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Research article

Caputo-Fabrizio fractional integro-differential equations: Existence, uniqueness, and β-Ulam stability results for the solutions in a Banach space

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Abstract: The study of Ulam stability for (functional, differential, difference, integral, integro-differential, and fractional differential) equations heavily relies on inequalities. In such scientific and engineering research, fixed point theorems (FPTs) are essential instruments. This article, on one hand, focused on the β-Ulam-Hyers stability (β-UHS) of non-instantaneous impulsive fractional integro-differential equations (N-IIFIDEs) involving the Caputo-Fabrizio fractional derivatives (C-FFDs) in a Banach space. On the other hand, we established the existence and uniqueness (E-UR) of solutions by employing the Banach contraction mapping principle (BCMP) and Krasnoselskii's fixed point theorem (KFPT). To validate the theoretical insights, a carefully crafted example was introduced. In this way, we generalized recent interesting results.

Keywords: β-Ulam stability; Caputo-Fabrizio fractional integro-differential equation; fractional boundary value problem; non-instantaneous impulsive; integro-differential equations
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1. Introduction

Derivatives are fundamental concepts in applied mathematics, defining a function's rate of change and serving as a cornerstone for developing mathematical models across various real-world applications [1–3]. The integer-order derivatives are less helpful and useful for characterizing the memory and heredity characteristics of various materials and processes than the fractional derivatives (FDs) and integrals of arbitrary order. Over the past three decades, FDs have gained significant attention due to their superior ability to model complex phenomena compared to classical derivatives. Researchers and engineers have increasingly acknowledged the importance of developing novel FDs featuring either singular or nonsingular kernels. These advancements aim to improve the modeling and simulation of a wide range of complex real-world phenomena [4].

Fractional differential equations (FDEs) have gained recognition as essential tools for mathematical modeling across various disciplines, including aerodynamics, physics, chemistry, engineering, and many more (see e.g., [5–7] and [8, 9]). Integro-differential equations (I-DEs) are a crucial research area due to their widespread applications in various fields, such as fractional power law and heat transfer phenomena [10, 11]. Many researchers have explored FDEs or fractional I-DEs under different boundary conditions (BCs), as evidenced by studies in [12–14].

In engineering, FDEs play a vital role in optimizing control systems, particularly in environments where traditional models fail to account for lingering effects. The study of N-IIs is especially relevant in electronic circuit design, fluid mechanics, and vibration analysis. Similarly, in physics, these equations assist in modeling viscoelastic materials, anomalous diffusion, and reaction-diffusion processes with delayed response times [15–17].

Moreover, financial mathematics has embraced fractional calculus as a powerful tool to predict market trends and quantify risk. By incorporating N-IIs, models can better reflect economic shocks that do not occur instantaneously but influence asset dynamics over time, leading to improved forecasting and risk mitigation strategies.

Analytic solutions for a viscous fluid incorporating the FDs of Caputo and Caputo-Fabrizio (C-F) are presented in [18]. In [19], researchers modeled a Maxwell fluid using an FD with a nonsingular kernel, obtaining semi-analytical solutions. A comparative study in [20] examined Atangana-Baleanu and C-FFD models for a generalized Casson fluid, leading to precise solutions. Furthermore, various nonlinear analytic techniques have been applied to investigate the existence of solutions for nonlinear DEs, as demonstrated in [21, 22].

In the last few decades, initial boundary value problem (IBVP) research has progressed and has been very helpful in creating a range of models of real processes in applications. Tian and Bai [23] reported some existence findings from an IBVP involving FDs of the Caputo type, and E-UR have been developed using the FPT (see e.g., [24]). Recent studies have shown that a significant portion of the literature on FDEs focuses predominantly on the Caputo and Riemann-Liouville types. These works explore a range of scenarios, including time-delay systems, impulsive effects, and various boundary value conditions (BVCs).

In [11], authors studied the existence and approximate periodic solution of a nonlinear fractional I-DE:

$$\begin{cases} {}_0^{CF}D^{\delta}x(\vartheta) = F(\vartheta, x(\vartheta), Bx(\vartheta)), & \vartheta \in [0, T], \\ x(0) = x_0, \end{cases}$$

where ${}_0^{CF}D^{\delta}$ denotes the C-FFD ($\delta \in (0, 1]$).

In [25], the authors examined the existence of solutions for the following class of hybrid fractional I-DEs involving the Caputo and Riemann-Liouville FD:

$$\begin{cases} {}^{C}D^{\theta,r}\Big[\dfrac{x(a) - \sum_{i=1}^{m^*} I^{w_i,\theta}k_i(a,x(a))}{A(a, x(a))}\Big] = B(a, x(a)), & 0 \leq a \leq 1, \\ x(0) = 0,\ x'(0) = 0,\ x''(0) = 0, \cdots, & x^{m^*-2}(0) = 0, \\ x(\mu) = a\displaystyle\int_0^{\rho} x(r)dr + b\int_{\theta}^{1} x(r)dr, & 0 < \rho < \mu < \theta < 1, \end{cases} \tag{1}$$

where ${}^{C}D^{\theta,r}$ is the θ-Caputo FD of order $r(m^* - 1 < r < m^*)$, $I^{\phi,\theta}$ is the θ-Riemann-Liouville fractional integral (FI) of order $\phi > 0, \phi \in \{w_1, w_2, \cdots, w_n\}, A \in C([0,T]\times\mathbb{R}, \mathbb{R}-0)$, and $B, k_i \in C([0,T]\times\mathbb{R}, \mathbb{R})$, with $k_i(0,0) = 0$, $i = 0, 1, \cdots, m^*$.

In [26], Legendre orthonormal polynomials and the least squares technique have been employed to examine the following fractional I-DEs:

$$\begin{cases} v''(x) + D^{\nu}v(x) + D^{-\zeta}v(x) = A(x), & x \in [0,q],\ \nu > 0, \text{ and } \zeta > 2, \\ v(0) = \eta_1, \quad v'(0) = \eta_2, \end{cases} \tag{2}$$

where q is a positive real number, D^{ν} is the ν-th Caputo FD, and $D^{-\zeta}$ is the ζ-th Riemann-Liouville FI.

In [27], the authors examined the E-UR of solutions for the following nonlinear fractional implicit I-DEs of Hadamard-Caputo type with fractional BCs:

$$\begin{cases} {}^{C}_{CH}D^{r}u(\rho) = h\Big(\rho, u(\rho), {}^{C}_{H}D^{r}u(\rho), \int_1^{\rho} K(\rho, t, v(t))dt\Big), \\ u(1) = 0, \quad \alpha_H I^{q}u(\eta) + \beta^{C}_{H}D^{\gamma}u(\psi)\lambda, \end{cases} \tag{3}$$

where ${}_{H}I^{q}$ is the standard Hadamard FI, ${}^{C}_{CH}D^{r}$ is the Hadamard-Caputo FD, $h : \xi \times \mathbb{R}^3 \to \mathbb{R}$, $K : \xi \times \xi \times \mathbb{R} \to \mathbb{R}$ are given functions, $\eta \in \xi =: (1, \psi], \psi > 1$, and α, λ, β are real constants.

In [28], the existence and approximate controllability of the following class of fractional I-DEs:

$$\begin{cases} {}^{C}D^{\alpha}_{0}u(t) + Gu(t) = \int_0^t \frac{h(u(s))}{(t-s)^{\beta}} g(s, u(s), u_s)ds + Bv(t), \ t \in \mathbb{R}^+, \\ u(t) = \phi(t), \quad t \in H := [-\lambda, 0], \end{cases} \tag{4}$$

were examined in some Fréchet spaces, where ${}^{C}D^{\alpha}_{0}$ is the Caputo FD of order $\alpha \in (0,1)$, $G : D(G) \subset F \to F$ is some operator defined on F, where F is the Fréchet space. The function g is some nonlinear function and v is some control function.

In the study of DEs, commensurate equations refer to systems in which all operators (such as FDs) share a common order. This structural uniformity often allows the use of standard analytical tools, including classical stability analysis and transform methods. In contrast, incommensurate equations involve operators of genuinely different, non-multiple orders, reflecting multiscale or heterogeneous dynamics. Such equations typically model complex phenomena with competing memory effects or temporal behaviors and require more sophisticated analytical frameworks, as well as tailored numerical schemes.

In [29, 30], the authors obtained the existence of a mild solution for an incommensurate FDE of the form

$$\begin{aligned} {}^{c}\mathfrak{D}^{\alpha_1}y_1 &= f_1(t, y_1, \ldots, y_\nu), \\ {}^{c}\mathfrak{D}^{\alpha_1}y_2 &= f_2(t, y_1, \ldots, y_\nu), \\ &\vdots \\ {}^{c}\mathfrak{D}^{\alpha_1}y_\nu &= f_\nu(t, y_1, \ldots, y_\nu), \end{aligned} \tag{5}$$

with the initial condition

$$[y_0(0), \ldots, y_\nu(0)] = [y_{0,1}, ..., y_{0,\nu}]$$

on $[0,\mathcal{T}]$, where $\mathcal{T}\in\mathbb{R}$ (real numbers), $y_i:[0,\mathcal{T}]\to\mathbb{R}$, $\alpha_i\in(0,1)$, and $\nu\in\mathbb{N}$ (natural numbers). The UHS of these equations in the spaces of continuous and piecewise continuous functions is studied in [30].

In [31], the authors examined the E-UR for the following fractional I-DEs:

$$\begin{cases}({}^CD^{\alpha_1}+\lambda_1^CD^{\alpha_1-1})A_1(t)=\Psi_1(t,A_1(t),A_2(t),{}^CD^{\sigma_1}A_2(t),I^{\eta_1}A_2(t))\\({}^CD^{\alpha_2}+\lambda_2^CD^{\alpha_2-1})A_2(t)=\Psi_2(t,A_1(t),A_2(t),{}^CD^{\sigma_2}A_1(t),I^{\eta_2}A_1(t)),\end{cases}\tag{6}$$

with the following BCs:

$$\begin{cases}A_1(\tau)=r_1A_2(a_1)+r_2\tau A_2'(a_2)\\\tau A_1'(\tau)=s_1A_2(b_1)+s_2\tau A_2'(b_2)\\A_2(t)=\sigma_1A_1(t_1)+\sigma_2\tau A_1'(t_2)\\\tau A_2'(\tau)=\eta_1A_1(w_1)+\eta_2\tau A_1'(w_2)\end{cases}\tag{7}$$

where ${}^CD^{\alpha_1},{}^CD^{\alpha_2}$ are the Caputo FDs of order α_1,α_2, $1<\alpha_1,\alpha_2<2$, respectively, $r_i,s_i,\sigma_i\in\{1,2\}$, $\Psi_i, i=1,2\in C(J\times\mathbb{R}\times\mathbb{R})$, and $a_i,b_i,w_i,t_i\in[0,T], i\in\{1,2\}$. In [32], the authors examined the E-UR and controllability for the following fractional neutral I-DEs and N-IIs with infinite delay:

$$\begin{cases}{}^CD_t^r[v(t)-A(t,v_t)]=Bu(t)+F\Big(t,v_t,\displaystyle\int_0^t H(t,s,v_s)ds\Big),\ t\in(s_k,t_{k+1}],\\k=0,1,2,\cdots,m\\v(t)=J_k(v(t_k))+g_k(t,v(t_k)),\ t\in(t_k,s_k],\ k=1,2,\cdots,m\\v_0=\Phi\in B_h,\ t\in(-\infty,0),\end{cases}\tag{8}$$

where ${}^CD_t^r$ is the Caputo FD of order $r\in(0,1)$ and $J=[0,T]$. The operator B is the infinitesimal generator of some semigroup.

Expanding further, our study on the initial value problem (IVP) for nonlinear implicit FDEs with non-instantaneous impulses (N-IIs) (see e.g., [33]) provides a crucial bridge between theoretical mathematics and practical applications. By incorporating the C-FFD, we enhance the capacity of fractional calculus to model systems with memory-dependent characteristics, allowing for a more nuanced understanding of dynamic processes.

One of the principal challenges associated with these equations stems from their implicit structure, wherein variable relationships are not defined through direct or explicit formulations. This inherent complexity demands the application of sophisticated analytical tools and numerical schemes to extract reliable and meaningful solutions. Moreover, the integration of N-IIs introduces additional layers of intricacy, as it necessitates a nuanced treatment of abrupt transitions that occur progressively over designated time intervals rather than at discrete moments.

The periodic solutions for the following fractional Volterra-Fredholm I-DE:

$$\begin{cases}{}_0^{CF}D^\delta x(\vartheta)=F\Big(\vartheta,x(\vartheta),\displaystyle\int_0^{\delta(\vartheta)}G_1(\kappa,x(\kappa))d\kappa,\int_0^T G_2(\kappa,x(\kappa))d\kappa\Big),\vartheta\in[0,T],\\x(0)=x_0,\ \ x(0)=x(T)+\displaystyle\int_0^T H_1[x(\vartheta)]d\vartheta,\end{cases}$$

have been established in [13], where ${}_0^{CF}D^\delta$ denotes the C-FFD ($\delta\in(0,1]$).

In essence, our research provides a comprehensive framework for analyzing the interplay between FDs and N-IIs, offering new perspectives on mathematical theory and practical implementation across multiple disciplines. By pushing the boundaries of contemporary analysis, we aim to equip researchers, engineers, and decision-makers with deeper insights into complex systems that evolve over time.

Motivated by earlier research, the project aims to explore N-IIs and C-F I-DEs with BCs (with $\mathrm{b} = \kappa_0 < \vartheta_1 \leq \vartheta_2 < \cdots < \vartheta_m \leq \kappa_m \leq \kappa_{m+1} = \mathfrak{T}$)

$$\begin{cases} {}^{CF}D^{\delta}\mathrm{z}(\vartheta) = F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)), \vartheta \in J : (\kappa_i, \vartheta_{i+1}], & 0 < \delta \leq 1, \\ \mathrm{z}(\vartheta) = H_i(\vartheta, \mathrm{z}(\vartheta)) \qquad \vartheta \in (\vartheta_i, \kappa_i] \;\; i = 1, ..., m, \\ \mathrm{z}(b) - \mathrm{z}'(b) = \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, z(\kappa)) d\kappa, \end{cases} \tag{9}$$

where the CF- FD of order δ is ${}^{CF}D^{\delta}$, $F : [\mathrm{b}, \mathfrak{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $G : [\mathrm{b}, \mathfrak{T}] \times \mathbb{R} \to \mathbb{R}$ & $H_i : [\vartheta_i, \kappa_i] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $B\mathrm{z}(\vartheta) = \int_0^{\vartheta} \mathrm{x}(\vartheta, \kappa, \mathrm{z}(\kappa)) d\kappa$, and $\mathrm{x} \in C(D, \mathbb{R}^+)$ with domain $D = \{(\vartheta, \kappa) \in \mathbb{R}^2 : \mathrm{b} \leq \kappa \leq \vartheta \leq \mathfrak{T}\}$.

Motivated by the aforementioned factors, our work aims to close the noted research gap. The following summarizes the main contributions of the study.

I) To examine β-UHS of the N-IIFIDE involving the C-FFD in a Banach space.

II) To establish the E-UR of the solutions for the N-IIFIDE involving the C-FFD using Banach and Krasnoselskii's FPT.

III) To validate our results, an example is provided.

The remainder of this article is organized as follows. Section 2 introduces essential lemmas and foundational notions that form the basis for the main theoretical results. Sections 3 and 4 investigate the E-UR and β-UHS of solutions for the problem formulations specified in Eq (9). Section 5 give an example to verify the results.

2. Preliminaries

We define the space of piecewise continuous functions as

$$\mathscr{PC}([\mathrm{b}, \mathfrak{T}], \mathbb{R}) := \{\mathrm{z} : [b, \mathfrak{T}] \to \mathbb{R} : \mathrm{z} \in C((\vartheta_k, \vartheta_{k+1}], \mathbb{R})\}.$$

Take $\mathrm{z}(\vartheta_k^-)$ and $\mathrm{z}(\vartheta_k^+)$ with $\mathrm{z}(\vartheta_k^-) = \mathrm{z}(\vartheta_k^+)$ satisfying $\|\mathrm{z}\|_{\mathscr{PC}} = \sup\{|\mathrm{z}(\vartheta)| : \mathrm{b} \leq \vartheta \leq \mathfrak{T}\}$.

Set $\mathscr{PC}^1([\mathrm{b}, \mathfrak{T}], \mathbb{R}) : \left\{\mathrm{z} \in \mathscr{PC}([\mathrm{b}, \mathfrak{T}], \mathbb{R}) : \mathrm{z}' \in \mathscr{PC}^1([\mathrm{b}, \mathfrak{T}], \mathbb{R})\right\}$ with norm $\|\mathrm{z}\|_{\mathscr{PC}} := \max\{\|\mathrm{z}\|_{\mathscr{PC}}, \|\mathrm{z}'\|_{\mathscr{PC}}\}$.

Definition 2.1. ([1]) The Caputo FD of order $\delta > 0$, $n - 1 < \delta < n$ $(n \in \mathbb{N})$, for F (a $C^n([0, \vartheta])$), is described as

$${}^{C}D_{0^+}^{\delta} F(\vartheta) = I^{n-\delta} D^n F(\vartheta) = \frac{1}{\Gamma(n-\delta)} \int_0^{\vartheta} (\vartheta - \kappa)^{n-\delta-1} F^{(n)}(\kappa) d\kappa.$$

Definition 2.2. ([34]) Let $\delta \in (0,1]$, fractional C-FD of order δ for a function F (a $C^1([0,\vartheta])$) is defined by

$$^{CF}D^{\delta}_{\vartheta}(\mathrm{F}(\vartheta)) = \frac{1}{1-\delta}\int_{\mathrm{b}}^{\vartheta} F'(\vartheta)exp\Big[-\delta\frac{\vartheta-\mathrm{z}}{1-\delta}\Big]d\vartheta,\ \vartheta > \mathrm{b},$$

Definition 2.3. Let $\delta \in (0,1]$, fractional C-F integral of order δ for a function F is defined by

$$^{CF}I^{\delta}_{\vartheta}(\mathrm{F}(\vartheta)) = \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}F(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_0^{\vartheta} F(\vartheta)d\vartheta,\ \vartheta > b,$$

where $\mathscr{P}(\cdot)$ is some normalization constant ($\mathscr{P}(0) = \mathscr{P}(1) = 1$).

Lemma 2.1. *Let $n < \delta \leq n+1$. Then $^{CF}D^{\delta}_{\mathrm{b}^+}\mathrm{z}(\vartheta) = 0$, if $\mathrm{z}(\vartheta)$ is a constant function.*

Lemma 2.2. *Given continuous functions $H_i(\cdot) : [\vartheta_i, \kappa_i] \to \mathbb{R}$, for a continuous function $g : \mathscr{PC}([\mathrm{b},\mathfrak{T}]) \to \mathbb{R}$, then*

$$\mathrm{z}(\vartheta) = \begin{cases} H_m(\kappa_m) + \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa,\mathrm{z}(\kappa))d\kappa + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}g(\vartheta) & \\ +\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta & \vartheta \in [\mathrm{b},\vartheta_1], \\ H_i(\vartheta), & \vartheta \in (\vartheta_i,\kappa_i], \\ H_i(\kappa_i) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta & \\ -\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\kappa_i} g(\vartheta)d\vartheta, & \vartheta \in (\kappa_i,\vartheta_{i+1}], \end{cases} \tag{10}$$

is a solution of the system given by

$$\begin{aligned} ^{CF}D^{\delta}\mathrm{z}(\vartheta) &= g(\vartheta), & \vartheta \in (\kappa_i,\vartheta_{i+1}], 0<\delta\leq 1, \\ \mathrm{z}(\vartheta) &= H_i(\vartheta,\mathrm{z}(\vartheta)) & \vartheta \in (\vartheta_i,\kappa_i],\quad i=1,...,m, \\ \mathrm{z(b)} - \mathrm{z}'(\mathrm{b}) &= \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa,\mathrm{z}(\kappa))d\kappa. \end{aligned} \tag{11}$$

Proof. Assume that $\mathrm{z}(\vartheta)$ satisfies (11). Integrating (11) for $\vartheta \in [\mathrm{b},\vartheta_1]$, we get

$$\mathrm{z}(\vartheta) = \mathrm{z}(\mathfrak{T}) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta. \tag{12}$$

On the other hand, if $\vartheta \in (\kappa_i,\vartheta_{i+1}], i = 1,2,\cdots,m$ and again integrating (11), we have

$$\mathrm{z}(\vartheta) = \mathrm{z}(\kappa_i) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\kappa_i}^{\mathfrak{T}} g(\vartheta)d\vartheta. \tag{13}$$

Next, employing the impulsive condition, $\mathrm{z}(\vartheta) = H_i(\vartheta),\ \vartheta \in (\vartheta_i,\kappa_i]$, we get

$$\mathrm{z}(\kappa_i) = H_i(\kappa_i). \tag{14}$$

Consequently, from (13) and (14), we get

$$\mathrm{z}(\vartheta) = H_i(\kappa_i) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta, \tag{15}$$

and

$$\mathrm{z}(\vartheta) = \mathscr{H}_{\mathrm{i}}(\kappa_{\mathrm{i}}) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta) d\vartheta \\ - \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} g(\vartheta) d\vartheta, \ \vartheta \in (\kappa_i, \vartheta_{i+1}]. \tag{16}$$

Now utilizing the condition $\mathrm{z}(\mathrm{b}) - \mathrm{z}'(\mathrm{b}) = \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa)) d\kappa$, we obtain

$$\mathrm{z}(\mathfrak{T}) = H_m(\kappa_m) + \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa)) d\kappa, \ \vartheta \in [\mathrm{b}, \mathfrak{T}]. \tag{17}$$

Hence, using the FDs and above lemmas, this shows that Eqs (12), (16) and (17) imply (10). Now, we recall one of the main tools in our study, see, e.g., [24]. □

Theorem 2.3. *(KFPT) For a closed bounded convex subset $\mathcal{S}$ of a Banach space $\mathcal{B}$, a contraction $O^* : \mathcal{S} \to \mathcal{B}$, and $O^{**} : \mathcal{S} \to \mathcal{B}$, a continuous mapping with $O^{**}(\mathcal{S})$ relatively compact. If $O^*(x) + O^{**}(y) \in \mathcal{S}$, $\forall x, y \in \mathcal{S}$, then the mapping $O^* + O^{**}$ has at least one FP.*

3. Main results

We utilize the following hypothesis to prove the main results:
(H_1): A continuous function F with some constants $K_1, K_2 > 0$:

$$|F(\vartheta, A_1, B_1) - F(\vartheta, A_2, B_2)| \leq K_1 |A_1 - A_2| + K_2 |B_1 - B_2|, \\ \forall A_1, B_1, A_2, B_2 \in \mathbb{R}, \ \vartheta \in J.$$

(H_2): A constant $N > 0$ exists: $|z(\vartheta, \varphi, A_1) - z(\vartheta, \varphi, B_1)| \leq N |A_1 - B_1|$.
(H_3): $|H_i(\vartheta, V_1) - H_i(\vartheta, V_2)| \leq L_{h_i} |V_1 - V_2|$, for $V_1, V_2 \in \mathbb{R}$.
(H_4): A constant $K_G > 0$ exists: $|G(\vartheta, U)| \leq K_G$, $\forall \ \vartheta \in J$ and $U \in \mathbb{R}$.
(H_5): A constant $K_F > 0$ exists: $|F(\vartheta, A_1, B_1)| \leq K_F$, $\forall \ \vartheta \in J$ and $A_1, B_1 \in \mathbb{R}$.
(H_6): A function $\Theta_i(\vartheta)$, exists with $i = 1, 2, \cdots, m$:

$$|H_i(\vartheta, W_1, \varsigma_1)| \leq \Theta_i(\vartheta), \ \ \vartheta \in [\vartheta_i, s_i], \forall \ \ W_1, \varsigma_1 \in \mathbb{R}.$$

Theorem 3.1. *Let $(H_1) - (H_4)$ hold consequently:*

$$M : \max\{ \max_{i=1,2,\cdots,m} L_{h_i} + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \right) (K_1 + K_2 N)(\vartheta_{i+1} + \kappa_i), \\ L_{h_i} + K_G(\mathfrak{T} - \mathrm{b}) + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \right) (K_1 + K_2 N) \} < 1, \tag{18}$$

then the problem (9) *admits a unique solution on the interval* $[\mathrm{b}, \mathfrak{T}]$.

Proof. Let the operator P be defined as follows:

$$(P\mathrm{z})(\vartheta)=\begin{cases} H_m(\kappa_m,\mathrm{z}(\kappa_m))+\int_{\mathrm{b}}^{\mathfrak{T}}G(\kappa,\mathrm{z}(\kappa))d\kappa+\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta)) & \\ +\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}}F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta)), & \vartheta\in[\mathrm{b},\vartheta_1],\\ H_i(\vartheta), & \vartheta\in(\vartheta_i,\kappa_i],\ i=1,2,...,m,\\ H_i(\kappa_i)+\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}F(\vartheta,\mathrm{z}(\vartheta)),B\mathrm{z}(\vartheta))+\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}}F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta))d\vartheta & \\ -\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\kappa_i}F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta))d\vartheta, & \vartheta\in(\kappa_i,\vartheta_{\mathrm{i}+1}],\ i=1,2,...,m.\end{cases}$$

We first show that P is a contraction.

Case 1: For $\mathrm{z},\mathrm{y}\in\mathscr{PC}([\mathrm{b},\mathfrak{T}],\mathbb{R})$ and $\vartheta\in[\mathrm{b},\vartheta_1]$, we have

$$\begin{aligned}\left|(P\mathrm{z})(\vartheta)-(P\mathrm{y})(\vartheta)\right| &\le L_{h_i}\left|\mathrm{z}(\kappa)-\mathrm{y}(\kappa)\right|d\kappa\\ &+K_G\int_{\mathrm{b}}^{\mathfrak{T}}\left|\mathrm{z}(\kappa)-\mathrm{y}(\kappa)\right|d\kappa\\ &+\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}(K_1+K_2N)\left|\mathrm{z}(\kappa)-\mathrm{y}(\kappa)\right|d\kappa\\ &+\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)}(K_1+K_2N)\int_{\mathrm{b}}^{\mathfrak{T}}\left|\mathrm{z}(\kappa)-\mathrm{y}(\kappa)\right|d\kappa,\\ &\le\left[L_{h_i}+K_G(\mathfrak{T}-\mathrm{b})+\left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}+\frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_1+K_2N)\right]\\ &\left\|\mathrm{z}(\vartheta)-\mathrm{y}(\vartheta)\right\|_{\mathscr{PC}}.\end{aligned}$$

Case 2: For $\vartheta\in(\vartheta_i,\kappa_i]$,

$$\begin{aligned}\left|(P\mathrm{z})(\vartheta)-(P\mathrm{y})(\vartheta)\right| &\le\left|H_i(\vartheta,\mathrm{z}(\vartheta))-H_i(\vartheta,\mathrm{y}(\vartheta))\right|,\\ &\le L_{h_i}\left\|\mathrm{z}-\mathrm{y}\right\|_{\mathscr{PC}}.\end{aligned}$$

Case 3: For $\vartheta\in(\kappa_i,\vartheta_{i+1}]$, we have

$$\begin{aligned}&\left|(P\mathrm{z})(\vartheta)-(P\mathrm{y})(\vartheta)\right|\\ &\le\left|H_i(\kappa_i,\mathrm{z}(\kappa_i)-H_i(\kappa_i,\mathrm{y}(\kappa_i)\right|\\ &+\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}\left|F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta))-F(\vartheta,\mathrm{y}(\vartheta),B\mathrm{y}(\vartheta))\right|\\ &+\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\mathfrak{T}}\left|F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta))-F(\vartheta,\mathrm{y}(\vartheta),B\mathrm{y}(\vartheta))\right|d\vartheta\\ &-\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}\int_{\mathrm{b}}^{\kappa_i}\left|F(\vartheta,\mathrm{z}(\vartheta),B\mathrm{z}(\vartheta))-F(\vartheta,\mathrm{y}(\vartheta),B\mathrm{y}(\vartheta))\right|d\vartheta,\\ &\le\left[L_{h_i}+\left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}+\frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_1+K_2N)(\vartheta_{i+1}^p+\kappa_i^p)\right]\left\|\mathrm{z}-\mathrm{y}\right\|_{\mathscr{PC}}.\end{aligned}$$

Therefore, from the cases above, P is a contraction because $M<1$. Consequently, the problem stated in (9) admits a unique solution on $[\mathrm{b},\mathfrak{T}]$. □

Theorem 3.2. *Let* $(H_4)-(H_6)$ *be satisfied. Let* $L_i := \sup_{\vartheta\in[\vartheta_i,\kappa_i]} \zeta_i(\vartheta) < \infty$*,* $\mathrm{y} := \max L_{h_i} < 1$ *(*$\forall$ $i = 1, 2, \cdots, m$*), and* $B_{p,r} = \{\mathrm{z} \in \mathscr{PC}(J,\mathbb{R}) : |\mathrm{z}|_{\mathscr{PC}} \leq r\}$*. Then problem* (9) *has at least one solution on* $[\mathrm{b}, \mathfrak{T}]$.

Proof. We define the operators Q and S on $B_{p,r}$ by (with $i = 1, 2, \cdots, m$)

$$(Q\mathrm{z})(\vartheta) = \begin{cases} H_m(\kappa_m, \mathrm{z}(\kappa_m)), & \vartheta \in [b, \vartheta_1], \\ H_i(\vartheta, \mathrm{z}(\vartheta)), & \vartheta \in (\vartheta_i, \kappa_i], \\ H_i(\kappa_i, \mathrm{z}(\kappa_i)), & \mathrm{z} \in (\kappa_i, \vartheta_{i+1}] \end{cases}$$

and (with $i = 1, 2, ..., m$)

$$(S\mathrm{z})(\vartheta) = \begin{cases} \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa))d\kappa + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} g(\vartheta) \\ + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta, & \vartheta \in (\mathrm{b}, \vartheta_1], \\ 0, & \vartheta \in (\vartheta_i, \kappa_i], \\ \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} g(\vartheta) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} g(\vartheta)d\vartheta \\ - \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} g(\vartheta)d\vartheta, & \vartheta \in (\kappa_i, \vartheta_{i+1}]. \end{cases}$$

We employ KFPT and prove in four steps as follows.

Step 1: The norm of the sum of $Q\mathrm{z}$ and $S\mathrm{y}$ is still in $B_{p,r}$.

Take $\mathrm{z}, \mathrm{y} \in B_{p,r}$:

Case 1: For $\vartheta \in [\mathrm{b}, \vartheta_1]$,

$$\begin{aligned} \left|Q\mathrm{z} + S\mathrm{y}\right| &\leq \left|H_m(\kappa_m, \mathrm{z}(\kappa_m))\right| + \int_{\mathrm{b}}^{\mathfrak{T}} |G(\kappa, \mathrm{z}(\kappa))|\, d\kappa + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} |F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))| \\ &\quad + \frac{2\delta(\mathfrak{T}-b)}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} |F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))d\vartheta|, \\ &\leq \left[K_F + K_G(\mathfrak{T}-\mathrm{b}) + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_F)\right], \\ &\leq r. \end{aligned}$$

Case 2: For each $\vartheta \in (\vartheta_i, \kappa_i]$,

$$\left|Q\mathrm{z} + S\mathrm{y}\right| \leq |H_i(\vartheta, W_1(\vartheta))| \leq M_i.$$

Case 3: For $\vartheta \in (\kappa_i, \vartheta_{i+1}]$,

$$\begin{aligned} \left|Q\mathrm{z} + S\mathrm{y}\right| &\leq \left|H_i(\kappa_i, \mathrm{z}(\kappa_i))\right| + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} |F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))| \\ &\quad + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} |F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa| \\ &\quad - \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} |F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa|, \\ &\leq L_i + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_F)(\vartheta_{i+1} + \kappa_i), \\ &\leq r. \end{aligned}$$

Thus, $Q\mathrm{z} + S\,\mathrm{y} \in B_{p,r}$.

Step 2: The operator Q is a contraction on $B_{p,r}$.

Case 1: For $\mathrm{z}_1, \mathrm{z}_2 \in B_{p,r}$ and $\vartheta \in [b, \vartheta_1]$,

$$|Q\mathrm{z}_1(\vartheta) - Q\mathrm{z}_2(\vartheta)| \leq L_i\,|\mathrm{z}_1(\kappa_m) - \mathrm{z}_2(\kappa_m)| \leq L_i\,\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathscr{PC}}.$$

Case 2: For $\vartheta \in (\vartheta_i, \kappa_i], i = 1, 2, \cdots, m$,

$$|Q\mathrm{z}_1(\vartheta) - Q\mathrm{z}_2(\vartheta)| \leq L_i\,\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathscr{PC}}.$$

Case 3: For $\vartheta \in (\kappa_i, \vartheta_{i+1}]$,

$$|Q\mathrm{z}_1(\vartheta) - Q\mathrm{z}_2(\vartheta)| \leq L_i\,\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathscr{PC}}.$$

We deduce:

$$|Q\mathrm{z}_1(\vartheta) - Q\mathrm{z}_2(\vartheta)| \leq L_i\,\|\mathrm{z}_1 - \mathrm{z}_2\|_{\mathscr{PC}},$$

which proves that Q is a contraction.

Step 3: The operator S is continuous.

Let z_n be a sequence: $\mathrm{z}_n \to \mathrm{z}$ in $\mathscr{PC}([\mathrm{b}, \mathfrak{T}], \mathbb{R})$.

Case 1: For $\vartheta \in [b, \vartheta_1]$,

$$|S\,\mathrm{z}_n(\vartheta) - S\,\mathrm{z}(\vartheta)| \leq \left[(\mathfrak{T} - \mathrm{b}) + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)\right] \left\|F(\vartheta, \mathrm{z}_n(\vartheta), Bz(\vartheta)) - F(\vartheta, \mathrm{z}(\vartheta), Bz(\vartheta))\right\|_{\mathscr{PC}}.$$

Case 2: For $\vartheta \in (\vartheta_i, \kappa_i]$,

$$|S\,\mathrm{z}_n(\vartheta) - S\,\mathrm{z}(\vartheta)| = 0.$$

Case 3: For $\vartheta \in (\kappa_i, \vartheta_{i+1}]$, with $(i = 1, 2, \cdots, m)$,

$$|S\,\mathrm{z}_n(\vartheta) - S\,\mathrm{z}(\vartheta)| \leq \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}(\vartheta_{i+1} - \kappa_i) \left\|F(\vartheta, \mathrm{z}_n(\vartheta), Bz(\vartheta)) - F(\vartheta, \mathrm{z}(\vartheta), Bz(\vartheta))\right\|_{\mathscr{PC}}.$$

Therefore, we conclude that $n \to \infty$, $\|S\,\mathrm{z}_n(\vartheta) - S\,\mathrm{z}(\vartheta)\|_{\mathscr{PC}} \longrightarrow 0$.

Step 4: Compactness of operator S.

First, note that S is uniformly bounded on $B_{p,r}$, i.e.,

$$\|S\,\mathrm{z}\| \leq \left(\frac{2(1-\delta)L_i}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})L_i}{(2-\delta)\mathscr{P}(\delta)}\right)(1 + r) < r.$$

Case 1: For $\vartheta \in [b, \vartheta_1], 0 \leq V_1 \leq V_2 \leq \vartheta_1, \mathrm{z} \in B_{p,r}$,

$$|S\,V_2 - S\,V_1| \leq \left(\frac{2(1-\delta)L_i}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})L_i}{(2-\delta)\mathscr{P}(\delta)}\right)(1 + r)(V_2 - V_1).$$

Case 2: For $\vartheta \in (\vartheta_i, \kappa_i], \vartheta_i < V_1 < V_2 \leq \kappa_i, \mathrm{z} \in B_{p,r}$,

$$|SV_2 - SV_1| = 0.$$

Case 3: For $\vartheta \in (\kappa_i, \vartheta_{i+1}], \kappa_i < V_1 < V_2 \leq \vartheta_{i+1}, \mathrm{z} \in B_{p,r}$,

$$|SV_2 - SV_1| \leq \left(\frac{2(1-\delta)L_i}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})L_i}{(2-\delta)\mathscr{P}(\delta)}\right)(1+r)(V_2 - V_1).$$

From the above cases, we get $|SV_2 - SV_1| \longrightarrow 0$ as $V_2 \longrightarrow V_1$ and S is equicontinuous. Then, S is relatively compact by equicontinuity and boundedness. Consequently, S is compact by the Ascoli-Arzela theorem. Hence, problem (9) admit at least one solution. □

4. Stability in the sense of β-Ulam

The stability of a system is fundamental in both theoretical and practical contexts, ensuring reliability in applications such as transportation systems, which demand smooth and safe operations. Mathematical stability plays a critical role in the analysis of differential equations governing such systems, which has been extensively studied by researchers, leading to valuable results (see [35–37]), see also [38, 39] for the Ulam stability of some FDEs. as well as some other kinds of stability, e.g., [40–42]. Ulam stability of FDEs has a long history and nice results (see, e.g., [43]). Roughly speaking, it answers the question of whether an approximate solution of an equation lies close (in some sense) to its exact solution or not.

Let $J = [\mathrm{b}, \mathfrak{T}]$ and $\mathscr{PC}(J, \mathbb{R})$ be the β-Banach space of continuous function form J into $\mathbb{R}$ with the β-norm $\|\mathrm{z}\|_\beta = \sup\left\{|\mathrm{z}(\vartheta)|^\beta : \vartheta \in J, 0 < \beta \leq 1\right\}$, $\forall \mathrm{z} \in C(J, \mathbb{R})$.
We also need the piecewise continuous β-Banach space

$$\mathscr{PC}([\mathrm{b}, \mathfrak{T}], \mathbb{R}) := \{F : [\mathrm{b}, \mathfrak{T}] \to \mathbb{R} : F \in C(\vartheta_k, \vartheta_{k+1}], \mathbb{R}\},$$

and there exists $\mathrm{z}(\vartheta_k^-)$ and $\mathrm{z}(\vartheta_k^+)$ with $\mathrm{z}(\vartheta_k^-) = \mathrm{z}(\vartheta_k^+)$ satisfying the $P\beta$-norm $\|F\|_{P\beta} = \sup\left\{|F(\vartheta)|^\beta : \vartheta \in J, 0 < \beta \leq 1\right\}$.

Definition 4.1. Equation (9) is β-UHS with respect to $\upsilon_r \in \mathscr{PC}(J, \mathbb{R}_+)$ and a real number $C_{f,\nu,\beta} > 0$ exists:

$$\begin{cases} \left|{}^{CF}D^\delta \mathrm{z}(\vartheta) - F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))\right| \leq \upsilon_r, \\ |\mathrm{z}(\vartheta) - H_i(\vartheta, \mathrm{z}(\vartheta))| \leq \upsilon_r, \qquad \vartheta \in (\vartheta_i, \kappa_i], \quad i = 1, \cdots, m, \end{cases} \tag{19}$$

and for $\mathrm{y} \in \mathscr{PC}$ of (19), there exists a mild solution $\mathrm{z} \in \mathscr{PC}$ of (9) with

$$\left|\mathrm{z}(\vartheta) - \mathrm{y}(\vartheta)\right|^\beta \leq C_{f,\nu,\beta}\upsilon_r^\beta, \quad \vartheta \in J.$$

Theorem 4.1. *Assuming that the hypotheses $(H_1) - (H_3)$ are satisfied, it follows that the problem defined by* (9) *possesses β-UHS.*

Proof. Denote by z the unique solution of

$$\begin{cases} {}^{CF}D^{\delta}\mathrm{z}(\vartheta) = F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)), \vartheta \in (\kappa_i, \vartheta_{i+1}], & 0 < \delta \leq 1, \\ \mathrm{z}(\vartheta) = H_i(\vartheta, \mathrm{z}(\vartheta)), \qquad \vartheta \in (\vartheta_i, \kappa_i] \;\; i = 1, ..., m, \\ \mathrm{z}(\mathrm{b}) - \mathrm{z}'(\mathrm{b}) = \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa))d\kappa, \end{cases} \tag{20}$$

and we get

$$\mathrm{z}(\vartheta) = \begin{cases} H_m(\kappa_m) + \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa))d\kappa + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)) \\ + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))d\vartheta, & \vartheta \in [\mathrm{b}, \vartheta_1], \\ H_i(\vartheta), & \vartheta \in (\vartheta_i, \kappa_i], \\ H_i(\kappa_i) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)) + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))d\vartheta \\ - \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta))d\vartheta, & \vartheta \in (\kappa_i, \vartheta_{i+1}]. \end{cases} \tag{21}$$

Let $\mathrm{y} \in \mathscr{PC}^1(J, \mathbb{R})$ be a solution of Eq (19). Then, in view of relation (21), it follows that for every $\vartheta \in (\kappa_i, \vartheta_{i+1}]$, we have

$$\begin{aligned} \Big| \mathrm{y}(\vartheta) - H_i(\kappa_i, \mathrm{y}(\kappa_i)) - \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\kappa, \mathrm{z}(\kappa), B\mathrm{x}(\kappa))d\kappa \\ - \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{z}(\kappa), B\mathrm{x}(\kappa)) \\ + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} F(\kappa, \mathrm{z}(\kappa), B\mathrm{x}(\kappa))d\kappa \Big|, \\ \leq \upsilon_r + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\vartheta_{i+1} - \kappa_i)}{(2-\delta)\mathscr{P}(\delta)} \upsilon_r, \end{aligned}$$

and for $(\vartheta_i, \kappa_i], i = 1, 2, ..., m$, we have

$$|\mathrm{z}(\vartheta) - H_i(\vartheta, \mathrm{z}(\vartheta))| \leq \upsilon_r.$$

For each $\vartheta \in [\mathrm{b}, \vartheta_1]$, we have

$$\begin{aligned} \Big| \mathrm{y}(\vartheta) - H_i(\kappa_m, \mathrm{y}(\kappa_m)) + \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa))d\kappa + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa)) \\ + \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa \Big|, \\ \leq \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \upsilon_r. \end{aligned}$$

Case 1: For each $\vartheta \in [\mathrm{b}, \vartheta_1]$, we get

$$\begin{aligned} \Big| \mathrm{y}(\vartheta) - \mathrm{z}(\vartheta) \Big| \leq \Big| \mathrm{y}(\vartheta) - H_i(\kappa_m, \mathrm{y}(\kappa_m)) \\ - \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, \mathrm{z}(\kappa))d\kappa - \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa)) \end{aligned}$$

$$
\begin{aligned}
&- \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa \Big| \\
&+ \Big| \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa \\
&- \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))d\kappa \Big|, \\
&\leq \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} \upsilon_r + L_{h_i} + \frac{2\delta(K_1 + K_2 N)(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \left|\mathrm{y} - \mathrm{z}\right|_{\mathscr{PC}}.
\end{aligned}
$$

Consequently,

$$
\left[1 - \left(\frac{2\delta(K_1 + K_2 N)(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}\right] \left|\mathrm{y} - \mathrm{z}\right|_{\beta} \leq \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} \upsilon_r + L_{h_i}\right)^{\beta}.
$$

$$
\left|\mathrm{y}(\vartheta) - \mathrm{z}(\vartheta)\right|^{\beta} \leq C_{f,\nu,\beta} \upsilon_r^{\beta}, \quad \vartheta \in [\mathrm{b}, \vartheta_1], \tag{22}
$$

where

$$
C_{f,\nu,\beta} := \frac{\left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} \upsilon_r + L_{h_i}\right)^{\beta}}{1 - \left(\frac{2\delta(K_1 + K_2 N)(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}}.
$$

Case 2: For $\vartheta \in (\vartheta_i, \kappa_i], i = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
\left|\mathrm{y}(\vartheta) - \mathrm{z}(\vartheta)\right|^{\beta} &\leq \left|\mathrm{y}(\vartheta) - H_i(\vartheta, \mathrm{y}(\vartheta))\right|^{\beta} \\
&\quad + \left|H_i(\vartheta, \mathrm{y}(\vartheta)) - H_i(\vartheta, \mathrm{z}(\vartheta))\right|^{\beta}, \\
&\leq \upsilon_r^{\beta} + \left(\frac{L_{h_i} \kappa_i}{\mathscr{P}(\delta)}\right)^{\beta} \left|\mathrm{y} - \mathrm{x}\right|_{P\beta},
\end{aligned}
$$

which implies that

$$
\left|\mathrm{y}(\vartheta) - \mathrm{z}(\vartheta)\right|^{\beta} \leq C_{f,\nu,\beta} \upsilon_r^{\beta}, \ \vartheta \in (\vartheta_i, \kappa_i], \ i = 1, 2, \cdots, m, \tag{23}
$$

under which

$$
C_{f,\nu,\beta} = \frac{1}{1 - \left(\frac{L_{h_i} \kappa_i^{\delta}}{\mathscr{P}(\delta)}\right)^{\beta}}.
$$

Case 3: For $\vartheta \in (\kappa_i, \vartheta_{i+1}], i = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
\left|\mathrm{y}(\vartheta) - \mathrm{z}(\vartheta)\right|^{\beta} \leq \Big| &H_i(\kappa_i) + \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} F(\kappa, \mathrm{y}(\kappa), B\mathrm{y}(\kappa)) \\
&+ \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} F(\kappa, \mathrm{y}(\kappa), B\mathrm{y}(\kappa))d\kappa
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} F(\kappa, \mathrm{y}(\kappa), B\mathrm{y}(\kappa))d\kappa\Big|^{\beta} \\
&+ \Big[\frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\mathfrak{T}} \Big|F(\kappa, \mathrm{y}(\kappa), B\mathrm{y}(\kappa)) - F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))\Big| d\kappa \\
&- \frac{2\delta}{(2-\delta)\mathscr{P}(\delta)} \int_{\mathrm{b}}^{\kappa_i} \Big|F(\kappa, \mathrm{y}(\kappa), B\mathrm{y}(\kappa)) - F(\kappa, \mathrm{z}(\kappa), B\mathrm{z}(\kappa))\Big| d\kappa\Big]^{\beta}, \\
&\leq \left(\upsilon_r + \frac{2(1-\delta)(\vartheta_{i+1} - \kappa_{i})}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta} + \left(\frac{2\delta(K_1 + K_2 N)(\vartheta_{i+1} - \kappa_i)}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta} \Big|\mathrm{y} - \mathrm{z}\Big|_{\mathscr{P}\beta}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\left[1 - \left(\frac{2\delta(K_1 + K_2 N)(\vartheta_{i+1} - \kappa_i)}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}\right] \Big|\mathrm{y} - \mathrm{z}\Big|_{\mathscr{P}\beta} \\
&\leq \left[1 + \left(\frac{2(1-\delta)(\vartheta_{i+1} - \kappa_i}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}\right] \upsilon_r^{\beta} \Big|\mathrm{y} - \mathrm{z}\Big|_{\mathscr{P}\beta}.
\end{aligned}
$$

Thus,

$$
\Big|\mathrm{y}(\vartheta) - \mathrm{z}(\vartheta)\Big|^{\beta} \leq C_{f,\nu,\beta} \upsilon_r^{\beta}, \tag{24}
$$

where

$$
C_{f,\nu,\beta} := \frac{1 + \left(\frac{2(1-\delta)(\vartheta_{i+1} - \kappa_{i})\upsilon_r}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}}{1 - \left(\frac{2\delta(K_1 + K_2 N)(\vartheta_{i+1} - \kappa_i)}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta}}.
$$

Summarizing Eqs (22)–(24) $\Rightarrow$ (9) is β-Ulam stable with respect to ϑ_r. □

5. Example

In this section, we consider some particular cases of the nonlinear FIDE to apply our results in the study of existence and Ulam stabilities, specifically, UHS and UHR. Consider the nonlinear C-F of the form:

$$
\begin{cases}
{}^{CF}D^{\delta}\mathrm{z}(\vartheta) = F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)), \vartheta \in J : (\kappa_i, \vartheta_{i+1}], \qquad 0 < \delta \leq 1, \\
\mathrm{z}(\vartheta) = H_i(\vartheta, \mathrm{z}(\vartheta)), \qquad \vartheta \in (\vartheta_i, \kappa_i], \ i = 1, ..., m, \\
\mathrm{z}(b) - \mathrm{z}'(b) = \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, z(\kappa))d\kappa.
\end{cases} \tag{25}
$$

The following examples are particular cases of the FIDE given by (24).

Consider the C-F of the form:

$$
\begin{cases}
{}^{CF}D^{1/2}\mathrm{z}(\vartheta) & = \frac{1}{4}\mathrm{z}(\vartheta) + \frac{1}{10}\int_1^{\vartheta} \frac{1}{\kappa exp(\vartheta^2 - 1) + 4}\mathrm{z}(\kappa)d\kappa \\
\mathrm{z}(\vartheta) & = \frac{|\mathrm{z}(\vartheta)|}{2(1 + |\mathrm{z}(\vartheta)|)}, \qquad \vartheta \in (1, 2], \\
\mathrm{z}(\mathrm{b}) - \mathrm{z}'(\mathrm{b}) & = \int_1^2 \frac{|\mathrm{z}(\kappa)|}{10 + |\mathrm{z}(\kappa)|}d\kappa.
\end{cases} \tag{26}
$$

Set

$$F(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)) = \frac{1}{4}z + B\mathrm{z}(\vartheta),$$

and

$$B\mathrm{z}(\vartheta) = \frac{1}{10}\int_1^{\vartheta} \frac{1}{\kappa exp(\vartheta^2-1)+4}\mathrm{z}(\kappa)d\kappa,$$
$$G(\vartheta, \mathrm{z}(\kappa)) = \int_1^2 \frac{|\mathrm{z}(\kappa)|}{10+|\mathrm{z}(\kappa)|},$$

which is a nonlinear FIDE involving the Hadamard FD. In this case, $\vartheta = \frac{1}{2}$. Set

$$F(\vartheta, \mathrm{z}, \eta) = \frac{1}{4}\mathrm{z} + \frac{1}{10}\eta$$

and $\vartheta \in [1, 2]$. Using the hypothesis (H_1), we get

$$|F(\vartheta, A_1, B_1) - F(\vartheta, A_2, B_2)| \leq \frac{1}{4}|A_1 - A_2| + \frac{1}{10}|B_1 - B_2|,$$
$$\forall A_1, B_1, A_2, B_2 \in \mathbb{R},$$

and

$$|z(\vartheta, \varphi, A_1) - z(\vartheta, \varphi, B_1)| \leq \frac{1}{\kappa exp(\vartheta^2-1)+4}|A_1 - B_1|$$
$$\leq \frac{1}{5}|A_1 - B_1|.$$

Hence the assumptions $(H_1 - H_3)$ are satisfied and $1 < \delta \leq 2$, $\mathrm{b} = 1$, $\mathfrak{T} = 2$, $K_1 = \frac{1}{4}$, $K_2 = \frac{1}{10}$, $K_G = \frac{1}{10}$, $N = \frac{1}{5}$, $\delta = \frac{1}{2}$, $\mathscr{P}(\delta) = 1$, and $L_{h_i} = \frac{1}{3}$. By using Theorem 3.1, we conclude that:

$$L_{h_i} + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_1 + K_2N)(\vartheta_{i+1} + \kappa_i) < 1$$

and

$$L_{h_i} + K_G(\mathfrak{T}-\mathrm{b}) + \left(\frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)} + \frac{2\delta(\mathfrak{T}-\mathrm{b})}{(2-\delta)\mathscr{P}(\delta)}\right)(K_1 + K_2N) < 1.$$

According to Theorem 3.1, there is a unique solution of (26) on $[1, 2]$. Further, we get the solution z of problem (26) below:

$$\begin{cases} \mathrm{z}(\vartheta) = \frac{|\mathrm{z}(\vartheta)|}{2(1+|\mathrm{z}(\vartheta)|)} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\vartheta}^{0}(\vartheta-\kappa)^{\frac{1}{2}} + \frac{1}{5e^{\vartheta+2}(1+|\mathrm{z}|)} + \int_0^{\vartheta}\frac{e^{-(\kappa-\vartheta)}}{10}\mathrm{z}(\kappa)d\kappa\int_1^2\frac{|\mathrm{z}(\kappa)|}{10+|\mathrm{z}(\kappa)|}d\kappa, & \vartheta \in (\mathrm{b}, 1], \\ \mathrm{z}(\vartheta) = \frac{|\mathrm{z}(\vartheta)|}{2(1+|\mathrm{z}(\vartheta)|)}, & \vartheta \in (1, 2], \\ \mathrm{z}(\mathrm{b}) - \mathrm{z}'(\mathrm{b}) = \frac{|\mathrm{z}(\vartheta)|}{2(1+|\mathrm{z}(\vartheta)|)} + \frac{1}{\Gamma(\frac{1}{2})}\int_{\vartheta}^{0}(\vartheta-\kappa)^{\frac{1}{2}} + \frac{1}{5e^{\vartheta+2}(1+|\mathrm{z}|)} & \\ + \int_0^{\vartheta}\frac{e^{-(\kappa-\vartheta)}}{10}\mathrm{z}(\kappa)d\kappa - \frac{1}{\Gamma(\frac{1}{2})}\int_{\vartheta}^{0}(\vartheta-\kappa)^{\frac{1}{2}} + \frac{1}{5e^{\vartheta+2}(1+|\mathrm{z}|)} & \\ + \int_0^{\vartheta}\frac{e^{-(\kappa-\vartheta)}}{10}\mathrm{z}(\kappa)d\kappa, & \vartheta \in (1, 2]. \end{cases} \tag{27}$$

For $\vartheta \in (0, 1]$, we obtain

$$\left|\mathrm{y} - \mathrm{z}\right|_{P\beta} = \frac{2(1-\delta)}{(2-\delta)\mathscr{P}(\delta)}\upsilon_r + L_{h_i} + \frac{2\delta(K_1 + K_2N)(\mathfrak{T} - \mathrm{b})}{(2-\delta)\mathscr{P}(\delta)} \leq 1.0784.$$

For $\vartheta \in (1, 2]$, we get

$$\begin{aligned}\left|\mathrm{y} - \mathrm{z}\right|_{P\beta} &= \left(\upsilon_r + \frac{2(1-\delta)(\vartheta_{i+1} - \kappa_{i})\upsilon_r}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta} \\ &\quad + \left(\frac{2\delta(K_1 + K_2N)(\vartheta_{i+1} - \kappa_i)}{(2-\delta)\mathscr{P}(\delta)}\right)^{\beta} \leq 1.5723.\end{aligned}$$

This establishes that the system of Eqs (27) exhibits β-UHS with respect to the norm $\upsilon_r = 1$.

6. Conclusions

This article investigates the β-UHS for the N-IIFIDE incorporating the C-FFD within a Banach space. Furthermore, findings regarding uniqueness and existence are proven. Banach's FPTs are used to display the uniqueness result, while KFPTs are used to examine the existence results. Lastly, we provide an example that demonstrates the consistency of the theoretical findings. In this way, we filled the gab existing in the literature for this particular system of FDE. In the future, we plan to do the following:

1) Investigate E-UR for a generalized version of the current C-F BVP with various FD types using a novel approach.
2) Study β-UHS for the following generalized version of (9):

$$\begin{cases} {}^{CF}D^{\delta}\mathrm{z}(\vartheta) = \sum_{i=1}^{n} A_i F_i(\vartheta, \mathrm{z}(\vartheta), B\mathrm{z}(\vartheta)), \vartheta \in (\kappa_i, \vartheta_{i+1}], & 0 < \delta \leq 1, \\ \mathrm{z}(\vartheta) = H_i(\vartheta, \mathrm{z}(\vartheta)), & \vartheta \in (\vartheta_i, \kappa_i], \quad i = 1, ..., m, \\ \mathrm{z}(b) - \mathrm{z}'(b) = \int_{\mathrm{b}}^{\mathfrak{T}} G(\kappa, z(\kappa))d\kappa. \end{cases} \tag{28}$$

3) Examine the E-UR and β-UHS for (9) involving different FDs.

Author contributions

Conceptualization, E.E., K.V. and E.A.; methodology, E.E., K.V.; software, E.E., K.V. and E.A.; validation, E.E., K.V. and E.A.; formal analysis, E.E., K.V. and E.A.; investigation, E.E., K.V. and E.A.; data curation, E.E., K.V. and E.A.; writing—original draft preparation, E.E; K.V.; writing—review and editing, E.E.; visualization, E.E., K.V. and E.A.; supervision, E.E., K.V.; project administration, E.E., K.V. and E.A. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no generative-AI tools are used in this article

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 2006. https://doi.org/10.1007/978-0-387-21593-8
2. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, **198** (1998).
3. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus: Models and numerical methods*, World Scientific, 2012.
4. G. S. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Switzerland: Gordon and Breach Science Publishers , 1993.
5. M. S. Abdo, T. Abdeljawad, S. M. Ali, K. Shah, On fractional boundary value problem involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions, *Adv. Differ. Equ.*, **2021** (2021), 37. https://doi.org/10.1186/s13662-020-03196-6
6. S. Abbas, M. Benchohra, J. J. Nieto, Caputo-Fabrizio fractional differential equations with instantaneous impulses, *AIMS Mathematics*, **6** (2021), 2932–2946. https://doi.org/10.3934/math.2021177
7. C. Long, J. Xie, G. Chen, D. Luo, Integral boundary value problem for fractional order Differential equations with non-instantaneous impulses, *Int. J. Math. Anal.*, **14** (2020), 251–266. https://doi.org/10.12988/ijma.2020.912110
8. D. Luo, Z. Luo, Existence of solutions for fractional differential inclusions with initial value condition and non-instantaneous impulses, *Filomat*, **33** (2019), 5499–5510.
9. A. Salim, M. Benchohra, J. R. Graef, J. E. Lazreg, Boundary value problem for fractional order generalized Hilfer-type fractional derivative with non-instantaneous impulses, *Fractal Fract.*, **5** (2021), 1. https://doi.org/10.3390/fractalfract5010001
10. W. Du, M. Feakan, M. Kostia, D. Velinov, β-Ulam-Hyers stability and existence of solutions for non-instantaneous impulsive fractional integral equations, *Fractal Fract.*, **8** (2024), 469. https://doi.org/10.3390/fractalfract8080469
11. A. S. Rafeeq, Periodic solution of Caputo-Fabrizio fractional integro-differential equation with periodic and integral boundary conditions, *Eur. J. Pure Appl. Math.*, **15** (2022), 144–157. https://doi.org/10.29020/nybg.ejpam.v15i1.4247

12. A. Benzahi, N. Abada, N. Arar, S. A. Idris, M. S. Abdo, W. Shatanawi, Caputo-Fabrizio type fractional differential equations with non-instantaneous impulses: Existence and stability results, *Alex. Eng. J.*, **87** (2024), 186–200. https://doi.org/10.1016/j.aej.2023.12.036

13. M. A. Mohammed, R. G. Metkar, On Caputo-Fabrizio fractional integro-differential equations, *TWMS J. Appl. Eng. Math.*, **15** (2025), 952–965.

14. K. Shah, R. Gul, Study of fractional integro-differential equations under Caputo-Fabrizio derivative, *Math. Method. Appl. Sci.*, **45** (2022), 7940–7953. https://doi.org/10.1002/mma.7477

15. V. Gupta, J. Dabas, Nonlinear fractional boundary value problem with not-instantaneous impulse, *AIMS mathematics*, **2** (2017), 365–376. https://doi.org/10.3934/Math.2017.2.365

16. D. Li, Y. Li, F. Chen, X. Feng, Instantaneous and non-instantaneous impulsive boundary value problem involving the generalized Ψ-Caputo fractional derivative, *Fractal Fract.*, **7** (2023), 206. https://doi.org/10.3390/fractalfract7030206

17. S. Liu, J. Wang, Y. Zhou, Optimal control of noninstantaneous impulsive differential equations, *J. Franklin I.*, **354** (2017), 7668–7698. https://doi.org/10.1016/j.jfranklin.2017.09.010

18. N. A. Shah, M. A. Imran, F. Miraj, Exact solutions of time fractional free convection flows of viscous fluid over an isothermal vertical plate with Caputo and Caputo-Fabrizio derivatives, *J. Prime Res. Math.*, **13** (2017), 56–74.

19. M. A. Imran, M. B. Riaz, N. A. Shah, A. A. Zafar, Boundary layer flow of MHD generalized Maxwell fluid over an exponentially accelerated infinite vertical surface with slip and Newtonian heating at the boundary, *Results Phys.*, **8** (2018), 1061–1067. https://doi.org/10.1016/j.rinp.2018.01.036

20. N. A. Sheikh, F. Ali, M. Saqib, I. Khan, S. A. A. Jan, A comparative study of Atangana-Baleanu and Caputo-Fabrizio fractional derivatives to the convective flow of a generalized Casson fluid, *Eur. Phys. J. Plus*, **132** (2017), 54. https://doi.org/10.1140/epjp/i2017-11326-y

21. R. Agarwal, S. Hristova, D. O'Regan, Non-instantaneous impulses in Caputo fractional differential equations, *Fract. Calc. Appl. Anal.*, **20** (2017), 595–622. https://doi.org/10.1515/fca-2017-0032

22. S. Abbas, M. Benchohra, G. M. N'Guerekata, Instantaneous and non-instantaneous impulsive integro-differential equations in Banach spaces, *Abstr. Appl. Anal.*, **2020** (2020), 269015. https://doi.org/10.1155/2020/2690125

23. Y. Tian, Z. Bai, Impulsive boundary value problem for differential equations with fractional order, *Differ. Equ. Dyn. Syst.*, **21** (2013), 253–260. https://doi.org/10.1007/s12591-012-0150-6

24. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003.

25. M. S. Algolam, M. Chebab, M. S. Abdo, F. Alshammari, K. Aldwoah, Analysis of hybrid fractional integro-differential equations with application to cholera dynamics, *Sci. Rep.*, **15** (2025), 33905. https://doi.org/10.1038/s41598-025-10159-y

26. Y. Jia, M. Xu, Y. Lin, D. Jiang, An efficient technique based on least-squares method for fractional integro-differential equations, *Alex. Eng. J.*, **64** (2023), 97–105. https://doi.org/10.1016/j.aej.2022.08.033

27. A. A. Hamoud, C. Kechar, A. Ardjouni, H. Emadifar, A. Kumar, L. Abualigah, On Hadamard-Caputo implicit fractional integro-differential equations with boundary fractional conditions, *Kragujev. J. Math.*, **50** (2026), 491–504.
28. F. Mesri, A. Salim, M. Benchohra, Controllability of fractional integro-differential equations with delays and singular kernels in Fréchet spaces, *Mathematics*, **13** (2025), 3685. https://doi.org/10.3390/math13223685
29. B. Shiri, Y. G. Shi, D. Baleanu, The well-posedness of incommensurate FDEs in the space of continuous functions, *Symmetry*, **16** (2024), 1058. https://doi.org/10.3390/sym16081058
30. B. Shiri, Y. G. Shi, D. Baleanu, Ulam-Hyers stability of incommensurate systems for weakly singular integral equations, *J. Comput. Appl. Math.*, **474** (2025), 116920. https://doi.org/10.1016/j.cam.2025.116920
31. F. S. Aldosary, M. Murugesan, H. Gundogdu, Nonlinear sequential fractional integro-differential systems: Caputo-type derivatives and boundary constraints, *Math. Method. Appl. Sci.*, **48** (2025), 15194–15218. https://doi.org/10.1002/mma.70009
32. k. Malar, R. Ilavarasi, Existence, uniqueness and controllability results for fractional neutral integro-differential equations with non-instantaneous impulses and delay, *Kragujev. J. Math.*, **50** (2026), 683–724.
33. X. Zhang, P. Agarwal, Z. Liu, X. Zhang, W. Ding, A. Ciancio, On the fractional differential equations with not instantaneous impulses, *Open Phys.*, **14** (2016), 676–684. https://doi.org/10.1515/phys-2016-0076
34. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. http://doi.org/10.12785/pfda/010201
35. X. Yu, Existence and β-Ulam-Hyers stability for a class of fractional differential equations with non-instantaneous impulses, *Adv. Differ. Equ.*., **2015** (2015), 104. https://doi.org/10.1186/s13662-015-0415-9
36. A. Zada, S. Ali, Y. Li, Ulam-type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition, *Adv. Differ. Equ.*, **2017** (2017), 317. https://doi.org/10.1186/s13662-017-1376-y
37. A. Zada, N. Ali, U. Riaz, Ulam's stability of multi-point implicit boundary value problem with non-instantaneous impulses, *Boll. Unione Mat. Ital.*, **13** (2020), 305–328. https://doi.org/10.1007/s40574-020-00219-8
38. A. B. Makhlouf, E. S. El-Hady, H. Arfaoui, S. Boulaaras, L. Mchiri, Stability of some generalized fractional differential equations in the sense of Ulam-Hyers-Rassias, *Bound. Value Probl.*, **2023** (2023), 8. https://doi.org/10.1186/s13661-023-01695-5
39. A. B. Makhlouf, E. El-Hady, S. Boulaaras, L. Mchiri, Stability results of some fractional neutral integro-differential equations with delay, *J. Funct. Space.*, **2022** (2022), 8211420. https://doi.org/10.1155/2022/8211420
40. Z. Yan, Y. Song, J. H. Park, Finite-time stability and stabilization for stochastic markov jump systems with mode-dependent time delays, *ISA T.*, **68** (2017), 141–149. https://doi.org/10.1016/j.isatra.2017.01.018

41. Z. Yan, X. Zhou, G. Chang, Z. Gao, Finite-time annular domain stability and stabilization of stochastic systems with semi-Markovian switching, *IEEE T. Automat. Contr.*, **68** (2023), 6247–6254. https://doi.org/10.1109/TAC.2022.3228202

42. Z. Yan, G. Chang, Z. Z. Xing, R. Xu, J. Cheng, Annular finite-time stability and stabilization in probability for nonlinear stochastic Markov jump systems with impulsive effects, *Int. J. Robust Nonlin.*, **36** (2026), 245–255. https://doi.org/10.1002/rnc.70137

43. D. Boucenna, A. Ben Makhlouf, E. S. El-hady, M. A. Hammami, Ulam-Hyers-Rassias stability for generalized fractional differential equations, *Math. Method. Appl. Sci.*, **44** (2021), 10267–10280. https://doi.org/10.1002/mma.7406