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Research article

Counting spanning trees in generalized K_n-chain/ring graphs

Sujing Cheng and Jun Ge*

School of Mathematical Sciences, Sichuan Normal University, Chengdu 610068, China

* **Correspondence:** Email: mathsgejun@163.com.

Abstract: Let K_n be the complete graph of order n. Very recently, the number of spanning trees (the $\mathcal{NST}$ for short) and the resistance distances in K_n-chain (ring) graphs were determined explicitly. We generalized the concept to the generalized K_n-chain (ring) graph $[L_n^s]^m$ ($[C_n^s]^m$). New formulae for the $\mathcal{NST}$ of $[L_n^s]^m$ and $[C_n^s]^m$ were given by a simple and more physical way with a novel technique of adding a pair of positive and negative edges, avoiding complicated linear algebraic computations.

Keywords: spanning tree; generalized K_n-chain/ring graph; graph transformation; vertex-weighted graph; complete split graph
Mathematics Subject Classification: 05C31, 05C70

1. Introduction

All graphs or networks discussed here are undirected, without loops, but parallel edges are permitted. Consider an edge-weighted graph G with the vertex set $V(G)$ and the edge set $E(G)$, equipped with an edge weight function that $\omega : E(G) \to \mathbb{R}$. This graph can be viewed as a resistor network where edge e represents a resistor with conductance $\omega(e)$. Then, the resistance of edge e is given by $r_e = \frac{1}{\omega(e)}$. For a vertex-weighted graph G with (vertex) weight function $\omega : V(G) \to \mathbb{R}$, the vertex weighting naturally induces an edge weighting where each edge $e = uv$ carries weight $\omega(u)\omega(v)$. Let $R_G(u, v)$ denote the resistance distance between u and v in G.

Let $t(G)$ denote the weighted spanning tree enumeration of graph G, defined as

$$t(G) = \sum_{T\in\mathcal{T}(G)} \prod_{e\in E(T)} \omega(e),$$

where $\mathcal{T}(G)$ represents the set of all spanning trees in G, and ω specifies the edge weight assignment. The weight of an individual spanning tree T corresponds to the multiplicative product of its constituent edge weights. If G is an unweighted graph (formally, $\omega \equiv 1$), the enumeration simplifies to $t(G) = |\mathcal{T}(G)|$, meaning the $\mathcal{NST}$ of G.

Counting spanning trees in graphs and networks constitutes a foundational research domain with wide-ranging applications across diverse disciplines, including Ising/Potts model [26] and sandpile dynamics [6] in statistical physics, network reliability [1] in network theory, random walk [7, 21] in probability, as well as a knot invariant named knot determinant in low dimensional topology [3]. This area has captivated the interest of mathematicians and physicists for over 170 years, as evidenced by the original works of Kirchhoff [14] and Cayley [4]. Recent advancements in this topic are fruitful, including Moon-tyle formulas for complete bipartite graphs [8, 9], complete multipartite graphs [16] and multiple complete split-like graphs [28], counting spanning trees in lattices or complex networks [18, 19, 29], line graphs [11], weighted graphs [30], K_n-complement of bipartite graphs [12], and counting spanning trees with a perfect matching [17].

The graph transformation method, which originated in the field of electrical networks, has emerged as a potent tool for calculating the $\mathcal{NST}$ in graphs. This technique traces its origin to the pioneering work of Kennelly in 1899 [13], who first introduced the triangle-star and star-triangle transformations in the context of electrical circuits. These transformations, along with their generalizations, the mesh-star, or star-mesh transformations [25], soon became fundamental tools in the theory of electrical networks. A significant advancement came in 2010 when Teufl and Wagner [23] demonstrated that such transformations have profound implications for determining the $\mathcal{NST}$ in graphs with specific structures.

Further enriching our understanding of the interconnectedness between different graph theoretical concepts and electrical network properties, Bollobás [2] explored intricate relationships linking the $\mathcal{NST}$ in graphs to effective resistance in electrical circuits, as well as to the behavior of random walks on graphs. It highlights the rich tapestry of connections that exist among these seemingly disparate areas, emphasizing the elegance and utility of graph theory in solving complex problems across disciplines.

Recently, Kosar et al. [15] derived an elegant formula for the $\mathcal{NST}$ in a K_5-chain graph denoted by K_5^m. In parallel, Yan, Kosar, Aslam, Zaman, and Ullah [27] computed various graph invariants for K_4-chain graphs. Furthermore, Sun, Sardar, Yang, and Xu [22] studied K_n-chain/ring graphs, determining both resistance distances and the Kirchhoff index. Additionally, Cheng and Ge [5] counted the $\mathcal{NST}$ in K_n-chain/ring graphs.

The generalized K_n-chain graph, denoted as $[L_n^s]^m$ (where $n \geq 2s \geq 4$, $s \geq 1$, and $m \geq 1$), is a simple graph consisting of m complete subgraphs $K_n^1, K_n^2, \ldots, K_n^m$ (K_n^i is the i-th complete graph of order n for $1 \leq i \leq m$) such that K_n^i and K_n^{i+1} share exactly s common vertices for $1 \leq i \leq m-1$, and each vertex in $[L_n^s]^m$ belongs to at most two complete subgraphs of order n.

If we identify K_n^{m+1} with K_n^1 in $[L_n^s]^{m+1}$ and also insure the properties above, then the resulting structure forms the generalized K_n-ring graph, denoted by $[C_n^s]^m$ (where $n \geq 4$, $s \geq 1$, and $m \geq 3$). For instance, when $n = 8$ and $s = 3$, Figure 1(a) represents the generalized K_8-chain graph $[L_8^3]^m$, and Figure 1(b) represents the generalized K_8-ring graph $[C_8^3]^m$.

In this paper, we employ the equivalence transformation along with the principles of elimination and substitution to derive novel formulae for calculating the $\mathcal{NST}$ in the generalized K_n-chain/ring graph $[L_n^s]^m$ and $[C_n^s]^m$.

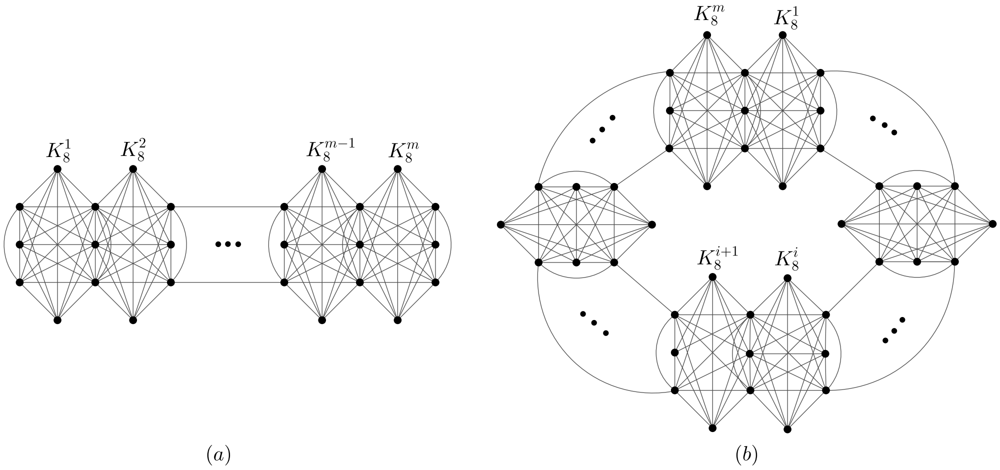

Figure 1. (a). the generalized K_8-chain graph $[L_8^3]^m$; (b). the generalized K_8-ring graph $[C_8^3]^m$.

2. Preliminaries

Now we provide some key notations and lemmas that are essential for understanding our approach.

Definition 1. Define N_1 and N_2 as two graphs with a common vertex subset S ($S \subseteq V(N_1) \cap V(N_2)$). We consider N_1 and N_2 to be S-electrically equivalent if for each pair u, v in S, the resistance distance $R_{N_1}(u, v) = R_{N_2}(u, v)$.

Principle of substitution Let H be a subgraph of G, and suppose that H is $V(H)$-electrically equivalent to H^*. Then, the resistance distance $R_G(u, v)$ is equal to $R_{G^*}(u, v)$ for any pair of vertices $\{u, v\}$ in $V(G)$, with G^* being the network created by replacing H in G with H^*.

By applying linear algebra and principles from electrical network theory, a robust graph transformation technique for enumerating spanning trees was established by Teufl and Wagner [24].

Lemma 2.1 ([24]). *Let G be an edge-weighted connected graph that can be divided into two edge-disjoint subgraphs X and H (where H is connected), with each inheriting weights naturally. These subgraphs satisfy $V(G) = V(X) \cup V(H)$ and $V(X) \cap V(H) = S$. Let H' be an edge-weighted graph such that $E(X) \cap E(H') = \emptyset$ and $V(X) \cap V(H') = S$. Furthermore, suppose that H and H' are electrically equivalent with respect to S. Define $G' = X \cup H'$. Then, we have*

$$\frac{t(G')}{t(G)} = \frac{t(H')}{t(H)}.$$

As a special case, we introduce the following mesh-star transformation for spanning tree enumeration which is important in proving our main results.

Lemma 2.2 ([24]). *Let K_n be a complete subgraph of the edge-weighted graph G, where each edge in K_n has weight ω. Define G' as the graph formed via substitution of the subgraph K_n in G with its electrically equivalent star graph $K_{1,n}$, where the weight of each edge is $n\omega$. Then, the $\mathcal{NST}$ in G' is given by:*

$$t(G) = \frac{1}{n^2\omega} t(G').$$

A split graph is a graph that the vertex set can be partitioned into a clique and an independent set. In the complete split graph $S_{m,n}$, the vertex set $V(S_{m,n})$ is divided into $X \cup Y$, where $X = \{u_1, u_2, \ldots, u_m\}$ is a clique and $Y = \{v_1, v_2, \ldots, v_n\}$ is an independent set, and each vertex in X is connected to each vertex in Y by a single edge.

Recently, Ge, Liao, and Zhang [10] derived the Moon-type formula for vertex-weighted complete split graphs. As a direct corollary of the main result in [10], the $\mathcal{NST}$ in vertex-weighted $S^{\omega}_{m,n}$ is determined as follows.

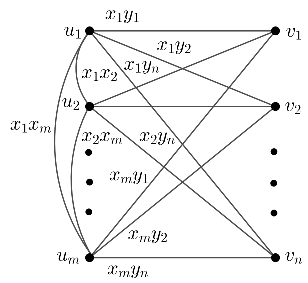

Figure 2. The edge-weighted complete split graph induced by the vertex-weighted $S^{\omega}_{m,n}$.

Theorem 2.1 ([10]). *Let $S^{\omega}_{m,n}$ be the vertex-weighted complete split graph with vertex set partitioned into X and Y, where $X = \{u_1, u_2, \ldots, u_m\}$ forms a complete subgraph (clique) and $Y = \{v_1, v_2, \ldots, v_n\}$ forms an independent set. A vertex in X is assigned a weight x_i, and a vertex in Y is assigned a weight y_i. This vertex weighting results in edge weights $\omega(u_iu_j) = x_ix_j$ for $u_i, u_j \in X$ and $\omega(u_iv_j) = x_iy_j$ for $u_i \in X$ and $v_j \in Y$, as illustrated in Figure 2.*

Define $x = \sum_{i=1}^{m} \omega(u_i)$, and $y = \sum_{i=1}^{n} \omega(v_i)$. Then the $\mathcal{NST}$ of $S^{\omega}_{m,n}$ is

$$t(S^{\omega}_{m,n}) = x^{n-1}(x+y)^{m-1} \prod_{i=1}^{m} x_i \prod_{j=1}^{n} y_j.$$

3. Main results

We start to prove our main results.

Theorem 3.1. *The number of spanning trees in the generalized K_n-chain graph $[L^s_n]^m$, where $n \geq 4$, $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, and $m \geq 1$, is given by*

$$t\left([L^s_n]^m\right) = s^{m-1} \cdot n^{m(n-2s)+2s-2} \cdot (2n-s)^{(m-1)(s-1)}.$$

Proof. In the network $[L^s_n]^m$, K^i_n and K^{i+1}_n share exactly s vertices, forming a complete subgraph denoted by K^i_s. Denote the vertex set of $[L^s_n]^m$ by

$$V([L^s_n]^m) = \left(\bigcup_{i=1}^{m-1} V(K^i_s)\right) \bigcup \left(\bigcup_{i=1}^{m} V^i\right),$$

where $V(K^i_s) = \{v_{i1}, v_{i2}, \ldots, v_{is}\}$ and

$$V^i = V(K^i_n) \setminus (V(K^{i-1}_s) \cup V(K^i_s)),$$

with $V(K_s^0) = V(K_s^m) = \emptyset$ for $1 \leq i \leq m$ and $s \geq 1$. By replacing each edge $e = v_{ik}v_{il}$ (where $1 \leq i \leq m-1$ and $1 \leq k \neq l \leq s$) in K_s^i with two edges of unit weight and one edge of weight -1 (as shown in Figure 3), we obtain a new graph $\widehat{[L_n^s]^m}$. This transformation does not alter the Laplacian matrix, hence the $\mathcal{NST}$ remains the same. Indeed, the triple of edges (two positive and one negative) between v_{ik} and v_{il} has total weight $1 + 1 - 1 = 1$, which is the same as the original edge weight. Consequently, the effective resistance between v_{ik} and v_{il} remains unchanged, making the subgraph electrically equivalent to the original edge. According to Definition 1 and Lemma 2.1, such an electrically equivalent substitution preserves the number of spanning trees of the whole graph. Thus,

$$t([L_n^s]^m) = t(\widehat{[L_n^s]^m}).$$

According to Lemma 2.2, replacing each K_n^i (for $1 \leq i \leq m$) in $\widehat{[L_n^s]^m}$ with a star graph $K_{1,n}$ (where each edge receives weight n and the center vertex is O_i) results in a new graph $\overline{[L_n^s]^m}$ (as depicted in Figure 4), which is $V([L_n^s]^m)$-equivalent to $\widehat{[L_n^s]^m}$.

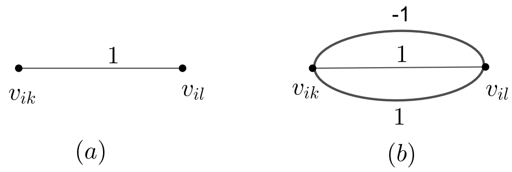

Figure 3. Operation on $[L_n^s]^m$.

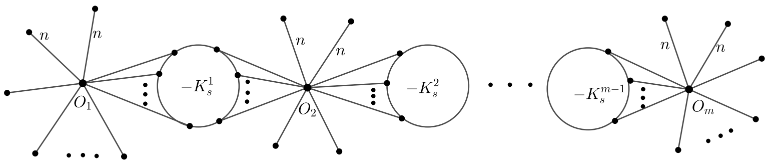

Figure 4. The graph $\overline{[L_n^s]^m}$.

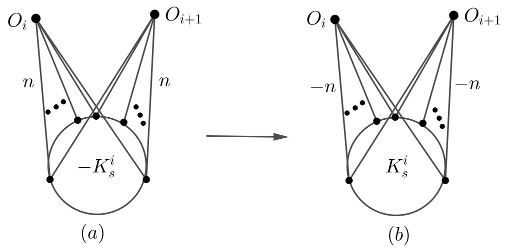

Figure 5. (a) The induced subgraph N_i of $\overline{[L_n^s]^m}$. (b) $-N_i$.

Denote by N_i the induced subgraph with vertex set $\{O_i, O_{i+1}, v_{i1}, v_{i2}, \ldots, v_{is}\}$ and edge set $E(N_i) = \{O_i v_{ik} \mid 1 \leq i \leq m-1, 1 \leq k \leq s, v_{ik} \in V(-K_s^i)\} \cup \{O_{i+1} v_{il} \mid 1 \leq i \leq m-1, 1 \leq l \leq s, v_{il} \in V(-K_s^i)\} \cup E(-K_s^i)$, where $V(K_s^i) = \{v_{i1}, v_{i2}, \ldots, v_{is}\}$ and $-K_s^i$ denotes the weighted graph obtained by taking the negative of each edge weight in K_s^i, as depicted in Figure 5 (a). It is evident that N_i is an edge-weighted complete split graph, and $-N_i$ (see Figure 5 (b)) is an edge-weighted complete split graph naturally induced by the vertex-weighted complete split graph, where the vertex weights satisfy $\omega(O_i) = \omega(O_{i+1}) = -n$ and $\omega(v_{ik}) = 1$ for $1 \le i \le m-1$ and $1 \le k \le s$. By Theorem 2.1, we have

$$t(-N_i) = (s-2n)^{s-1} \cdot s^{2-1} \cdot 1^s \cdot (-n)^2 = sn^2(s-2n)^{s-1}.$$

Then, for N_i, we have

$$t(N_i) = (-1)^{s+1} t(-N_i) = sn^2(2n-s)^{s-1}. \tag{3.1}$$

$\overline{[L_n^s]^m}$ has $m-1$ induced N_i and $mn - 2ms + 2s$ pendant edges (the weight of which is n). Each O_i in $\overline{[L_n^s]^m}$ is a cut vertex. Thus,

$$t(\overline{[L_n^s]^m}) = n^{mn-2ms+2s}[sn^2(2n-s)^{s-1}]^{m-1}. \tag{3.2}$$

By Lemma 2.2 and Eq (3.2), we have

$$\begin{aligned} t([L_n^s]^m) &= \left(\frac{1}{n^2}\right)^m n^{mn-2ms+2s}[sn^2(2n-s)^{s-1}]^{m-1} \\ &= s^{m-1} n^{m(n-2s)+2s-2}(2n-s)^{(m-1)(s-1)}. \end{aligned}$$

□

Remark 1. For the case $s = 2$, we have $t([L_n^2]^m) = 4^{m-1} n^{m(n-4)+2}(n-1)^{m-1}$, which is consistent with the one derived in [5].

We now prove the formula of the $\mathcal{NST}$ in the generalized K_n-ring graph $[C_n^s]^m$.

Theorem 3.2. *The number of spanning trees in the generalized K_n-ring graph $[C_n^s]^m$, where $n \ge 4$, $m \ge 3$, and $1 \le s \le \lfloor \frac{n}{2} \rfloor$, is given by*

$$t\left([C_n^s]^m\right) = 2m \cdot s^{m-1} \cdot n^{m(n-2s)-1} \cdot (2n-s)^{m(s-1)}.$$

Proof. In the network $[C_n^s]^m$, K_n^i and K_n^{i+1} share exactly s vertices, forming a complete subgraph denoted by K_s^i. Denote the vertex set of $[C_n^s]^m$ by

$$V([C_n^s]^m) = \left(\bigcup_{i=1}^{m} V(K_s^i)\right) \bigcup \left(\bigcup_{i=1}^{m} V^i\right),$$

where $V(K_s^i) = \{v_{i1}, v_{i2}, \dots, v_{is}\}$, $V(K_s^{m+1}) = V(K_s^1)$, and $V^i = V(K_n^i) \setminus (V(K_s^i) \cup V(K_s^{i+1}))$ for $1 \le i \le m$ and $s \ge 1$.

Similarly, by substituting each edge in K_s^i with two positive edges (of unit weight) and one negative edge (of weight -1), and applying the mesh-star transformation to the m copies of K_n, we derive a new graph $\overline{[C_n^s]^m}$ (Figure 6). By Lemma 2.2,

$$t([C_n^s]^m) = \left(\frac{1}{n^2}\right)^m t\left(\overline{[C_n^s]^m}\right). \tag{3.3}$$

The graph $\overline{[C_n^s]^m}$ consists of m induced subgraphs N_i (Figure 5(a)) arranged in a cycle, together with $(n-2s)m$ pendant edges, each of weight n. Therefore, a spanning tree in $\overline{[C_n^s]^m}$ consists of three kinds of substructures: a spanning 2-forest (a *spanning 2-forest* of a connected graph G is an acyclic spanning subgraph of G with exactly two connected components) of N_i separating O_i and O_{i+1}; $m-1$ spanning trees of $N_1, \ldots, N_{i-1}, N_{i+1}, \ldots, N_m$; and all $(n-2s)m$ pendant edges (each of weight n).

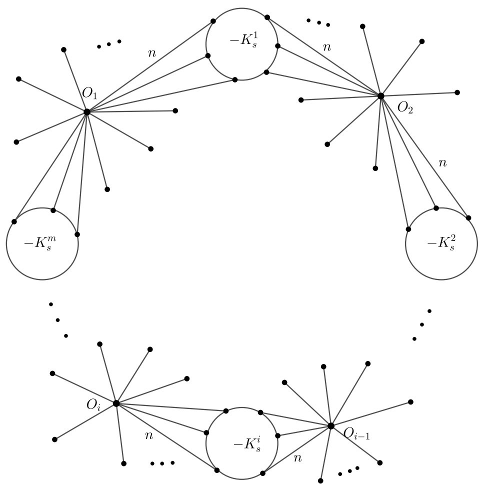

Figure 6. The graph $\overline{[C_n^s]^m}$.

Now, $\overline{[C_n^s]^m}$ can be viewed as a *2-separable graph* in the sense of Li and Yan [20]. Specifically, the base graph G is a cycle C_m with vertices $O_1, O_2, \ldots, O_m$ and edges $e_i = (O_i, O_{i+1})$ for $i = 1, \ldots, m$ (with $O_{m+1} = O_1$). For each edge e_i, the corresponding replacement graph is N_i with root vertices O_i and O_{i+1}. The graphs N_i are connected and share only the root vertices with the base graph and with each other.

By Theorem 1.1 of Li and Yan [20], the weighted spanning tree enumeration of a 2-separable graph is given by

$$t\left(\overline{[C_n^s]^m}\right) = n^{(n-2s)m} \sum_{T \in \mathcal{T}(G)} \left(\prod_{e_i \in E(T)} t(N_i) \right) \left(\prod_{e_j \in E(G) \setminus E(T)} t(N_j/(O_j, O_{j+1})) \right), \tag{3.4}$$

where $t(N_i/(O_i, O_{i+1}))$ denotes the weighted spanning tree number of N_i after contracting O_i and O_{i+1} into a single vertex.

Since G is a cycle C_m, its spanning trees are obtained by deleting exactly one edge. Let T_k be the spanning tree of G obtained by deleting edge e_k. Then, Eq (3.4) simplifies to

$$t\left(\overline{[C_n^s]^m}\right) = n^{(n-2s)m} \sum_{k=1}^{m} \left(t(N_k/(O_k, O_{k+1})) \cdot \prod_{i \neq k} t(N_i) \right). \tag{3.5}$$

$t(N_i/(O_i, O_{i+1}))$ is equal to the determinant of the matrix obtained from the Laplacian matrix of N_i after deleting the rows and columns corresponding to the vertices O_i and O_{i+1}. That is,

$$t(N_i/(O_i, O_{i+1})) = \det((2n-s)I_s + J_s) = 2n(2n-s)^{s-1}, \tag{3.6}$$

where I_s denotes the identity matrix of order s and J_s denotes the all-ones matrix of order s.

Substituting Eqs (3.1) and (3.6) into Eq (3.5) yields

$$t\left(\overline{[C_n^s]^m}\right) = n^{(n-2s)m} \sum_{k=1}^{m} \left(2n(2n-s)^{s-1} \cdot \prod_{i \neq k} \left(sn^2(2n-s)^{s-1} \right) \right)$$

$$= n^{(n-2s)m} \cdot m \cdot 2n(2n-s)^{s-1} \cdot \left(sn^2(2n-s)^{s-1}\right)^{m-1}$$
$$= 2ms^{m-1}n^{m(n-2s+2)-1}(2n-s)^{m(s-1)}.$$

Finally, using Eq (3.3), we obtain

$$\begin{aligned} t([C_n^s]^m) &= \left(\frac{1}{n^2}\right)^m t\left(\overline{[C_n^s]^m}\right) \\ &= 2ms^{m-1}n^{m(n-2s+2)-1-2m}(2n-s)^{m(s-1)} \\ &= 2ms^{m-1}n^{m(n-2s)-1}(2n-s)^{m(s-1)}. \end{aligned}$$

This completes the proof. □

Remark 2. For the case $s = 2$, we have $t([C_n^2]^m) = 4^m mn^{m(n-4)-1}(n-1)^m$, which is consistent with the formula derived in [5].

Corollary 3.1.

$$\frac{t([C_n^s]^m)}{t([L_n^s]^m)} = \frac{2m(2n-s)^{s-1}}{n^{2s-1}}.$$

4. Discussion

This paper develops a graph transformation framework for enumerating spanning trees in complex networks. Central to our approach is an innovative proof technique employing counterbalanced edge pairs to streamline the proof process. Our results provide explicit formulae for the number of spanning trees in generalized K_n-chain and ring graphs.

Our method offers a distinct advantage over the classical Kirchhoff's matrix tree theorem for these specific graph classes. While Kirchhoff's theorem requires computing the determinant of a large Laplacian matrix—computationally expensive for large networks—our transformation technique reduces the problem to simpler structures such as vertex-weighted complete split graphs, where explicit formulae are available. This not only simplifies computation but also provides physical intuition from electrical network theory, bypassing heavy linear algebraic manipulations. The approach relies on electrical equivalence and graph transformations tailored for graphs with overlapping complete subgraphs. For more complex overlapping patterns (e.g., non-complete subgraphs or higher-order overlaps), similar transformations may require additional considerations to ensure electrical equivalence. Moreover, the introduction of negative edge weights, while mathematically sound, may lack direct physical interpretation in some contexts.

5. Conclusions

In one of our subsequent works, we calculate the resistance distances in the generalized K_n-chain/ring graphs. Furthermore, the number of spanning trees of a graph emerges as the thermodynamic limit ($q \to 0$) of the partition function in the q-state Potts model, mathematically equivalent to specific evaluations of the Tutte polynomial. Calculating the Tutte polynomial is generally ♯**P**-hard. A natural open problem is whether generalized K_n-chain/ring graphs admit an explicit expression for the Tutte polynomial.

Author contributions

Sujing Cheng: Conceptualization, Methodology, Formal analysis, Writing–original draft, Writing–review & editing; Jun Ge: Conceptualization, Methodology, Supervision, Validation, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

There are no conflicts of interests or competing interests.

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