \leq q$ with $(u, q) = 1$ and $(v, q) = 1$. In this paper, by combining estimates for general Kloosterman sums and Gauss sums with properties of trigonometric sums, we derive nontrivial bounds for Gauss sums over the sets $\aleph(n, q)$, $\hbar_{u,v}(A, B, H)$, and $\aleph(n, q) \cap \hbar_{u,v}(A, B, H)$, where

$$\aleph(n, q) = \{a \in \mathbb{Z} \mid (a, q) = 1,\ n \nmid a + \overline{a}\},$$

and

$$\hbar_{u,v}(A, B, H) = \{a \in \mathbb{Z} | (a, q) = 1, ab \equiv 1 \bmod q, 1 \leq a \leq A, 1 \leq b \leq B, |ua - vb| \leq H\}.$$

Keywords: Gauss sum; Lehmer number; upper bound
Mathematics Subject Classification: 11L05, 11L40

1. Introduction

1.1. Motivation and background

For any Dirichlet character χ modulo q, the classical Gauss sums are defined by

$$G(m, \chi) = \sum_{a=1}^{q} \chi(a) e\Big(\frac{ma}{q}\Big), \tag{1.1}$$

where $e(y) = e^{2\pi i y}$. Various properties of $G(m, \chi)$ can be found in standard texts on analytic number theory; see, for example, [8]. Perhaps the most famous properties of $G(m, \chi)$ are the following identities:

$$G(m, \chi^*) = \overline{\chi}^*(m)\, \tau(\chi^*) \qquad \text{and} \qquad |\tau(\chi^*)| = \sqrt{q},$$

where χ^* is a primitive character modulo q, $\overline{\chi^*}$ is the conjugate character of χ^*, and $\tau(\chi^*) = G(1, \chi^*)$. If χ is a nonprimitive character modulo q, then the value distribution of $\tau(\chi)$ is very irregular; in fact, it can even be zero. The properties of $G(m, \chi)$ have been generalized by many scholars (see [12, 17]).

When $a \in [N+1, N+H]$, the sum (1.1) takes the form

$$\sum_{a=N+1}^{N+H} \chi(a) e\Big(\frac{ma}{q}\Big). \tag{1.2}$$

By extending his well-known work on character sums, Burgess [1–3] obtained several estimates for (1.2).

For an integer a with $(a, q) = 1$, we denote by $\bar{a}$ the multiplicative inverse of a modulo q. Put

$$\aleph(n, q) = \{a \in \mathbb{Z} \mid (a, q) = 1,\ n \nmid a + \overline{a}\}.$$

The elements of $\aleph(n, q)$ are called the generalized Lehmer numbers modulo q. In the special case $n = 2$, we have $\aleph(2, q) = N(q)$, the set of classical Lehmer numbers. Let $|N(q)|$ denote the cardinality of the set $N(q)$. D. H. Lehmer asked us to find $|N(p)|$ or at least to say something nontrivial about it for an odd prime p (see Problem F12 of [4]). W. P. Zhang [15, 16] obtained several asymptotic estimates for $|N(p)|$ and $|N(q)|$. The classical problem has been generalized by many scholars (see [6, 10, 11, 18]).

Han and Zhang [5] obtained an upper bound estimate of the character sums over classical Lehmer numbers as

$$\sum_{a \in N(p)} \chi(a) = \sum_{\substack{a=1 \\ 2 \nmid (a+\bar{a})}}^{p-1} \chi(a) \ \ll\ p^{1/2} \ln^2 p,$$

where χ is an arbitrary non-principal character modulo an odd prime p. Later, Ma, Chen, Qin, and Zhang [7] extended the result of [5] by studying the character sums over $\aleph(n, q)$.

In contrast to these works, which are devoted to character sums, the present paper turns to the study of Gauss sums over the set $\aleph(n, q)$, namely,

$$\sum_{a \in \aleph(n,q)} \chi(a) e\Big(\frac{ma}{q}\Big).$$

Using estimates for Gauss sums, together with properties of trigonometric sums, we obtain the following result.

1.2. Main results

Theorem 1.1. *Let $q \geq 3$ be an integer and $n \geq 2$ be a fixed integer with $(n, q) = 1$. For any non-principal character χ modulo q, we obtain*

$$\sum_{a \in \aleph(n,q)} \chi(a) e\Big(\frac{ma}{q}\Big) \ll n q^{1/2} d^5(q) \log^2 q.$$

On the other hand, let $A = A(q) \leq q$, $B = B(q) \leq q$, $(u, q) = 1$, $(v, q) = 1$, and $H = H(q) \leq q$. Define

$$\hbar_{u,v}(A, B, H) = \Big\{a \in \mathbb{Z} \,\Big|\, (a, q) = 1, ab \equiv 1 \bmod q, 1 \leq a \leq A, 1 \leq b \leq B, |ua - vb| \leq H\Big\}.$$

It directly generalizes the set of so-called H-flat numbers modulo q, which was studied extensively by Xi and Yi [13] and the references therein. Moreover, Ren, He, and Zhang [9] dealt with general partial Gauss sums over the set $\hbar_{1,1}(A, B, H)$.

The present paper extends earlier work in two directions. We first derive nontrivial bounds for Gauss sums over $\hbar_{u,v}(A,B,H)$ with weights u and v. We then study Gauss sums over the intersection $\aleph(n,q) \cap \hbar_{u,v}(A,B,H)$, which has not been considered before. More precisely, we consider Gauss sums of the form

$$\sum_{a\in\hbar_{u,v}(A,B,H)} \chi(a)e\Big(\frac{ma}{q}\Big) \qquad \text{and} \qquad \sum_{a\in\aleph(n,q)\cap\hbar_{u,v}(A,B,H)} \chi(a)e\Big(\frac{ma}{q}\Big),$$

and our aim is to prove nontrivial upper bounds for these sums. Our main results may now be stated as follows.

Theorem 1.2. *Let $q \geq 3$ be an integer and χ a non-principal Dirichlet character modulo q. Then,*

$$\sum_{a\in\hbar_{u,v}(A,B,H)} \chi(a)e\Big(\frac{ma}{q}\Big) \ll d(q)q^{\frac{1}{2}}\Big(\frac{umABd(q)\log H}{q^2} + \frac{vBd(q)\log q\log H}{q} + \log^3 q\Big),$$

which is uniformly nontrivial for any positive integer m such that $m < q^{\frac{1}{2}}$.

Remark 1.3. *The restriction $m < q^{1/2}$ is imposed to ensure nontriviality. The bound holds for all m, but its leading term depends linearly on m. A comparison with the trivial bound shows that the estimate is nontrivial only when $m < q^{1/2}$.*

In addition, in the special case $u = v = 1$, Theorem 1.2 coincides with the corresponding result in [9]. Similarly, we obtain the following result.

Theorem 1.4. *Let $q \geq 3$ be an integer, $n \geq 2$ a fixed integer with $(n,q) = 1$, and χ a non-principal Dirichlet character modulo q. Then,*

$$\sum_{a\in\aleph(n,q)\cap\hbar_{u,v}(A,B,H)} \chi(a)e\Big(\frac{ma}{q}\Big) \ll nd(q)q^{\frac{1}{2}}\Big(\frac{umABd(q)\log H}{q^2} + \frac{vBd(q)\log q\log H}{q} + \log^3 q\Big),$$

which is uniformly nontrivial for any positive integer m such that $m < q^{\frac{1}{2}}$.

1.3. Notations

The following notations will be used throughout this paper:

- $e(x) = \exp(2\pi ix) = e^{2\pi ix}$;
- (a,b) denotes the greatest common divisor of a and b;
- $f \ll g$ means $|f| \leqslant cg$ for some unspecified positive constant c;
- $\phi(q)$ and $d(q)$ denote the Euler function and divisor function, respectively; $\mu(q)$ denotes the Möbius function;
- $\max\{a,b\}$ denotes the greatest number among a, b;
- $\min\{a,b\}$ denotes the least number among a, b;
- $\{x\}$ denotes the fractional part of x, and $\langle x\rangle = \min\{\{x\}, 1-\{x\}\}$;
- $\bar{x}$ denotes the multiplicative inverse of x modulo q; and
- $\sum'$ denotes the sum restricted to a reduced residue system mod q.

2. Preliminaries

Lemma 2.1. *Let Q, n, t, r be integers with $Q > \max\{2, n\}$ and $t \geq 0$. Define $h(r, t; n) = \sum_{a=1}^{n} a^t e\Big(\frac{ra}{Q}\Big)$. Then, we have*

$$h(r,t;n)\begin{cases} = \frac{n^{t+1}}{t+1} + O(n^t), & \text{if } Q \mid r, \\ \ll \frac{n^t}{\left|\sin\left(\frac{\pi s}{Q}\right)\right|}, & \text{if } Q \nmid r, \end{cases}$$

where $s = \min\{r,\ Q - r\}$ with $1 \leq r \leq Q - 1$.

Proof. See Lemma 3 of [14]. □

We need the following lemma by Xi and Yi; see [13, Lemma 1].

Lemma 2.2. *Let q be a positive integer. Then, we have*

$$|K_\chi(m, n, q)| \leq q^{\frac{1}{2}}(m, n, q)^{\frac{1}{2}} d(q),$$

where $K_\chi(m, n, q) = \sum_{a \bmod q} \chi(a) e(\frac{ma+n\bar{a}}{q})$ is the general Kloosterman sum.

Lemma 2.3. *Let $q \geq 3$ be an integer, and let χ, ψ be Dirichlet characters modulo q such that $\chi \neq \psi_0$ and $\chi\overline{\chi} = \psi_0$. Then, we have the estimate*

$$\sum_{\substack{\psi \bmod q \\ \psi \neq \psi_0 \\ \psi \neq \overline{\chi}}} G(r_1, \chi\psi)\, G(r_2, \psi) \ \ll\ \phi(q)\, q^{1/2}\, (r_1, q)^{1/2}\, (r_2, q)^{1/2}\, d(q).$$

Proof. See Lemma 3 of [7]. □

Lemma 2.4. *Let $0 < \rho \leq \frac{1}{2}$, $x_0, x_1, \ldots, x_k$ be a sequence of real numbers such that*

$$\langle x_k - x_{k'} \rangle \geq \rho, \qquad x_k \neq x_{k'},$$

and $\langle x_0 \rangle = \min\{\langle x_1 \rangle, \ldots, \langle x_k \rangle\}$. Then, we have

$$\sum_{k=1}^{K} \frac{1}{\langle x_k \rangle} \ \ll\ \rho^{-1} \log(K + 1).$$

Proof. See Lemma 2 of Chapter 5.1 in [8]. □

Lemma 2.5. *Let $q \geq 3$ be an integer, χ be a character modulo q, $n \geq 2$ be a fixed integer with $(n, q) = 1$, and l be an integer with $1 \leq l \leq n$. Then, we have*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \chi(a) e\Big(\frac{ma}{q}\Big) e\Big(\frac{(a+b)l}{n}\Big) \ll q^{1/2} \phi(q) d^2(q) \log q.$$

Proof. The exponential sums can be separated into two parts depending only on a and b, respectively, which allows us to rewrite the double sum as

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \chi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{(a+b)l}{n}\Big) \\
&= \sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{al}{n}\Big)\sum_{b=1}^{q}{}' e\Big(\frac{bl}{n}\Big).
\end{aligned} \tag{2.1}$$

Note that

$$\chi(a) = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)e\Big(-\frac{ar}{q}\Big).$$

Then, by (2.1), we obtain

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \chi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{(a+b)l}{n}\Big) \\
&= \frac{1}{q}\sum_{r=1}^{q-1} G(r,\chi)\sum_{b=1}^{q}{}' e\Big(\frac{bl}{n}\Big)\sum_{a=1}^{q} e\Big(\Big(\frac{l}{n}-\frac{(r-m)}{q}\Big)a\Big) \\
&= \frac{1}{q}\sum_{b=1}^{q}{}' e\Big(\frac{bl}{n}\Big)\Big(\sum_{r=1}^{q-1} G(r,\chi)\frac{f_m(l,r,n,q)}{e(\frac{l}{n}-\frac{(r-m)}{q})-1}\Big),
\end{aligned} \tag{2.2}$$

where

$$f_m(l,r,n,q) = 1 - e\Big(\Big(\tfrac{l}{n}-\tfrac{(r-m)}{q}\Big)q\Big).$$

Applying the upper bound

$$|G(r,\chi)| \le q^{\frac{1}{2}}(r,q),$$

we have

$$\sum_{r=1}^{q-1} G(r,\chi)\frac{f_m(l,r,n,q)}{e\Big(\frac{l}{n}-\frac{(r-m)}{q}\Big)-1} \ll q^{\frac{1}{2}}\sum_{r=1}^{q-1}\frac{(r,q)}{\Big|e\Big(\frac{l}{n}-\frac{(r-m)}{q}\Big)-1\Big|}. \tag{2.3}$$

Let $u \equiv r-m \pmod q$, $1 \le u \le q-1$, then

$$\sum_{r=1}^{q-1}\frac{(r,q)}{\big|e(\frac{l}{n}-\frac{(r-m)}{q})-1\big|} = \sum_{u=1}^{q-1}\frac{(u+m,q)}{\big|e(\frac{l}{n}-\frac{u}{q})-1\big|}. \tag{2.4}$$

We group the terms according to the value $d = (u+m,q)$. For each such $d \mid q$, write $u = dt-m$ with $1 \le t \le q/d$ and $|e(x)-1| = 2|\sin(\pi x)| \asymp \langle x\rangle$, and we obtain

$$\sum_{u=1}^{q-1}\frac{(u+m,q)}{\big|e(\frac{l}{n}-\frac{u}{q})-1\big|} = \sum_{d\mid q} d \sum_{\substack{1\le t\le q/d\\(t,q/d)=1}}\frac{1}{\langle\frac{l}{n}-\frac{dt-m}{q}\rangle}. \tag{2.5}$$

Applying the Möbius inversion, the inner sum of (2.5) becomes

$$\sum_{k|q/d}\mu(k)\sum_{t\le q/(kd)}\frac{1}{\left\langle \frac{l}{n}-\frac{tkd-m}{q}\right\rangle}. \tag{2.6}$$

The points

$$x_t=\frac{tkd-m}{q}-\frac{l}{n}$$

form an arithmetic progression with spacing kd/q

$$\langle x_{t_i}-x_{t_j}\rangle=\left\langle\frac{kd}{q}(t_i-t_j)\right\rangle\ge\frac{kd}{q},$$

then by Lemma 2.4, we obtain

$$\sum_{t\le q/(kd)}\frac{1}{\langle x_t\rangle}\ll\frac{q}{kd}\log q. \tag{2.7}$$

It follows from (2.4)–(2.7) that

$$\sum_{r=1}^{q-1}\frac{(r,q)}{\left|e(\frac{l}{n}-\frac{(r-m)}{q})-1\right|}\ll\sum_{d|q}d\sum_{k|q/d}\frac{q}{kd}\log q\ll q\,d^2(q)\log q. \tag{2.8}$$

Combining (2.2), (2.3), and (2.8), we obtain

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{(a+b)l}{n}\Big)\ll q^{1/2}\phi(q)d^2(q)\log q.$$

This completes the proof of Lemma 2.5. □

3. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. By the definition of $\aleph(n,q)$, we obtain

$$\begin{aligned}
&\sum_{a\in\aleph(n,q)}\chi(a)e\Big(\frac{ma}{q}\Big)\\
&=\sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)-\sum_{\substack{a=1\\ n|(a+\bar{a})}}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)\\
&=\sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)-\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{\substack{b=1\\ n|(a+b),ab\equiv 1 \bmod q}}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)\\
&=\sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big)-\frac{1}{\phi(q)}\sum_{\psi \bmod q}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{\substack{b=1\\ n|(a+b)}}^{q}\chi(a)\psi(ab)e\Big(\frac{ma}{q}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big) - \frac{1}{n\phi(q)}\sum_{\psi \bmod q}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\chi(a)\psi(ab)e\Big(\frac{ma}{q}\Big)\sum_{l=1}^{n}e\Big(\frac{(a+b)l}{n}\Big)\\
&= \sum_{a=1}^{q}\chi(a)e\Big(\frac{ma}{q}\Big) - \frac{1}{n\phi(q)}\sum_{\substack{\psi \bmod q\\ \psi\neq\psi_0\\ \psi\neq\bar{\chi}}}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\chi(a)\psi(ab)e\Big(\frac{ma}{q}\Big)\sum_{l=1}^{n}e\Big(\frac{(a+b)l}{n}\Big)\\
&\quad - \frac{1}{n\phi(q)}\sum_{l=1}^{n}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\chi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{(a+b)l}{n}\Big)\\
&\quad - \frac{1}{n\phi(q)}\sum_{l=1}^{n}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\overline{\chi}(b)e\Big(\frac{ma}{q}\Big)e\Big(\frac{(a+b)l}{n}\Big)\\
&= S_1 - S_2 - S_3 - S_4.
\end{aligned}
\tag{3.1}
$$

We now proceed to estimate S_2.

$$
\begin{aligned}
S_2 &= \frac{1}{n\phi(q)}\sum_{\substack{\psi \bmod q\\ \psi\neq\psi_0\\ \psi\neq\bar{\chi}}}\sum_{l=1}^{n}\sum_{a=1}^{q}{}'\chi\psi(a)e\Big(\frac{ma}{q}\Big)e\Big(\frac{al}{n}\Big)\sum_{b=1}^{q}{}'\psi(b)\,e\Big(\frac{bl}{n}\Big)\\
&= \frac{1}{n\phi(q)}\sum_{\substack{\psi \bmod q\\ \psi\neq\psi_0\\ \psi\neq\bar{\chi}}}\sum_{l=1}^{n}\sum_{a=1}^{q}\frac{1}{q}\sum_{r_1=1}^{q-1}G(r_1,\chi\psi)e\Big(-\frac{ar_1}{q}\Big)e\Big(\frac{al}{n}\Big)e\Big(\frac{ma}{q}\Big)\\
&\qquad\times\sum_{b=1}^{q}\frac{1}{q}\sum_{r_2=1}^{q-1}G(r_2,\psi)e\Big(-\frac{br_2}{q}\Big)e\Big(\frac{bl}{n}\Big)\\
&= \frac{1}{n\phi(q)q^2}\sum_{\substack{\psi \bmod q\\ \psi\neq\psi_0\\ \psi\neq\bar{\chi}}}\sum_{l=1}^{n}\sum_{r_1=1}^{q-1}G(r_1,\chi\psi)\sum_{r_2=1}^{q-1}G(r_2,\psi)\\
&\qquad\times\sum_{a=1}^{q}e\Big(\Big(\frac{l}{n}-\frac{(r_1-m)}{q}\Big)a\Big)\sum_{b=1}^{q}e\Big(\Big(\frac{l}{n}-\frac{r_2}{q}\Big)b\Big)\\
&= \frac{1}{n\phi(q)q^2}\sum_{\substack{\psi \bmod q\\ \psi\neq\psi_0\\ \psi\neq\bar{\chi}}}\sum_{l=1}^{n}\sum_{s=1}^{q-1}\sum_{r_2=1}^{q-1}G(s+m,\chi\psi)G(r_2,\psi)\\
&\qquad\times\sum_{a=1}^{q}e\Big(\Big(\frac{l}{n}-\frac{s}{q}\Big)a\Big)\sum_{b=1}^{q}e\Big(\Big(\frac{l}{n}-\frac{r_2}{q}\Big)b\Big)\\
&= \frac{1}{n\phi(q)q^2}\sum_{l=1}^{n}\sum_{s=1}^{q-1}\sum_{r_2=1}^{q-1}\frac{f_1(l,s,n,q)f_2(l,r_2,n,q)}{\Big(e\big(\frac{l}{n}-\frac{s}{q}\big)-1\Big)\Big(e\big(\frac{l}{n}-\frac{r_2}{q}\big)-1\Big)}
\end{aligned}
$$

$$\times \sum_{\substack{\psi \bmod q \\ \psi \neq \psi_0 \\ \psi \neq \bar{\chi}}} G(s+m, \chi\psi)\, G(r_2, \psi).$$

Then, from Lemma 2.3, we have

$$S_2 \ll \frac{1}{n\phi(q)q^2} \sum_{l=1}^{n} \sum_{s=1}^{q-1} \sum_{r_2=1}^{q-1} \frac{\phi(q) q^{1/2} (s+m, q)^{1/2} (r_2, q)^{1/2} d(q)}{\left| e\left(\frac{l}{n} - \frac{s}{q}\right) - 1 \right| \left| e\left(\frac{l}{n} - \frac{r_2}{q}\right) - 1 \right|}. \tag{3.2}$$

Similar to (2.8), we have

$$\sum_{s=1}^{q-1} \frac{(s+m, q)^{1/2}}{\left| e\left(\frac{l}{n} - \frac{s}{q}\right) - 1 \right|} \ll q d^2(q) \log q. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$S_2 \ll n \frac{d(q)}{q^{3/2}}\, q^2 d^4(q) \log^2 q = n q^{1/2}\, d^5(q) \log^2 q. \tag{3.4}$$

Now, we estimate S_3. By Lemma 2.5, we have

$$S_3 \ll n \frac{1}{\phi(q)}\, q^{1/2} \phi(q) d^2(q) \log q = n q^{1/2} d^2(q) \log q. \tag{3.5}$$

Similarly, we obtain the estimate

$$S_4 \ll n q^{1/2} d^2(q) \log q. \tag{3.6}$$

Combining (3.1) and (3.4)–(3.6), we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

From the orthogonality relation for character sums modulo q and the definition of $\hbar_{u,v}(A, B, H)$, we have

$$\begin{aligned}
&\sum_{a \in \hbar_{u,v}(A,B,H)} \chi(a) e\Big(\frac{ma}{q}\Big) \\
&= \sum_{t \le H} \sideset{}{'}\sum_{\substack{a \le A,\ b \le B \\ ua - vb \equiv t \bmod q \\ ab \equiv 1 \bmod q}} \chi(a) e\Big(\frac{ma}{q}\Big) \\
&= \frac{1}{q} \sum_{t \le H} \sum_{s \le q} e\Big(-\frac{st}{q}\Big) \sideset{}{'}\sum_{\substack{a \le A,\ b \le B \\ ab \equiv 1 \bmod q}} \chi(a) e\Big(\frac{ma}{q}\Big) e\Big(\frac{(ua - vb)s}{q}\Big) \\
&= \frac{1}{q^3} \sum_{t \le H} \sum_{s \le q} e\Big(-\frac{st}{q}\Big) \sideset{}{'}\sum_{\substack{a,b \le q \\ ab \equiv 1 \bmod q}} \chi(a) e\Big(\frac{(m + us + r)a - (vs - w)b}{q}\Big) \\
&\quad \times \sum_{c \le A} \sum_{r \le q} e\Big(\frac{-rc}{q}\Big) \sum_{d \le B} \sum_{w \le q} e\Big(\frac{-wd}{q}\Big)
\end{aligned}$$

$$= \frac{1}{q^3}\sum_{r,w\le q}\sum_{s\le q}\sum_{t\le H} e\Big(-\frac{st}{q}\Big)K_\chi(m+us+r,\ -vs+w;\ q)\,h(-r,0;A)\,h(-w,0;B).$$

Then, we obtain

$$\begin{aligned}
&\sum_{a\in\hbar_{u,v}(A,B,H)}\chi(a)e\Big(\frac{ma}{q}\Big)\\
&= \frac{1}{q^3}\sum_{s\le q}\sum_{t\le H} e\Big(-\frac{st}{q}\Big)K_\chi(m+us,\ -vs;\ q)\,h(-q,0;A)\,h(-q,0;B)\\
&\quad+\frac{1}{q^3}\sum_{r\le q-1}\sum_{s\le q}\sum_{t\le H} e\Big(-\frac{st}{q}\Big)K_\chi(m+us+r,\ -vs;\ q)\,h(-r,0;A)\,h(-q,0;B)\\
&\quad+\frac{1}{q^3}\sum_{w\le q-1}\sum_{s\le q}\sum_{t\le H} e\Big(-\frac{st}{q}\Big)K_\chi(m+us,\ -vs+w;\ q)\,h(-q,0;A)\,h(-w,0;B)\\
&\quad+\frac{1}{q^3}\sum_{r,w\le q-1}\sum_{s\le q}\sum_{t\le H} e\Big(-\frac{st}{q}\Big)K_\chi(m+us+r,\ -vs+w;\ q)\,h(-r,0;A)\,h(-w,0;B)\\
&= \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4.
\end{aligned}\tag{4.1}$$

First, we estimate $\mathcal{T}_1$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\mathcal{T}_1 &\ll \frac{1}{q^3}\sum_{s\le q}\min\Big\{H,\ \Big\langle\frac{s}{q}\Big\rangle^{-1}\Big\}|K_\chi(m+us,-vs;q)|\,|h(-q,0;A)|\,|h(-q,0;B)|\\
&\ll HABq^{-5/2}d(q)\sum_{s\le q/H}(m+us,q)^{1/2}+ABq^{-3/2}d(q)\sum_{q/H<s\le q-1}\frac{(m+us,q)^{1/2}}{s}.
\end{aligned}$$

Note the estimates

$$\begin{aligned}
\sum_{s\le q/H}(m+us,q)^{1/2} &= \sum_{d|q}d^{1/2}\sum_{\substack{s\le q/H\\ d|(m+us)}}1\\
&\ll \sum_{d|q}d^{1/2}\Big(\frac{m}{d}+\frac{uq}{dH}\Big)\ll md(q)+H^{-1}qud(q)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{q/H<s\le q-1}\frac{(m+us,q)^{1/2}}{s} &= \sum_{d|q}d^{1/2}\sum_{\substack{q/H<s\le q-1\\ us\equiv -m \mod d}}\frac{1}{s}\\
&= \sum_{d|q}d^{1/2}\frac{(u,d)}{d}\log H\ll ud(q)\log H.
\end{aligned}$$

Then, we have

$$\mathcal{T}_1\ll uABq^{-3/2}d^2(q)\log H+mABq^{-3/2}d^2(q).\tag{4.2}$$

Now, we estimate $\mathcal{T}_2$. It follows from Lemmas 2.1 and 2.2 that

$$\mathcal{T}_2\ll\frac{1}{q^3}\sum_{r\le q-1}\sum_{s\le q}\min\Big\{H,\ \Big\langle\frac{s}{q}\Big\rangle^{-1}\Big\}|K_\chi(m+us+r,\ -vs;q)|\,|h(-r,0;A)|\,|h(-q,0;B)|$$

$$
\begin{aligned}
&\ll HBq^{-5/2}d(q)\sum_{r\le q-1}\sum_{s\le q/H}(m+us+r,-vs,q)^{1/2}\frac{1}{|\sin(\frac{\pi r}{q})|}\\
&\quad+\ Bq^{-3/2}d(q)\sum_{r\le q-1}\sum_{q/H<s\le q-1}\frac{(m+us+r,-vs,q)^{1/2}}{s}\frac{1}{|\sin(\frac{\pi r}{q})|}\\
&\ll HBq^{-3/2}d(q)\sum_{r\le q-1}\frac{1}{r}\sum_{s\le q/H}(m+us+r,-vs,q)^{1/2}\\
&\quad+Bq^{-1/2}d(q)\sum_{r\le q-1}\frac{1}{r}\sum_{q/H<s\le q-1}\frac{(m+us+r,-vs,q)^{1/2}}{s}.
\end{aligned}
$$

Combine

$$
\begin{aligned}
&\sum_{r\le q-1}\frac{1}{r}\sum_{s\le q/H}(m+us+r,-vs,q)^{1/2}\\
&\quad=\sum_{d|q}d^{1/2}\sum_{\substack{s\le q/H\\ d|vs}}\ \sum_{\substack{r\le q-1\\ r\equiv-(m+us)\mod d}}\frac{1}{r}\\
&\quad\ll\sum_{d|q}d^{1/2}\frac{q(v,d)}{dH}\sum_{\substack{r\le q-1\\ r\equiv-(m+us)\mod d}}\frac{1}{r}\\
&\quad\ll\sum_{d|q}d^{1/2}\frac{q(v,d)}{dH}\frac{\log q}{d}\ll vH^{-1}qd(q)\log q,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{r\le q-1}\frac{1}{r}\sum_{q/H<s\le q-1}\frac{(m+us+r,-vs,q)^{1/2}}{s}\\
&\quad=\sum_{d|q}d^{1/2}\sum_{\substack{q/H<s\le q-1\\ d|vs}}\frac{1}{s}\sum_{\substack{r\le q-1\\ r\equiv-(m+us)\mod d}}\frac{1}{r}\\
&\quad\ll\sum_{d|q}d^{1/2}\frac{\log q}{d}\sum_{\substack{q/H<s\le q-1\\ d|vs}}\frac{1}{s}\\
&\quad\ll\log q\sum_{d|q}d^{-1/2}\frac{(v,d)}{d}\log H\\
&\quad\ll vd(q)\log H\log q.
\end{aligned}
$$

Then, we have

$$\mathcal{T}_2\ll vBq^{-1/2}d^2(q)\log H\log q. \tag{4.3}$$

Similarly, we get the estimate

$$\mathcal{T}_3\ll vAq^{-3/2}d^2(q)\log q. \tag{4.4}$$

Next, we consider $\mathcal{T}_4$. By Lemmas 2.1 and 2.2, we have

$$\mathcal{T}_4\ll q^{-5/2}d(q)\sum_{r,w\le q-1}\sum_{s\le q}\min\Big\{H,\ \Big\langle\frac{s}{q}\Big\rangle^{-1}\Big\}(m+us+r,w-vs,q)^{1/2}\frac{1}{|\sin(\frac{\pi r}{q})|}\frac{1}{|\sin(\frac{\pi w}{q})|}$$

$$\ll Hq^{-1/2}d(q)\sum_{r,\,w\le q-1}\frac{1}{rw}\sum_{s\le q/H}(m+us+r,w-vs,q)^{1/2}$$
$$+\,q^{1/2}d(q)\sum_{r,\,w\le q-1}\frac{1}{rw}\sum_{q/H<s\le q-1}\frac{(m+us+r,w-vs,q)^{1/2}}{s}.$$

By the estimates

$$\begin{aligned}\sum_{r,w\le q-1}\frac{1}{rw}\sum_{s\le q/H}(m+us+r,w-vs,q)^{1/2}&=\sum_{d|q}d^{1/2}\sum_{s\le q/H}\sum_{\substack{r\le q-1\\ d|m+us+r}}\frac{1}{r}\sum_{\substack{w\le q-1\\ d|w-vs}}\frac{1}{w}\\ &\ll\sum_{d|q}d^{1/2}\sum_{s\le q/H}\sum_{\frac{m+us+1}{d}<t\le\frac{m+us+q-1}{d}}\frac{1}{dt-(m+us)}\sum_{\frac{1-vs}{d}<t'\le\frac{q-1-vs}{d}}\frac{1}{vs+dt'}\\ &\ll H^{-1}qd(q)\log^2 q,\end{aligned}$$

and

$$\begin{aligned}\sum_{r,w\le q-1}\frac{1}{rw}\sum_{q/H<s\le q-1}\frac{(m+us+r,w-vs,q)^{1/2}}{s}&=\sum_{d|q}d^{1/2}\sum_{q/H<s\le q-1}\frac{1}{s}\sum_{\substack{r\le q-1\\ d|m+us+r}}\frac{1}{r}\sum_{\substack{w\le q-1\\ d|w-vs}}\frac{1}{w}\\ &=\sum_{d|q}d^{1/2}\sum_{q/H<s\le q-1}\frac{1}{s}\sum_{\frac{m+us+1}{d}<t\le\frac{m+us+q-1}{d}}\frac{1}{dt-(m+us)}\sum_{\frac{1-vs}{d}<t'\le\frac{q-1-vs}{d}}\frac{1}{vs+dt'}\\ &\ll d(q)\log^3 q,\end{aligned}$$

we have

$$\mathcal{T}_4\ll q^{\frac{1}{2}}d^2(q)\log^3 q. \tag{4.5}$$

By (4.1)–(4.5), we complete the proof of Theorem 1.2.

5. Proof of Theorem 1.4

From the orthogonality relation for character sums modulo q and the definition of $\aleph(n,q)$, we have

$$\begin{aligned}&\sum_{a\in\aleph(n,q)\cap\hbar_{uv}(A,B,H)}\chi(a)e\Big(\frac{ma}{q}\Big)\\ &=\sum_{a\in\hbar_{uv}(A,B,H)}\chi(a)e\Big(\frac{ma}{q}\Big)-\sum_{\substack{a\in\hbar_{uv}(A,B,H)\\ n|(a+\bar a)}}\chi(a)e\Big(\frac{ma}{q}\Big)\\ &=\mathcal{M}_1-\mathcal{M}_2.\end{aligned} \tag{5.1}$$

By Theorem 1.2, we obtain

$$\mathcal{M}_1\ll q^{\frac{1}{2}}d(q)\Big(\frac{umABd(q)\log H}{q^2}+\frac{vBd(q)\log q\log H}{q}+\log^3 q\Big). \tag{5.2}$$

Then, we estimate $\mathcal{M}_2$.

$$\begin{aligned}
\mathcal{M}_2 &= \frac{1}{n} \sum_{\substack{t\le H \\ ua-vb\equiv t \bmod q, ab\equiv 1 \bmod q}} \sideset{}{'}\sum_{a\le A, b\le B} \chi(a) \sum_{l=1}^{n} e\Big(\frac{(a+b)l}{n}\Big) e\Big(\frac{ma}{q}\Big) \\
&= \frac{1}{nq} \sum_{s\le q} \sum_{t\le H} e\Big(\frac{-st}{q}\Big) \sideset{}{'}\sum_{\substack{a\le A, b\le B \\ ab\equiv 1 \bmod q}} \chi(a) e\Big(\frac{(m+us)a - vsb}{q}\Big) \sum_{l=1}^{n} e\Big(\frac{(a+b)l}{n}\Big) \\
&= \frac{1}{nq^3} \sum_{s\le q} \sum_{t\le H} e\Big(\frac{-st}{q}\Big) \sum_{\substack{a\le A, b\le B \\ ab\equiv 1 \bmod q}} \chi(a) e\Big(\frac{(m+us)a - vsb}{q}\Big) \\
&\quad \times \sum_{c\le A} \sum_{r\le q} e\Big(\frac{r(a-c)}{q}\Big) \sum_{d\le B} \sum_{w\le q} e\Big(\frac{w(b-d)}{q}\Big) \sum_{l=1}^{n} e\Big(\frac{(a+b)l}{n}\Big) \\
&= \frac{1}{nq^3} \sum_{r,w\le q} \sum_{s\le q} \sum_{t\le H} e\Big(\frac{-st}{q}\Big) \sum_{\substack{a,b\le q \\ ab\equiv 1 \bmod q}} \chi(a) e\Big(\frac{(m+us+r)a - (vs-w)b}{q}\Big) \\
&\quad \times \sum_{l=1}^{n} \sum_{c\le A} e\Big(\Big(\frac{l}{n} - \frac{r}{q}\Big)c\Big) \sum_{d\le B} e\Big(\Big(\frac{l}{n} - \frac{w}{q}\Big)d\Big) \\
&= \frac{1}{nq^3} \sum_{r,w\le q} \sum_{s\le q} \sum_{t\le H} e\Big(\frac{-st}{q}\Big) \sum_{l=1}^{n} K_\chi(m+us+r, w-vs, q) h(lq-nr, 0, A) h(lq-nw, 0, B).
\end{aligned}$$

Similarly, following the proof of Theorem 1.2, we obtain

$$\mathcal{M}_2 \ll nd(q)q^{\frac{1}{2}} \Big(\frac{umABd(q)\log H}{q^2} + \frac{vBd(q)\log q \log H}{q} + \log^3 q \Big). \tag{5.3}$$

Then, by (5.1)–(5.3), we complete the proof of Theorem 1.4.

Author contributions

Xiaoying Liu: Conceptualization, theoretical analysis, writing–original draft; Zhefeng Xu: Supervision, conceptualization, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 12471006, 12371007), and the Natural Science Foundation of Shaanxi Province (No. 2025JC-YBQN-015).

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. D. A. Burgess, Partial Gaussian sums, *Bull. London Math. Soc.*, **20** (1988), 589–592. https://doi.org/10.1112/blms/20.6.589
2. D. A. Burgess, Partial Gaussian sums II, *Bull. London Math. Soc.*, **21** (1989), 153–158. https://doi.org/10.1112/blms/21.2.153
3. D. A. Burgess, Partial Gaussian sums III, *Glasgow Math. J.*, **34** (1992), 253–261. https://doi.org/10.1017/S001708950000879X
4. R. K. Guy, *Unsolved problems in number theory*, 3 Eds., New York: Springer, 2004. https://doi.org/10.1007/978-0-387-26677-0
5. D. Han, W. P. Zhang, Upper bound estimate of character sums over Lehmer's numbers, *J. Inequal. Appl.*, **2013** (2013), 392. https://doi.org/10.1186/1029-242x-2013-392
6. D. Han, Z. F. Xu, Y. Yi, T. P. Zhang, A note on high-dimensional D. H. Lehmer problem, *Taiwanese J. Math.*, **25** (2021), 1137–1157. https://doi.org/10.11650/tjm/210705
7. Y. K. Ma, H. Chen, Z. Z. Qin, T. P. Zhang, Character sums over generalized Lehmer numbers, *J. Inequal. Appl.*, **2016** (2016), 270. https://doi.org/10.1186/s13660-016-1213-y
8. C. D. Pan, *Goldbach conjecture*, Beijing: Science Press, 1981.
9. G. L. Ren, D. D. He, T. P. Zhang, On general partial Gaussian sums, *J. Inequal. Appl.*, **2016** (2016), 295. https://doi.org/10.1186/s13660-016-1238-2
10. I. E. Shparlinski, On a generalization of a Lehmer problem, *Math. Z.*, **263** (2009), 619–631. https://doi.org/10.1007/s00209-008-0434-2
11. J. K. Wang, Z. F. Xu, Partitions into two Lehmer numbers in $\mathbb{Z}_q$, *Math. Slovaca*, **75** (2025), 45–54. https://doi.org/10.1515/ms-2025-0004
12. X. Wang, L. Wang, G. H. Chen, The fourth power mean of the generalized quadratic Gauss sums associated with some Dirichlet characters, *AIMS Mathematics*, **9** (2024), 17774–17783. https://doi.org/10.3934/math.2024864
13. P. Xi, Y. Yi, On character sums over flat numbers, *J. Number Theory*, **130** (2010), 1234–1240. https://doi.org/10.1016/j.jnt.2009.10.011
14. Z. F. Xu, Distribution of the difference of an integer and its m-th power mod n over incomplete intervals, *J. Number Theory*, **133** (2013), 4200–4223. https://doi.org/10.1016/j.jnt.2013.06.006
15. W. P. Zhang, On a problem of D. H. Lehmer and its generalization, *Compos. Math.*, **86** (1993), 307–316.
16. W. P. Zhang, A problem of D. H. Lehmer and its generalization (II), *Compos. Math.*, **91** (1994), 47–56.
17. W. P. Zhang, L. Wang, New product formulas for classical Gauss sums, *Acta. Math. Sin.-English Ser.*, **41** (2025), 1580–1590. https://doi.org/10.1007/s10114-025-3543-5

18. M. X. Zhong, T. P. Zhang, Partitions into three generalized D. H. Lehmer numbers, *AIMS Mathematics*, **9** (2024), 4021–4031. https://doi.org/10.3934/math.2024196