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Research article

Sequential mixing condition for continuous-time entangled Markov chains

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Abstract: We established a continuous-time framework for entangled quantum Markov chains derived from classical continuous-time Markov processes. Unlike previous studies focused on discrete-time dynamics, we demonstrated that a robust mixing property emerges directly from the structural architecture of the underlying continuous-time process. Our analysis showed that inhomogeneous entangled chains exhibited path-independent convergence to a limiting distribution determined by asymptotic temporal parameters. The results motivated future study of these entangled Markov chains in connection with open quantum system dynamics and quantum master equations.

Keywords: entanglement; continuous-time; Markov chains; mixing; quantum theory; inhomogeneous
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1. Introduction

Continuous-time quantum Markov dynamics have attracted considerable attention for their ability to provide a rigorous and more realistic representation of complex open quantum systems. The foundational work on quantum dynamical semigroups [1, 2] established a robust mathematical framework for describing the irreversible evolution of such systems. This formalism finds its most concrete and powerful expression in the theory of Markovian master equations, pioneered by Davies [3, 4] and leading to the now-standard Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form [5, 6]. The Lindblad equation has proven to be remarkably versatile, allowing for the recovery of a wide range of physically relevant models and capturing intricate phenomena such as the generation and decay of quantum entanglement. The mathematical underpinnings of these models are profound, synthesizing concepts from the theory of continuous-time stochastic processes [7] and operator algebras [8]. Recent theoretical advances, particularly in resonance theory, have further solidified the validity of the Markovian approximation, demonstrating its applicability across all relevant time scales [9]. Ongoing research continues to refine our understanding of the delicate interplay between correlation decay, non-Markovianity, and the emergence of Markovian dynamics [10, 11], making the subject a cornerstone of modern quantum theory, as comprehensively treated in seminal texts like that of Breuer and Petruccione [12].

Quantum Markov chains (QMCs) [13, 14] are states defined on discrete infinite tensor products of matrix algebras, characterized by Markovian correlations generated by Markov operators [15, 16]. Discrete-time QMCs have demonstrated their robustness in accurately describing quantum systems, offering a solid framework for analyzing their properties. Within the realm of QMCs, finitely correlated states [17, 18] stand out as an important subclass, playing a pivotal role in the characterization of valence bond states. Since their introduction in the context of quantum random walks [19], entangled Markov chains have developed into a rich subject of study. Early work established their foundational entanglement measures, which were later refined to support deeper analysis [20]. More recently, researchers have turned to the chains' long-term behavior, examining properties like ψ-mixing in homogeneous systems [21] and exploring how entanglement transitions relate to certain stochastic operators [22]. This line of inquiry highlights how classical probabilistic concepts can adapt to quantum settings, opening fresh perspectives on complex systems. For instance, one application models a quantum switch as a continuous-time Markov chain [23]. Together, these advances illustrate the growing dialogue between probability theory and quantum information, suggesting new pathways to understand entanglement through dynamical processes.

Hidden quantum Markov models are a quantum extension of classical hidden Markov models. Recently, a solid mathematical foundation has been established for both Markovian and non-Markovian hidden quantum processes [24]. Specifically, entangled Hidden Markov Models were introduced in [25], providing new insights into the role of entanglement in quantum systems. Furthermore, a recent study has explored inhomogeneous entangled dynamics within the context of elephant random walks [26], expressing the memory dependence of these dynamics through local mean entropy formulas. These advancements deepen our understanding of quantum memory effects and their implications for complex systems.

This paper develops a continuous-time framework for entangled Markov chains. In contrast to previous works [21, 22] where the Markov-Dobrushin inequality [27] played a central role, we show that in the continuous-time setting the mixing property emerges directly from the structural properties of the underlying continuous-time Markov process. Specifically, we establish a robust mixing property for a class of inhomogeneous entangled Markov chains derived from a common finite-state continuous-time Markov chain. We demonstrate the stability of this mixing with respect to various consistent realization paths and prove that the resulting limiting distribution is path-independent but depends on the asymptotic time.

Our results extend the ψ-mixing property previously established for the homogeneous discrete-time case [21] and for entangled chains associated with F-stochastic operators [22]. This continuous-time extension not only broadens the applicability of the theory, but also reveals a deeper connection between the ergodic features of the base Markov process and the entanglement architecture. A natural question arising from this work is whether the dynamics of such entangled chains can be described by a Markovian master equation [3, 12]. Furthermore, the study of entanglement quantification within this continuous-time framework remains an open direction. These promising avenues will be addressed in future work.

We begin this paper with a preliminary section for the embedding of continuous-Markov chains into quantum dynamics in Section 2. In Section 3, we introduce the concept of entangled Markov chains associated with continuous-time Markov dynamics, laying the groundwork for our analysis. Section 4 presents the core results of this study, where we prove the main theorem. Finally, in Section 5, we conclude with a discussion of the implications and potential extensions of our findings.

2. Quantum embedding of Markovian dynamics

A continuous-time Markov chain is a stochastic process $\{X(t), t \geq 0\}$ that satisfies the Markov property:

$$\mathbb{P}(X_{t+s} \in A \mid \mathcal{F}_{\leq t}) = \mathbb{P}(X_{t+s} \in A \mid X_t), \tag{2.1}$$

where $\mathcal{F}_{\leq t}$ is the filtration representing the information up to time t. This implies that the future evolution of X_t depends only on the current state and not on the past trajectory.

If the right hand side of (2.1) does not depend on t the CTMC is called homogeneous.

In the sequel, $\{X(t), t \geq 0\}$ denotes a homogeneous continuous time Markov chain (HCTMC) with finite state space $\mathcal{S}$. For every states $(i, j) \in \mathcal{S}^2$, let

$$\Pi_{ij}(t) = \mathbb{P}(X(t) = j \mid X(0) = i),$$

denote the transition probability from the state i to state j in time t.

The HCTMC is characterized by a *rate matrix* $Q = (Q_{ij})$, where Q_{ij} represents the rate of transition from state i to state j.

The transition probabilities satisfy the Kolmogorov forward equations (or Chapman-Kolmogorov equations):

$$\frac{d}{dt}\Pi_{ij}(t) = \sum_{k\in\mathcal{S}} \Pi_{ik}(t)Q_{kj} \tag{2.2}$$

Similarly, the Kolmogorov backward equations are given by:

$$\frac{d}{dt}\Pi_{ij}(t) = \sum_{k\in\mathcal{S}} Q_{ik}\Pi_{kj}(t), \tag{2.3}$$

The rate matrix Q (also called the *generator matrix*) has the following properties:

$$Q_{ij} \geq 0 \quad \text{for} \quad i \neq j, \tag{2.4}$$

$$Q_{ii} = -\sum_{j\neq i} Q_{ij} \tag{2.5}$$

The relationship between Q and $\Pi(t)$ is characterized by the matrix exponential function. The transition matrix $\Pi(t)$ is obtained from the Q-matrix Q by

$$\Pi(t) = e^{tQ},$$

This formulation ensures that $\Pi(t)$ satisfies the Kolmogorov forward (2.2) and backward (2.3) equations, describing the evolution of transition probabilities over time. Thus, the Q-matrix provides the infinitesimal transition rates, while $\Pi(t)$ captures the cumulative transition probabilities over time t.

Let $\mathcal{S}$ be a finite state space. Let $\mathcal{H}$ be a Hilbert space of dimension $|\mathcal{S}|$, equipped with an orthonormal basis $\{|i\rangle : i \in \mathcal{S}\}$.

Consider a time-homogeneous Markov process $\{X(t)\}_{t\geq 0}$ on $\mathcal{S}$ with transition probability matrix $\Pi(t) = (\Pi_{ij}(t))_{i,j\in\mathcal{S}}$, where $\Pi_{ij}(t) = \mathbb{P}(X(t) = j \mid X(0) = i)$. Let $Q = (Q_{ij})$ be the associated rate matrix (generator), satisfying the forward Kolmogorov equation (2.2). We construct a quantum representation of this stochastic process in the doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$. For a given initial distribution $\pi = (\pi_i)_{i\in\mathcal{S}}$, we define the time-dependent state vector:

$$|\psi(t)\rangle := \sum_{i,j\in\mathcal{S}} \sqrt{\pi_i}\,\sqrt{\Pi_{ij}(t)}\,|i\rangle \otimes |j\rangle. \tag{2.6}$$

This state is normalized since

$$\langle\psi(t)|\psi(t)\rangle = \sum_{i,j} \pi_i\,\Pi_{ij}(t) = \sum_j \left(\sum_i \pi_i\,\Pi_{ij}(t)\right) = 1.$$

The individual transition amplitudes are encoded in the elementary vectors:

$$|\psi_{ij}(t)\rangle := \sqrt{\Pi_{ij}(t)}\,|i\rangle \otimes |j\rangle. \tag{2.7}$$

We introduce a time-dependent, diagonal operator $H(t)$ on $\mathcal{H}\otimes\mathcal{H}$ defined by:

$$H(t) := \sum_{i,j\in\mathcal{S}} \lambda_{ij}(t)\,|i\rangle\langle i| \otimes |j\rangle\langle j|, \tag{2.8}$$

where the coefficients $\lambda_{ij}(t)$ are specified in terms of the classical Markov data:

$$\lambda_{ij}(t) := \frac{1}{2}\,\frac{[\Pi(t)Q]_{ij}}{\Pi_{ij}(t)}. \tag{2.9}$$

This definition is well-defined wherever $\Pi_{ij}(t) > 0$. Using the forward Kolmogorov equation, $[\Pi(t)Q]_{ij} = \frac{d}{dt}\Pi_{ij}(t)$, the coefficient can be expressed equivalently as:

$$\lambda_{ij}(t) = \frac{1}{2}\,\frac{\frac{d}{dt}\Pi_{ij}(t)}{\Pi_{ij}(t)} = \frac{d}{dt}\log\sqrt{\Pi_{ij}(t)}.$$

The quantum state $|\psi(t)\rangle$, constructed from the classical transition probabilities, satisfies a simple first-order differential equation driven by the operator $H(t)$.

Theorem 2.1. *Let $\Pi(t)$ be the transition matrix of a Markov process with generator Q, and let π be an initial distribution. Let $|\psi(t)\rangle$ be defined as in* (2.6) *and $H(t)$ as in* (2.8) *with coefficients given by* (2.9). *Then,*

$$\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{2.10}$$

Proof. We compute the time derivative of $|\psi(t)\rangle$:

$$\frac{d}{dt}|\psi(t)\rangle = \sum_{i,j} \sqrt{\pi_i}\, \frac{d}{dt} \sqrt{\Pi_{ij}(t)}\; |i\rangle \otimes |j\rangle.$$

Since $\frac{d}{dt}\sqrt{\Pi_{ij}(t)} = \frac{1}{2}\frac{\frac{d}{dt}\Pi_{ij}(t)}{\sqrt{\Pi_{ij}(t)}}$, and using $\frac{d}{dt}\Pi_{ij}(t) = [\Pi(t)Q]_{ij}$, we have:

$$\frac{d}{dt}\sqrt{\Pi_{ij}(t)} = \frac{1}{2}\frac{[\Pi(t)Q]_{ij}}{\sqrt{\Pi_{ij}(t)}}.$$

Substituting the expression for $\lambda_{ij}(t)$ from (2.9),

$$\frac{d}{dt}\sqrt{\Pi_{ij}(t)} = \lambda_{ij}(t)\sqrt{\Pi_{ij}(t)}.$$

Therefore,

$$\frac{d}{dt}|\psi(t)\rangle = \sum_{i,j} \sqrt{\pi_i}\, \lambda_{ij}(t) \sqrt{\Pi_{ij}(t)}\; |i\rangle \otimes |j\rangle.$$

Now, applying $H(t)$ to $|\psi(t)\rangle$:

$$H(t)|\psi(t)\rangle = \sum_{i,j} \lambda_{ij}(t)\Big(\sqrt{\pi_i}\sqrt{\Pi_{ij}(t)}\Big)|i\rangle \otimes |j\rangle,$$

which is identical to the expression for $\frac{d}{dt}|\psi(t)\rangle$. This proves (2.10). □

For a composite quantum system, a description based solely on a wave function is insufficient to capture the complete physical state. A more general framework, using density matrices, is required. The pure-state density matrix of our embedded system is given by

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|,$$

where $|\psi(t)\rangle$ is defined in Eq (2.6). Unlike a wave function, $\rho(t)$ can also describe statistical mixtures, making it indispensable for modeling open quantum systems that interact with an environment.

An open quantum system is defined as one that interacts with an external environment, such that the joint evolution of the system and environment is unitary. The total Hilbert space is bipartite, $\mathcal{H}_S \otimes \mathcal{H}_E$, where $\mathcal{H}_S$ and $\mathcal{H}_E$ denote the system and environment spaces, respectively [12]. For the reduced system density operator, the most general form of Markovian, completely positive dynamics is given by the Lindblad master equation [5, 6]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \sum_i \gamma_i\Big(L_i\rho(t)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho(t)\}\Big),$$

here, H is the effective Hamiltonian governing coherent evolution, L_i are the Lindblad (jump) operators encoding dissipative interactions with the environment, $\gamma_i \geq 0$ are the corresponding dissipation rates, and $\{a, b\} = ab + ba$ denotes the anticommutator.

A key feature of our quantum embedding is the emergence of entanglement between the "history" (subsystem A) and "outcome" (subsystem B) registers. The entanglement entropy [28], defined as the von Neumann entropy of either reduced density matrix,

$$S_{\text{ent}}(t) = -\operatorname{Tr}[\rho_A(t)\log_2 \rho_A(t)],$$

quantifies these non-classical correlations, with $\rho_A(t) = \operatorname{Tr}_B[\rho(t)]$.

Deriving explicit Lindblad operators that exactly reproduce the dynamics of $\rho(t)$ from the classical transition data, and characterizing how the corresponding dissipative terms relate to the entanglement entropy $S_{\text{ent}}(t)$, present substantial theoretical challenges. The time-dependent, highly constrained structure of the generator, along with the functional complexity of $\rho(t)$, may obstruct an exact Lindblad representation. A rigorous analysis of these questions—including the existence of a Lindblad form for the embedding and the dynamical behavior of the entanglement entropy—is reserved for future investigation.

3. EMCs associated with time-continuous dynamics

In the notations of the previous sections, $X_t \equiv (\pi, (\Pi(t))_t, Q)$ is a continuous-time Markov chain with a countable state space $\mathcal{S}$ and π is the distribution of X_0. Let $\mathcal{H}$ be a Hilbert space with dimension $|\mathcal{S}|$, with orthonormal basis $e = \{\,|i\rangle\,\}_i$. Denote $\mathcal{B} = \mathcal{B}(\mathcal{H})$ its associated algebra of bounded operators.

Definition 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. A linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is* completely positive *if for every $n \in \mathbb{N}$, the ampliated map*

$$\Phi^{(n)} : M_n(\mathfrak{A}) \to M_n(\mathfrak{B}), \quad [a_{ij}] \mapsto [\Phi(a_{ij})],$$

sends positive elements to positive elements. If Φ preserves the unit ($\Phi(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathfrak{B}}$), it is called completely positive and identity-preserving (CPIP).

Definition 3.2. *A* transition expectation *is a CPIP map $\mathcal{E} : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B}$. Physically, $\mathcal{E}$ describes a one-step interaction that couples two consecutive systems and outputs an observable of the second system conditioned on the first.*

Following Accardi [13], a transition expectation $\mathcal{E}$ is the *fundamental building block* for constructing a homogeneous quantum Markov chain on $\mathcal{B}^{\otimes\mathbb{N}}$. Given an initial state φ_0 on $\mathcal{B}$, the quantum Markov chain is uniquely determined by its finite-dimensional marginals: for any $n \geq 1$ and $a_0, \ldots, a_n \in \mathcal{B}$,

$$\varphi(a_0 \otimes \cdots \otimes a_n) = \varphi_0\Big(\mathcal{E}(a_0 \otimes \mathcal{E}(a_2 \otimes \cdots \otimes \mathcal{E}(a_n \otimes \mathbf{1})\cdots))\Big). \tag{3.1}$$

This recursive formula shows explicitly how $\mathcal{E}$ encodes the transition mechanism between consecutive time steps.

At each time t, the algebra of observables for the system is represented by $\mathcal{B}_t$, which is isomorphic to $\mathcal{B}$, meaning that $\mathcal{B}_t$ is a copy of $\mathcal{B}$ corresponding to the observable at time t. Let $\mathcal{D}_e$ be the diagonal subalgebra of $\mathcal{B}$ associated with the basis e, which is spanned by the atomic projections $|i\rangle\langle i|, i \in \mathcal{S}$.

For any two times s and t, we define a map $\mathcal{E}_s^{s+t} : \mathcal{B}_s \otimes \mathcal{B}_{s+t} \to \mathcal{B}_s$, which acts as a linear extension of the following expression:

$$\mathcal{E}_s^{s+t}(a_s \otimes a_{s+t}) = \sum_{i,j,k,l} \sqrt{\Pi_{ik}(t)\Pi_{jl}(t)}\, a_{s;ij} a_{s+t;kl} |i\rangle\langle j|, \tag{3.2}$$

where $a_s = \sum_{i,j} a_{s;ij}|i\rangle\langle j| \in \mathcal{B}_s$ and $a_{s+t} = \sum_{k,l} a_{s+t;kl}|k\rangle\langle l| \in \mathcal{B}_{s+t}$. Here, $\Pi_{ik}(t)$ represents the transition probability of the Markov process over time t.

The backward entangled Markov operator associated with $\mathcal{E}_s^{s+t}$ is defined as

$$\mathcal{P}_s^{s+t}(a) = \mathcal{E}_s^{s+t}(\mathbf{1}_s \otimes a_{s+t}) = \sum_{i,k,l} \sqrt{\Pi_{ik}(t)\Pi_{il}(t)}\, a_{s+t;kl}|i\rangle\langle i|, \tag{3.3}$$

The forward entangled Markov operator associated with $\mathcal{E}_s^{s+t}$ is expressed as

$$\mathcal{T}_s^{s+t}(a_s) = \mathcal{E}_s^{s+t}(a_s \otimes \mathbf{1}_{s+t}) = \sum_{i,j,k} \sqrt{\Pi_{ik}(t)\Pi_{jk}(t)}\, a_{s;ij}|i\rangle\langle j|, \tag{3.4}$$

To represent this transformation more compactly, we introduce the partial isometry $V(t) : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, which acts on the basis vectors $|i\rangle$ by:

$$V(t)|i\rangle \;=\; \sum_{j\in\mathcal{S}} \sqrt{\Pi_{ij}(t)}\, |i\rangle \otimes |j\rangle.$$

The adjoint operator $V^*(t) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is defined by the linear extension:

$$V^*(t)\, |i\rangle \otimes |j\rangle = \sqrt{\Pi_{ij}(t)}\, |i\rangle.$$

It can be shown that $V^*(t)V(t) = \mathbf{1}$, ensuring that the map $\mathcal{E}_t(\cdot)$ can be equivalently written in terms of $V(t)$ and $V^*(t)$ as:

$$\mathcal{E}_t(a) := V^*(t)aV(t), \qquad \forall a \in \mathcal{B}_s \otimes \mathcal{B}_{s+t}. \tag{3.5}$$

This formulation encapsulates the dynamics of the system as an interplay between the isometry $V(t)$ and its adjoint $V^*(t)$. Consider the initial state

$$|\psi_0\rangle = \sum_{i\in\mathcal{S}} \sqrt{\pi_i}\, |i\rangle.$$

Definition 3.3. *A sequence of times $\tau = \{t_n\}_{n\geq 0}$, where $t_0 = 0$, and the cumulative sums $s_n = \sum_{k=0}^{n} t_k$ form a strictly increasing sequence with $\sum_{n=0}^{\infty} t_n = +\infty$, is called a persistent sequence. Furthermore, if $\lim_{n\to\infty} t_n = t_\infty > 0$, the sequence is said to be asymptotically constant.*

Let φ_0 be the state on $\mathcal{A}$ defined by

$$\varphi_0(a) = \langle\psi_0|a|\psi_0\rangle.$$

For a given sequence of times $\tau := (t_n)_n$ such that $t_0 = 0$ and $\sum_n t_n = \infty$. Let $s_n = \sum_j t_j$ for every n. The sequence $(\mathcal{E}_{s_n}^{s_{n+1}})_n$ define a time-inhomogeneous entangled Markov chain in the sense of [22], on the quasi-local algebra

$$\mathcal{B}_\tau = \bigotimes_n \mathcal{B}_{s_n}$$

where the algebra $\mathcal{B}_{s_n}$ is the embedding of $\mathcal{B}$ into the position s_n is denoted by σ_{s_n} given by

$$\mathcal{B}_{s_n} := \sigma_{s_n}(\mathcal{B}) = \mathbf{1}_0 \otimes \mathbf{1}_{s_1} \otimes \cdots \otimes \mathbf{1}_{s_{n-1}} \otimes \mathcal{B} \otimes \mathbf{1}_{s_{n+1}} \cdots \tag{3.6}$$

The left shift operator σ_τ on the space $\mathcal{B}_\tau$ is defined as an endomorphism, which shifts the index of the sequence $\{s_m\}_{m\in\mathbb{N}}$ by one step to the left. Specifically, it satisfies the relation:

$$\sigma_\tau(\sigma_{s_m}) = \sigma_{s_{m+1}}. \tag{3.7}$$

This means that applying σ_τ to σ_{s_m} results in the next element $\sigma_{s_{m+1}}$ in the sequence.

Additionally, the n^{th} power of the left shift operator σ_τ corresponds to shifting the sequence by n steps. That is, for any $n \in \mathbb{N}$, the following holds:

$$\sigma_\tau^n(\sigma_{s_m}) = \sigma_{s_{m+n}}. \tag{3.8}$$

This indicates that applying the operator σ_τ^n to σ_{s_m} results in $\sigma_{s_{m+n}}$, which is n steps ahead in the sequence.

The finite correlations of the entangled Markov chain can be described in terms of shifted operators and nested conditional expectations. For $a_i \in \mathcal{B}$ and $a_{s_i} = \sigma_{s_i}(a_i)$, where $0 \le i \le n$, the correlation functional $\varphi(\tau;\cdot)$ is expressed as:

$$\begin{aligned} &\varphi(\tau; \sigma_{s_0}(a_0)\sigma_{s_1}(a_1)\cdots\sigma_{s_n}(a_{t_n})) \\ &= \varphi_0\left(\mathcal{E}_{s_0}^{s_1}\left(a_0 \otimes \mathcal{E}_{s_1}^{s_2}\left(a_{s_1} \otimes \cdots \mathcal{E}_{s_{n-1}}^{s_n}\left(a_{s_{n-1}} \otimes \mathcal{E}_{s_n}^{s_{n+1}}\left(a_{s_n} \otimes \mathbf{1}_{s_{n+1}}\right)\right)\right)\right)\right). \end{aligned} \tag{3.9}$$

The state $\varphi(\tau;\cdot)$, whose expression is given by (3.9), captures the intrinsic correlations through a family of nested entangled transition expectations $\mathcal{E}_{s_i}^{s_{i+1}}$. The identity operator $\mathbf{1}_{s_{n+1}}$ serves as a (trivial) boundary condition, in general it can be replaced by a family positive operators $(h_n)_n$ satisfying the compatibility condition $\sigma_{s_n}(h_n) = \mathcal{E}_{s_n}^{s_{n+1}}(\mathbf{1}_{s_n} \otimes \sigma_{s_{n+1}}(h_{n+1}))$.

4. Mixing for inhomogeneous EMCs

Recall in the context of quasi-local algebra, such as the infinite tenor product algebra $\mathcal{B}^{\otimes\mathbb{N}}$, ergodic states are the extreme points of the set of all translation invariant states [8]. Analytically, The ergodicity of a state φ on $\mathcal{B}^{\otimes\mathbb{N}}$ is expressed through the convergence of the ergodic mean:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k(a)) = \varphi(a), \tag{4.1}$$

for all local observable $X \in \mathcal{B}^{\otimes\mathbb{N}}$, where σ^k denotes the k-step shift.

The mixing condition (known also clustering property [8]) is a stronger property and it is characterizing through $a, b \in \mathcal{B}^{\otimes\mathbb{N}}$, the correlations decay asymptotically:

$$\lim_{n\to\infty} \varphi(\sigma^n(a)b) = \varphi(a)\varphi(b). \tag{4.2}$$

This represents asymptotic statistical independence, where distant observables become completely decorrelated. In the context of the present paper, since the state is not translation invariant we adopt the notions of ergodic and mixing in the following

Definition 4.1. *Let $\tau = \{t_n\}_n$ be a persistent sequence with $s_n = t_n - t_{n-1}$. The state $\varphi(\tau, \cdot)$ on $\mathcal{B}_{\tau;\mathbb{N}} = \bigotimes \mathcal{B}_{s_n}$ is:*

- *ergodic if there exists a state ψ on $\mathcal{B}^{\otimes\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \varphi\left(\tau; \sigma_{s_k}(b_0)\sigma_{s_{k+1}}(b_{k+1}) \cdots \sigma_{s_{m+k}}(b_m)\right) = \psi\left(b_0 \otimes b_1 \cdots b_m\right), \tag{4.3}$$

  for all $m \in \mathbb{N}$ and for any $b_i \in \mathcal{B}$, where $i = 0, 1, \ldots, m$.
- *ψ-mixing if there exists a state ψ on $\mathcal{B}^{\otimes\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \varphi\left(\tau; \sigma_{s_0}(a_0)\sigma_{s_1}(a_1) \cdots \sigma_{s_m}(a_m)\sigma_{s_n}(b_0)\sigma_{s_{n+1}}(b_{n+1}) \cdots \sigma_{s_{m+n}}(b_m)\right) \tag{4.4}$$

$$= \varphi\left(\tau; \sigma_{s_0}(a_0)\sigma_{s_1}(a_1) \cdots \sigma_{s_m}(a_m)\right) \psi\left(b_0 \otimes b_1 \otimes \cdots \otimes b_m\right),$$

  for all $m \in \mathbb{N}$ and for any $a_i, b_i \in \mathcal{B}$, where $i = 0, 1, \ldots, m$.

This condition ensures that correlations between past and future observables vanish asymptotically, meaning that the state φ exhibits long-term statistical independence in the limit.

The relationship between mixing and ergodicity reveals a fundamental hierarchy in the behavior of quantum dynamical systems. Mixing, as defined in (4.2), is a strictly stronger property than ergodicity (4.1).

Theorem 4.2. *Let $(X(t))_t \equiv (\pi, \Pi(t))$ be a time-continuous Markov chain. If $(X(t))_t$ is ergodic with a limiting distribution $\pi_\infty = (\pi_{\infty;i})_{i\in\mathcal{S}}$, then for any persistent sequence $\tau = \{t_n\}_n$ that is asymptotically constant, the entangled Markov chain $\varphi \equiv (\pi, (\mathcal{E}_{s_n}^{s_{n+1}}))$, where $s_n = \sum_{j=0}^{n} t_j$, is mixing. Moreover, for every $m \in \mathbb{N}$ and for every $a_i, b_i \in \mathcal{B}$, $i = 0, 1, \ldots, m$, we have*

$$\lim_{n\to\infty} \varphi\left(\tau; \sigma_{s_0}(a_0)\sigma_{s_1}(a_1) \cdots \sigma_{s_m}(a_m)\sigma_{s_n}(b_0)\sigma_{s_{n+1}}(b_{n+1}) \cdots \sigma_{s_{m+n}}(b_m)\right)$$

$$= \varphi\left(\tau; \sigma_{s_0}(a_0)\sigma_{s_1}(a_1) \cdots \sigma_{s_m}(a_m)\right) \psi\left(b_0 \otimes b_1 \cdots b_m\right), \tag{4.5}$$

where ψ_∞ is a homogeneous entangled Markov chain defined on the quasi-local algebra $\mathcal{B}_{\mathbb{N}} = \bigotimes_{n\in\mathbb{N}} \mathcal{B}$, with

$$\psi\left(b_0 \otimes b_1 \otimes \cdots \otimes b_m\right) = \sum_{k} \sum_{\substack{j_0,\ldots,j_{m+1} \\ i_0,\ldots,i_{m+1}}} \pi_{\infty;k} \sqrt{\Pi_{ki_0}(t_\infty)\Pi_{kj_0}(t_\infty)} \tag{4.6}$$

$$\times \prod_{\ell=0}^{m} \sqrt{\Pi_{i_\ell i_{\ell+1}}(t_\infty)\Pi_{j_\ell j_{\ell+1}}(t_\infty)}\, b_{\ell;i_\ell j_\ell}\delta_{i_{m+1},j_{m+1}}.$$

Proof. Let $m \in \mathbb{N}$. Consider the tensors $a = a_{s_0} \otimes a_{s_1} \otimes \cdots \otimes a_{s_m}$ and $b = b_{s_0} \otimes b_{s_1} \otimes \cdots \otimes b_{s_m} \in \mathcal{B}_{s_0} \otimes \mathcal{B}_{s_1} \otimes \cdots \otimes \mathcal{B}_{s_m}$, where each $a_{s_i} = \sigma_{s_i}(a_i)$ and $b_{s_i} = \sigma_{s_i}(b_i)$ belong to the algebra $\mathcal{B}_{s_i}$ at each site s_i.

Consider the following expression for $\hat{b}_{s_n}$, which involves the recursive application of quantum operations:

$$\hat{b}_{s_n} = \mathcal{E}_{s_n}^{s_{n+1}}\left(\sigma_{s_n}(b_{s_0}) \otimes \cdots \otimes \mathcal{E}_{s_{n+m}}^{s_{n+m+1}}\left(\sigma_{s_{n+m}}(b_{s_m}) \otimes \mathbf{1}_{\sigma_{s_{n+m+1}}}\right) \cdots\right),$$

where $\mathcal{E}_{s_n}^{s_{n+1}}$ is a completely positive map between the states at time s_n and s_{n+1}.

By applying Eq (3.4), we derive the recursive relation:

$$
\begin{aligned}
&\mathcal{E}^{s_{n+m}}_{s_{n+m-1}}\left(\sigma_{s_{n+m-1}}(b_{s_{m-1}})\otimes\mathcal{E}^{s_{n+m+1}}_{s_{n+m}}\left(\sigma_{s_{n+m}}(b_{s_m})\otimes\mathbf{1}_{\sigma_{s_{n+m+1}}}\right)\right)\\
&=\sum_{i_m,j_m,j}\sqrt{\Pi_{i_mj}(t_{m+n+1})\Pi_{j_mj}(t_{m+n+1})}\,b_{s_m;i_mj_m}\,\mathcal{E}^{s_{n+m}}_{s_{n+m-1}}\left(\sigma_{s_{n+m-1}}(b_{s_{m-1}})\otimes\sigma_{s_{n+m}}(|i_m\rangle\langle j_m|)\right)\\
&=\sum_{\substack{j_{m-1},j_m,j\\ i_{m-1},i_m}}\sqrt{\Pi_{i_{m-1}i_m}(t_{m+n})\Pi_{j_{m-1}j_m}(t_{m+n})}\sqrt{\Pi_{i_mj}(t_{m+n+1})\Pi_{j_mj}(t_{m+n+1})}\\
&\times b_{s_{m-1};i_{m-1}j_{m-1}}\,b_{s_m;i_mj_m}\,\sigma_{s_{n+m-1}}\left(|i_{m-1}\rangle\langle j_{m-1}|\right).
\end{aligned}
$$

By iterating this process and applying Eq (3.2), we obtain:

$$
\hat{b}_{s_n}=\sum_{\substack{j_0,j_1,\dots,j_{m+1}\\ i_0,i_1,\dots,i_{m+1}}}\prod_{\ell=0}^{m}\sqrt{\Pi_{i_\ell i_{\ell+1}}(t_{n+\ell+1})\Pi_{j_\ell j_{\ell+1}}(t_{n+\ell+1})}\,b_{\ell;i_\ell j_\ell}\,\delta_{i_{m+1},j_{m+1}}\,\sigma_{s_n}\left(|i_0\rangle\langle j_0|\right), \tag{4.7}
$$

where $\delta_{\cdot,\cdot}$ represents the Kronecker delta, ensuring that $i_{m+1}=j_{m+1}$.

From the expression (3.3), the action of the backward entangled Markov operator $\mathcal{P}^{s_n}_{s_{n-1}}$ on $\hat{b}_{s_n}$ leads to:

$$
\mathcal{P}^{s_n}_{s_{n-1}}(\hat{b}_{s_n})=\sum_k c_k(b;n)\sigma_{s_{n-1}}\left(|k\rangle\langle k|\right),
$$

where

$$
c_k(b;n):=\sum_{\substack{j_0,\dots,j_{m+1}\\ i_0,\dots,i_{m+1}}}\sqrt{\Pi_{ki_0}(t_n)\Pi_{kj_0}(t_n)}\prod_{\ell=0}^{m}\sqrt{\Pi_{i_\ell i_{\ell+1}}(t_{n+\ell+1})\Pi_{j_\ell j_{\ell+1}}(t_{n+\ell+1})}\,b_{s_\ell;i_\ell j_\ell}\,\delta_{i_{m+1},j_{m+1}}.
$$

As $t_n\to t_\infty$, the coefficients $c_k(b;n)$ converge to:

$$
c_k(b):=\sum_{\substack{j_0,\dots,j_{m+1}\\ i_0,\dots,i_{m+1}}}\sqrt{\Pi_{ki_0}(t_\infty)\Pi_{kj_0}(t_\infty)}\prod_{\ell=0}^{m}\sqrt{\Pi_{i_\ell i_{\ell+1}}(t_\infty)\Pi_{j_\ell j_{\ell+1}}(t_\infty)}\,b_{s_\ell;i_\ell j_\ell}\,\delta_{i_{m+1},j_{m+1}}. \tag{4.8}
$$

For any diagonal operator $c=\sum_k c_k|k\rangle\langle k|\in\mathcal{D}_e$ and time $s\geq 0$, we have:

$$
\mathcal{P}^{s+t}_{s}(j_{s+t}(c))=\sum_{i,k}\Pi_{ik}(t)c_k\,\sigma_s\left(|i\rangle\langle i|\right). \tag{4.9}
$$

By recursively applying this over several time steps, we get:

$$
\begin{aligned}
&\mathcal{P}^{s_{m+2}}_{s_{m+1}}(\mathcal{P}^{s_{m+3}}_{s_{m+2}}(\cdots\mathcal{P}^{s_{n-1}}_{s_{n-2}}(\mathcal{P}^{s_n}_{s_{n-1}}(\hat{b}_{s_n}))\cdots))\\
&=\sum_{i,k}\left[\Pi(t_{m+2})\Pi(t_{m+3})\cdots\Pi(t_{n-1})\right]_{ik}c_k(b;n)\,\sigma_{s_{m+1}}(|i\rangle\langle i|).
\end{aligned}
$$

Thanks to the semi-group property of $\Pi(t)$, this simplifies to:

$$
\mathcal{P}^{s_{m+2}}_{s_{m+1}}(\mathcal{P}^{s_{m+3}}_{s_{m+2}}(\cdots\mathcal{P}^{s_{n-1}}_{s_{n-2}}(\mathcal{P}^{s_n}_{s_{n-1}}(\hat{b}_{s_n})\cdots)))=\sum_{i,k}\left[\Pi(s_{n-1}-s_{m+1})\right]_{ik}c_k(b;n)\sigma_{s_{m+1}}(|i\rangle\langle i|).
$$

Since the Markov chain $(X(t))_t$ is ergodic, the limiting behavior of this process is given by the following bound:

$$\begin{aligned}&\left|[\Pi(s_{n-1}-s_{m+1})]_{ik}c_k(b;n)-[\Pi_\infty]_{ik}c_k(b)\right|\\ &\le \left|[\Pi(s_{n-1}-s_{m+1})]_{ik}-[\Pi_\infty]_{ik}\right|c_k(b;n)+[\Pi_\infty]_{ik}\left|c_k(b;n)-c_k(b)\right|,\end{aligned}$$

where $[\Pi_\infty]_{ik}$ denotes the limiting transition matrix as $s\to\infty$, and the inequality provides an upper bound on the difference between the finite-time behavior and the asymptotic behavior of the process.

As $s_n\to\infty$ when $n\to\infty$, we have the convergence:

$$\lim_{n\to\infty}\left|[\Pi(s_{n-1}-s_{m+1})]_{ik}-[\Pi_\infty]_{ik}\right|=0,$$

which implies that the transition matrix $\Pi(s_{n-1}-s_{m+1})$ converges to its asymptotic value Π_∞ as the time difference $s_{n-1}-s_{m+1}$ increases indefinitely.

Since the sequence $(c_k(b;n))_n$ converges to $c_k(b)$, we conclude that for every $i,k\in\mathcal{S}$, the following limit holds:

$$\lim_{n\to\infty}[\Pi(s_{n-1}-s_{m+1})]_{ik}c_k(b;n)=[\Pi_\infty]_{ik}c_k(b).$$

This shows that the combined transition and coefficient sequence converges to a well-defined asymptotic value, describing the long-term behavior of the system.

Now, consider the expression for the entangled state evolution:

$$\begin{aligned}\varphi(a\sigma_\tau^n(b)) &= \varphi\left(a\otimes\mathbf{1}_{s_{m+1}}\otimes\cdots\otimes\mathbf{1}_{s_{n-1}}\otimes\hat{b}_{s_n}\right)\\ &= \phi_0\left(\mathcal{E}_{s_0}^{s_1}\left(a_{s_0}\otimes\cdots\mathcal{E}_{s_m}^{s_{m+1}}\left(a_m\otimes\mathcal{E}_{s_{m+1}}^{s_{m+2}}\left(\mathbf{1}_{s_{m+1}}\otimes\cdots\otimes\mathcal{E}_{s_{n-1}}^{s_n}\left(\mathbf{1}_{s_{n-1}}\otimes\hat{b}_n\right)\cdots\right)\cdots\right)\right)\right)\\ &= \phi_0\left(\mathcal{E}_{s_0}^{s_1}\left(a_{s_0}\otimes\mathcal{E}_{s_1}^{s_2}\left(a_{s_1}\otimes\cdots\mathcal{E}_{s_m}^{s_{m+1}}\left(a_{s_m}\otimes\mathcal{P}_{s_{m+1}}^{s_{m+2}}\left(\mathcal{P}_{s_{m+2}}^{s_{m+3}}\left(\cdots\mathcal{P}_{s_{n-1}}^{s_n}\left(\hat{b}_n\right)\cdots\right)\right)\right)\cdots\right)\right)\right).\end{aligned}$$

By continuity of the initial state ϕ_0 and the maps $\mathcal{E}_{s_j}^{s_{j+1}}$, we obtain the following limiting behavior:

$$\begin{aligned}\lim_{n\to\infty}\varphi(a\sigma_\tau^n(b)) &= \phi_0\left(\mathcal{E}_{s_0}^{s_1}\left(a_{s_0}\otimes\mathcal{E}_{s_1}^{s_2}\left(a_{s_1}\otimes\cdots\mathcal{E}_{s_m}^{s_{m+1}}\left(a_{s_m}\otimes\mathbf{1}_{s_{m+1}}\psi_\infty(b)\right)\cdots\right)\right)\right)\\ &= \varphi(\tau;a)\psi_\infty(b_\infty).\end{aligned}$$

where $b_\infty=\sigma_0(b_0)\sigma_{t_\infty}(b_1)\cdots\sigma_{mt_\infty}(a_m)$.

Finally, the definition of $\psi(b)$, which describes the asymptotic behavior of b, is given by:

$$\psi(b):=\psi_0\left(\mathcal{E}_0^{t_\infty}\left(b_0\otimes\cdots\otimes\mathcal{E}_{(m-1)t_\infty}^{mt_\infty}\left(b_{m-1}\otimes\mathcal{E}_{mt_\infty}^{(m+1)t_\infty}\left(b_m\otimes\mathbf{1}\right)\right)\right)\right).\tag{4.10}$$

and

$$\psi_0(a_0)=\sum_{k,i}\pi_{\infty;k}\sqrt{\Pi_{ki}(t_\infty)\Pi_{kj}(t_\infty)}a_{0;ij}.$$

This finishes the proof. □

Remark 4.3. *The limiting distribution π_∞ is interpreted as the asymptotic invariant distribution of the sequence of backward Markov operator $\mathcal{P}_{s_n}^{s_{n+1}}$, as explained through (4.9). Theorem 4.2 shows that the ergodicity of the classical continuous-time Markov chain implies the mixing of the associated EMC independently of the persistent sequence $(t_n)_n$. For a periodic sequence (i.e., $t_{n+1}=t_n+T$ for some fixed period $T>0$), Theorem 4.2 reproduces the result of [21], where a stronger assumption—positivity of the Markov–Dobrushin constant—was required.*

4.1. *Extended analysis of two-state dynamics*

Consider a simple two-state Markov chain with state space $\mathcal{S} = \{0, 1\}$, where the Hilbert space $\mathcal{H} \equiv \mathbb{C}^2$ is spanned by the orthonormal basis vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The algebra of observables is given by $\mathcal{B} \equiv M_2(\mathbb{C})$, the set of all 2×2 complex matrices.

The transition rates between the two states are defined as follows:

- Transition from state 0 to state 1 occurs at rate $q_{01} = \alpha$.
- Transition from state 1 to state 0 occurs at rate $q_{10} = \beta$.

Here, α and β are positive real numbers. The transition rate matrix Q for this system is:

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The transition matrix $\Pi(t)$ over time t is the matrix exponential $\Pi(t) = \exp(tQ)$, which can be computed as:

$$\Pi(t) = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta + \alpha e^{-t(\alpha+\beta)} & \alpha(1 - e^{-t(\alpha+\beta)}) \\ \beta(1 - e^{-t(\alpha+\beta)}) & \alpha + \beta e^{-t(\alpha+\beta)} \end{pmatrix}.$$

In the long-time limit, as $t \to \infty$, the system converges to a stationary distribution, with the limiting transition matrix given by:

$$\Pi_\infty = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}.$$

The stationary distribution π_∞ corresponding to this limiting behavior is:

$$\pi_\infty = \frac{\beta}{\alpha + \beta}|0\rangle + \frac{\alpha}{\alpha + \beta}|1\rangle.$$

According to Theorem 4.2, for any persistent sequence $\tau = \{t_n\}_n$ that is asymptotically constant with $\lim t_n = t_\infty > 0$, the associated entangled Markov chain $\varphi \equiv (\pi, (\mathcal{E}_{S_n}^{S_{n+1}}))$ exhibits the ψ-mixing property. This means that the limiting distribution of the entangled Markov chain is independent of the system's initial configuration, confirming its ergodic nature.

These results generalize the ψ-mixing property previously established in the homogeneous case and provide a robust framework for understanding the long-term behavior of entangled dynamics in continuous-time Markov chains.

5. Discussion

This work establishes a mixing property of a continuous-time quantum system. The central result, Theorem 4.2, provides a concrete manifestation of the so-called ψ-mixing property. It confirms the decorrelation of temporally distant observables a and b, a hallmark of quantum systems exhibiting mixing behavior. In the asymptotic limit, the joint state factorizes into independent contributions from distinct temporal regions. This decoupling indicates that the long-term dynamics are dominated by separate influences: the state's history, encoded in $\varphi(\tau; a)$, and its future asymptotic behavior, captured by $\psi_\infty(b)$. The semi-group property of the transition matrices $\Pi(t)$ was instrumental in simplifying this analysis, revealing a cumulative evolutionary structure that clarifies the system's long-term trajectory. Furthermore, our treatment of diagonal operators—showing they remain diagonal under time evolution with coefficients evolving via $\Pi(t)$—provided a significantly simplified framework for probing the asymptotic limit.

Theorem 4.2 reveals two physically distinct limiting scenarios, corresponding to the extremes of the time-scale parameter t_∞, which warrant separate consideration.

Case 1: $t_\infty = 0$ Here, $\Pi(t_\infty)$ reduces to the identity matrix. The limiting state ψ_∞ (4.10) simplifies to:

$$\psi\left(j_0(b_0)j_{t_\infty}(b_1)\cdots j_{mt_\infty}(b_m)\right) = \sum_i \pi_{\infty;i} \prod_{\ell=0}^{m} b_{\ell;ii}. \tag{5.1}$$

This expression describes a *degenerate* state with support confined strictly to the diagonal algebra $\mathcal{D}_{e;n} = \bigotimes_n \mathcal{D}_e$. The dynamics in this limit becomes trivial, freezing the system's correlations and suppressing any off-diagonal (coherent) contributions.

Case 2: $t_\infty = \infty$ In this opposite extreme, the transition matrix converges to $\Pi_\infty = \pi_\infty \otimes \mathbf{1}$, yielding $\Pi_{\infty;ij} = \pi_{\infty;j}$. The state ψ_∞ takes the form:

$$\psi\left(b_0 \otimes b_1 \otimes \cdots \otimes b_m\right) = \sum_{\substack{j_0,\ldots,j_m \\ i_0,\ldots,i_m}} \prod_{\ell=0}^{m} \sqrt{\pi_{\infty;i_\ell}\pi_{\infty;j_\ell}}\, b_{\ell;i_\ell j_\ell}. \tag{5.2}$$

Crucially, this state retains off-diagonal terms (through the independent sums over i_ℓ and j_ℓ), reflecting a richer structure that can preserve quantum coherences. Notably, the state for $t_\infty = 0$ can be viewed as the diagonal restriction of the state for $t_\infty = \infty$, highlighting a hierarchical relationship between these limits. Despite their structural differences, the underlying Markovian process is trivial in both cases.

This work opens several compelling avenues for further investigation, naturally extending from the established framework.

First, a fundamental question is whether the continuous-time dynamics of these entangled chains admit a description via a Markovian master equation in the GKSL form [5, 6]. Deriving such a master equation from our discrete-step recursive maps would not only deepen the connection to the established theory of open quantum systems [3, 12] but also provide powerful tools for analyzing relaxation rates, decoherence channels, and non-Markovian departures.

Second, our study invites a rigorous quantification of entanglement within this continuous-time framework. Developing measures—such as entanglement entropy or negativity—that can track the generation, propagation, and asymptotic decay of quantum correlations in these inhomogeneous chains is a crucial next step. This would allow us to compare the entanglement structure of the two limiting cases directly and explore how mixing behavior influences long-range quantum correlations.

Finally, the mathematical structure revealed here suggests potential applications beyond foundational theory. These include the design of novel quantum algorithms that leverage controlled mixing, the study of thermalization in complex quantum systems, and the analysis of information scrambling in disordered models.

Author contributions

Abdessatar Souissi: Conceptualization, Investigation, Resources, Methodology, Writing–original draft, Writing–review & editing; Abdessatar Barhoumi: Visualization, Project administration, Methodology, Resources.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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