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Research article

The generalized k-connectivity of conditional recursive networks

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Abstract: The generalized k-connectivity $\kappa_k(G)$ of graph G is defined as the maximum number of internally disjoint Steiner trees in G, which is a generalization of classical connectivity $\kappa(G)$ of G just for $k = 2$. Conditional recursive networks (CRNs) form a new family of composite networks constructed from complete graphs. In this paper, we determine the generalized k-connectivity of CRNs for $k = 4$.

Keywords: the generalized connectivity; the generalized edge-connectivity; conditional recursive networks
Mathematics Subject Classification: 05C05, 05C40

1. Introduction

We consider finite, simple, and undirected graphs; for notations and definitions not described here, we refer to the book [1, 2]. As usual, we by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ respectively denote the vertex set, the edge set, the maximum degree, the minimum degree of graph G and by $[n]$ denote the set of positive integers $\{1, 2, \ldots, n\}$. Use $E(S_1, S_2)$ to denote the set of edges whose ends are respectively in $S_1 \subset V(G)$ and $S_2 \subset V(G)$, which is simplified by $E(S)$ when $S_1 = S_2$.

For a graph $G = (V, E)$ of order n and a vertex set $S \subseteq V$ with at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $V(T) \cap V(T') = S$ and $E(T) \cap E(T') = \emptyset$ (or *edge disjoint* if $E(T) \cap E(T') = \emptyset$). The *generalized local connectivity* $\kappa(S)$ (or the *generalized local edge-connectivity* $\lambda(S)$) is the maximum number of internally (or edge) disjoint S-trees connecting S in G. For an integer k with $2 \le k \le n$, the *generalized k-connectivity* $\kappa_k(G)$ (or the *generalized k-edge-connectivity* $\lambda_k(G)$) of G is defined as $\kappa_k(G) = \min\{\kappa(S) \mid S \subseteq V(G), |S| = k\}$ and $\kappa_2(G) = \kappa(G)$ (or $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G)\ ,\ |S| = k\}$ and $\lambda_2(G) = \lambda(G)$.) For exact values, further research can be found in [4, 12]; for generalized values, see [6, 7, 10, 11]. Clearly, the generalized k-connectivity and generalized k-edge-connectivity measure the fault tolerance of an interconnection network more accurately than the classical connectivity, evaluates the fault tolerance in terms of the number of disjoint paths. Unfortunately, computing the exact values of generalized k-(edge) connectivity for general k is NP-complete, even for special graphs.

In [8], S. Li determined the generalized 3-connectivity of graphs such as star graphs S_n and bubble-sort graphs B_n. J. Wang [15] considered the generalized 3-connectivity of burnt pancake graphs BP_n and godan graphs. S. Zhao [19] determined the generalized 3-connectivity of alternating group graphs and (n, k)-star graphs. More recently, several authors have discussed the generalized 4-connectivity of graphs such as burnt pancake graphs BP_n [14], godan graphs [13], alternating Group Graphs AG_n [5], pancake graphs [18], and hierarchical cubic networks [20].

Conditional recursive networks (CRNs) form a new family of composite networks based on the complete graph, which includes many common networks and shares the same structural properties as alternating group networks. For further research on these networks, see [16, 21].

Definition 1.1. *[17] Let $l(\geq 3)$ be an integer and K_l be a complete graph. The l-order 1-dimensional conditional recursive networks (simplified CRNs), denoted by $G_{l,1}$, is isomorphic to K_l. For $m \geq 2$, the l-order m-dimensional conditional recursive networks $G_{l,m} \cong K_l$ for $1 \leq m \leq l$ and for $m \geq l+1$, $G_{l,m}$ can be divided into m disjoint subnetworks $G^j_{l,m-1}$ for $j \in \{1, 2, \dots, m\}$, where $G^j_{l,m-1} \cong G_{l,m-1}$. $G_{l,m}$ usually denoted by $G^1_{l,m-1} \oplus G^2_{l,m-1} \oplus \dots \oplus G^m_{l,m-1}$ and satisfies the following conditions. $G_{3,m}$ for $1 \leq m \leq 5$ illustrated in Figure 1.*
(1) For any $1 \leq i \neq j \leq m$, $|E(G^i_{l,m-1}, G^j_{l,m-1})| = \frac{(m-2)!}{(l-1)!}$.
(2) $G_{l,m}$ has a perfect matching $M = E(G_{l,m}) - E(\cup^m_{i=1} G^i_{l,m-1})$.
(3) For any vertex $v \in V(G^i_{l,m-1})$, v has only one external neighbor in $G_{l,m} - G^i_{l,m-1}$.
(4) $G_{l,m}$ can be decomposed into $\frac{m!}{r!}$ disjoint subnetworks $G_{l,r}$ when $m \geq l+1$.

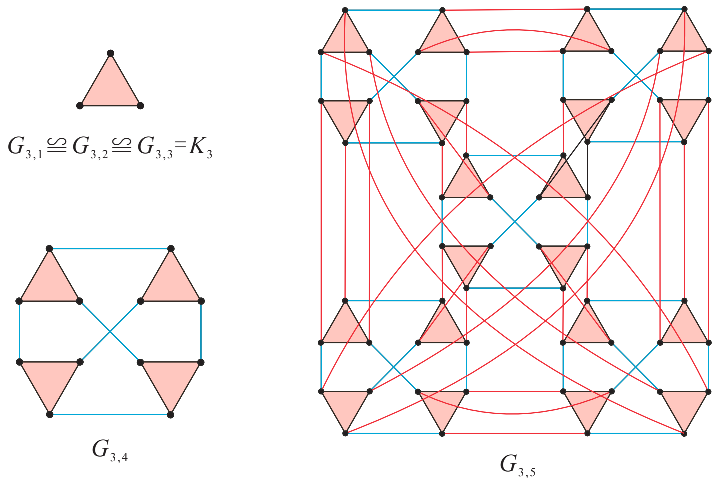

Figure 1. The CRNs $G_{3,m}$ for $m \in \{1, 2, 3, 4, 5\}$.

In this paper, we determine the generalized 4-connectivity of CRNs and show that $\kappa_4(G_{l,m}) = \lambda_4(G_{l,m}) = m - 2$.

2. Preliminary

Lemma 2.1. *[17] Let $G_{l,m}$ be l-order m-dimensional CRNs for $m \geq l \geq 3$. Then, (1) $G_{l,m}$ is a $(m-1)$-regular graph with $\frac{m!}{(l-1)!}$ vertices. (2) $\kappa(G_{l,m}) = m - 1$.*

Lemma 2.2. *[2] Let G be a k-connected graph, let x be a vertex of G and let $Y \subseteq V(G)\backslash\{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y, that is, there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

Lemma 2.3. *[2] Let G be a k-connected graph, and let $X, Y \subseteq V(G)$ with $|X|, |Y| \geq k$. Then there exists a set of k pairwise vertex-disjoint (X, Y)-paths in G.*

Lemma 2.4. *[3] For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$.*

Lemma 2.5. *[9] Let G be a connected graph of order n with minimum degree δ. If there are two adjacent vertices of degree δ, then $\kappa_k(G) \leq \lambda_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.*

3. Main results

In this section, we determine the generalized 4-connectivity and the generalized 4-edge-connectivity of conditional recursive networks $G_{l,m}$.

Theorem 3.1. *Let $G_{l,m}$ be l-order m-dimensional CRNs for $l \geq 3$. Then*

$$\kappa_4(G_{l,m}) = \begin{cases} l-2, & \text{If } 1 \leq m \leq l; \\ m-2, & \text{If } m \geq l+1. \end{cases}$$

Proof. First, consider $G_{l,m} \cong K_l$ for $1 \leq m \leq l$, by Lemma 2.4, we directly get $\kappa_4(G_{l,m}) = \kappa_4(K_l) = l - \lceil \frac{4}{2} \rceil = l - 2$. Here we mainly discuss the case for $m \geq l + 1 \geq 4$; since $G_{l,m}$ is $(m-1)$-regular, by Lemma 2.5, we have $\kappa_4(G_{l,m}) \leq \delta - 1 = m - 2$. Now we need only show $\kappa_4(G_{l,m}) \geq m - 2$, this suffice to prove that there exist $m - 2$ internally disjoint S-trees in $G_{l,m}$ for any 4-element set $S \subseteq V(G_{l,m})$. For convenience to narrate, let $G_{l,m} = G^1 \oplus G^2 \oplus \ldots \oplus G^m$, where $G^i \cong G_{l,m-1}$ for $i \in [m]$, and suppose $S = \{x, y, z, w\}$ is any 4-element subset of $V(G_{l,m})$. First, we consider $m = 4$; this means $G_{l,m} = G_{3,4}$ and $G^i = G_{3,3} = K_3$. By simple checking, there always exist 2 internally disjoint S-trees in $G_{3,4}$, this implies $\kappa_4(G_{l,m}) \geq 2 = m - 2$ holds. The following considers the case for $m \geq 5$.

Case 1 $|S \cap V(G^i)| = 4$ for some $i \in [m]$

Without loss of generality, suppose $x, y, z, w \in V(G^1)$. We proceed by induction on m. For the case $m = 5$, it is clear that $G_{l,m}$ is $G_{3,5}$ with $G^1 = G_{3,4}$ or $G_{4,5}$ with $G^1 = K_4$. Consider $x, y, z, w \in V(G^1)$; by similar checking, we find that whether it is $G_{3,5}$ or $G_{4,5}$, it always contains 3 internally disjoint S-trees. This implies $\kappa_4(G_{l,m}) \geq m-2$ holds for $m = 5$. Now we assume that the conclusion holds for $m = k(\geq 6)$ and go on to verify the conclusion holds for $m = k+1$. Notice $G_{l,k+1} = G^1 \oplus G^2 \oplus \ldots \oplus G^{k+1}$ with $G^i \cong G_{l,k}$ for $i \in [k + 1]$ and $x, y, z, w \in V(G^1)$, by induction hypothesis, we have $\kappa_4(G^1) \geq k - 2$ for $k \geq l + 1$. This implies that there are at least $k - 2$ internally disjoint S-trees in G^1, named $T_1, T_2, \ldots, T_{k-2}$. By Definition 1.1(3), we know x, y, z, w have only one external neighbor in $G_{l,k+1} - G^1$, named x', y', z' and w', respectively. Consider $G_{l,k+1}$ is k-regular and G^1 is $(k-1)$-regular; we can construct a $\{x', y', z', w'\}$-tree T in $G_{l,k+1}$. Now let $T_{k-1} = xx' + yy' + zz' + T$. Thus, $T_1, T_2, \ldots, T_{k-1}$ are $k - 1$ internally disjoint S-trees in $G_{l,k+1}$. This follows $\kappa_4(G_{l,k+1}) \geq k-1 = (k+1)-2$, and the conclusion holds for $m = k+1 (k \geq 6)$. By the above argument, we have $\kappa_4(G_{l,m}) \geq m-2$ for $m \geq k+1$.

Case 2 $|S \cap V(G^i)| = 3, |S \cap V(G^j)| = 1$ for some $i \neq j \in [m]$

Without loss of generality, suppose $x, y, z \in V(G^1)$ and $w \in V(G^2)$. First, we consider the case $m = l+1$. Notice that $G_{l,l+1} = G^1 \oplus G^2 \oplus \ldots \oplus G^{l+1}$, where $G^i \cong K_l$ for $i \in [l+1]$. Now we show that $\kappa_4(G_{l,l+1}) \geq l-1$. Consider $x, y, z \in V(G^1) = V(K_l)$ and $w \in V(G^2) = V(K_l)$. By Lemma 2.4, we can get $\kappa_3(G^1) = \kappa_3(K_l) = l - \lceil \frac{3}{2} \rceil = l-2$. This means there are $l-2$ internally disjoint $\{x, y, z\}$-trees $T_i (1 \leq i \leq l-2)$ in G^1. Further, we find that one edge of triangle xyz is always missed in the process of forming $\{x, y, z\}$-trees T_i, without loss of generality, suppose xy is the missing edge. Now we turn our attention to the external neighbors of x, y, z, w, named x', y', z', w'. Consider $x, y, z \in V(G^1)$, then x', y', z' are in distinct G^i for $i \neq 1$.

If $\{x', y', z'\} \cap V(G^2) = \emptyset$, without loss of generality, suppose $x' \in V(G^5)$, $y' \in V(G^3)$, and $z' \in V(G^4)$. Then we form $l-1$ internally disjoint S-trees in $G_{l,l+1}$ such as $T_i + G^j + w$ for $1 \leq i \leq l-3$ and $j \in [l+1] \setminus \{1, 3, 4, 5\}$, $T_{l-2} + y' + G^3 + w$ and $xy + x' + G^5 + w + G^4 + z' + z$. Otherwise, if $\{x', y', z'\} \cap V(G^2) \neq \emptyset$, suppose $y' \in V(G^2)$, then $l-1$ internally disjoint S-trees are $T_i + G^j + w$ for $1 \leq i \leq l-3$ and $j \in [l+1] \setminus \{1, 2, 4, 5\}$, $T_{l-2} + y' + w$ and $xy + x' + G^5 + w + G^4 + z' + z$. By now, we show that $\kappa_4(G_{l,m}) \geq m-2$ for $m = l+1$. An example of $G_{4,5}$ illustrated in Figure 2.

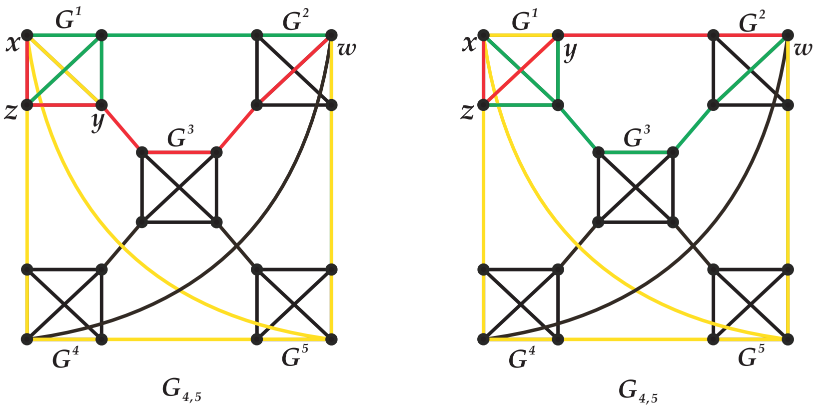

Figure 2. Internally disjoint S-trees in $G_{4,5}$ illustrated by distinct color.

Next, we consider the case for $m > l+1$. Recall $G_{l,m} = G^1 \oplus G^2 \oplus \ldots \oplus G^m$, $x, y, z \in V(G^1)$, $w \in V(G^2)$ for $G^1 = G^2 = G_{l,m-1}$, let $G^1 = G^{11}_{l,m-1} \oplus G^{12}_{l,m-1} \oplus \ldots \oplus G^{1(m-1)}_{l,m-1}$, $G^2 = G^{21}_{l,m-1} \oplus G^{22}_{l,m-1} \oplus \ldots \oplus G^{2(m-1)}_{l,m-1}$ where $G^{1i}_{l,m-1} = G^{2i}_{l,m-1} = G_{l,m-2}$ for $1 \leq i \leq m-1$. Suppose $x \in V(G^{1r}_{l,m-1}), y \in V(G^{1s}_{l,m-1}), z \in V(G^{1t}_{l,m-1})$ for $r, s, t \in [m-1]$. Further, we find that the external neighbor $u' \in V(G^m)$ of vertex $u \in V(G^{11}_{l,m-1})$ and the external neighbor $u' \in V(G^b)$ of $u \in V(G^{1b}_{l,m-1})$ for $1 \leq b \leq m$.

Subcase 2.1 $r \neq s \neq t \in [m-1]$

Clearly, x, y, and z are in distinct parts, this means the external neighbors x', y' z' are also in distinct parts. Consider $\kappa(G^1) = m-2$ and $x, y, z \in V(G^1)$; G^1 contains at least $m-2$ internally disjoint (x, y)-paths, which are denoted by $xP_1y, xP_2y, \ldots, xP_{m-2}y$. Now we select $m-3$ vertices $x_i \in V(P_i) \cap N_{G^{1r}_{l,m-1}}(x) \setminus \{y\}$ and their internal neighbors $\widehat{x_i} \in N_{G^1}(x_i) \setminus \{x, y\}$ for $1 \leq i \leq m-3$. By Lemma 2.2, there exists an $(m-3)$-fan from z to $Z = \{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_{m-3}\}$, which are internally disjoint (z, Z)-paths denoted by $\widehat{x}_1Q_1z, \widehat{x}_2Q_2z, \ldots, \widehat{x}_{m-3}Q_{m-3}z$. Further, we discuss as follows.

Subcase 2.1.1 $r \neq s \neq t \neq 2$

Consider $x \in V(G^{1r}_{l,m-1}), y \in V(G^{1s}_{l,m-1}), z \in V(G^{1t}_{l,m-1})$ for $r \neq s \neq t \neq 2$, suppose $x \in V(G^{11}_{l,m-1})$, $y \in V(G^{14}_{l,m-1})$ and $z \in V(G^{13}_{l,m-1})$. Then the external neighbors of x, y, z and x_i are $x' \in V(G^m)$, $y' \in V(G^4)$, $z' \in V(G^3)$ and $x'_i \in V(G^m)$ for $i \in [m-3]$. If $w \notin V(G^{2m}_{l,m-1})$, we choose $m-2$ vertices $w_i \in V(G^{2m}_{l,m-1})$ such that their external neighbors $w'_i \in V(G^m)$. Consider $\kappa(G^2) = m-2$, there exists an $(m-2)$-fan from w to $W = \{w_1, w_2, \ldots, w_{m-2}\} \subseteq V(G^2)\backslash\{w\}$, and denoted by $wW_1w_1, wW_2w_2, \ldots, wW_{m-2}w_{m-2}$. Let $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$ and $W' = \{w'_1, w'_2, \ldots, w'_{m-2}\}$. By Lemma 2.3, G^m contains a set of $m-2$ pairwise vertex disjoint (X', W')-paths, which are denoted by $x'_1L_1w'_1, x'_2L_2w'_2, \ldots, x'_{m-3}L_{m-3}w'_{m-3}, x'L_{m-2}w'_{m-2}$. Now choose $\widehat{x'} \in V(L_{m-2})$ such that its external neighbor $(\widehat{x'})' \in V(G_{l,m}\backslash(G^1 \cup G^2 \cup G^m))$, and then we get a $\{y', z', (\widehat{x'})'\}$-tree T in $G_{l,m}\backslash(G^1 \cup G^2 \cup G^m)$ by using these external neighbors $y', z', (\widehat{x'})'$. Now let $T_i = xP_iy + x_i\hat{x}_i + \hat{x}_iQ_iz + x_ix'_i + x'_iL_iw'_i + w_iW_iw$ for $i \in [m-3]$ and $T_{m-2} = xx' + x'L_{m-2}w'_{m-2} + w_{m-2}W_{m-2}w + x'\widehat{x'} + \widehat{x'}(\widehat{x'})' + T + y'y + z'z$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 3. Otherwise, if $w \in V(G^{2m}_{l,m-1})$, we have $T^*_{m-2} = xx' + x'L_{m-2}ww' + w'w + x'\widehat{x'} + \widehat{x'}(\widehat{x'})' + T + y'y + z'z$ replace T_{m-2} for case $w \notin V(G^{2m}_{l,m-1})$ to get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-3}, T^*_{m-2}$ in $G_{l,m}$.

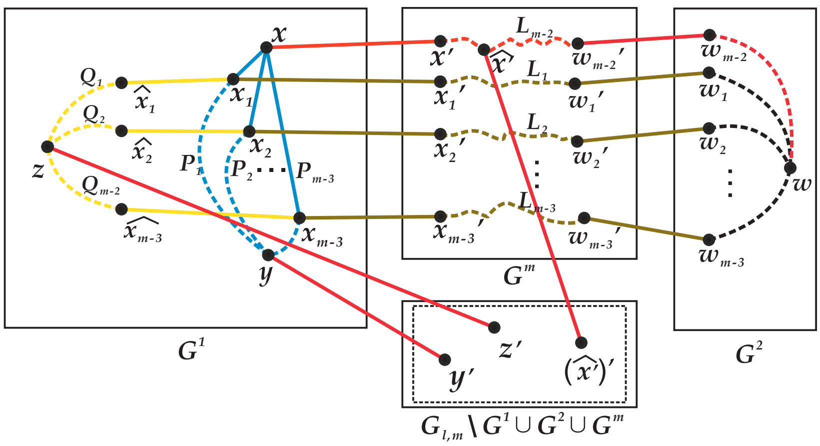

Figure 3. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Subcase 2.1.1.

In particular, if $L_{m-2} = P_2$, we choose a vertex $\widehat{w} \in N_{G^{2m}_{l,m-1}}(w)$ such that its external neighbor $\widehat{w}' \in V(G_{l,m}\backslash(G^1 \cup G^2 \cup G^m))$ and use these external neighbors $y', z', \widehat{w}'$ to construct a $\{y', z', \widehat{w}'\}$-tree T in $G_{l,m}\backslash(G^1 \cup G^2 \cup G^m)$. And then, we by $xx' + x'L_{m-2}w'_{m-2} + w_{m-2}W_{m-2}w + w\widehat{w} + \widehat{w}\widehat{w}' + T + y'y + z'z$ replace T_{m-2} for case $w \notin V(G^{2m}_{l,m-1})$ and by $xx' + x'L_{m-2}ww' + w'w + w\widehat{w} + \widehat{w}\widehat{w}' + T + y'y + z'z$ replace T^*_{m-2} for case $w \in V(G^{2m}_{l,m-1})$ to get $m-2$ internally disjoint S-trees in $G_{l,m}$.

Subcase 2.1.2 One of r, s, t is 2

Suppose $x \in V(G^{12}_{l,m-1})$, $y \in V(G^{14}_{l,m-1})$ and $z \in V(G^{13}_{l,m-1})$. Then external neighbors of x, y, z and x_i are $x' \in V(G^2)$, $y' \in V(G^4)$, $z' \in V(G^3)$ and $x'_i \in V(G^2)$ for $i \in [m-3]$. Consider $\kappa(G^2) = m-2$, there exists an $(m-2)$-fan from w to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$, which denoted by $wW_1x'_1, wW_2x'_2, \ldots, wW_{m-3}x'_{m-3}, wW_{m-2}x'$. Similar discussion as before, if $w' \notin V(G^1)$, then external neighbors of y, z, w are $y', z', w' \in V(G_{l,m}\backslash(G^1 \cup G^2))$ and we use these external neighbors to construct a $\{y', z', w'\}$-tree T in $G_{l,m}\backslash(G^1 \cup G^2)$. Now we let $T_i = xP_iy + x_i\hat{x}_i + \hat{x}_iQ_iz + x_ix'_i + x'_iW_iw$ for $i \in [m-3]$, $T_{m-2} = xx' + x'W_{m-2}w + ww' + T + y'y + z'z$ and thus obtain $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 4.

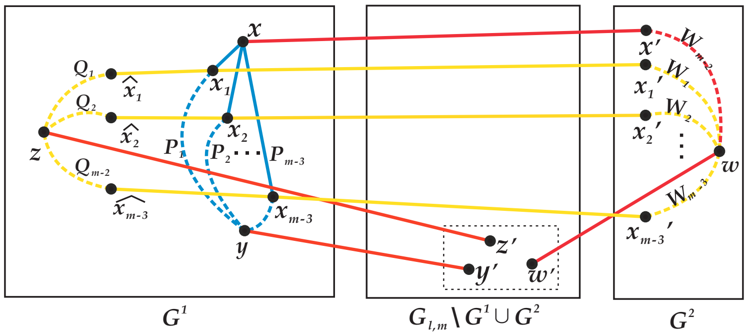

Figure 4. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Subcase 2.1.2.

Otherwise, if $w' \in V(G^1)$, we choose an internal neighbor $\widehat{x'}$ of x' such that its external neighbor $(\widehat{x'})'$ is in $G_{l,m} \backslash (G^1 \cup G^2)$ and by tree $T^*_{m-2} = xx' + x'W_{m-2}w + x'\widehat{x'} + \widehat{x'}(\widehat{x'})' + T + y'y + z'z$ replace tree T_{m-2} for case $w' \notin V(G^1)$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-3}, T^*_{m-2}$ in $G_{l,m}$.

Subcase 2.2 Two of r, s, t are equal

Without loss of generality, suppose $s = t$ and let $x \in V(G^{1r}_{l,m-1})$ and $y, z \in V(G^{1s}_{l,m-1})$ for some $r \neq s \in [m-1]$. By Lemma 2.1(2), G^1 contains at least $m-2$ internally disjoint (y, z)-paths, named $yP_1z, yP_2z, \ldots, yP_{m-2}z$. Now we select $m-3$ vertices $y_i \in V(P_i) \cap N_{G^1}(y) \setminus \{y, z\}$ and let $Y = \{y_1, y_2, \ldots, y_{m-3}\}$. By Lemma 2.2, there exists an $(m-3)$-fan from x to Y in G^1, which denoted by $y_1Q_1x, y_2Q_2x, \ldots, y_{m-3}Q_{m-3}x$. Now we choose $m-3$ distinct vertices $x_i \in V(Q_i) \cap N_{G^{1i}_{l,m-1}}(x)$. Further, we discuss according to r, s. Cases for $r, s \neq 2$ and $r = 2$, by similar argument as Subcase 2.1.1 and Subcase 2.1.2, respectively, we would get $m-2$ internally disjoint S-trees in $G_{l,m}$, details omitted. As for case $s = 2$, let $x \in V(G^{11}_{l,m-1})$ and $y, z \in V(G^{12}_{l,m-1})$ and their external neighbors $x', x'_i \in V(G^m)$, $y', z' \in V(G^2)$. Similarly discuss as before, if $w \notin V(G^{2m}_{l,m-1})$, choose $m-2$ distinct vertices $w_i \in V(G^{2m}_{l,m-1})$ with the external neighbors $w'_i \in V(G^m)$ and construct a such $(m-2)$-fan from w to $W = \{w_1, w_2, \ldots, w_{m-2}\} \subseteq V(G^2) \backslash \{w\}$ as $wW_1w_1, wW_2w_2, \ldots, wW_{m-2}w_{m-2}$. Then construct a set of $m-2$ pairwise vertex disjoint paths between $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$ and $W' = \{w'_1, w'_2, \ldots, w'_{m-2}\}$ as $x'_1L_1w'_1, x'_2L_2w'_2, \ldots, x'_{m-3}L_{m-3}w'_{m-3}, x'L_{m-2}w'_{m-2}$. Notice $w_{m-2}, y', z' \in V(G^2)$, suppose the path from w_{m-2} to z' (or y') is R. Now let $T_i = yP_iz + y_iQ_ix + x_ix'_i + x'_iL_iw'_i + w'_iw_i + w_iW_iw$ for $i \in [m-3]$, $T_{m-2} = xx' + x'L_{m-2}w'_{m-2} + w_{m-2}W_{m-2}w + w_{m-2}Rz' + z'z + yP_{m-2}z$ and thus we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 5.

Otherwise, if $w \in V(G^{2m}_{l,m-1})$, by the Definition 1.1(3), the external neighbors of y, z, w are $y', z', w' \in V(G^2)$. We use these external neighbors to construct a new $\{y', z', w'\}$-tree T in G^2 and by $T^*_{m-2} = xx' + x'L_{m-2}w' + w'w + T + yy' + zz'$ to replace T_{m-2} for case $w \notin V(G^{2m}_{l,m-1})$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-3}, T^*_{m-2}$ in $G_{l,m}$.

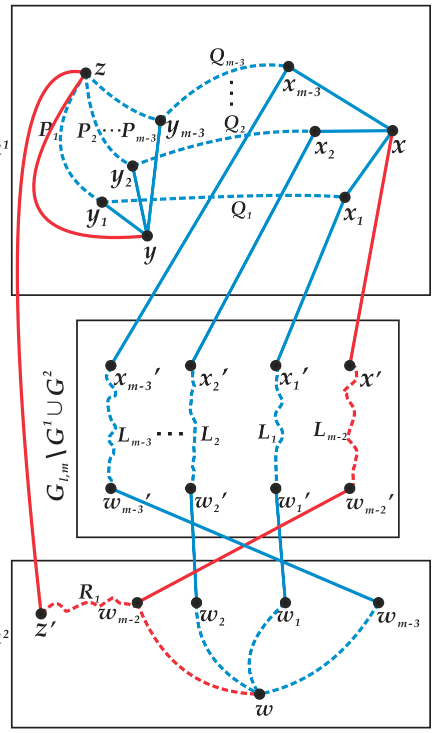

Figure 5. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Subcase 2.2.

Subcase 2.3 $r = s = t \in [m-1]$

Subcase 2.3.1 $r \neq 2$

Suppose $r = 1$ and let $x, y, z \in V(G^{11}_{l,m-1})$. Then the external neighbors of x, y and z are $x', y', z' \in V(G^m)$. Similarly, choose $m-3$ distinct vertices $x_i \in V(G^{12}_{l,m-1})$ such that their external neighbors $x'_i \in V(G^2)$. Consider $x, y, z, x_i \subset V(G^1)$, by similar argument as Case 1, we would get $m-3$ internally disjoint $\{x, y, z, x_i\}$-trees $T'_1, T'_2, \ldots, T'_{m-3}$. Then choose $w_1 \in V(G^{2m}_{l,m-1})\backslash\{w\}$ such that its external neighbor $w'_1 \in V(G^m)$. By Lemma 2.2, there exist and can be constructed $(m-2)$-fan from w to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, w_1\}$ such as $x'_1 Q_1 w, x'_2 Q_2 w, \ldots, x'_{m-3} Q_{m-3} w, w_1 Q_{m-2} w$. Consider the external neighbors of x, y, z and w_1 are $x', y', z', w'_1 \in V(G^m)$, we use these external neighbors to construct a $\{w'_1, x', y', z'\}$-tree T in G^m. Let $T_i = T'_i + x_i x'_i + x'_i Q_i w$ for $i \in [m-3]$, and $T_{m-2} = xx' + yy' + zz' + T + w'_1 w_1 + w_1 Q_{m-2} w$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Subcase 2.3.2 $r = 2$

Consider $x, y, z \in V(G^{12}_{l,m-1})$ and $x', y', z' \in V(G^2)$; we discuss by w. If $w \notin V(G^{21}_{l,m-1})$, then choose $m-3$ distinct vertices $x_i \in V(G^{11}_{l,m-1})$ such that their external neighbors $x'_i \in V(G^m)$. By similar argument as Case 1, we get $m-3$ internally disjoint $\{x, y, z, x_i\}$-trees $T'_1, T'_2, \ldots, T'_{m-3}$. Then choose $m-3$ vertices $w_i \in V(G^{2m}_{l,m-1})\backslash\{w\}$ such that their external neighbors $w'_i \in V(G^m)$ and construct a $(m-3)$-fan from w to $W = \{w_1, w_2, \ldots, w_{m-3}\}$ such as $w_1 Q_1 w, w_2 Q_2 w, \ldots, w_{m-3} Q_{m-3} w$. Then, by Lemma 2.3, we construct a set of $m-3$ pairwise vertex disjoint paths between $X' = \{x'_1, x'_2, \ldots, x'_{m-3}\}$ and $W' = \{w'_1, w'_2, \ldots, w'_{m-3}\}$ such as $x'_1 L_1 w'_1, x'_2 L_2 w'_2, \ldots, x'_{m-3} L_{m-3} w'_{m-3}$. Notice $x', y', z' \in V(G^2)$, then use these external neighbors to construct a $\{x', y', z', w\}$-tree T in G^2. Finally, let $T_i = T'_i + x_i x'_i + x'_i L_i w'_i + w'_i w_i + w_i W_i w$ for $i \in [m-3]$, $T_{m-2} = xx' + yy' + zz' + T$, and thus get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Otherwise, if $w \in V(G^{21}_{l,m-1})$, choose $m-4$ distinct vertices $x_i \in V(G^{11}_{l,m-1})$ such that their external neighbors $x'_i \in V(G^m)$ and similarly form $m-4$ internally disjoint $\{x, y, z, x_i\}$-trees $T'_1, T'_2, \ldots, T'_{m-4}$. Let T'_{m-3} be xyz(or xzy, or yxz), and $P_{m-2} = xz$(or xy, or yz). Then choose $m-4$ vertices $w_i \in V(G^{2m}_{l,m-1})$ such that their external neighbors $w'_i \in V(G^m)$ to construct a $(m-3)$-fan in G^2 from w to $W = \{w_1, w_2, \ldots, w_{m-4}\}$ such as $w_1Q_1w, w_2Q_2w, \ldots, w_{m-4}Q_{m-4}w$. And construct a set of $m-3$ pairwise vertex disjoint paths between $X' = \{x'_1, x'_2, \ldots, x'_{m-4}\}$ and $W' = \{w'_1, w'_2, \ldots, w'_{m-4}\}$ such as $x'_1L_1w'_1, x'_2L_2w'_2, \ldots, x'_{m-4}L_{m-4}w'_{m-4}$. Follow these, we use external neighbors of x, y and z to construct a $\{x', y', w\}$-tree T and a new $\{z', w\}$-path R in G^2. Finally, let $T_i = T'_i + x_ix'_i + x'_iL_iw'_i + w'_iw_i + w_iW_iw$ for $i \in [m-4]$, $T_{m-3} = T'_{m-3} + zz' + R$, and $T_{m-2} = P_{m-2} + xx' + yy' + T$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Case 3 $|S \cap V(G^i)| = 2, |S \cap V(G^j)| = 2$ for some $i \neq j \in [m]$

Without loss of generality, let $x, y \in V(G^1)$ and $z, w \in V(G^2)$. Recall $G^1 = G^{11}_{l,m-1} \oplus G^{12}_{l,m-1} \oplus \ldots \oplus G^{1(m-1)}_{l,m-1}$, $G^2 = G^{21}_{l,m-1} \oplus G^{22}_{l,m-1} \oplus \ldots \oplus G^{2(m-1)}_{l,m-1}$ where $G^{1i}_{l,m-1} = G^{2i}_{l,m-1} = G_{l,m-2}$ for $1 \leq i \leq m-1$, we suppose $x \in V(G^{1r}_{l,m-1}), y \in V(G^{1s}_{l,m-1}), z \in V(G^{2t}_{l,m-1}), w \in V(G^{2o}_{l,m-1})$ for $r, s, t, o \in [m-1]$.

Subcase 3.1 One of r, s is 2

Without loss of generality, suppose $x \in V(G^{12}_{l,m-1}), y \in V(G^{13}_{l,m-1})$. By Lemma 2.1(2), G^1 contains at least $m-2$ internally disjoint (x, y)-paths $xP_1y, xP_2y, \ldots, xP_{m-2}y$. Now select $m-3$ vertices $x_i \in V(P_i) \cap N_{G^{12}_{l,m-1}}(x)$ such that their external neighbors $x'_i \in V(G^2)$ and construct a $(m-2)$-fan from w to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\} \subseteq V(G^2)$ such as $x'_1W_1w, x'_2W_2w, \ldots, x'_{m-3}W_{m-3}w, x'W_{m-2}w$. Then select $m-2$ vertices $w_i \in V(W_i)$ and construct a $(m-2)$-fan from z to $W = \{w_1, w_2, \ldots, w_{m-2}\}$ such as $w_1Q_1z, w_2Q_2z, \ldots, w_{m-2}Q_{m-2}z$. Now let $T_i = xP_iy + x_ix'_i + x'_iW_iw + w_iQ_iz$ for $i \in [m-3]$, $T_{m-2} = xP_{m-2}y + xx' + x'W_{m-2}w + w_{m-2}Q_{m-2}z$ and thus we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 6.

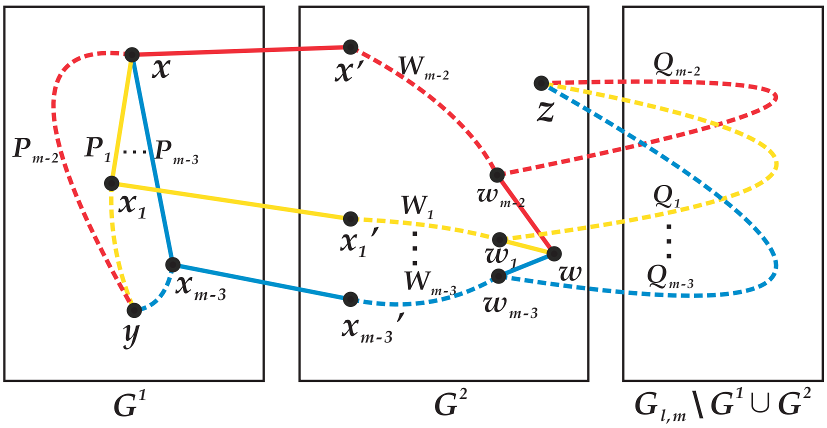

Figure 6. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Subcase 3.1.

Subcase 3.2 $r = s = 2$

Clearly, suppose $x, y \in V(G^{12}_{l,m-1})$, and thus G^1 contains at least $m-2$ internally disjoint (x, y)-paths $xP_1y, xP_2y, \ldots, xP_{m-2}y$, where $V(P_{m-3}) \cap V(G^{12}_{l,m-1}) = \{x, y\}$ and P_{m-2} is an (x, y)-path of length 1. Now select $m-4$ vertices $x_i \in V(P_i) \cap N_{G^{12}_{l,m-1}}(x)$ such that x_i are the internal neighbors of x. Let $x'_i \in V(G^{21}_{l,m-1})$ be the external neighbors of x_i and construct a $(m-2)$-fan in G^2 from w to $X' = \{x'_1, x'_2, \ldots, x'_{m-4}, y', x'\}$ denoted by $x'_1W_1w, x'_2W_2w, \ldots, x'_{m-4}W_{m-4}w, y'W_{m-3}w, x'W_{m-2}w$. Then select $m-$

2 vertices $w_i \in V(W_i)$ and construct a $(m-2)$-fan in $G_{l,m} - G^1$ from z to $W = \{w_1, w_2, \ldots, w_{m-2}\}$ such as $w_1Q_1z, w_2Q_2z, \ldots, w_{m-2}Q_{m-2}z$. Now let $T_i = xP_iy + x_ix_i' + x_i'W_iw + w_iQ_iz$ for $i \in [m-4]$, $T_{m-3} = xP_{m-3}y + yy' + y'W_{m-3}w + w_{m-3}Q_{m-3}z$, $T_{m-2} = xP_{m-2}y + xx' + x'W_{m-2}w + w_{m-2}Q_{m-2}z$. Then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Subcase 3.3 $r, s \neq 2$

Considering $r, s \neq 2$, we discuss t, o. If one of t, o is 1, then z or w is in $G^{21}_{l,m-1}$. By similar arguments as Subcase 3.1 and Subcase 3.2, we would get $m-2$ internally disjoint S-trees in $G_{l,m}$, the details omitted. Otherwise, if $t, o \neq 1$, then $z, w \notin V(G^{21}_{l,m-1})$. Without loss of generality, suppose $x \in V(G^{11}_{l,m-1}), y \in V(G^{13}_{l,m-1})$ and $z \in V(G^{23}_{l,m-1})$. By Lemma 2.1(2), G^1 contains at least $m-2$ internally disjoint (x, y)-paths such as $xP_1y, xP_2y, \ldots, xP_{m-2}y$, where $V(P_{m-2}) \cap V(G^{11}_{l,m-1}) = \{x\}$. Now select $m-3$ internal neighbors of x such as $x_i \in V(P_i) \cap N_{G^{11}_{l,m-1}}(x)$ with external neighbors $x_i' \in V(G^m)$. Similarly, there are $m-2$ internally disjoint (z, w)-paths in G^2 such as $zQ_1w, zQ_2w, \ldots, zQ_{m-2}w$. Then select $m-3$ vertices $z_i \in V(Q_i) \cap N_{G^{23}_{l,m-1}}(z)$ such that z_i are the internal neighbors of z with external neighbors $z_i' \in V(G^3)$. Now form a set of $m-2$ pairwise vertex disjoint (X', Z') paths $x_1'L_1z_1', x_2'L_2z_2', \ldots, x_{m-3}'L_{m-3}z_{m-3}', x'L_{m-2}z'$ between $X' = \{x_1', x_2', \ldots, x_{m-3}', x'\}$ and $Z' = \{z_1', z_2', \ldots, z_{m-3}', z'\}$. Thus, let $T_i = xP_iy + x_ix_i' + x_i'L_iz_i' + z_i'z_i + zQ_iw$ for $i \in [m-3]$, $T_{m-2} = xP_{m-2}y + xx' + x'L_{m-2}z' + z'z + zQ_{m-2}w$ and then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Case 4 $|S \cap V(G^i)| = 2, |S \cap V(G^j)| = |S \cap V(G^k)| = 1$ for some $i \neq j \neq k \in [m]$

Without loss of generality, suppose $x, y \in V(G^1)$, $z \in V(G^2)$ and $w \in V(G^3)$ and further let $x \in V(G^{1r}_{l,m-1}), y \in V(G^{1s}_{l,m-1})$ for some $r, s \in [m-1]$.

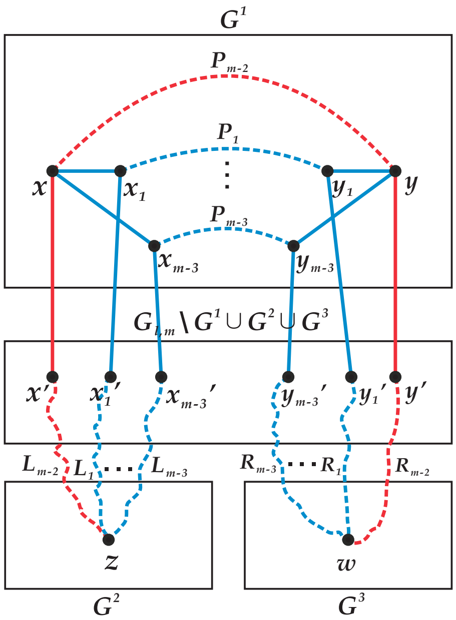

Figure 7. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Subcase 4.1.

Subcase 4.1 $r \neq s \in [m-1]$

Without loss of generality, suppose $x \in V(G^{11}_{l,m-1}), y \in V(G^{14}_{l,m-1})$ and $m-2$ internally disjoint (x, y)-paths in G^1 are $xP_1y, xP_2y, \ldots, xP_{m-2}y$. Then select $m-3$ distinct vertices $x_i \in V(P_i) \cap N_{G^{11}_{l,m-1}}(x)$.

Similarly, select $m-3$ distinct vertices $y_i \in V(P_i) \cap N_{G^{14}_{l,m-1}}(y)$, Notice the external neighbors of x_i and y_i are $x'_i \in V(G^m)$ and $y'_i \in V(G^4)$, we construct a $(m-2)$-fan from z to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$ such as $zL_1x'_1, zL_2x'_2, \ldots, zL_{m-3}x'_{m-3}, zL_{m-2}x'$, and a $(m-2)$-fan from w to $Y' = \{y'_1, y'_2, \ldots, y'_{m-3}, y'\}$ such as $wR_1y'_1, wR_2y'_2, \ldots, wR_{m-3}y'_{m-3}, wR_{m-2}y'$. Now let $T_i = xP_iy + x_ix'_i + x'_iL_iz + y_iy'_i + y'_iR_iw$ for $i \in [m-3]$, $T_{m-2} = xP_{m-2}y + xx' + x'L_{m-2}z + yy' + y'R_{m-2}w$, and then we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 7.

Subcase 4.2 $r = s \in [m-1]$

Without loss of generality, suppose $x, y \in V(G^{11}_{l,m-1})$ and $m-2$ internally disjoint (x,y)-paths in G^1 are $xP_1y, xP_2y, \ldots, xP_{m-2}y$. Now select $m-4$ distinct vertices $x_i \in V(P_i) \cap N_{G^{11}_{l,m-1}}(x)$ and $y_i \in N_{G^{11}_{l,m-1}}(x_i) \setminus V(P_i)$. Similarly, we construct a $(m-2)$-fan from z to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$ such as $zL_1x'_1, zL_2x'_2, \ldots, zL_{m-3}x'_{m-3}, zL_{m-2}x'$ and a $(m-2)$-fan from w to $Y' = \{y'_1, y'_2, \ldots, y'_{m-3}, y'\}$ such as $wR_1y'_1, wR_2y'_2, \ldots, wR_{m-3}y'_{m-3}, wR_{m-2}y'$. Now let $T_i = xP_iy + x_ix'_i + x'_iL_iz + x_iy_i + y_iy'_i + y'_iR_iw$ for $i \in [m-4]$, $T_{m-3} = xP_{m-3}y + xx_{m-3} + x_{m-3}x'_{m-3} + x'_{m-3}L_{m-3}z + yy_{m-3} + y_{m-3}y'_{m-3} + y'_{m-3}R_{m-3}w$, and $T_{m-2} = xP_{m-2}y + xx' + x'L_{m-2}z + yy' + y'R_{m-2}w$ and thus we get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$.

Case 5 $|S \cap V(G^i)| = |S \cap V(G^j)| = |S \cap V(G^k)| = |S \cap V(G^l)| = 1$ for some $i \neq j \neq k \neq l \in [m]$

Without loss of generality, suppose $x \in V(G^1)$, $y \in V(G^2)$, $z \in V(G^3)$, $w \in V(G^4)$, and their external neighbors are x', y', z', w', respectively. If $|\{x', y', z', w'\} \cap (G_{l,m} \setminus (G^1 \cup G^2 \cup G^3 \cup G^4))| \geq 2$, suppose $x', y' \in V(G_{l,m} \setminus (G^1 \cup G^2 \cup G^3 \cup G^4))$. Then by Lemma 2.1(2), there exist $m-2$ internally disjoint (x,z)-paths in $G^1 \cup G^3$, denoted by $xP_1z, xP_2z, \ldots, xP_{m-2}z$, and $m-2$ internally disjoint (y,w)-paths in $G^2 \cup G^4$, denoted by $yQ_1w, yQ_2w, \ldots, yQ_{m-2}w$. By a similar argument as Case 3, we can obtain $m-2$ internally disjoint S-trees in $G_{l,m}$, the details omitted.

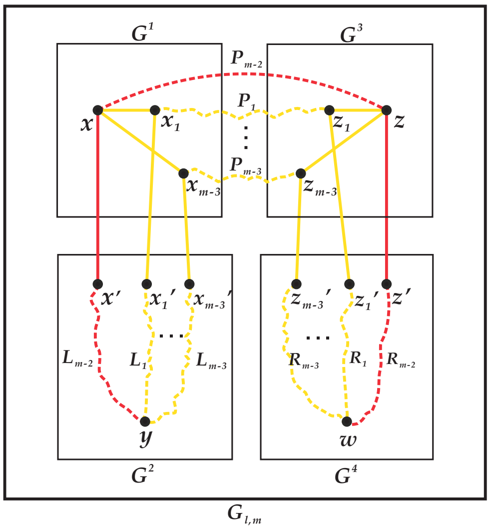

Figure 8. Internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in Case 5.

If $|\{x', y', z', w'\} \cap (G_{l,m} \setminus (G^1 \cup G^2 \cup G^3 \cup G^4))| \leq 1$, then $|\{x', y', z', w'\} \cap (G^1 \cup G^2 \cup G^3 \cup G^4)| \geq 3$. Suppose $x', y', z' \in V(G^1 \cup G^2 \cup G^3 \cup G^4)$ and $G^i = G^{i1}_{l,m-1} \oplus G^{i2}_{l,m-1} \oplus \ldots \oplus G^{i(m-1)}_{l,m-1}$ for $i \in [m]$. Clearly, $x \in V(G^{12}_{l,m-1}), V(G^{13}_{l,m-1}), V(G^{14}_{l,m-1})$, $y \in V(G^{21}_{l,m-1}), V(G^{23}_{l,m-1}), V(G^{24}_{l,m-1})$ and $z \in V(G^{31}_{l,m-1}), V(G^{32}_{l,m-1}), V(G^{34}_{l,m-1})$. Without loss of generality, suppose $x \in V(G^{12}_{l,m-1})$, $y \in V(G^{21}_{l,m-1})$, $z \in V(G^{34}_{l,m-1})$. Then $z' \in V(G^4)$ and $G^1 \cup G^3$ contains at least $m-2$ internally disjoint (x,z)-paths

$xP_1z, xP_2z, \ldots, xP_{m-2}z$. Now select $m-3$ distinct vertices $x_i \in V(P_i) \cap N_{G^{12}_{l,m-1}}(x)$, $z_i \in V(P_i) \cap N_{G^{34}_{l,m-1}}(z)$ and suppose x'_i and z'_i are the external neighbor of x_i and z_i, respectively. Now we construct a $(m-2)$-fan from y to $X' = \{x'_1, x'_2, \ldots, x'_{m-3}, x'\}$ such as $yL_1x'_1, yL_2x'_2, \ldots, yL_{m-3}x'_{m-3}, yL_{m-2}x'$ and $(m-2)$-fan from w to $Z' = \{z'_1, z'_2, \ldots, z'_{m-3}, z'\}$ such as $wR_1z'_1, wR_2z'_2, \ldots, wR_{m-3}z'_{m-3}, wR_{m-2}z'$, and then let $T_i = xP_iz + x_ix'_i + x'_iL_iy + z_iz'_i + z'_iR_iw$ for $i \in [m-3]$, $T_{m-2} = xP_{m-2}z + xx' + x'L_{m-2}y + zz' + z'R_{m-2}w$ and get $m-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{m-2}$ in $G_{l,m}$. See Figure 8. For other cases for distinct distribution of x, y, z, we can similarly construct $m-2$ internally disjoint S-trees in $G_{l,m}$, here omitting details.

By now, we show that $G_{l,m}$ always contains $m-2$ internally disjoint S-trees and thus we get $\kappa_4(G_{l,m}) \geq m-2$ for $m \geq l \geq 3$. This completes the proof. □

By Lemma 2.5 and Theorem 3.1, we directly get the generalized 4-edge-connectivity of CRNs.

Corollary 3.1. *Let m be an integer and $G_{l,m}$ be l-order m-dimensional CRNs for $l \geq 3$. Then*

$$\lambda_4(G_{l,m}) = \begin{cases} l-2, & \text{If } 1 \leq m \leq l; \\ m-2, & \text{If } m \geq l+1. \end{cases}$$

Author contributions

Yinkui Li: Conceptualization, Funding acquisition, Writing–review and editing; Yilin Song: Formal analysis, Investigation, Writing–original draft; Zhuomo An: Data curation, Methodology, Project administration, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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