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Research article

Digital power generalized exchange option pricing considering liquidity risk

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Abstract: Traditional option pricing models mostly assume that the market is frictionless, ignoring the impact of liquidity on option price. In response, this paper considers a digital power generalized exchange option pricing problem when the underlying asset has liquidity risk. We obtained a closed-form digital power option pricing formula in a incomplete market by measure transformation. Finally, numerical experiments were conducted by comparing the prices computed by the new formula with those from Monte-Carlo simulations, thereby validating the accuracy of the new formula. Building on this, the impact of liquidity on option prices was further investigated.

Keywords: digital power exchange options; liquidity; measure transformation; numerical analysis
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1. Introduction

The classical Black-Scholes model [1] has been applied extensively to the valuation of financial derivatives. However, a growing body of empirical evidence suggests that the assumptions of the BS model are overly restrictive. In order to relax these assumptions, many scholars have conducted extended research based on this model. For example, Merton [2] proposed an option pricing model based on a jump-diffusion process, yielding results that align more closely with real financial markets. Vasicek [3] pioneered the extension of the Black-Scholes formula from a constant to a stochastic interest rate. Amihud and Mendelson [4, 5] first proposed the concept of a liquidity premium, demonstrating the relationship between market liquidity and the returns of risky assets.

Since then, many scholars have incorporated the impact of liquidity factors on the underlying asset price into their option pricing research. In 2018, Li et al. investigated the impact of liquidity factors on quanto options [6], Asian options [7], and discrete barrier options [8]. Recently, He et al. [9, 10] investigated the impact of stochastic liquidity factors on exchange option pricing through numerical simulations. In addition to accounting for the influence of liquidity factors, He and Mittal et al. [11, 12]

also incorporated the effect of default risk in option valuation. Further studies on the effect of liquidity on option prices can be found in [13, 14]. Collectively, these findings demonstrate that liquidity is a significant factor influencing option prices. The results summarized above are all based on the partial differential equation framework for option pricing.

When market liquidity is insufficient, multiple risk-neutral measures exist in incomplete markets, and thus classic pricing theory cannot be directly applied. Therefore, it is crucial for researchers to identify a suitable pricing measure applicable in incomplete markets. To address this, Gerber et al. [15] suggested employing the Esscher measure transformation in order to identify a suitable equivalent martingale measure for pricing options. By utilizing the Esscher measure transformation, Feng et al. [16] proposed a liquidity-adjusted European option pricing model. Compared to models that ignore liquidity effects, their model yields smaller pricing errors and greater stability. Gao et al. derives closed-form solutions for liquidity-adjusted European options [17], exchange options [18], and quanto options [19] using the Esscher measure transformation. Li [20] introduced the digital power exchange option, an extension of the power exchange option, designed to mitigate losses caused by a significant price deviation between the two underlying assets.

A novel contribution of this paper is the consideration of liquidity risk's effect on the underlying asset price in the valuation of digital power exchange options. In order to derive a closed-form expression for option prices within the risk-neutral measure framework, the Esscher measure transform is employed to derive the equivalent martingale measure Q from the real-world probability measure P. Numerical simulations confirm the accuracy of the derived pricing formula and further investigate the influence of various parameters on the option's value. The results show that the option premium increases significantly when liquidity risk is taken into account.

The rest of this paper is constructed as follows. Section 2 presents some fundamental theories. Section 3 derived the pricing formula of digital power generalized exchange based on liquidity adjustment. Section 4 elaborates on empirical analysis and numerical simulation results. Section 5 presents a conclusion.

2. Fundamental theory

First, define a probability space $(\Omega, \mathcal{F}_t, P)$ to characterize the uncertainty in the financial market, where Ω denotes the set of all possible states, $\mathcal{F}_t$ represents the information available at time t, and P denotes the real-world probability measure. Then, let $S(t) = S(0)\mathrm{e}^{X(t)}$, $0 \leq t \leq T$, and let $\{X(t)\}_{t\geq 0}$ be a stochastic process with independent and stationary increments. Let $F(x,t) = P[X(t) \leq x]$. The corresponding moment generating function is defined as $M(z,t) = \mathrm{E}[\mathrm{e}^{zX(t)}]$, and $X(t)$ satisfies condition: $X(0) = 0$, $M(z,t) = [M(z,1)]^t$.

Definition 2.1. *[15] Let h be an arbitrary real number. Define the Esscher transform of the probability density function $f(x,t)$ of $X(t)$ with parameter h as*

$$f(x,t;h) = \frac{\mathrm{e}^{hx} f(x,t)}{\int_{-\infty}^{+\infty} \mathrm{e}^{hx} f(x,t)\mathrm{d}x} = \frac{\mathrm{e}^{hx} f(x,t)}{M(h,t)}.$$

Definition 2.2. *[15] The Esscher transform of the moment generating function $M(z,t)$ of $X(t)$ with* *parameter h is*

$$M(z,t;h)=\int_{-\infty}^{+\infty}\mathrm{e}^{hx}f(x,t;h)\mathrm{d}x.$$

The essence of applying the Esscher transform to a function is a measure transformation. Specifically, it involves introducing the Radon-Nikodym derivative on the original probability space P, so that the new probability measure Q satisfies

$$\frac{\mathrm{d}Q}{\mathrm{d}P}=\frac{\mathrm{e}^{hx}}{M[h,t]}.$$

Property 2.1. *[15]*

$$M(z,t;h)=\frac{M(z+h,t)}{M(h,t)}. \tag{2.1}$$

Property 2.2. *[15] Let* $S_t=S_0\mathrm{e}^{X_t}$, $X_t=\mu t+\sigma W_t$, $0\le t\le T$, $\{W_t\}_{0\le t\le T}$ *is the standard Brownian motion in probability space* $(\Omega,\mathcal{F}_t,P)$, *and* μ, σ *are constants.* X_t *is a normal distribution with mean* μ *and variance* σ^2 *per unit time. Then,*

$$M(z,t)=\mathrm{E}\left[\mathrm{e}^{zX_t}\right]=\mathrm{e}^{(z\mu+\frac{1}{2}z^2\sigma^2)t}.$$

According to Eq (2.1)*, the moment-generating function of* X_t *under the Esscher measure (parameter h) is*

$$M(z,t;h)=E^Q\left[e^{zX_t}\right]=\mathrm{E}\left[e^{zX_t};h\right]=e^{\left[z(\mu+h\sigma^2)+\frac{1}{2}z^2\sigma^2\right]t}.$$

Therefore, the drift rate of X_t *per unit time changes from* μ *to* $\mu+h\sigma^2$*, and the volatility remains unchanged under the Esscher measure (parameter h).*

Lemma 2.1. *(Bayes' theorem) [21] Let P and Q be probability measures on the measurable space* $(\Omega,\mathcal{F}_t)$*. Let* $\mathcal{G}$ *be a sub-*σ*-algebra of* $\mathcal{F}_t$*, and* $\Lambda=\frac{\mathrm{d}Q}{\mathrm{d}P}$ *denote the Radon-Nikodym derivative of Q with respect to P. Then, for any measurable random variable X, we have*

$$\mathrm{E}_Q[X\mid\mathcal{G}]=\frac{\mathrm{E}_P[X\Lambda\mid\mathcal{G}]}{\mathrm{E}_P[\Lambda\mid\mathcal{G}]}. \tag{2.2}$$

3. Main contents

The pricing of generalized digital power generalized exchange options, considering the effects of liquidity, will be presented in this section. The valuation will be conducted under incomplete market conditions using a measure transformation technique.

3.1. Model specification

In order to derive the stock price process under illiquidity conditions, Brunetti and Caldarera [22] assume that the stock demand function $D\equiv D(S(t),L(t),I(t))$ is related to the stock price $S(t)$, liquidity discount factor $L(t)$, and information process $I(t)$, and it takes the following form

$$D(S(t),L(t),I(t))=F\left(\frac{(I(t))^{\iota}}{L(t)S(t)}\right) \tag{3.1}$$

where, $F(\cdot)$ is a smooth and strictly increasing function with parameters $\iota > 0$. The stock information process $I(t)$ follows

$$\frac{\mathrm{d}I(t)}{I(t)} = \alpha_I \mathrm{d}t + \sigma_I \mathrm{d}W^{I,P}(t) \tag{3.2}$$

where, the parameter $\alpha_I > 0, \sigma_I > 0$, $W^{I,P}(t)$ represents a standard Brownian motion under the actual probability measure P. Both Eqs (3.1) and (3.2) demonstrate that the impact of liquidity on stock prices can be captured by a liquidity discount factor $L(t)$ defined by

$$\frac{\mathrm{d}L(t)}{L(t)} = \left(\frac{1}{2}\xi^2\omega^2(t) - \xi\omega(t)\right)\mathrm{d}t - \xi\omega(t)\mathrm{d}W^{L,P}(t) \tag{3.3}$$

where $\omega(t)$ represents the stock liquidity level at time t, $\omega(t) > 0(\omega(t) < 0)$ means that the market is in shortage(surplus), $\omega(t) = 0$ indicates a perfectly liquid market, and ξ is the sensitivity of stock prices $S(t)$ to liquidity levels. The standard Brownian motion processes $W^{I,P}(t)$ and $W^{L,P}(t)$ are independent of each other, that is $\mathrm{d}W^{I,P}(t)\mathrm{d}W^{L,P}(t) = 0$. Under the market clearing condition, Brunetti and Caldarera solved the equilibrium equation (3.1) by using the chain rule of implicit function derivation, and obtained a liquidity-adjusted stock pricing model

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(\mu + \xi\omega(t) + \frac{1}{2}\xi^2\omega^2(t)\right)\mathrm{d}t + \xi\omega(t)\mathrm{d}W^{L,P}(t) + \lambda\mathrm{d}W^{I,P}(t) \tag{3.4}$$

where, $\mu = \alpha_I\iota + \frac{1}{2}\sigma_I^2\iota(\iota - 1)$, $\lambda = \iota\sigma_I$.

3.2. *Risk-neutral dynamics*

In the probability space $(\Omega, \mathcal{F}_t, P)$, we assume that the two risky assets ($i = 1, 2$) follow the following processes:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \left(\mu_i + \xi_i\omega_i(t) + \frac{1}{2}\xi_i^2\omega_i^2(t)\right)\mathrm{d}t + \xi_i\omega_i(t)\mathrm{d}W_i^{L,P}(t) + \lambda_i\mathrm{d}W_i^{I,P}(t) \tag{3.5}$$

where, $W_i^{L,P}(t)$ and $W_i^{I,P}(t)$ are pairwise independent standard Brownian motions. Furthermore, $W_1^{L,P}(t)$ and $W_2^{L,P}(t)$ are mutually independent, while $W_1^{I,P}(t)$ and $W_2^{I,P}(t)$ have a correlation coefficient of ρ, i.e., $\mathrm{d}W_1^{I,P}(t) \cdot \mathrm{d}W_2^{I,P}(t) = \rho\mathrm{d}t$. To derive a closed-form solution, this paper assumes $\omega_i(t)$ to be constant. By solving stochastic differential Eq (3.5), we obtain the closed-form expression for the underlying stock

$$S_i(t) = S(0)\exp\left[\int_0^t (\mu_i + \xi_i\omega_i(s) - \frac{1}{2}\lambda_i^2)\mathrm{d}s + \int_0^t \xi_i\omega_i(s)\mathrm{d}W_i^{L,P}(s) + \int_0^t \lambda_i\mathrm{d}W_i^{I,P}(s)\right]. \tag{3.6}$$

Let

$$X_i(t) = \int_0^t (\mu_i + \xi_i\omega_i(s) - \frac{1}{2}\lambda_i^2)\mathrm{d}s + \int_0^t \xi_i\omega_i(s)\mathrm{d}W_i^{L,P}(s) + \int_0^t \lambda_i\mathrm{d}W_i^{I,P}(s). \tag{3.7}$$

When market liquidity is insufficient, the equivalent martingale measure in incomplete markets is not unique. According to the martingale pricing principle, this renders the pricing of options impossible. In incomplete markets, it has been proposed by Gerber et al. [15] to employ the

Esscher transform to identify an equivalent martingale measure suitable for option pricing. Therefore, according to Definition 2.2, we introduce a new measure Q with respect to P. It is defined by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} \mid \mathcal{F}_t = \exp\left\{\sum_{i=1}^{2}\left[\int_0^t -\frac{1}{2}h_i^2\left(\xi_i^2\omega_i^2(u)+\lambda_i^2\right)\mathrm{d}u + \int_0^t h_i\xi_i\omega_i(u)\mathrm{d}W_i^{L,P}(u)\int_0^t h_i\lambda_i\mathrm{d}W_i^{I,P}(u)\right]\right\}. \quad (3.8)$$

To ensure that the transformed measure Q is a risk-neutral measure, the parameter h_i must satisfy

$$h_i = -\frac{\mu_i + \xi_i\omega_i(t) + \frac{1}{2}\xi_i^2\omega_i^2(t) - r}{\xi_i^2\omega_i^2(t) + \lambda_i^2},\ i = 1, 2 \quad (3.9)$$

where r is the risk-free interest rate. By Girsanov's theorem, the Brownian motion under the risk-neutral measure Q satisfies

$$\mathrm{d}W_i^{L,Q}(t) = \mathrm{d}W_i^{L,P}(t) - h_i\xi_i\omega_i(t)\mathrm{d}t,$$
$$\mathrm{d}W_i^{I,Q}(t) = \mathrm{d}W_i^{I,P}(t) - h_i\lambda_i\mathrm{d}t$$

where $W_1^{L,Q}(t)$ and $W_2^{L,Q}(t)$ are independent and the correlation coefficient between $W_1^{I,Q}(t)$ and $W_2^{I,Q}(t)$ is ρ.

Under the risk-neutral measure Q, the dynamics of two risky assets prices can be rewritten by

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = r\mathrm{d}t + \xi_i\omega_i(t)\mathrm{d}W_i^{L,Q}(t) + \lambda_i\mathrm{d}W_i^{I,Q}(t).$$

3.3. Pricing model for digital power generalized exchange options

This section is devoted to the formulation of digital power generalized exchange options under the consideration of liquidity factors and provides a closed-form solution for their valuation.

Definition 3.1. *[20] Let a_1, a_2, b_1, b_2, K_1, and K_2 be constants. $\chi_{\{.\}}$ represents the indicative function. At the maturity T, if the payoff of an option satisfies*

$$C(T) = \left[b_1S_1^{a_1}(T) - b_2S_2^{a_2}(T)\right]^+\chi_{\left\{K_1\le\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}\le K_2\right\}}, (K_2 \ge K_1 > 0). \quad (3.10)$$

We refer to this type of option as a digital power generalized exchange option. Where $[K_1, K_2]$ represents the exercise interval of the option, a_1,a_2 are exponents, and b_1, b_2 are the coefficients for the two assets.

Compared to the standard power exchange option, this model incorporates an additional indicator function, denoted by

$$\chi_{\left\{K_1\le\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}\le K_2\right\}}.$$

This indicator function $\chi_{\{.\}}$ assumes the value one provided that the power ratio $\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}$ satisfies the interval condition $[K_1, K_2]$ and the value zero if this condition is not met. If the deviation between

$S_1^{a_1}(T)$ and $S_2^{a_2}(T)$ is too large, the resulting ratio will become excessively high or low. For the conventional power exchange option, the option price can be excessively high or fall to zero, which presents a significant risk to investors. Consequently, by defining an appropriate interval, the model hedges against the risk associated with overly large deviations in the prices of the two assets.

To derive a closed-form expression for the option under the risk-neutral measure Q, now we introduce an equivalent martingale measure Q_2 with respect to Q, whose Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}Q_2}{\mathrm{d}Q} = \frac{S_2^{a_2}(T)}{\mathrm{E}_Q\left[S_2^{a_2}(T)\right]}. \tag{3.11}$$

Denote by $V(0,T)$ the digital power generalized option price, which is represented by

$$\begin{aligned} V(0,T) =& \mathrm{E}_Q\left[\mathrm{e}^{-rT}C(T)\right] \\ =& \mathrm{E}_Q\left[\mathrm{e}^{-rT}\left[b_1 S_1^{a_1}(T) - b_2 S_2^{a_2}(T)\right]^+ \chi_{\left\{K_1 \le \frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} \le K_2\right\}}\right] \\ =& \mathrm{e}^{-rT} b_1 \mathrm{E}_Q\left[\mathrm{E}_Q(S_2^{a_2}(T)) \cdot \frac{S_2^{a_2}(T)}{\mathrm{E}_Q\left[S_2^{a_2}(T)\right]} \cdot \left(\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} - \frac{b_2}{b_1}\right)^+ \chi_{\left\{K_1 \le \frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} \le K_2\right\}}\right] \\ =& \mathrm{e}^{-rT} b_1 \mathrm{E}_Q(S_2^{a_2}(T)) \cdot \mathrm{E}_{Q_2}\left[\left(\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} - \frac{b_2}{b_1}\right)^+ \chi_{\left\{K_1 \le \frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} \le K_2\right\}}\right]. \end{aligned} \tag{3.12}$$

Let

$$G(T) = \frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}$$

and

$$C^*(T) = \left(G(T) - \frac{b_2}{b_1}\right)^+ \chi_{\left\{K_1 \le \frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} \le K_2\right\}}.$$

Then, Eq (3.12) can be rewritten as

$$V(0,T) = \mathrm{e}^{-rT} b_1 \mathrm{E}_Q(S_2^{a_2}(T)) \cdot \mathrm{E}_{Q_2}(C^*(T)). \tag{3.13}$$

Property 3.1. *(1) If* $\frac{b_2}{b_1} > K_2$, *then* $C^*(T) = 0$.
(2) If $K_1 \le \frac{b_2}{b_1} \le K_2$, *then* $C^*(T) = \left(G(T) - \frac{b_2}{b_1}\right)^+ \chi_{\left\{\frac{b_2}{b_1} \le G(T) \le K_2\right\}}$.
(3) If $\frac{b_2}{b_1} \le K_1$, *then* $C^*(T) = \left(G(T) - \frac{b_2}{b_1}\right)^+ \chi_{\{K_1 \le G(T) \le K_2\}}$.

Theorem 3.1. *Assuming that the stock with liquidity risk is considered as the underlying asset of the digital power generalized exchange option, its price process satisfies the Eq* (3.5). *Combining Property 3.1, the price of the option at the initial time is obtained as follows:*

(1) If $\frac{b_2}{b_1} > K_2$, *then* $C^*(T) = 0$.

(2) *If* $K_1 \le \frac{b_2}{b_1} \le K_2$, *then*

$$V(0,T) = b_1 \cdot \mathrm{e}^{-rT+M_2+\frac{1}{2}\Sigma_2} \cdot \left\{ \mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot [N(d_2)-N(d_1)] - \frac{b_2}{b_1} \cdot [N(d_4)-N(d_3)] \right\} \tag{3.14}$$

where

$$M_2 = \ln S_2^{a_2}(0) + a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T$$

$$M_1 = \ln \frac{S_1^{a_1}(0)}{S_2^{a_2}(0)} + a_1(r - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\xi_1^2\omega_1^2(T))T - a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T$$

$$\Sigma_2 = \left[a_2^2\lambda_2^2 + a_2^2\xi_1^2\omega_2^2(t)\right]T, \quad \Sigma_1 = \left[a_1^2\lambda_1^2 + a_1^2\xi_1^2\omega_1^2(T) - 2\rho a_1 a_2 \lambda_1 \lambda_2 + a_2^2\lambda_2^2 + a_2^2\xi_2^2\omega_2^2(T)\right]T$$

$$\bar{\rho} = \frac{\left[\rho a_1 a_2 \lambda_1 \lambda_2 - a_2^2\lambda_2^2 - a_2^2\xi_2^2\omega_2^2(T)\right]T}{\sqrt{\Sigma_1}\sqrt{\Sigma_2}}$$

$$d_2 = \frac{\ln K_2 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}, \quad d_1 = \frac{\ln \frac{b_2}{b_1} - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}$$

$$d_4 = d_2 + \sqrt{\Sigma_1}, \quad d_3 = d_1 + \sqrt{\Sigma_1}.$$

(3) *If* $\frac{b_2}{b_1} \le K_1$, *then*

$$V(0,T) = b_1 \cdot e^{-rT+M_2+\frac{1}{2}\Sigma_2} \cdot \left\{ e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot [N(d_2')-N(d_1')] - \frac{b_2}{b_1} \cdot [N(d_4')-N(d_3')] \right\} \tag{3.15}$$

where

$$M_2 = \ln S_2^{a_2}(0) + a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T$$

$$M_1 = \ln \frac{S_1^{a_1}(0)}{S_2^{a_2}(0)} + a_1(r - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\xi_1^2\omega_1^2(T))T - a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T$$

$$\Sigma_2 = \left[a_2^2\lambda_2^2 + a_2^2\xi_1^2\omega_2^2(t)\right]T, \quad \Sigma_1 = \left[a_1^2\lambda_1^2 + a_1^2\xi_1^2\omega_1^2(T) - 2\rho a_1 a_2 \lambda_1 \lambda_2 + a_2^2\lambda_2^2 + a_2^2\xi_2^2\omega_2^2(T)\right]T$$

$$\bar{\rho} = \frac{\left[\rho a_1 a_2 \lambda_1 \lambda_2 - a_2^2\lambda_2^2 - a_2^2\xi_2^2\omega_2^2(T)\right]T}{\sqrt{\Sigma_1}\sqrt{\Sigma_2}}$$

$$d_2' = \frac{\ln K_2 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}, \quad d_1' = \frac{\ln K_1 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}$$

$$d_4' = d_2 + \sqrt{\Sigma_1}, \quad d_3' = d_1 + \sqrt{\Sigma_1}.$$

Proof. (1) As shown in Property 3.1(1).

(2) According to Eq (3.13), to compute the option price, one only needs to calculate $\mathrm{E}_Q(S_2^{a_2}(T))$ and $\mathrm{E}_{Q_2}(C^*(T))$. By *Itô* formula, we have

$$\ln S_i(T) = \ln S_i(0) + (r - \frac{1}{2}\lambda_i^2 - \frac{1}{2}\xi_i^2\omega_i^2(T))T + \lambda_i W_i^{I,Q}(T) + \xi_i\omega_i(T)W_i^{L,Q}(T). \quad (3.16)$$

From Eq (3.16), we can obtain

$$\ln S_2^{a_2}(T) = \ln S_2^{a_2}(0) + a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T + a_2\lambda_2 W_2^{I,Q}(T) + a_2\xi_2\omega_2(T)W_2^{L,Q}(T)$$

$$\begin{aligned}\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} =& \ln\frac{S_1^{a_1}(0)}{S_2^{a_2}(0)} + a_1(r - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\xi_1^2\omega_1^2(T))T - a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T \\ &+ a_1\lambda_1 W_1^{I,Q}(T) + a_1\xi_1\omega_1(T)W_1^{L,Q}(T) - a_2\lambda_2 W_2^{I,Q}(T) - a_2\xi_2\omega_2(T)W_2^{L,Q}(T).\end{aligned}$$

The above equation states that $\ln S_2^{a_2}(T)$ and $\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}$ are normally distributed random variables, with means of

$$\mathrm{E}\left[\ln S_2^{a_2}(T)\right] = \ln S_2^{a_2}(0) + a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T \triangleq M_2$$

$$\mathrm{E}\left[\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}\right] = \ln\frac{S_1^{a_1}(0)}{S_2^{a_2}(0)} + a_1(r - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\xi_1^2\omega_1^2(T))T - a_2(r - \frac{1}{2}\lambda_2^2 - \frac{1}{2}\xi_2^2\omega_2^2(T))T \triangleq M_1.$$

According to the definition of covariance, their covariance is

$$\begin{aligned}\mathrm{Cov}\left(\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}, \ln S_2^{a_2}(T)\right) =& \mathrm{Cov}\left(\ln S_1^{a_1}(T) - \ln S_2^{a_2}(T), \ln S_2^{a_2}(T)\right) \\ =& \mathrm{Cov}\left(\ln S_1^{a_1}(T), \ln S_2^{a_2}(T)\right) - \mathrm{Var}\left(\ln S_2^{a_2}(T)\right) \\ =& \left[\rho a_1 a_2\lambda_1\lambda_2 - a_2^2\lambda_2^2 - a_2^2\xi_2^2\omega_2^2(T)\right]T.\end{aligned}$$

Since $\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}$ and $\ln S_2^{a_2}(T)$ have a bivariate normal distribution, they can be expressed as

$$\ln S_2^{a_2}(T) = M_2 + \sqrt{\Sigma_2}\varepsilon_2,$$

$$\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)} = M_1 + \sqrt{\Sigma_1}\varepsilon_1,$$

where Σ_2 is the variance of $\ln S_2^{a_2}(T)$ and Σ_1 is the variance of $\ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}$, with their respective values being

$$\Sigma_2 = \left[a_2^2\lambda_2^2 + a_2^2\xi_1^2\omega_2^2(T)\right]T$$

$$\Sigma_1 = \left[a_1^2\lambda_1^2 + a_1^2\xi_1^2\omega_1^2(T) - 2\rho a_1 a_2\lambda_1\lambda_2 + a_2^2\lambda_2^2 + a_2^2\xi_2^2\omega_2^2(T)\right]T.$$

Here, ε_1 and ε_2 are two random variables following a standard normal distribution, with a correlation coefficient of

$$\bar{\rho} = \frac{\mathrm{Cov}\left(\ln S_2^{a_2}(T), \ln\frac{S_1^{a_1}(T)}{S_2^{a_2}(T)}\right)}{\sqrt{\Sigma_1}\sqrt{\Sigma_2}} = \frac{\left[\rho a_1 a_2\lambda_1\lambda_2 - a_2^2\lambda_2^2 - a_2^2\xi_2^2\omega_2^2(T)\right]T}{\sqrt{\Sigma_1}\sqrt{\Sigma_2}}.$$

According to Property 2.1, it is easy to get that

$$\mathrm{E}_Q\left[S_2^{a_2}(T)\right] = \mathrm{E}_Q\left[e^{M_2+\sqrt{\Sigma_2}\varepsilon_2}\right] = e^{M_2+\frac{1}{2}\Sigma_2}.$$

Then, Eq (3.11) can be simplified to an expression involving only ε_2

$$\frac{\mathrm{d}Q_2}{\mathrm{d}Q} = \frac{S_2^{a_2}(T)}{\mathrm{E}_Q\left[S_2^{a_2}(T)\right]} = \frac{\mathrm{e}^{M_2+\sqrt{\Sigma_2}\varepsilon_2}}{\mathrm{e}^{M_2+\frac{1}{2}\Sigma_2}} = e^{\sqrt{\Sigma_2}\varepsilon_2-\frac{1}{2}\Sigma_2}. \tag{3.17}$$

From the definition of the Esscher measure transformation, the parameter h is given by $\sqrt{\Sigma_2}$. From Property 2.1, we know that under the new probability measure Q_2, $\varepsilon_2 \sim N(\sqrt{\Sigma_2}, 1)$. Under the original measure Q, ε_1 and ε_2 follow a bivariate standard normal distribution with correlation coefficient $\bar{\rho}$. Using the property of the bivariate normal distribution, ε_1 can be decomposed into a part correlated with ε_2 and an independent part

$$\varepsilon_1 = \bar{\rho}\varepsilon_2 + \sqrt{1-\bar{\rho}^2}\varepsilon. \tag{3.18}$$

By

$$\begin{aligned}
\mathrm{E}_{Q_2}\left[C^*(T)\right] =& \mathrm{E}_{Q_2}\left[\left(G(T) - \frac{b_2}{b_1}\right)^+ \chi_{\left\{\frac{b_2}{b_1}\le G(T)\le K_2\right\}}\right] \\
=& \mathrm{E}_{Q_2}\left[G(T)\chi_{\left\{\frac{b_2}{b_1}\le G(T)\le K_2\right\}}\right] - \frac{b_2}{b_1}\mathrm{E}_{Q_2}\left[\chi_{\left\{\frac{b_2}{b_1}\le G(T)\le K_2\right\}}\right] \\
=& \Lambda_1 - \Lambda_2.
\end{aligned}$$

Next, we calculate Λ_1 and Λ_2

$$\begin{aligned}
\Lambda_1 =& \mathrm{E}_{Q_2}\left[G(T)\chi_{\left\{\frac{b_2}{b_1}\le G(T)\le K_2\right\}}\right] \\
=& \mathrm{E}_{Q_2}\left[\mathrm{e}^{M_1+\sqrt{\Sigma_1}\varepsilon_1} \cdot \chi_{\left\{\frac{b_2}{b_1}\le e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\sqrt{\Sigma_1}\varepsilon}\le K_2\right\}}\right] \\
=& \mathrm{E}_{Q_2}\left[\mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\sqrt{\Sigma_1}\varepsilon} \cdot \chi_{\left\{\frac{b_2}{b_1}\le e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\sqrt{\Sigma_1}\varepsilon}\le K_2\right\}}\right] \\
=& \mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot \left[N(d_2) - N(d_1)\right]
\end{aligned}$$

$$\Lambda_2 = \frac{b_2}{b_1}\mathrm{E}_{Q_2}\left[\chi_{\left\{\frac{b_2}{b_1}\le e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\Sigma_1\varepsilon}\le K_2\right\}}\right] = \frac{b_2}{b_1} \cdot \left[N(d_4) - N(d_3)\right]$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and

$$d_2 = \frac{\ln K_2 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}, \quad d_1 = \frac{\ln \frac{b_2}{b_1} - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}$$

$$d_4 = d_2 + \sqrt{\Sigma_1}, \quad d_3 = d_1 + \sqrt{\Sigma_1}.$$

Hence, it follows that the option price is given by

$$V(0,T) = b_1 \cdot \mathrm{e}^{-rT+M_2+\frac{1}{2}\Sigma_2} \cdot \left\{ \mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot [N(d_2) - N(d_1)] - \frac{b_2}{b_1} \cdot [N(d_4) - N(d_3)] \right\}.$$

(3) The proof of Eq (3.15) is the same as that of Eq (3.14). Here, only the differences from Theorem 3.1(2) are presented. When $\frac{b_2}{b_1} \leq K_1$, we need to calculate the option price $V(0,T)$. By

$$\begin{aligned}
\mathrm{E}_{Q_2}[C^*(T)|\mathcal{F}_t] =& \mathrm{E}_{Q_2}\left[\left(G(T) - \frac{b_2}{b_1}\right)^+ \chi_{\{K_1 \leq G(T) \leq K_2\}}\right] \\
=& \mathrm{E}_{Q_2}[G(T)\chi_{\{K_1 \leq G(T) \leq K_2\}}] - \frac{b_2}{b_1}\mathrm{E}_{Q_2}[\chi_{\{K_1 \leq G(T) \leq K_2\}}] \\
=& \Lambda_1 - \Lambda_2.
\end{aligned}$$

Next, we calculate Λ_1 and Λ_2

$$\begin{aligned}
\Lambda_1 =& \mathrm{E}_{Q_2}[G(T)\chi_{\{K_1 \leq G(T) \leq K_2\}}] \\
=& \mathrm{E}_{Q_2}\left[\mathrm{e}^{M_1+\sqrt{\Sigma_1}\varepsilon_1} \cdot \chi_{\left\{K_1 \leq e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\Sigma_1\varepsilon} \leq K_2\right\}}\right] \\
=& \mathrm{E}_{Q_2}\left[\mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\Sigma_1\varepsilon} \cdot \chi_{\left\{K_1 \leq e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\Sigma_1\varepsilon} \leq K_2\right\}}\right] \\
=& \mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot [N(d_2) - N(d_1)]
\end{aligned}$$

$$\Lambda_2 = \frac{b_2}{b_1}\mathrm{E}_{Q_2}\left[\chi_{\left\{K_1 \leq e^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\Sigma_1\varepsilon} \leq K_2\right\}}|\mathcal{F}_t\right] = \frac{b_2}{b_1} \cdot [N(d_4) - N(d_3)]$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, and

$$d_2' = \frac{\ln K_2 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}, \quad d_1' = \frac{\ln K_1 - M_1 - \bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2} - \Sigma_1}{\sqrt{\Sigma_1}}$$

$$d_4' = d_2' + \sqrt{\Sigma_1}, \quad d_3' = d_1' + \sqrt{\Sigma_1}.$$

Hence, it follows that the option price is given by

$$V(0,T) = b_1 \cdot \mathrm{e}^{-rT+M_2+\frac{1}{2}\Sigma_2} \cdot \left\{ \mathrm{e}^{M_1+\bar{\rho}\sqrt{\Sigma_1}\sqrt{\Sigma_2}+\frac{1}{2}\Sigma_1} \cdot [N(d_2) - N(d_1)] - \frac{b_2}{b_1} \cdot [N(d_4) - N(d_3)] \right\}.$$

4. Numerical experiments

This section will present the properties of the new formula Eq (3.14) through numerical experiments. The accuracy of the pricing formula is verified by a comparison between the analytical solution and the Monte-Carlo simulation price. To examine the influence of key parameters on the option price, a sensitivity analysis is then carried out. For simplicity, we assume that the market liquidity level $\omega_i(T)$ is a given constant and it is represented as ω_1 and ω_2 in the figure. The power exponent (a_1, a_2) and coefficient (b_1, b_2) adjust the leverage effect and nonlinearity of option pricing. Here, we only consider the general case and set them both to 1. The values of core parameters $(S_1(0), S_2(0), \lambda_1, \lambda_2, \xi_1, \xi_2, \omega_1(t), \omega_2(t))$ are strictly based on relevant classic literature in this field [7]. In the following, unless otherwise stated, the parameter values are summarized in Table 1.

Table 1. Parameter values.

Parameter	Value	Parameter	Value
a_1	1	a_2	1
b_1	1	b_2	1
$S_1(0)$	100	$S_2(0)$	100
λ_1	0.2	λ_2	0.25
ξ_1	0.7	ξ_2	0.75
$\omega_1(t)$	0.4	$\omega_2(t)$	0.5
K_2	5	ρ	0.8
r	0.05	T	1

The analytical and simulated prices of the digital power exchange option for various times to maturity are shown in Figure 1. Figure 2 shows the relative error between the two prices, all of which are below 1%. This demonstrates the accuracy of the pricing formula. Furthermore, the primary objective of this paper is to investigate the impact of liquidity risk on the pricing of digital power exchange options.

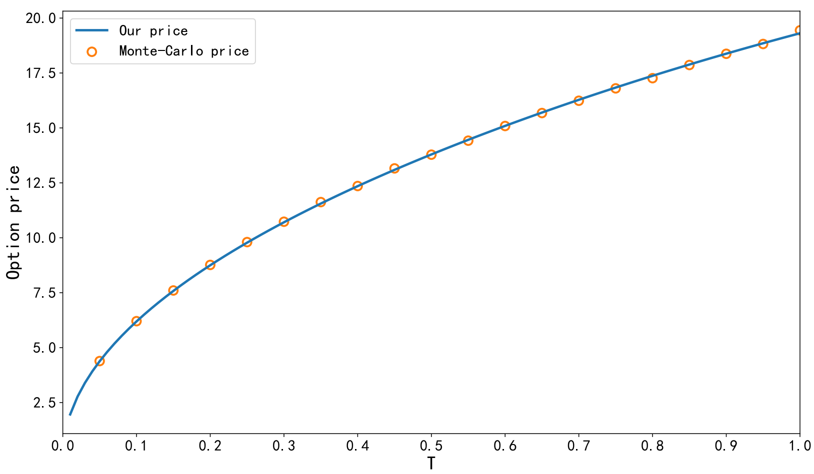

Figure 1. Our price vs. MC price.

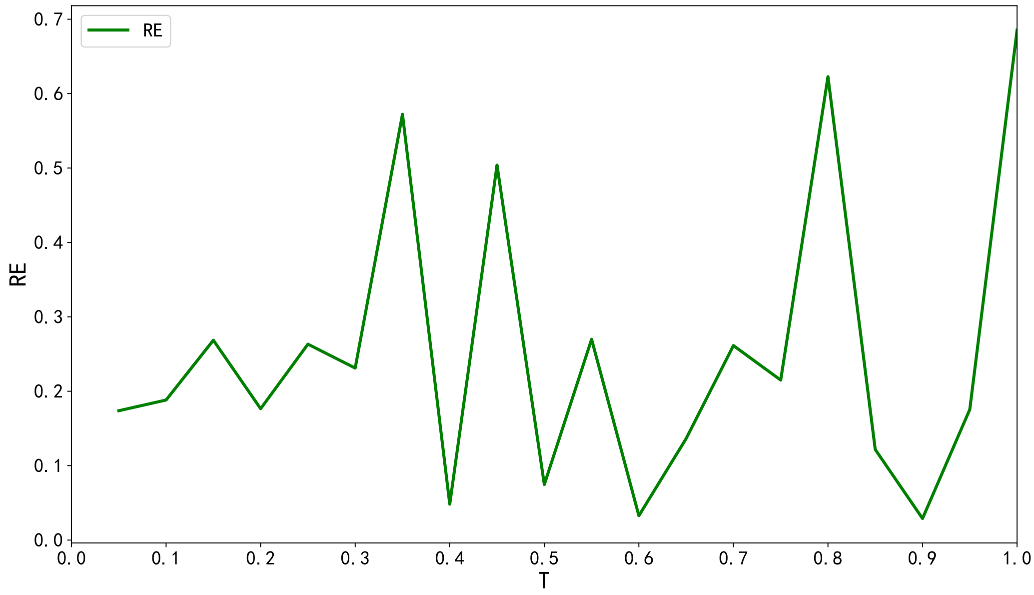

Figure 2. Relative errors.

From Figures 3 and 4, it can be observed that the price considering liquidity factors is consistently higher, demonstrating a positive liquidity premium. This is primarily because an increase in the liquidity level leads to a corresponding increase in the return of the underlying asset.

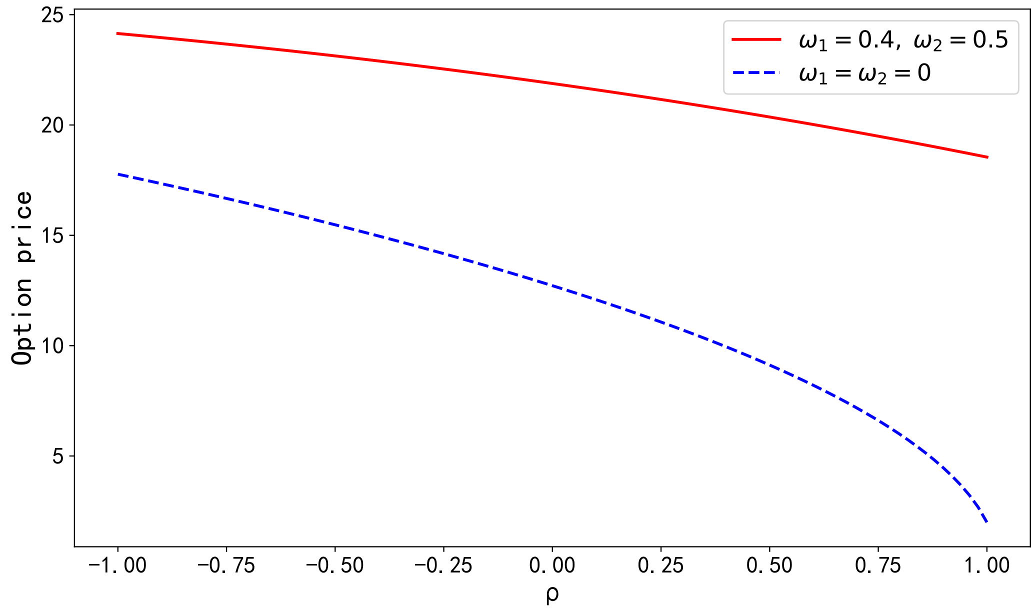

Figure 3. Sensitivity of option price to correlation coefficient ρ.

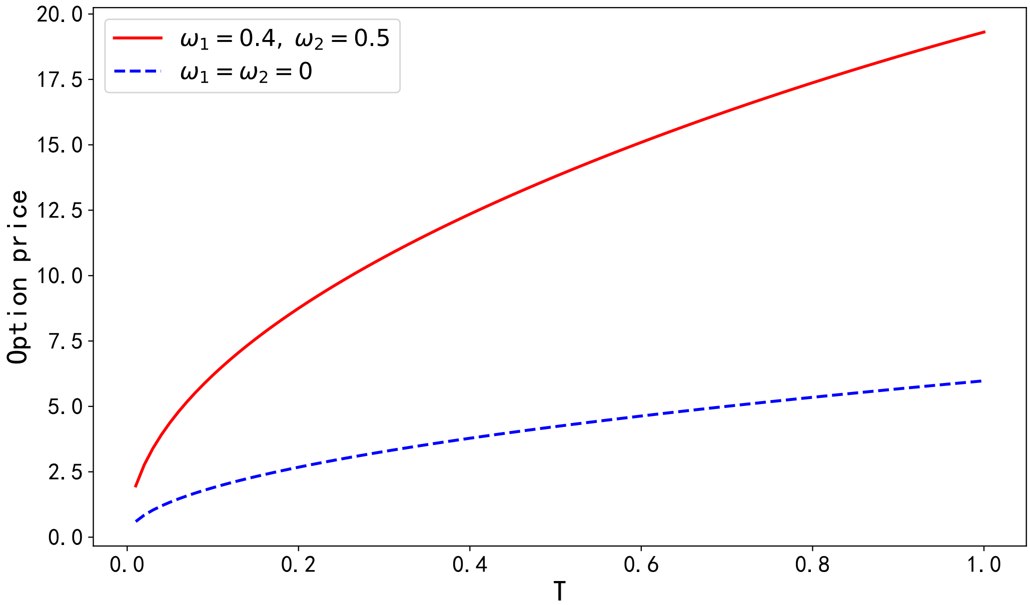

Figure 4. With vs. without liquidity effects.

As can be observed from Figures 5 and 6, the higher the liquidity of an asset, the higher the corresponding option price. These findings jointly establish liquidity as a critical factor in option pricing. In practical applications, the leverage effect of the option can be adjusted by setting different power parameters.

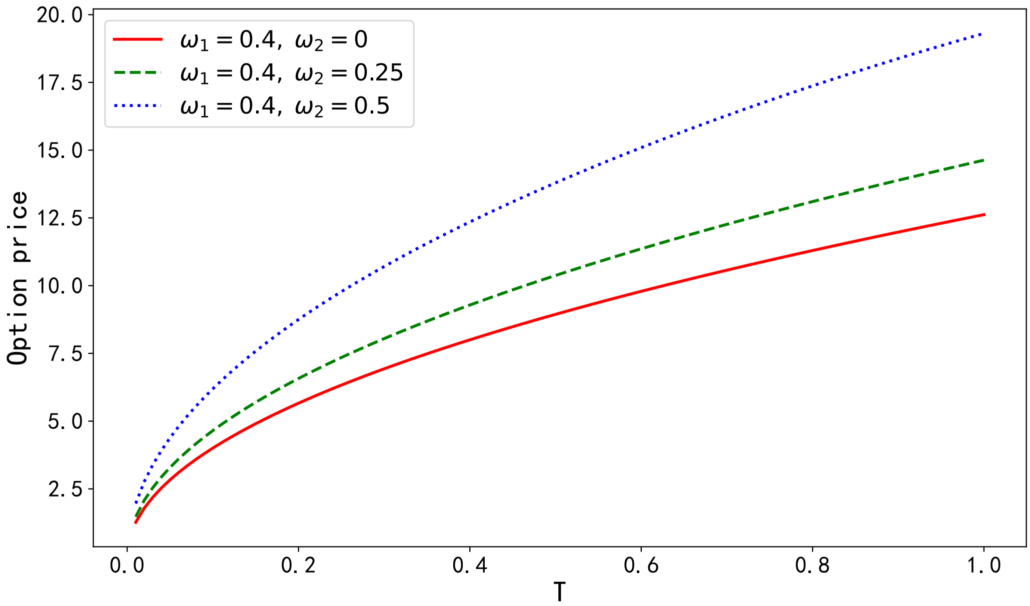

Figure 5. Option price versus ω_2 ($\omega_1 = 0.4$).

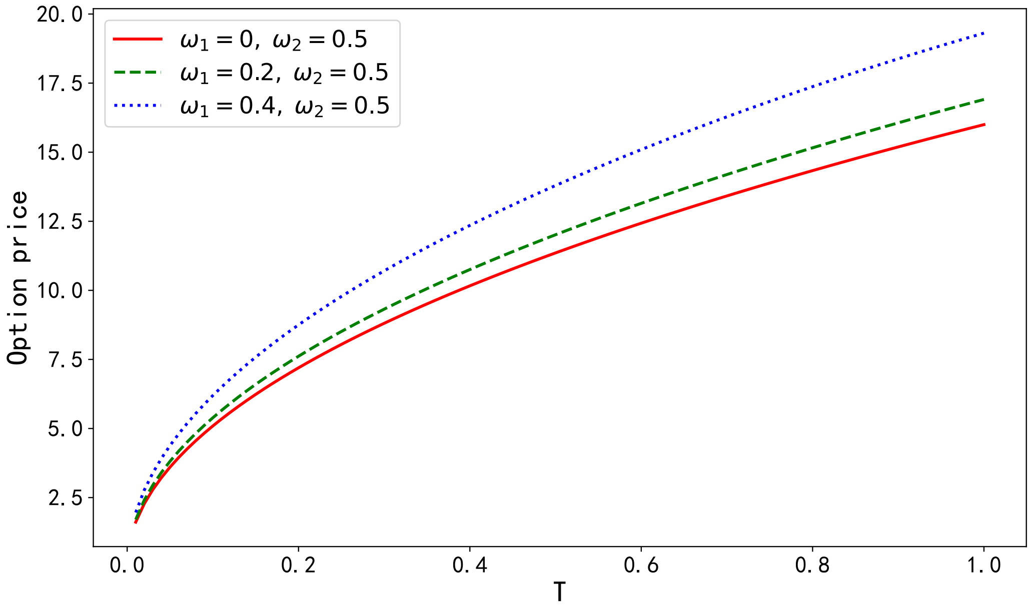

Figure 6. Option price versus ω_1 ($\omega_2 = 0.5$).

5. Conclusions

This paper investigates the pricing of liquidity-adjusted generalized digital power exchange options under incomplete market conditions. The original use of the measure transformation method yields an exact closed-form pricing formula for the option. The flexibility of the model was further demonstrated through numerical experiments, demonstrating how the price of digital power exchange option will be influenced by liquidity factors.

Appendix

A. Practical applications of digital power exchange options

Hedging the "Safety Margin" of Commodity Processing Margins

Scenario: A refinery's profit is simplified as "refined product price (e.g., gasoline) - crude oil price". However, this relationship is typically non-linear and exists within a historical normal range.

Asset 1: Gasoline price $S_1(t)$. The power a_1 might be set slightly greater than 1 (e.g., 1.1) to reflect that the price elasticity of the refined product is higher than that of crude oil during periods of strong demand.

Asset 2: Crude oil price $S_2(t)$. The power a_2 is set to 1.

Execution Interval: Based on historical data and operational costs, a tolerable profit margin interval is determined. For instance, a refinery operates at a reasonable profit when the power ratio of gasoline to crude oil prices ranges from 1.2 to 1.4.

Application:

◦ The refinery purchases a digital power exchange option.

∘ If the ratio falls within interval [1.2, 1.4] and the price of the refined product is higher than that of crude oil, the option pays a floating payoff.

∘ If the ratio exceeds 1.4, meaning the profit margin is excessively high and likely unsustainable due to factors, such as policy interventions, then the option does not pay out; however, the company retains the excess profits earned.

∘If the ratio falls below 1.2, meaning the profit margin is too low, the company is compelled to implement emergency measures, such as production cuts, and the option does not pay out.

Critically, the removal of these tail events leads to a lower option price, which efficiently contains the overall cost of hedging.

Author contributions

Kaihang Zhang: Methodology, conceptualization, writing – original draft, writing – review & editing; Liting Gao: Software, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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