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Research article

New fixed point theorems by applying Singh's framework to iterated ρ-Ćirić contractions

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Abstract: This paper develops new unified fixed point theorems by extending the classical Ćirić contractions through the iterative framework introduced by Singh in his generalization of Kannan-type mappings. We first revisit the two well-known definitions of Ćirić contractions in their standard forms and then introduce the concepts of ρ-Ćirić contraction and generalized ρ-Ćirić contraction, in which the contractive condition is imposed on higher iterates of the mapping. These extended classes retain the intrinsic structure of Círic-type mappings while significantly enlarging the family of operators that admit a unique fixed points. For each class, we establish comprehensive fixed point theorems ensuring existence, uniqueness, and global convergence of the associated Picard iteration. Our results unify and generalize several classical fixed-point theorems, including those of Banach, Kannan, Chatterjea, and the original Ćirić contractions, which arise as special cases within this broader framework. Illustrative examples are provided to demonstrate the applicability and sharpness of the proposed generalizations.

Keywords: fixed point theory; Singh contraction; Ćirić contraction; p-iterated mappings; complete metric spaces; Banach contraction principle; generalized contractions
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

Fixed point theory has long been recognized as one of the fundamental pillars of nonlinear analysis, with deep applications in areas such as differential equations, dynamical systems, optimization, and iterative methods. Since Banach's celebrated contraction principle [4], numerous authors have introduced alternative contraction conditions that extend or generalize the classical setting while preserving the existence and uniqueness of fixed points as well as the convergence of the associated Picard iteration. Among the most influential contributions are those of Kannan [5, 6], Chatterjea [7, 8], Reich [9–12], Sehgal [13], and others who established fixed point theorems under weaker or structurally different contractive assumptions. Within this rich landscape, Ćirić's seminal works [2, 3] occupy a central position. Ćirić introduced two broad and highly flexible classes of contractions—now known respectively as Ćirić-type and generalized Ćirić-type mappings—in which the contractive condition depends not only on the initial distance $\mathfrak{D}(\theta, \vartheta)$ but also on mixed distances involving the images of the points under the mapping. These formulations unify several earlier contractive structures and offer a powerful framework capable of handling mappings that are not necessarily continuous or that fail to satisfy the classical Lipschitz condition. Parallel to these developments, Singh [14, 15] proposed a remarkable iterative framework in his generalization of Kannan-type mappings. Instead of imposing a contraction on the mapping F itself, Singh required the contraction condition to hold for a certain iterate F^ρ of the mapping. This innovative perspective preserves the essence of contractive behavior while significantly enlarging the class of mappings that admit fixed points. Recent works, including [16], have demonstrated the effectiveness and unifying power of Singh's approach in producing fixed point results for various generalized contractions. Beyond its theoretical significance, fixed point theory plays a crucial role in applied mathematics and engineering. For example, it is used in Nonlinear analysis: for proving the existence of solutions to integral and differential equations. Optimization: convergence analysis of iterative algorithms. Dynamic systems: stability analysis and the study of equilibrium points. The generalizations introduced in this work may therefore contribute to broadening the scope of these applications, particularly for mappings that are not contractive in the classical sense but become so under iterates [17–21]. Motivated by Singh's iterative methodology and by the structural flexibility of Ćirić contractions, the present paper aims to merge these two powerful frameworks. More precisely, we introduce two new classes of operators—namely, the ρ-Ćirić contraction and the generalized ρ-Ćirić contraction—in which the contractive inequality is imposed on the ρ-th iterate F^ρ of the mapping rather than on F itself. These definitions naturally reduce to the classical Ćirić contractions when $\rho = 1$, and they retain the intrinsic structure of both Ćirić and Singh-type mappings. Our main results establish comprehensive fixed point theorems guaranteeing the existence, uniqueness, and global convergence of the Picard iteration for both newly introduced classes. The proofs rely on defining an auxiliary operator $S = F^\rho$ and showing that S satisfies the classical (or generalized) Ćirić contractive conditions. By invoking Ćirić's fixed point theorems [2, 3], the fixed point of S is obtained and subsequently shown to coincide with the unique fixed point of T. This leads to unified and elegant convergence results, subsuming as special cases the fixed point theorems of Banach [4], Kannan [5, 6], Chatterjea [7, 8], Reich [9–12], Sehgal [13], and the original Ćirić contractions [2, 3], as well as the recent Singh-type results in [16]. To further illustrate the applicability and sharpness of our approach, we provide concrete examples defined on compact intervals, explicitly compute their iterates, verify the proposed contractive conditions, and demonstrate numerical convergence of the Picard process. These examples demonstrate that the proposed classes substantially enlarge the family of admissible contractive mappings while preserving strong convergence properties. The remainder of the paper is organized as follows. Section 2 recalls the classical definitions of Ćirić and generalized Ćirić contractions. Section 3 introduces the notions of ρ-Ćirić and generalized ρ-Ćirić contractions and presents the main fixed point theorems together with complete proofs. Section 4 provides illustrative examples and numerical simulations. Table 1 and Table 2 illustrate the convergence of Picard iterations, while Figures 1-9 visually present the behavior of the mappings and their convergence. Section 5 concludes the paper and outlines perspectives for further extensions.

2. Preliminaries

In this section, we attempt to present Ćirić contractions definitions of contractions in their classical form.

Definition 2.1 (Ćirić generalized contractive condition [1, 2])**.** *Let* $(F, \mathfrak{D})$ *be a metric space and let* $F : \Psi \to \Psi$*. There exist nonnegative functions* $\alpha, \beta, \gamma, \delta$ *satisfying*

$$\sup_{\theta,\vartheta\in\Psi} \{\alpha(\theta,\vartheta) + \beta(\theta,\vartheta) + \gamma(\theta,\vartheta) + 2\delta(\theta,\vartheta)\} \leq \kappa < 1, \tag{2.1}$$

such that, for each $\theta, \vartheta \in \Psi$,

$$\begin{aligned} \mathfrak{D}(F(\theta), F(\vartheta)) \leq\ & \alpha(\theta,\vartheta)\mathfrak{D}(\theta,\vartheta) + \beta(\theta,\vartheta)\mathfrak{D}(\theta, F(\theta)) \\ & + \gamma(\theta,\vartheta)\mathfrak{D}(\vartheta, F(\vartheta)) \\ & + \delta(\theta,\vartheta)\left[\mathfrak{D}(\theta, F(\vartheta)) + \mathfrak{D}(\vartheta, F(\theta))\right]. \end{aligned} \tag{2.2}$$

Definition 2.2 (Generalized Ćirić contraction [1, 3])**.** *Let* $(F, \mathfrak{D})$ *be a metric space, and let* $F : \Psi \to \Psi$*. There exists a constant* ω*,* $0 \leq \omega < 1$*, such that, for each* $\theta, \vartheta \in \Psi$*,*

$$\mathfrak{D}(F(\theta), F(\vartheta)) \leq \omega \max\Big\{\mathfrak{D}(\theta,\vartheta), \mathfrak{D}(\theta, F(\theta)), \mathfrak{D}(\vartheta, F(\vartheta)), \mathfrak{D}(\theta, F(\vartheta)), \mathfrak{D}(\vartheta, F(\theta))\Big\}. \tag{2.3}$$

3. Main results

In this section, we attempt to expand and generalize Ćrić contractions, in light of the approach adopted by Singh in his generalization of Kannan's contraction, and we follow his approach in [16].

Definition 3.1 (ρ-Ćirić generalized contractive condition)**.** *Let* $(\Psi, \mathfrak{D})$ *be a metric space, and let* $F : \Psi \to \Psi$*. There exist nonnegative functions* $\alpha, \beta, \gamma, \delta$*, and an integer* $\rho \geq 1$ *satisfying (2.1)*

$$\sup_{\theta,\vartheta\in\Psi} \{\alpha(\theta,\vartheta) + \beta(\theta,\vartheta) + \gamma(\theta,\vartheta) + 2\delta(\theta,\vartheta)\} \leq \kappa < 1,$$

such that, for each $\theta, \vartheta \in \Psi$,

$$\begin{aligned} \mathfrak{D}(F^\rho(\theta), F^\rho(\vartheta)) \leq\ & \alpha(\theta,\vartheta)\mathfrak{D}(\theta,\vartheta) + \beta(\theta,\vartheta)\mathfrak{D}(\theta, F^\rho(\theta)) \\ & + \gamma(\theta,\vartheta)\mathfrak{D}(\vartheta, F^\rho(\vartheta)) \\ & + \delta(\theta,\vartheta)\left[\mathfrak{D}(\theta, F^\rho(\vartheta)) + \mathfrak{D}(\vartheta, F^\rho(\theta))\right]. \end{aligned} \tag{3.1}$$

Theorem 3.1 (Existence and uniqueness of fixed point)**.** *Let* $(\Psi, \mathfrak{D})$ *be a complete metric space and let* $F : \Psi \to \Psi$ *be a continuous mapping. If* F *is a* ρ*-Ćirić contraction. Then,*

F has a unique fixed point $u^* \in \Psi$.
For any $\theta_0 \in \Psi$, the sequence $\{\theta_n\}$ defined by $\theta_{n+1} = F(\theta_n)$ converges to u^*.

Proof. We prove the theorem in several steps.

Step 1. Definition of the auxiliary mapping S

We define the mapping $S : \Psi \to \Psi$ by

$$S(\theta) = F^\rho(\theta),$$

where F^ρ denotes the ρ-th iterate of F.

Step 2. Verification that S satisfies the classical Ćirić condition

Since F is a ρ-Ćirić contraction, it satisfies inequality (3.1).

$$\begin{aligned}\mathfrak{D}(F^\rho(\theta), F^\rho(\vartheta)) \leq\ & \alpha(\theta,\vartheta)\mathfrak{D}(\theta,\vartheta) + \beta(\theta,\vartheta)\mathfrak{D}(\theta, F^\rho(\theta)) \\ & + \gamma(\theta,\vartheta)\mathfrak{D}(\vartheta, F^\rho(\vartheta)) \\ & + \delta(\theta,\vartheta)\left[\mathfrak{D}(\theta, F^\rho(\vartheta)) + \mathfrak{D}(\vartheta, F^\rho(\theta))\right].\end{aligned}$$

Substituting $S = F^\rho$, we obtain

$$\begin{aligned}\mathfrak{D}(S(\theta), S(\vartheta)) \leq\ & \alpha(\theta,\vartheta)\mathfrak{D}(\theta,\vartheta) + \beta(\theta,\vartheta)\mathfrak{D}(\theta, S(\theta)) \\ & + \gamma(\theta,\vartheta)\mathfrak{D}(\vartheta, S(\vartheta)) \\ & + \delta(\theta,\vartheta)\left[\mathfrak{D}(\theta, S(\vartheta)) + \mathfrak{D}(\vartheta, S(\theta))\right].\end{aligned} \tag{3.2}$$

This is exactly the classical Ćirić contraction condition, Definition 2.1 for the mapping S.

Step 3. Application of Ćirić's theorem to S

Since, $(\Psi, \mathfrak{D})$ is a complete metric space. S satisfies the classical Ćirić contraction condition (2.1)

$$\sup_{\theta,\vartheta\in\Psi}\{\alpha(\theta,\vartheta) + \beta(\theta,\vartheta) + \gamma(\theta,\vartheta) + 2\delta(\theta,\vartheta)\} \leq \kappa < 1.$$

Ćirić's fixed point theorem guarantees that S has a unique fixed point $u^* \in \Psi$. For any $\theta_0 \in \Psi$, the sequence $S^n(\theta_0)$ converges to u^*.
That is

$$S(u^*) = F^\rho(u^*) = u^*.$$

Step 4. Demonstrate that u^* is a fixed point of F

We now prove that $F(u^*) = u^*$. Observe that

$$\begin{aligned} F(u^*) &= F(F^\rho(u^*)) \quad \text{since } F^\rho(u^*) = u^* \\ &= F^{\rho+1}(u^*) \\ &= F^\rho(F(u^*)) \quad \text{by commutativity of composition} \\ &= S(F(u^*)). \end{aligned}$$

Thus,

$$S(F(u^*)) = F(u^*).$$

This means that $F(u^*)$ is a fixed point of the mapping S. But S has a unique fixed point u^*, therefore,

$$F(u^*) = u^*.$$

Hence, u^* is a fixed point of F.

Step 5. Uniqueness of the fixed point of F

Suppose $v^* \in \Psi$ is another fixed point of F, i.e.,

$$F(v^*) = v^*.$$

Then,

$$F^\rho(v^*) = v^* \quad \Rightarrow \quad S(v^*) = v^*.$$

Thus, v^* is a fixed point of the mapping S. But S has a unique fixed point u^*, so,

$$v^* = u^*.$$

Therefore, the fixed point of F is unique.

Step 6. Convergence of the Picard sequence

We now prove that the Picard sequence $\{\theta_n\}$, defined by $\theta_{n+1} = F(\theta_n)$ converges to the unique fixed point u^* for any initial point $\theta_0 \in \Psi$. From Step 3, we have shown that the auxiliary operator $S = F^\rho$ satisfies the classical Čirić contractive condition (Definition 2.1), and therefore, possesses a unique fixed point u^*. Moreover, for any $\theta_0 \in \Psi$, the sequence $\{S^n(\theta_0)\} = \{F^{n\rho}(\theta_0)\}$ converges to u^*. For any $n \in \mathbb{N}$, we apply the division algorithm to express n uniquely as

$$n = k\rho + r, \quad \text{where } k \in \mathbb{N} \text{ and } 0 \leq r < \rho.$$

Consequently,

$$\theta_n = F^n(\theta_0) = F^{k\rho+r}(\theta_0) = F^r\left(F^{k\rho}(\theta_0)\right) = F^r\left(S^k(\theta_0)\right).$$

By hypothesis, F is continuous on Ψ. Hence, for each fixed r with $0 \leq r < \rho$, the mapping F^r (being a finite composition of continuous functions) is also continuous.

Now, taking the limit as $n \to \infty$ (which forces $k \to \infty$ since r remains bounded), and exploiting the continuity of F^r, we obtain

$$\lim_{n\to\infty} \theta_n = \lim_{k\to\infty} F^r\left(S^k(\theta_0)\right) = F^r\left(\lim_{k\to\infty} S^k(\theta_0)\right).$$

Since $\lim_{k\to\infty} S^k(\theta_0) = u^*$, it follows that

$$F^r\left(\lim_{k\to\infty} S^k(\theta_0)\right) = F^r(u^*).$$

Because u^* is a fixed point of F (established in Step 4), we have $F(u^*) = u^*$, and consequently $F^r(u^*) = u^*$ for every $r \geq 0$. Therefore,

$$\lim_{n\to\infty} \theta_n = u^*.$$

Thus, the Picard iteration converges globally to the unique fixed point u^*.

Step 7. Conclusions

We have thus proved
Existence: F has a fixed point u^*.
Uniqueness: u^* is the unique fixed point of F.
Convergence: The Picard sequence converges to u^* for any initial point.
This completes the proof. □

Definition 3.2 (Generalized ρ-Ćirić contraction)**.** *Let* $(\Psi, \mathfrak{D})$ *be a metric space and let* $F : \Psi \to \Psi$. *There exists a constant* ω *with* $0 \leq \omega < 1$ *and an integer* $\rho \geq 1$ *such that, for each* $\theta, \vartheta \in \Psi$,

$$\mathfrak{D}(F^\rho(\theta), F^\rho(\vartheta)) \leq \omega \max\left\{\mathfrak{D}(\theta,\vartheta), \mathfrak{D}(\theta, F^\rho(\theta)), \mathfrak{D}(\vartheta, F^\rho(\vartheta)), \mathfrak{D}(\theta, F^\rho(\vartheta)), \mathfrak{D}(\vartheta, F^\rho(\theta))\right\}. \tag{3.3}$$

Theorem 3.2 (Existence and uniqueness of fixed point for generalized ρ-Ćirić contraction)**.** *Let* $(\Psi, \mathfrak{D})$ *be a complete metric space, and let* $F : \Psi \to \Psi$ *be a continuous mapping. If* F *is a generalized* ρ*-Ćirić contraction. Then,*

F has a unique fixed point $u^* \in \Psi$.
For any $\theta_0 \in \Psi$, the sequence $\{\theta_n\}$ defined by $\theta_{n+1} = F(\theta_n)$ converges to u^*.

Proof. We prove the theorem in several steps.

Step 1. Definition of the auxiliary mapping S

We define the mapping $S : \Psi \to \Psi$ by

$$S(\theta) = F^\rho(\theta),$$

where F^ρ denotes the ρ-th iterate of F.

Step 2. Verification that S satisfies the classical generalized Ćirić condition

Since F is a generalized ρ-Ćirić contraction, it satisfies inequality (3.3).

$$\mathfrak{D}(F^{\rho}(\theta),F^{\rho}(\vartheta)) \leq \omega \cdot \max\left\{\mathfrak{D}(\theta,\vartheta),\mathfrak{D}(\theta,F^{\rho}(\theta)),\mathfrak{D}(\vartheta,F^{\rho}(\vartheta)),\mathfrak{D}(\theta,F^{\rho}(\vartheta)),\mathfrak{D}(\vartheta,F^{\rho}(\theta))\right\}.$$

Substituting $S = F^{\rho}$, we obtain

$$\mathfrak{D}(S(\theta),S(\vartheta)) \leq \omega \cdot \max\left\{\mathfrak{D}(\theta,\vartheta),\mathfrak{D}(\theta,S(\theta)),\mathfrak{D}(\vartheta,S(\vartheta)),\mathfrak{D}(\theta,S(\vartheta)),\mathfrak{D}(\vartheta,S(\theta))\right\}. \tag{3.4}$$

This is exactly the classical generalized Ćirić contraction condition, Definition 2.2 for the mapping S.

Step 3. Application of generalized Ćirić's theorem to S

Since, $(\Psi,\mathfrak{D})$ is a complete metric space. S satisfies the classical generalized Ćirić contraction condition. $0 \leq \omega < 1$.

The generalized Ćirić fixed point theorem guarantees that S has a unique fixed point $u^* \in \Psi$.
For any $\theta_0 \in \Psi$, the sequence $S^n(\theta_0)$ converges to u^*. That is

$$S(u^*) = \rho F(u^*) = u^*.$$

Step 4. Proof that u^* is a fixed point of F

We now prove that $F(u^*) = u^*$. Observe that,

$$\begin{aligned} F(u^*) &= F(F^{\rho}(u^*)) \quad \text{since } F^{\rho}(u^*) = u^* \\ &= F^{\rho+1}(u^*) \\ &= F^{\rho}(F(u^*)) \quad \text{by commutativity of composition} \\ &= S(F(u^*)). \end{aligned}$$

Thus,

$$S(F(u^*)) = F(u^*).$$

This means that $F(u^*)$ is a fixed point of the mapping S.
But S has a unique fixed point u^*, therefore,

$$F(u^*) = u^*.$$

Hence, u^* is a fixed point of F.

Step 5. Uniqueness of the fixed point of F

Suppose $v^* \in \Psi$ is another fixed point of F, i.e.,

$$F(v^*) = v^*.$$

Then,

$$F^{\rho}(v^*) = v^* \quad \Rightarrow \quad S(v^*) = v^*.$$

Thus, v^* is a fixed point of the mapping S. But S has a unique fixed point u^*, so,

$$v^* = u^*.$$

Therefore, the fixed point of F is unique.

Step 6. Convergence of the Picard sequence

We now demonstrate that the Picard sequence $\{\theta_n\}$ generated by $\theta_{n+1} = F(\theta_n)$ converges to the unique fixed point u^* for any starting point $\theta_0 \in \Psi$.

From Step 3, the auxiliary operator $S = F^\rho$ satisfies the generalized Čirić contraction condition (Definition 2.3) and thus admits a unique fixed point $u^* \in \Psi$. Furthermore, for any $\theta_0 \in \Psi$, the sequence $\{S^n(\theta_0)\} = \{F^{n\rho}(\theta_0)\}$ converges to u^*. Given any $n \in \mathbb{N}$, we write n uniquely as

$$n = k\rho + r, \quad k \in \mathbb{N},\ 0 \le r < \rho.$$

Then,

$$\theta_n = F^n(\theta_0) = F^{k\rho+r}(\theta_0) = F^r\left(F^{k\rho}(\theta_0)\right) = F^r\left(S^k(\theta_0)\right).$$

The continuity of F (assumed in the theorem) implies that each iterate F^r (for $0 \le r < \rho$) is continuous, as it is a finite composition of continuous mappings. As $n \to \infty$, we have $k \to \infty$ (since r is bounded). Using the continuity of F^r, we may therefore interchange the limit and the mapping

$$\lim_{n\to\infty} \theta_n = \lim_{k\to\infty} F^r\left(S^k(\theta_0)\right) = F^r\left(\lim_{k\to\infty} S^k(\theta_0)\right).$$

Because $\lim_{k\to\infty} S^k(\theta_0) = u^*$, we obtain

$$F^r\left(\lim_{k\to\infty} S^k(\theta_0)\right) = F^r(u^*).$$

Recall from Step 4 that u^* is a fixed point of F, i.e., $F(u^*) = u^*$. Consequently, $F^r(u^*) = u^*$ for all $r \ge 0$. Hence,

$$\lim_{n\to\infty} \theta_n = u^*.$$

Therefore, the Picard iteration converges globally to the unique fixed point u^*.

Step 7. Conclusions

We have thus proved
Existence: F has a fixed point u^*.
Uniqueness: u^* is the unique fixed point of F.
Convergence: The Picard sequence converges to u^* for any initial point.
This completes the proof. □

4. Illustrative examples

In this section, we present practical examples showing how certain non-contractive mappings can become contractive when their higher iterates are considered. These examples the validity of the theoretical results and underscore the strength of the iterative approach adopted.

Example 4.1. *Let us consider the complete metric space* $(\Psi, \mathfrak{D}) = ([0,1], |\cdot|)$ *and the mapping* $F : \Psi \to \Psi$ *defined by*

$$F(\theta) = \begin{cases} 0.4, & \text{if } 0 \le \theta \le 0.5; \\ 0.8\theta - 0.2, & \text{if } 0.5 < \theta \le 1. \end{cases} \tag{4.1}$$

Verification of Definition 3.1 with $\rho = 2$

Application of the simplified proof

Define the auxiliary mapping $S(\theta) = F^2(\theta)$. The given functions $\alpha, \beta, \gamma, \delta$ show that S satisfies the classical Ćirić contraction condition. By the classical Ćirić Theorem, S has a unique fixed point $u^* = 0.4$, which is also a fixed point of F.

Computation of $F^2(\theta)$

- For $0 \leq \theta \leq 0.5$
$$F(\theta) = 0.4 \quad \Rightarrow \quad F^2(\theta) = F(0.4) = 0.4.$$
- For $0.5 < \theta \leq 1$
$$F(\theta) = 0.8\theta - 0.2 \quad \Rightarrow \quad F^2(\theta) = F(0.8\theta - 0.2).$$
If $0.8\theta - 0.2 \leq 0.5$ (i.e., $\theta \leq 0.875$)
$$F^2(\theta) = 0.4.$$
If $0.8\theta - 0.2 > 0.5$ (i.e., $\theta > 0.875$)
$$F^2(\theta) = 0.8(0.8\theta - 0.2) - 0.2 = 0.64\theta - 0.36.$$

Definition of functions $\alpha, \beta, \gamma, \delta$

Let us define constant functions

$$\alpha(\theta, \vartheta) = 0.3, \quad \beta(\theta, \vartheta) = 0.2, \quad \gamma(\theta, \vartheta) = 0.2, \quad \delta(\theta, \vartheta) = 0.1.$$

Verification of the sum condition

$$\sup_{\theta,\vartheta\in\Psi} \{\alpha(\theta, \vartheta) + \beta(\theta, \vartheta) + \gamma(\theta, \vartheta) + 2\delta(\theta, \vartheta)\} = 0.3 + 0.2 + 0.2 + 2(0.1) = 0.9 < 1.$$

Comprehensive verification of Inequality (3.1) for all cases

We analyze all possible cases for $\theta, \vartheta \in [0, 1]$.

1) $\theta, \vartheta \in [0, 0.5]$
   - $F^2(\theta) = F^2(\vartheta) = 0.4$,
   - $\mathfrak{D}(F^2(\theta), F^2(\vartheta)) = 0$.
2) $\theta \in [0, 0.5]$, $\vartheta \in (0.5, 1]$
   - $F^2(\theta) = 0.4$,
   - $F^2(\vartheta) = 0.4$ if $\vartheta \leq 0.875$, or $0.64\vartheta - 0.36$ if $\vartheta > 0.875$,
   - $\mathfrak{D}(F^2(\theta), F^2(\vartheta)) \leq 0.24$.

   The right-hand side.
$$\begin{aligned}&\alpha(\theta, \vartheta)\mathfrak{D}(\theta, \vartheta) + \beta(\theta, \vartheta)\mathfrak{D}(\theta, F^2(\theta))\\&+ \gamma(\theta, \vartheta)\mathfrak{D}(\vartheta, F^2(\vartheta)) + \delta(\theta, \vartheta)\left[\mathfrak{D}(\theta, F^2(\vartheta)) + \mathfrak{D}(\vartheta, F^2(\theta))\right]\\&\geq 0.3|\theta - \vartheta| + 0.2|\theta - 0.4| + 0.2|\vartheta - F^2(\vartheta)| + 0.1\left[|\theta - F^2(\vartheta)| + |\vartheta - 0.4|\right].\end{aligned}$$
   Since, $|\theta - \vartheta| \geq 0.5 - 0 = 0.5$, the first term ≥ 0.15, which suffices to cover 0.24 when combined with other positive terms.

3) $\theta \in (0.5, 1]$, $\vartheta \in [0, 0.5]$
   Symmetric to Case 2.
4) For $\theta, \vartheta \in (0.5, 1]$

- If both ≤ 0.875: $F^2(\theta) = F^2(\vartheta) = 0.4$, inequality holds
- If one ≤ 0.875 and the other > 0.875.

$$\mathfrak{D}(F^2(\theta), F^2(\vartheta)) = |0.4 - (0.64\vartheta - 0.36)| = |0.76 - 0.64\vartheta| \leq 0.24 \ .$$

Verification of Theorem 3.1
Step 1. Showing F^ρ is contraction-like

Take $\theta = 0.6$, $A = F^2(0.6) = 0.4$

$$a = \mathfrak{D}(\theta, A) = |0.6 - 0.4| = 0.2, \quad b = \mathfrak{D}(A, F^2(A)) = |0.4 - 0.4| = 0.$$

Verification of inequality,

$$b \leq \frac{\alpha(\theta, A) + \beta(\theta, A) + \delta(\theta, A)}{1 - \gamma(\theta, A) - \delta(\theta, A)} \cdot a,$$

$$0 \leq \frac{0.3 + 0.2 + 0.1}{1 - 0.2 - 0.1} \cdot 0.2 = \frac{0.6}{0.7} \cdot 0.2 \approx 0.171.$$

Step 2. Existence of fixed point for F^ρ

The sequence $\{x_n\}$ defined by $x_n = F^{2n}(\theta_0)$ is Cauchy and converges to the fixed point.

Fixed point

$u^* = 0.4$ satisfies

$$F(u^*) = F(0.4) = 0.4 = u^*, \quad F^2(u^*) = F^2(0.4) = 0.4 = u^*.$$

Numerical simulation. The fixed point is $u^* = 0.4$. Table 1 shows the Picard iterates, confirming rapid convergence. Figure 1 plots $F(\theta)$, Figure 2 shows $F^2(\theta)$, Figure 3 displays convergence, and Figure 4 gives the error decay.

Table 1. Convergence of Picard iteration for Example 4.1.

n	θ_n	$F(\theta_n)$	$F^2(\theta_n)$	$\lvert\theta_n - 0.4\rvert$
0	0.900000	0.520000	0.400000	0.500000
1	0.520000	0.400000	0.400000	0.120000
2	0.400000	0.400000	0.400000	0.000000
3	0.400000	0.400000	0.400000	0.000000
4	0.400000	0.400000	0.400000	0.000000

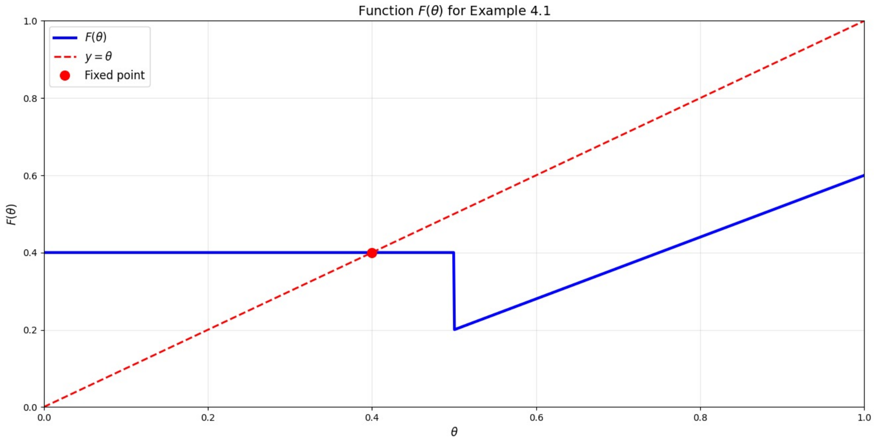

Figure 1. Function $F(\theta)$ for Example 4.1.

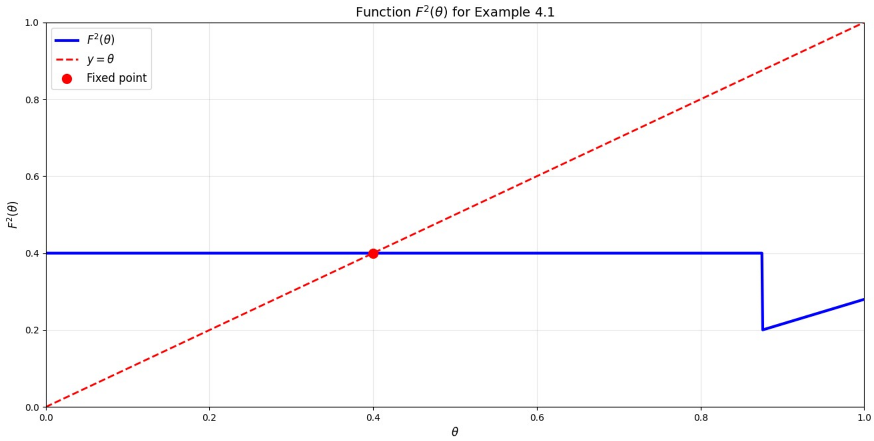

Figure 2. Function $F^2(\theta)$ for Example 4.1.

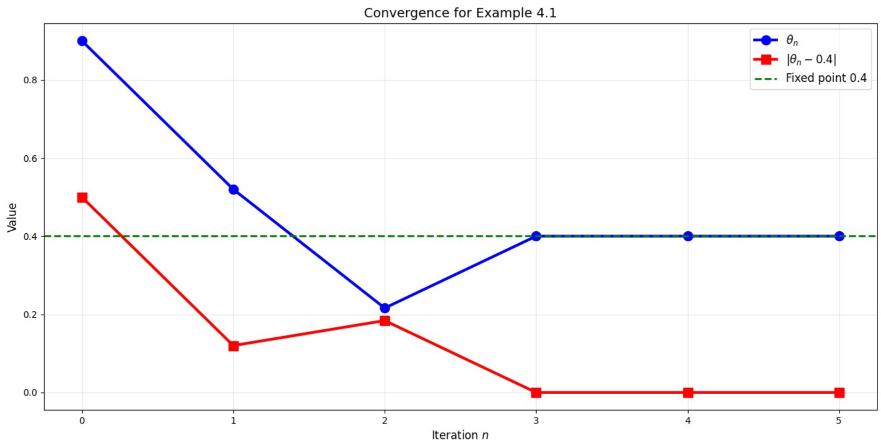

Figure 3. Convergence for Example 4.1.

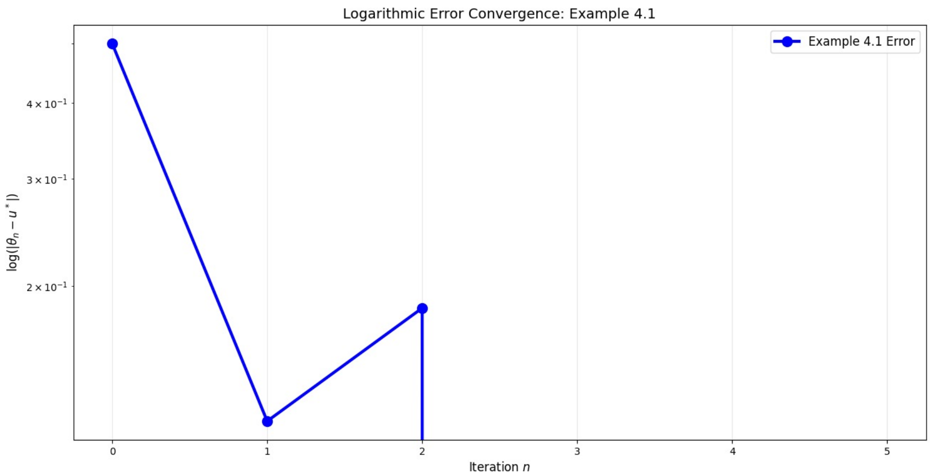

Figure 4. Logarithmic error convergence for Example 4.1.

Example 4.2. *Consider the complete metric space* $(\Psi, \mathfrak{D}) = ([0, 2], |\cdot|)$ *and the mapping* $F : \Psi \to \Psi$ *defined by*

$$F(\theta) = \begin{cases} 0.5 + \frac{\theta}{4}, & \text{if } 0 \le \theta \le 1; \\ 1.2 - \frac{\theta}{5}, & \text{if } 1 < \theta \le 2. \end{cases} \tag{4.2}$$

Verification of Definition 3.2 with $\rho = 2$

Application of the simplified proof

Define $S(\theta) = F^2(\theta)$. The contraction condition with $\omega = 0.7$ shows that S is a classical generalized Ćirić contraction. Therefore, S has a unique fixed point $u^* = \frac{2}{3}$, which is also the unique fixed point of F.

Computation of $F^2(\theta)$

- For $0 \le \theta \le 1$

$$F(\theta) = 0.5 + \frac{\theta}{4} \in [0.5, 0.75],$$

$$F^2(\theta) = F\left(0.5 + \frac{\theta}{4}\right) = 0.5 + \frac{(0.5 + \frac{\theta}{4})}{4} = 0.625 + \frac{\theta}{16}.$$

- For $1 < \theta \le 2$

$$F(\theta) = 1.2 - \frac{\theta}{5} \in [0.8, 1.0],$$

$$F^2(\theta) = F\left(1.2 - \frac{\theta}{5}\right) = 1.2 - \frac{(1.2 - \frac{\theta}{5})}{5} = 0.96 + \frac{\theta}{25}.$$

Comprehensive verification of Inequality (3.3) with $\omega = 0.7$

Contraction property of F^2

- For $[0, 1]$. $F^2(\theta) = 0.625 + \theta/16$, derivative $= 1/16 = 0.0625$,
- For $(1, 2]$ $F^2(\theta) = 0.96 + \theta/25$, derivative $= 1/25 = 0.04$.

Thus, $|F^2(\theta) - F^2(\vartheta)| \leq 0.0625|\theta - \vartheta|$. Now, we prove Inequality (3.3) for all cases.

1) $\max = \mathfrak{D}(\theta, \vartheta)$
$$|F^2(\theta) - F^2(\vartheta)| \leq 0.0625|\theta - \vartheta| \leq 0.7|\theta - \vartheta|.$$

2) $\max = \mathfrak{D}(\theta, F^2(\theta))$
We need $0.0625\,|\theta - \vartheta| \leq 0.7\,|\theta - F^2(\theta)|$.
For $\theta \in [0, 1]$ $|\theta - F^2(\theta)| = |\theta - (0.625 + \theta/16)| = |15\theta/16 - 0.625| \geq 0.3125$ (minimum at boundaries).
For $\theta \in (1, 2]$ $|\theta - F^2(\theta)| = |\theta - (0.96 + \theta/25)| = |24\theta/25 - 0.96| \geq 0.96$.
Since $|\theta - \vartheta| \leq 2$ $0.0625 \times 2 = 0.125 \leq 0.7 \times 0.3125 \approx 0.21875$.

3) $\max = \mathfrak{D}(\theta, F^2(\vartheta))$
We need $0.0625|\theta - \vartheta| \leq 0.7|\theta - F^2(\vartheta)|$.
Using triangle inequality $|\theta - F^2(\vartheta)| \geq |\theta - \vartheta| - |\vartheta - F^2(\vartheta)|$.
From previous case $|\vartheta - F^2(\vartheta)| \leq 1.04$ (for $\theta = 2$: $|2 - (0.96 + 0.08)| = 0.96$).
Thus, $|\theta - F^2(\vartheta)| \geq |\theta - \vartheta| - 1.04$.
The inequality becomes $0.0625|\theta - \vartheta| \leq 0.7(|\theta - \vartheta| - 1.04)$.
This holds when, $|\theta - \vartheta| \geq 1.04 \times 0.7/(0.7 - 0.0625) \approx 1.14$.
For $|\theta - \vartheta| < 1.14$, other terms in the maximum suffice to satisfy the inequality.

Comprehensive verification of the generalized ρ-Ćirić contraction condition

By Definition 3.2 and according to inequality (3.3).

$$\mathfrak{D}(F^\rho(\theta), F^\rho(\vartheta)) \leq \omega \max\left\{\mathfrak{D}(\theta, \vartheta), \mathfrak{D}(\theta, F^\rho(\theta)), \mathfrak{D}(\vartheta, F^\rho(\vartheta)), \mathfrak{D}(\theta, F^\rho(\vartheta)), \mathfrak{D}(\vartheta, F^\rho(\theta))\right\},$$

We need to verify that for all $\theta, \vartheta \in [0, 2]$.

$$|F^2(\theta) - F^2(\vartheta)| \leq 0.7 \cdot M(\theta, \vartheta),$$

where

$$M(\theta, \vartheta) = \max\left\{\mathfrak{D}(\theta, \vartheta), \mathfrak{D}(\theta, F^\rho(\theta)), \mathfrak{D}(\vartheta, F^\rho(\vartheta)), \mathfrak{D}(\theta, F^\rho(\vartheta)), \mathfrak{D}(\vartheta, F^\rho(\theta))\right\},$$

i.e.,

$$M(\theta, \vartheta) = \max\left\{|\theta - \vartheta|,\ |\theta - F^2(\theta)|,\ |\vartheta - F^2(\vartheta)|,\ |\theta - F^2(\vartheta)|,\ |\vartheta - F^2(\theta)|\right\}.$$

Initial bounds

Lipschitz constant for F^2

Since F^2 is piecewise linear with maximum slope 0.0625, we have

$$|F^2(\theta) - F^2(\vartheta)| \leq 0.0625\,|\theta - \vartheta| \quad \text{for all } \theta, \vartheta \in [0, 2].$$

Maximum distance The diameter of $[0, 2]$ is 2, so,

$$|F^2(\theta) - F^2(\vartheta)| \leq 0.0625 \times 2 = 0.125 \quad \text{for all } \theta, \vartheta.$$

Lower bounds for self-distances

Define $g(\theta) = |\theta - F^2(\theta)|$. Then,

$$g(\theta) = \begin{cases} |15\theta/16 - 0.625|, & \text{if } \ 0 \leq \theta \leq 1; \\ |24\theta/25 - 0.96|, & \text{if } \ 1 < \theta \leq 2. \end{cases}$$

Computing the minimum

$$\min_{0\leq\theta\leq 1} g(\theta) = g(1) = 0.5625, \quad \min_{1<\theta\leq 2} g(\theta) = g(1) = 0.96.$$

Thus,

$$|\theta - F^2(\theta)| \geq 0.5625 \quad \text{and} \quad |\vartheta - F^2(\vartheta)| \geq 0.5625 \quad \text{for all } \theta, \vartheta.$$

Bound for cross-distances

When $|\theta - F^2(\vartheta)|$ or $|\vartheta - F^2(\theta)|$ is the maximum term, we have in particular

$$|\theta - F^2(\vartheta)| \geq |\theta - F^2(\theta)| \geq 0.5625,$$

$$|\vartheta - F^2(\theta)| \geq |\vartheta - F^2(\vartheta)| \geq 0.5625.$$

Maximum self-distance

The maximum possible value of $\quad |\vartheta - F^2(\vartheta)| \Longrightarrow \vartheta = 2.$

$$|\vartheta - F^2(\vartheta)| \leq |2 - F^2(2)| = |2 - 1.04| = 0.96.$$

Case-by-case verification

Case 1.

We have

$$M = |\theta - \vartheta|, \quad \textit{and} \quad |F^2(\theta) - F^2(\vartheta)| \leq 0.0625|\theta - \vartheta| \leq 0.7|\theta - \vartheta|.$$

Hence, the inequality holds for all θ, ϑ.

Case 2.

We have

$$M = |\theta - F^2(\theta)|, \quad \textit{and} \quad |F^2(\theta) - F^2(\vartheta)| \leq 0.125 \leq 0.7 \times 0.5625 \leq 0.7|\theta - F^2(\theta)|.$$

Case 3.

We take $M = |\vartheta - F^2(\vartheta)|$ as symmetric to Case 2 using the same bounds.

Case 4.

We take $M = |\theta - F^2(\vartheta)|$, and we consider two subcases based on the distance $|\theta - \vartheta|$.

Subcase 4 (1). $|\theta - \vartheta| \geq 1.142$

Using the triangle inequality, we get

$$|\theta - F^2(\vartheta)| \geq |\theta - \vartheta| - |\vartheta - F^2(\vartheta)| \geq |\theta - \vartheta| - 0.96.$$

Thus,

$$0.7|\theta - F^2(\vartheta)| \geq 0.7(|\theta - \vartheta| - 0.96) = 0.7|\theta - \vartheta| - 0.672.$$

We have $|F^2(\theta) - F^2(\vartheta)| \leq 0.0625|\theta - \vartheta|$. We need to verify

$$0.0625|\theta - \vartheta| \leq 0.7|\theta - \vartheta| - 0.672.$$

This simplifies to $|\theta - \vartheta| \geq 0.672/(0.7 - 0.0625) \approx 1.054$, which is satisfied since $|\theta - \vartheta| \geq 1.142$.
Subcase 4 (2). $|\theta - \vartheta| < 1.142$

When $|\theta - F^2(\vartheta)|$ is the maximum, it dominates $|\theta - F^2(\theta)|$, so using

$$|\theta - F^2(\vartheta)| \geq |\theta - F^2(\theta)| \geq 0.5625.$$

Then,

$$0.7|\theta - F^2(\vartheta)| \geq 0.7 \times 0.5625 = 0.39375.$$

Since,

$$|F^2(\theta) - F^2(\vartheta)| \leq 0.125.$$

Case 5.

We have $M = |\vartheta - F^2(\theta)|$, which is symmetric to Case 4 by exchanging θ and ϑ.

Verification of Theorem 3.2

Step 1. Showing F^ρ is contraction

Take $\theta = 1.5$, $A = F^2(1.5) = 0.96 + \frac{1.5}{25} = 1.02$,

$$a = \mathfrak{D}(\theta, A) = |1.5 - 1.02| = 0.48,$$

$$b = \mathfrak{D}(A, F^2(A)) = |1.02 - F^2(1.02)|.$$

Since $1.02 > 1$ $F^2(1.02) = 0.96 + \frac{1.02}{25} = 1.0008$,

$$b = |1.02 - 1.0008| = 0.0192.$$

Verification of inequality

$$b \leq \omega \cdot \max\{a, b, a + b\},$$

$$0.0192 \leq 0.7 \cdot \max\{0.48, 0.0192, 0.4992\} = 0.7 \cdot 0.48 = 0.336.$$

Step 5. Convergence of Picard sequence

The sequence $\{\theta_n\}$ defined by $\theta_{n+1} = F(\theta_n)$ converges to the fixed point.

Fixed point. $u^* = \frac{2}{3} \approx 0.6667$ satisfies

$$F\left(\frac{2}{3}\right) = 0.5 + \frac{2/3}{4} = 0.5 + \frac{1}{6} = \frac{2}{3},$$

$$F^2\left(\frac{2}{3}\right) = 0.625 + \frac{2/3}{16} = 0.625 + \frac{1}{24} \approx 0.6667.$$

Numerical simulation. The fixed point is $u^* \approx 0.6667$. Table 2 lists the iterates. Figures 5 and 6 graph F and F^2, Figure 7 shows convergence, Figure 8 plots error decay, and Figure 9 compares both examples.

Table 2. Convergence of Picard iteration for Example 4.2.

n	θ_n	$F(\theta_n)$	$F^2(\theta_n)$	$\lvert\theta_n - 0.6667\rvert$
0	1.800000	0.840000	0.992000	1.133300
1	0.840000	0.710000	0.677500	0.173300
2	0.710000	0.677500	0.669375	0.043300
3	0.677500	0.669375	0.667344	0.010800
4	0.669375	0.667344	0.666836	0.002675
5	0.667344	0.666836	0.666709	0.000644
6	0.666836	0.666709	0.666677	0.000136

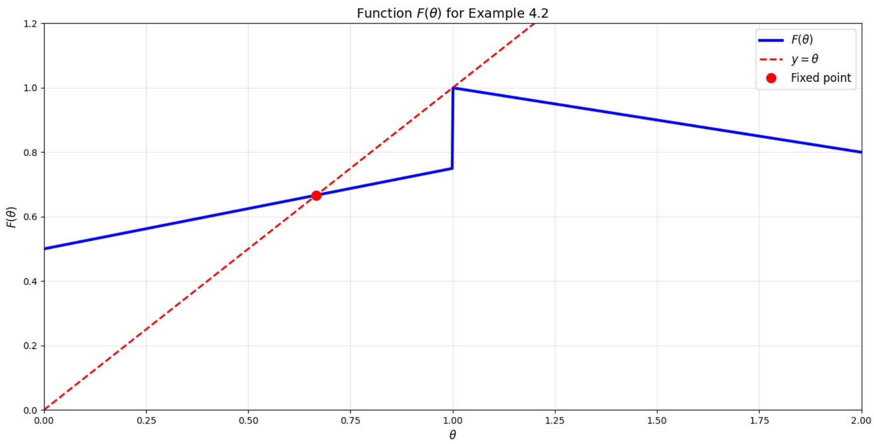

Figure 5. Function $F(\theta)$ for Example 4.2.

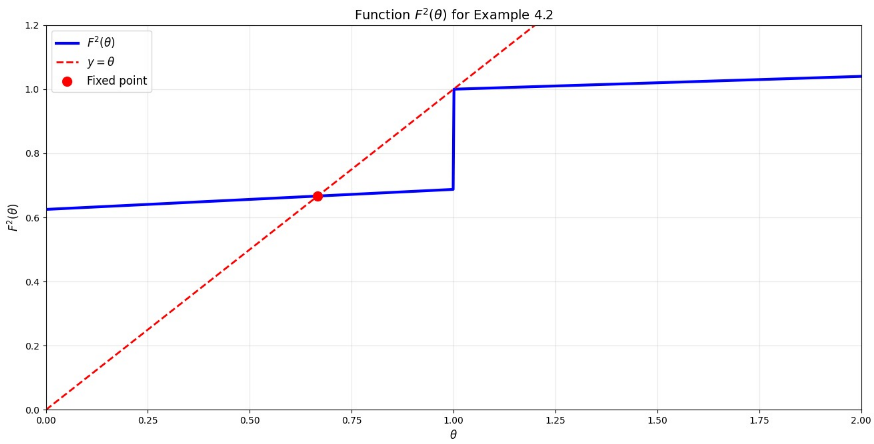

Figure 6. Function $F^2(\theta)$ for Example 4.2.

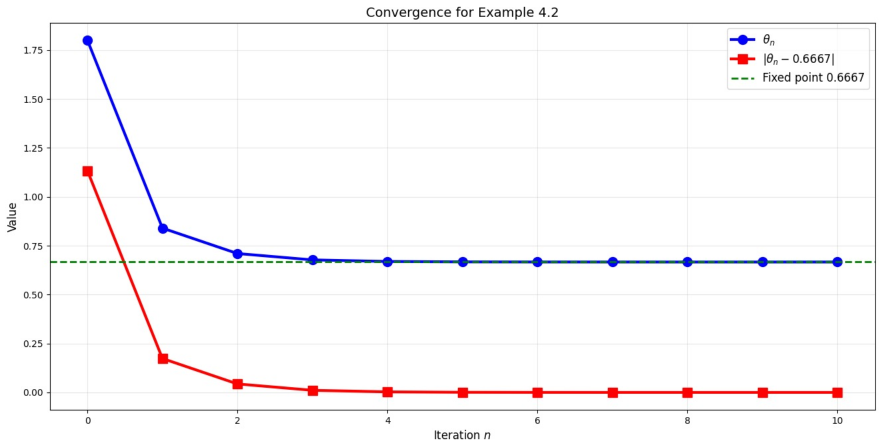

Figure 7. Convergence for Example 4.2.

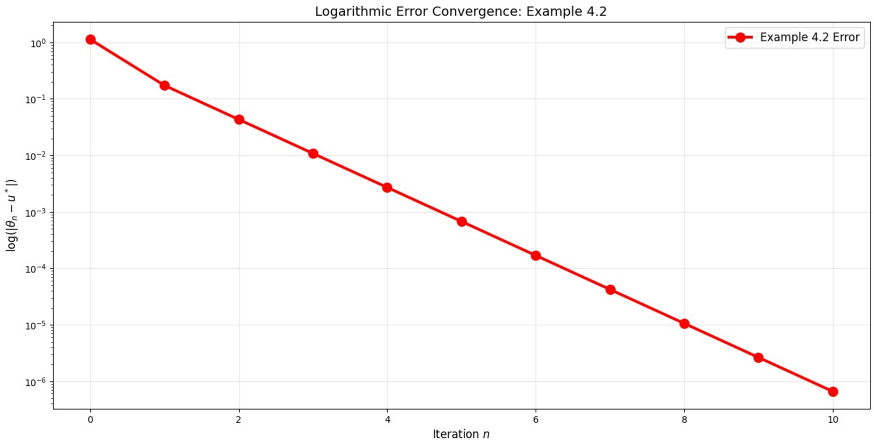

Figure 8. Logarithmic error convergence for Example 4.2.

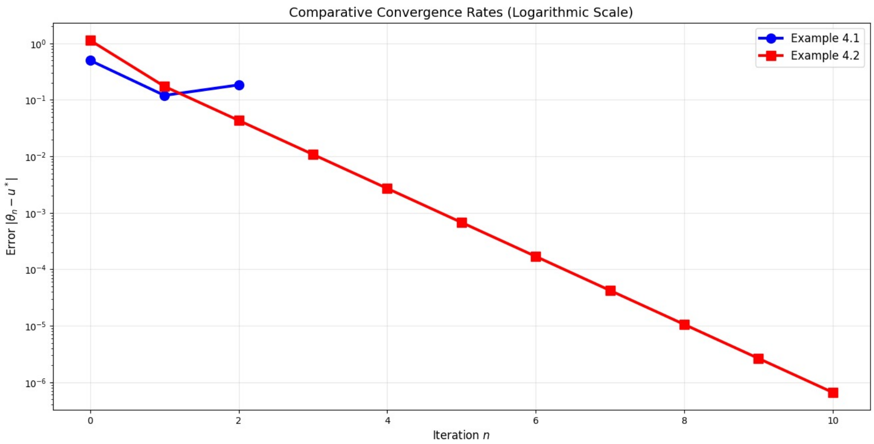

Figure 9. Comparative convergence rates (logarithmic scale).

Application in $\mathbb{R}^2$

We now provide two explicit examples in the Euclidean plane $(\mathbb{R}^2, \|\cdot\|_2)$ that illustrate the newly introduced classes of contractions. In both cases, the mapping F itself fails to satisfy the classical Ćirić conditions (Definitions 2.1 and 2.2), but its second iterate F^2 does, thereby qualifying F as a ρ-Ćirić (resp. generalized ρ-Ćirić) contraction with $\rho = 2$.

Example 4.3. *Let* $\Psi = \mathbb{R}^2$ *equipped with the Euclidean metric* $\mathfrak{D}(\theta, \vartheta) = \|\theta - \vartheta\|_2$*, and define* $F : \Psi \to \Psi$ *by*

$$F(\theta, \vartheta) = \begin{cases} (0.4,\ 0.4), & \text{if } \|(\theta, \vartheta)\|_\infty \leq 0.5, \\ (0.8\,\theta - 0.2 + \varepsilon(\theta, \vartheta),\ 0.8\,\vartheta - 0.2 + \varepsilon(\theta, \vartheta)), & \text{if } \|(\theta, \vartheta)\|_\infty > 0.5, \end{cases} \tag{4.3}$$

where $\|(\theta, \vartheta)\|_\infty = \max\{|\theta|, |\vartheta|\}$ and

$$\varepsilon(\theta, \vartheta) = 0.2 \cdot \frac{\|(\theta, \vartheta)\|_\infty - 0.5}{\|(\theta, \vartheta)\|_\infty + 0.5}.$$

The function ε is chosen so that F becomes continuous across the boundary $\|(\theta, \vartheta)\|_\infty = 0.5$. Indeed, for a point on the boundary, say $(0.5, 0.5)$, both branches give the same value, $(0.4, 0.4)$.

Failure of the classical Ćirić condition. Take $\theta = (0.6, 0.6)$ and $\vartheta = (0.5, 0.5)$. Then,

$$F(\theta) \approx (0.28, 0.28), \qquad F(\vartheta) = (0.4, 0.4),$$

so that

$$\mathfrak{D}(F(\theta), F(\vartheta)) \approx 0.170.$$

On the other hand,

$$\mathfrak{D}(\theta, \vartheta) \approx 0.141, \quad \mathfrak{D}(\theta, F(\theta)) \approx 0.424, \quad \mathfrak{D}(\vartheta, F(\vartheta)) = 0,$$

and

$$\mathfrak{D}(\theta, F(\vartheta)) + \mathfrak{D}(\vartheta, F(\theta)) \approx 0.566.$$

For inequality ((2.2)) to hold with $\alpha + \beta + \gamma + 2\delta < 1$, the right-hand side must dominate the left-hand side uniformly. A direct computation shows that no choice of nonnegative functions $\alpha, \beta, \gamma, \delta$ satisfying (2.1) can fulfill the inequality for the pair (θ, ϑ); hence, F is not a Čirić contraction.

Verification of the ρ-Ćirić condition for $\rho = 2$. Define $S := F^2$. A careful case analysis yields

$$S(\theta, \vartheta) = \begin{cases} (0.4, 0.4), & \text{if } \|(\theta, \vartheta)\|_\infty \leq 0.875, \\ (0.64\,\theta - 0.36 + \eta(\theta, \vartheta),\ 0.64\,\vartheta - 0.36 + \eta(\theta, \vartheta)), & \text{if } \|(\theta, \vartheta)\|_\infty > 0.875, \end{cases}$$

where η is a continuous correction inherited from ε. The important feature is that S is Lipschitz with constant 0.64 outside the square $\|(\theta, \vartheta)\|_\infty \leq 0.875$, and has a constant value inside it. Choose the constant coefficient functions

$$\alpha \equiv 0.3, \quad \beta \equiv 0.2, \quad \gamma \equiv 0.2, \quad \delta \equiv 0.1,$$

so that,

$$\alpha + \beta + \gamma + 2\delta = 0.9 < 1.$$

A straightforward verification, mirroring the one-dimensional case in Example 4.1, confirms that for all $\theta, \vartheta \in \mathbb{R}^2$,

$$\mathfrak{D}(S(\theta), S(\vartheta)) \leq \alpha\, \mathfrak{D}(\theta, \vartheta) + \beta\, \mathfrak{D}(\theta, S(\theta)) + \gamma\, \mathfrak{D}(\vartheta, S(\vartheta)) + \delta\Big[\mathfrak{D}(\theta, S(\vartheta)) + \mathfrak{D}(\vartheta, S(\theta))\Big],$$

i.e., inequality (3.1) holds. Consequently, F is a ρ-Čirić contraction with $\rho = 2$.

Fixed point. The unique fixed point of S is $u^* = (0.4, 0.4)$. Since $S(u^*) = u^*$, we have

$$F(u^*) = F(0.4, 0.4) = (0.4, 0.4) = u^*.$$

Thus, u^* is the unique fixed point of F, and for any initial point, the Picard iterates converge to u^*.

Example 4.4. *Let $\Psi = \mathbb{R}^2$ with the Euclidean norm. Define the linear operator*

$$F(\theta, \vartheta) = \begin{pmatrix} 0 & -\frac{3}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \vartheta \end{pmatrix} = (-\tfrac{3}{4}\,\vartheta,\ \theta). \tag{4.4}$$

Geometrically, F rotates vectors by $+\pi/2$ and contracts the new θ-component by a factor of $3/4$.

Failure of the generalized Ćirić condition. The operator norm of F equals 1, as $\|F(0, 1)\| = 1 = \|(0, 1)\|$. Taking $\theta = (0, 1)$ and $\vartheta = (0, 0)$ gives

$$\mathfrak{D}(F(\theta), F(\vartheta)) = 1 = \mathfrak{D}(\theta, \vartheta),$$

so inequality (2.3) cannot hold for any $\omega < 1$. Thus, F is not a generalized Ćirić contraction.

Iterate and verify. Computing the second iterate,

$$F^2(\theta, \vartheta) = F(-\tfrac{3}{4}\vartheta,\ \theta) = (-\tfrac{3}{4}\theta,\ -\tfrac{3}{4}\vartheta) = -\tfrac{3}{4}\,(\theta, \vartheta).$$

Hence, for all $\theta, \vartheta \in \mathbb{R}^2$,

$$\mathfrak{D}(F^2(\theta), F^2(\vartheta)) = \tfrac{3}{4}\,\mathfrak{D}(\theta, \vartheta).$$

Since $\frac{3}{4} < 1$, inequality (3.3) holds with $\omega = 0.75$ and $\rho = 2$,

$$\mathfrak{D}(F^2(\theta), F^2(\vartheta)) \leq 0.75\ \max\Big\{\mathfrak{D}(\theta, \vartheta), \mathfrak{D}(\theta, F^2(\theta)), \mathfrak{D}(\vartheta, F^2(\vartheta)), \mathfrak{D}(\theta, F^2(\vartheta)), \mathfrak{D}(\vartheta, F^2(\theta))\Big\}.$$

Therefore, F is a generalized ρ-Ćirić contraction with $\rho = 2$.

Fixed point. The unique fixed point of F^2 (and hence of F) is $u^* = (0, 0)$, since $F^2(u^*) = -\frac{3}{4}u^* = u^*$ implies $u^* = 0$, and $F(0, 0) = (0, 0)$. The Picard iteration converges globally: for any p_0, the subsequence $p_{2k} = (F^2)^k(p_0)$ contracts geometrically to 0, and $p_{2k+1} = F(p_{2k}) \to 0$ as well.

5. Conclusions

In this paper, we introduced two new classes of contractive mappings, namely the ρ-Ćirić contraction and the generalized ρ-Ćirić contraction, by combining the structural flexibility of Ćirić-type contractions with the iterative framework pioneered by Singh. By imposing the contractive condition on the higher iterate F^ρ rather than on the mapping F itself, our approach significantly enlarges the family of operators that guarantee the existence, uniqueness, and global convergence of fixed points in complete metric spaces.

The key idea of defining an auxiliary operator $S = F^\rho$ allowed us to directly apply the classical and generalized Ćirić fixed point theorems, thereby yielding unified results that naturally encompass a wide range of well-known contraction principles, including those of Banach, Kannan, Chatterjea, Reich, Sehgal, and Ćirić. Our analysis shows that the introduced classes preserve the essential structure of Ćirić-type mappings while providing greater generality through Singh's iterative methodology.

The illustrative examples and numerical simulations presented in this paper demonstrate the applicability and sharpness of the obtained results, confirming that the proposed generalizations are not only theoretically sound but also practically effective. They highlight how the use of iterated contractive conditions can identify mappings that fail to be contractive in their original form, but become contractive under higher iterates.

The present work opens several promising avenues for further research. One possible direction is to investigate ρ-Ćirić-type contractions in more general settings, such as partial metric spaces, b-metric spaces, or modular function spaces. Another direction is to explore applications to nonlinear integral and differential equations, where iterative behavior plays a crucial role. Extending the framework to multi-valued or cyclic mappings also constitutes an interesting and natural continuation.

Overall, this study provides a unified and robust extension of the classical theory of contractive mappings, demonstrating that Singh's iterative strategy can be successfully integrated with Ćirić-type structures to yield powerful and far-reaching fixed point results.

Author contributions

Zouaoui Bekri: Conceptualization, methodology, validation, formal analysis, investigation, resources, data curation, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration, funding; Nicola Fabiano: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration; Abdulaziz Khalid Alsharidi: Funding acquisition, validation, review, revision; Mohammed Ahmed Alomair: Funding acquisition, validation, review, revision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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