Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1463–1488.
DOI: 10.3934/math.2026062
Received: 23 October 2025
Revised: 14 December 2025
Accepted: 19 December 2025
Published: 19 January 2026

Research article

A higher-order Riemann-Hilbert problem for inhomogeneous complex partial differential equations

Yanyan Cui*and Chaojun Wang

College of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, Henan, China

* **Correspondence:** Email: cui9907081@163.com.

Abstract: In this paper, we mainly discussed an inhomogeneous Riemann-Hilbert boundary value problem for complex partial differential operators of higher order on the bicylinder D^2 in $\mathbb{C}^2$. Applying the classical Cauchy-Pompeiu formula and the Gauss theorem, we found out the solvable condition and obtained the specific solution of the inhomogeneous boundary value problem on the bicylinder D^2. The conclusion lays a solid foundation for further research on other types of boundary value problems concerning complex partial differential equations, such as mixed boundary value problems.

Keywords: boundary value problems; Cauchy-Pompeiu formula; Riemann-Hilbert problems; complex partial differential equations; Cauchy-Pompeiu formula
Mathematics Subject Classification: 32A30, 30C45

1. Introduction

The Riemann-Hilbert problem is a fundamental type of boundary value problem. The Riemann-Hilbert problems for inhomogeneous complex partial differential equations are widely used in the research of mechanics, physics, chemical cycling systems, laser and fiber optic communication, electrical engineering, epidemiology, and other fields. For example, inhomogeneous Riemann-Hilbert problems were applied in studying crack problems in engineering [1], Markov processes in queueing system theory [2], and non-linear wave equations [3]. However, the research on these problems often focuses on low-order, partial differential equations or discussions in low dimensional spaces. Therefore, we hope to obtain relevant conclusions on higher-order Riemann-Hilbert problems or in higher dimensional spaces, which is also the focus of this study. The conclusions drawn in this article will generalize the existing results of boundary value problems, and would provide new methods for high-dimensional signal processing in the engineering field.

The formation and development of the theory for partial differential equations are closely related to the development of physics and other natural sciences. With the development of science and technology, the research on inhomogeneous Riemann-Hilbert problems has run through many fields such as biological science, information science, geography science, astronomical science, and environmental science. Therefore, the study of Riemann-Hilbert problems for inhomogeneous complex partial differential equations not only has important scientific significance, but also has a positive promoting effect on the progress and development of society, as well as promoting the development of mathematics in function theory, algebra and topology, differential equations, differential geometry and other aspects.

With the development of modern hydrodynamics, physics and engineering technology, many new boundary value problems have emerged, which often lead to some related problems such as high-order nonlinear partial differential equations, degenerate type equations and mixed-type equations. These equations are usually very complex and difficult to solve, and have also attracted many scholars to study them. Li, Tian and Yang [4] systematically studied the general n-component nonlinear Schrödinger equations by the Riemann-Hilbert method. They obtained the corresponding solutions and presented the interactions between solitons from weak to strong. They also proposed a conjecture concerning the dynamical behaviors. Wang, Tian and Cheng [5] successfully derived the three-component coupled Hirota equations by the $\bar{\partial}$-dressing method, and first obtained the soliton solutions of the equations. Yang, Tian and Li [6] successfully obtained the solutions of the Riemann-Hilbert problem concerning the focusing nonlinear Schrödinger equation with multiple high-order poles under nonzero boundary conditions for the first time, by solving the corresponding algebraic system. Wu and Tian [7] successfully obtained the long-time asymptotic solution of the nonlocal short pulse equation concerning a suitable Riemann-Hilbert problem, by using the nonlinear steepest descent method. In addition, Li, Tian, Yang and Fan [8–10] have done some interesting work in investigating the Cauchy problems of the complex short pulse equation and the Wadati-Konno-Ichikawa equation with finite density initial data in the space-time solitonic regions and weighted Sobolev space. The problems are associated with the corresponding Riemann-Hilbert problems with the initial boundary value conditions. By the nonlinear $\bar{\partial}$-steepest descent method, they derived the long time asymptotic behaviors of the solutions of these equations in a fixed space-time cone. Based on the behaviors they proved the soliton resolution conjectures and the asymptotic stability of the solutions of these equations. Applying a method of the Riemann-Hilbert problem, Suzuki and Zhang [11, 12] discussed Zhang's conjectures 1 and 2 in a ferromagnetic 3D Ising model: trivialization of topological structure and topological phases. They constructed the solutions to Zhang's conjectures 1 and 2 by use of the monoidal transform. The above results are the latest achievements on the solutions and applications of Riemann-Hilbert boundary value problems.

The Gauss theorem and Pompeiu formula [13] play an important role in the function theory of generalized analytic functions. Many partial differential equations were studied on the basis of the Pompeiu formula, see, for example, [14–17]. The classical Pompeiu formula was generalized to different forms which are closely linked to the corresponding singular integral operators. Applying the properties of the singular integral operators, many boundary value problems for higher order complex partial differential equations or systems have been resolved [18–21].

Boundary value problems for functions of one complex variable, involving homogeneous or inhomogeneous problems, have been widely investigated on different domains in the complex plane $\mathbb{C}$ [22–26]. With the advent of explicit integral representation formulas for functions of several complex variables, some corresponding boundary value problems were studied in $\mathbb{C}^n$ [27–30]. For example, Suzuki [31] discussed Riemann and Hilbert problems on Stein manifolds and deduced the relations of Fuchs on projective algebraic manifolds of high dimensions in several complex variables by computing the characteristic polynomials. Çelebi [14] developed a unified method to derive integral representations for functions in $\mathbb{C}^n$, based on which the Schwarz and Dirichlet problems for higher-order model and linear equations were discussed. Applying the properties of higher-order Neumann functions, Çelebi [32] studied the Neumann problems for higher-order model equations in the unit polydisc in $\mathbb{C}^2$. Mohammed [33] studied the Riemann-Hilbert problem for holomorphic functions in higher- dimensional poly domains. Although different types of boundary value problems have been discussed in $\mathbb{C}^n$, the above achievements are basically related to analytic functions. The passive and irrotational physical fields have prompted people to study a wider range of function classes than analytical functions, such as polyanalytic functions. There have been lots of results on the study of polyanalytic functions in $\mathbb{C}$, but there are few in $\mathbb{C}^n$.

In 2020, Qiao, Cui, Li and Wang [34] obtained the Cauchy integral expression of k-holomorphic functions on the bounded domain $G = G_1 \times G_2 \times \cdots \times G_n$ in $\mathbb{C}^n$:

$$W(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 G} \sum_{k_1,\cdots,k_n=0}^{k-1} \prod_{j=1}^{n} \frac{\overline{z_j - \tau_j}^{k_j}}{k_j!(\tau_j - z_j)} \frac{\partial^{k_1+\cdots+k_n} W(\tau)}{\partial \overline{\tau_1}^{k_1} \cdots \partial \overline{\tau_n}^{k_n}} d\tau.$$

Applying this integral expression, we can discuss some boundary value problems for higher-order complex partial differential equations. In [35] Cui discussed the Riemann boundary value problem on the polydisc in $\mathbb{C}^2$, applying the properties of the Cauchy-type singular integral operators with k-holomorphic kernels. By the above Cauchy integral expression, the Riemann-Hilbert problem (homogeneous and non-homogeneous) for k-holomorphic functions can be discussed in the following, in which the methods, the area, and boundary conditions are different from what is in the above literatures.

In this paper, we mainly study a kind of Riemann-Hilbert problem for complex partial differential operators of higher order on the bicylinder, and obtain the specific solutions of the boundary value problem. The conclusions generalize the existing results of the corresponding Riemann-Hilbert boundary value problem for lower-order partial differential equations.

Throughout this paper, D is the unit disk, $D^2 = D \times D$ is the bicylinder in $\mathbb{C}^2$, and the characteristic boundary of D^2 is $\partial_0 D^2$.

2. Some lemmas

To get the main results, we need the following lemmas:

Lemma 2.1. *[36] Let $w \in C^1(G; \mathbb{C}) \bigcap C(\overline{G}; \mathbb{C})$, where G is a bounded smooth domain in the complex plane. Then*

$$w(z) = \frac{1}{2\pi i} \int_{\partial G} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_G w_{\bar{\zeta}}(\zeta) \frac{d\sigma_\zeta}{\zeta - z}, \tag{2.1}$$

$$\int_G w_{\bar{z}}(z) d\sigma_z = \frac{1}{2i} \int_{\partial G} w(z) dz, \int_G w_z(z) d\sigma_z = -\frac{1}{2i} \int_{\partial G} w(z) d\bar{z}. \tag{2.2}$$

Lemma 2.2. *Let D^2 be the bicylinder in $\mathbb{C}^2$ and $\gamma \in C(\partial_0 D^2)$, and let*

$$
\begin{aligned}
W(z) = &\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\Big]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\frac{1}{(2\pi)^2 i}\int_{\partial_0 D^2}\Im\Big\{\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}-(-\overline{\zeta_1})^{k_1}(-\overline{\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
\tag{2.3}
$$

Then $\partial_{\bar z_1}^k W(z) = 0 = \partial_{\bar z_2}^k W(z)$ and $\Re W(\zeta) = \gamma(\zeta)(\zeta \in \partial_0 D^2)$ if and only if

$$
\begin{aligned}
&\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar z_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
\tag{2.4}
$$

Proof. Suppose that $W(z)$ is the solution to $\Re W(\zeta) = \gamma(\zeta)(\zeta \in \partial_0 D^2)$ and $\partial_{\bar z_1}^k W(z) = 0 = \partial_{\bar z_2}^k W(z)$. Applying the Cauchy integral expression of k-holomorphic functions, $W(z)$ can be expressed as

$$
W(z) = \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!(\zeta_1-z_1)(\zeta_2-z_2)}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)d\zeta_1 d\zeta_2. \tag{2.5}
$$

On the other hand, as $(\frac{1}{\bar z_1}, \frac{1}{\bar z_2}) \notin \overline{D}^2$, then we have that

$$
0 = \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\frac{\overline{z_1}d\zeta_1}{1-\overline{z_1}\zeta_1}\frac{\overline{z_2}d\zeta_2}{1-\overline{z_2}\zeta_2},
$$

which leads to

$$
0 = \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{k_1,k_2=0}^{k-1}\frac{(z_1-\zeta_1)^{k_1}(z_2-\zeta_2)^{k_2}}{k_1!k_2!}\overline{\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)}\frac{z_1 z_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}. \tag{2.6}
$$

Adding (2.5) and (2.6), we get that

$$
\begin{aligned}
W(z) = &\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\frac{\zeta_1\zeta_2+z_1z_2}{(\zeta_1-z_1)(\zeta_2-z_2)\zeta_1\zeta_2}\Re\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\\
&+\frac{\zeta_1\zeta_2-z_1z_2}{(\zeta_1-z_1)(\zeta_2-z_2)\zeta_1\zeta_2}i\Im\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar\zeta_1}^{k_1}\partial_{\bar\zeta_2}^{k_2}W(\zeta)\Big\}d\zeta_1 d\zeta_2\\
= &\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1-\frac{z_1(\zeta_2-z_2)+z_2(\zeta_1-z_1)}{(\zeta_1-z_1)(\zeta_2-z_2)}\Big]
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\cdot \Re \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \frac{\partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W}{\zeta_1\zeta_2} \\
&+ \Big[\frac{z_1(\zeta_2-z_2)+z_2(\zeta_1-z_1)}{(\zeta_1-z_1)(\zeta_2-z_2)\zeta_1\zeta_2} + \frac{1}{\zeta_1\zeta_2}\Big] i\Im \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta)\Big\} d\zeta_1 d\zeta_2 \\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - 1\Big] \Re \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&- \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \Big(\frac{z_1}{\zeta_1-z_1} + \frac{z_2}{\zeta_2-z_2}\Big) \overline{\sum_{k_1,k_2=0}^{k-1} \frac{(z_1-\zeta_1)^{k_1}(z_2-\zeta_2)^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta)} \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&+ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} i\Im \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
$$

As

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \overline{\Big(\frac{z_1}{\zeta_1-z_1} + \frac{z_2}{\zeta_2-z_2}\Big)} \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{\overline{d\zeta_1}\,\overline{d\zeta_2}}{\overline{\zeta_1\zeta_2}} \\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{1}{\frac{1}{\bar{z}_1}-\zeta_1} d\zeta_1 \frac{d\zeta_2}{\zeta_2} \\
&+ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{1}{\frac{1}{\bar{z}_2}-\zeta_2} d\zeta_2 \frac{d\zeta_1}{\zeta_1} = 0+0=0
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} i\Im \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta) \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&= \frac{1}{(2\pi)^2 i}\int_{\partial_0 D^2} \Im\Big\{ \sum_{k_1,k_2=0}^{k-1} \frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2} - (-\overline{\zeta_1})^{k_1}(-\overline{\zeta_2})^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta)\Big\} \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} + i\Im W(0),
\end{aligned}
\tag{2.9}
$$

which is due to (2.5), plugging (2.8) and (2.9) into (2.7), we get $W(z) = (2.3) + ic$ (where $c = \Im W(0)$). Equation (2.3) follows that

$$
\begin{aligned}
\Re W(z) = &\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} 2\gamma(\zeta) \Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - \frac{1}{2}\Big] \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&+ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1} 2\Re\Big[\frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!} \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta)\Big] \cdot \Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - \frac{1}{2}\Big] \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
\tag{2.10}
$$

Applying the properties of the Poisson kernel of D^2,

$$
\lim_{z\to\partial_0 D^2} \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \gamma(\zeta) \frac{1-|z_1|^2}{|\zeta_1-z_1|^2} \frac{1-|z_2|^2}{|\zeta_2-z_2|^2} \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} = \gamma(z),
$$

i.e.,

$$\lim_{z\to\partial_0 D^2}\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}2\gamma(\zeta)\Re\Big\{\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]+\frac{z_1\overline{z_2}}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\Big\}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}=\gamma(z). \tag{2.11}$$

Equations (2.10) and (2.11) lead to (2.4) if $\Re W(\zeta)=\gamma(\zeta)(\zeta\in\partial_0 D^2)$.

On the other hand, (2.3) follows $\partial^k_{\bar z_1}W(z)=0=\partial^k_{\bar z_2}W(z)$ obviously. In addition, by (2.10) and (2.11), we get $\lim_{z\to\partial_0 D^2}\Re W(z)=\gamma(z)$ on the condition of (2.4). Therefore $W(z)$ satisfies $\Re W(\zeta)=\gamma(\zeta)(\zeta\in\partial_0 D^2)$, so we get the desired conclusion. □

Lemma 2.3. *[37] Let $\rho\in C_0^\infty(\mathbb{C})$, G be a bounded domain in the complex plane, and*

$$T_{m,n}\rho(z)=\int_G K_{m,n}(z-\zeta)\rho(\zeta)d\sigma_\zeta,$$

where

$$K_{m,n}(z)=\begin{cases}\frac{(-m)!(-1)^m}{(n-1)!\pi}z^{m-1}\bar z^{n-1}, & m\le 0;\\ \frac{(-n)!(-1)^n}{(m-1)!\pi}z^{m-1}\bar z^{n-1}, & n\le 0;\\ \frac{1}{(m-1)!(n-1)!\pi}z^{m-1}\bar z^{n-1}\Big[\log|z|^2-\sum_{k=1}^{m-1}\frac{1}{k}-\sum_{l=1}^{n-1}\frac{1}{l}\Big], & m,n\ge 1.\end{cases}$$

Then $\partial_{\bar z}(T_{m,n}\rho)=T_{m,n-1}\rho$ and $\partial_z(T_{m,n}\rho)=T_{m-1,n}\rho$ if $m+n\ge 1$.

In particularly, for $m=0,n=1$ in Lemma 2.3, $T_{0,1}\rho(z)=\frac{-1}{\pi}\int_G\frac{\rho(\zeta)}{\zeta-z}d\sigma_\zeta$ is usually denoted by the T-operator $T\rho(z)$, and therefore $\partial_{\bar z}T\rho(z)=T_{0,0}\rho(z)=\rho(z)$.

3. The Riemann-Hilbert BVP

Problem 3.1 (Problem RH). *For $\gamma, f_{lk}\in C(\partial_0 D^n)$ $(k\ge 1$, $redk\in\mathbb{Z}^+)$ being given functions with $\frac{\partial^{k_q}f_{lk_l}(z)}{\partial\bar z_q^{k_q}}=\frac{\partial^{k_l}f_{qk_q}(z)}{\partial\bar z_l^{k_l}}$ $(k_l,k_q\ge 1,\ k_l,k_q\in\mathbb{Z}^+$ for $l,q=1,\cdots,n)$, find a function $W(z)$ on D^n such that*

$$\begin{cases}\Re[\bar\zeta^p W(\zeta)]=\gamma(\zeta) & (\zeta\in\partial_0 D^n),\\ \frac{\partial^k W(z)}{\partial\bar z_l^k}=f_{lk}(z) & (z\in D^n),\end{cases}$$

where $p=(p_1,p_2,\cdots,p_n)$, $p_1,p_2,\cdots,p_n\in\mathbb{Z}$, and the mixed partials of the compatibility condition are the given functions.

Theorem 3.1. *Let $\gamma, f_{lk}\in C(\partial_0 D^2)$ $(k\ge 1$, $k\in\mathbb{Z}^+)$ with $\frac{\partial^{k_q}f_{lk_l}(z)}{\partial\bar z_q^{k_q}}=\frac{\partial^{k_l}f_{qk_q}(z)}{\partial\bar z_l^{k_l}}$ $(k_l,k_q\ge 1,\ k_l,k_q\in\mathbb{Z}^+$ for $l,q=1,2)$, and let*

$$W_0=(T_{0,k})_1 f_{1k}+(T_{0,k})_2 f_{2k}-(T_{0,k})_2(T_{0,k})_1\partial^k_{\bar\zeta_2}f_{1k}\ \ (k=1,2,\cdots), \tag{3.1}$$

$$\begin{aligned}A(\zeta)=\sum_{m_1=0}^{k_2}\sum_{m_2=0}^{k_1}C_{k_2}^{m_1}C_{k_1}^{m_2}\{&\partial^{m_2}_{\bar\zeta_1}f_{2m_1}-[(T_{0,k-m_2})_1\partial^{m_1}_{\bar\zeta_2}f_{1k}+(T_{0,k-m_1})_2\partial^{m_2}_{\bar\zeta_1}f_{2k}\\ &-(T_{0,k-m_1})_2(T_{0,k-m_2})_1\partial^k_{\bar\zeta_2}f_{1k}]\}\partial^{k_1-m_2}_{\bar\zeta_1}\bar\zeta_1^{p_1}\partial^{k_2-m_1}_{\bar\zeta_2}\bar\zeta_2^{p_2}\ \ (k_1,k_2=0,1,\cdots,k-1),\end{aligned} \tag{3.2}$$

with

$$\begin{cases} (T_{0,k})_i f(z_i) = \frac{-1}{\pi}\int_{D_i}\frac{1}{(k-1)!}\frac{\overline{(z_i-\zeta_i)}^{k-1}}{\zeta_i-z_i}f(\zeta_i)d\sigma_{\zeta_i} \ \ (i=1,2),\\ (T_{-1,k})_2 f(z_2) = \frac{-1}{\pi}\int_{D_2}\frac{1}{(k-1)!}\frac{\overline{(z_2-\zeta_2)}^{k-1}}{(z_2-\zeta_2)^2}f(\zeta_2)d\sigma_{\zeta_2}, \end{cases} \tag{3.3}$$

where $D_1 = D_2 = D$. Then Problem RH on D^2 is solvable, and the solution is as follows:

(i) In the case of $p_1, p_2 \geq 0$,

$$W(z) = z_1^{p_1}z_2^{p_2}\widetilde{\varphi}(z) + \sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{(v_1-p_1)(v_2-p_2)s_1l_1s_2l_2}z_1^{v_1}z_2^{v_2}\bar{z}_1^{\,l_1}\bar{z}_2^{\,l_2} + W_0(z), \tag{3.4}$$

on the condition that

$$\begin{aligned}&\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} + \frac{2}{(k-1)!}\Re\Big\{\int_{D^2}\Big[\bar{\zeta}_1^{p_1}\bar{z}_2^{p_2}\partial_{\zeta_1}f_{2k}(\zeta)\overline{(z_2-\zeta_2)}^{k-1}\\ &+\zeta_2^{p_2}z_1^{p_1}\overline{\partial_{\zeta_2}f_{1k}(\zeta)}(z_1-\zeta_1)^{k-1}\Big]\frac{z_1}{1-z_1\bar{\zeta}_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\ &=\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2},\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\widetilde{\varphi}(z)=&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\ &-\frac{z_1^{p_1}z_2^{p_2}}{\pi(k-1)!}\Big[\int_{D_1}\overline{f_{1k}(\zeta_1,z_2)}\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}d\sigma_{\zeta_1}+\int_{D_2}\overline{f_{2k}(z_1,\zeta_2)}\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2}\Big]\\ &-\frac{1}{\pi^2(k-1)!}\int_{D_1}\int_{D_2}\Big[z_1^{p_1}\zeta_2^{p_2}\overline{\partial_{\zeta_2}f_{1k}(\zeta)}\frac{z_1(z_1-\zeta_1)^{k-1}}{(1-z_1\bar{\zeta}_1)(\zeta_2-z_2)}+\zeta_1^{p_1}z_2^{p_2}\overline{\partial_{\zeta_1}f_{2k}(\zeta)}\frac{z_2(z_2-\zeta_2)^{k-1}}{(1-z_2\bar{\zeta}_2)(\zeta_1-z_1)}\\ &-\frac{z_1^{p_1}z_2^{p_2}}{(k-1)!}\overline{\partial_{\bar{\zeta}_2}^{k}f_{1k}(\zeta)}\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\ &+\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\ &+\frac{1}{(2\pi)^2 i}\int_{\partial_0 D^2}\Im\Big[\sum_{k_1,k_2=0}^{k-1}\frac{\overline{(z_1-\zeta_1)}^{k_1}\overline{(z_2-\zeta_2)}^{k_2}-(-\bar{\zeta}_1)^{k_1}(-\bar{\zeta}_2)^{k_2}}{k_1!k_2!}A(\zeta)\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2},\end{aligned} \tag{3.6}$$

and $\alpha_{(v_1-p_1)(v_2-p_2)s_1l_1s_2l_2}$ are arbitrary complex constants satisfying

$$\begin{cases}\sum_{l_2=\gamma}^{k-1}\sum_{s_2=l_2}^{k-1}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(-p_2-\gamma+l_2)s_1l_1s_2l_2}=0 \ \ (v_1\geq p_1+k, \gamma=1,2,\cdots,k-1),\\ \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2}=0 \ \ (v_1\geq p_1+k, v_2\geq -p_2),\end{cases} \tag{3.7}$$

$$\begin{cases} \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2} = 0 \ \ (-p_1 \le v_1 \le p_1, v_2 \ge p_2+k), \\ \sum_{s_1,s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{l_1=0}^{s_1}\left[\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2}+\overline{\alpha_{(l_1-v_1)(l_2-v_2)s_1l_1s_2l_2}}\right]=0 \ \ (-p_2\le v_2\le p_2), \\ \sum_{s_1,s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(p_2+1+t+l_2)s_1l_1s_2l_2}+\sum_{s_2=t+1}^{k-1}\sum_{l_2=t+1}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\overline{\alpha_{(l_1-v_1)(l_2-p_2-1-t)s_1l_1s_2l_2}}=0, \end{cases} \tag{3.8}$$

$$\begin{cases} \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=t+1}^{k-1}\sum_{l_1=t+1}^{s_1}\overline{\alpha_{(l_1-p_1-1-t)(l_2-v_2)s_1l_1s_2l_2}} = 0 \ \ (v_2 \le -p_2-k), \\ \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(p_1+1+t+l_1)(v_2+l_2)s_1l_1s_2l_2} = 0 \ \ (v_2 \ge p_2+k), \\ \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\left[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(p_1+1+t+l_1)(v_2+l_2)s_1l_1s_2l_2}+\sum_{s_1=t+1}^{k-1}\sum_{l_1=t+1}^{s_1}\overline{\alpha_{(l_1-p_1-1-t)(l_2-v_2)s_1l_1s_2l_2}}\right]=0(-p_2\le v_2\le p_2), \\ \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(p_1+1+t+l_1)(p_2+1+t+l_2)s_1l_1s_2l_2}+\sum_{s_2=t+1}^{k-1}\sum_{l_2=t+1}^{s_2}\sum_{s_1=t+1}^{k-1}\sum_{l_1=t+1}^{s_1}\overline{\alpha_{(l_1-p_1-1-t)(l_2-p_2-1-t)s_1l_1s_2l_2}}=0, \\ \sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{s_1=t+1}^{k-1}\sum_{l_1=t+1}^{s_1}\overline{\alpha_{(l_1-p_1-1-t)(l_2-p_1-1-t)s_1l_1s_2l_2}}+\sum_{s_2=t+1}^{k-1}\sum_{l_2=t+1}^{s_2}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(p_1+1+t+l_1)(l_2-p_2-1-t)s_1l_1s_2l_2}=0, \end{cases} \tag{3.9}$$

where $-p_1 \le v_1 \le p_1$ in (3.8) and $t = 0, 1, \cdots, k-2$ ($k \ge 2$).

(ii) In the case of $p_1 < 0$ and $p_2 \ge 0$,

$$W(z) = z_1^{p_1} z_2^{p_2}\widetilde{\varphi}(z)|_{p_1=0}+\sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{v_1(v_2-p_2)s_1l_1s_2l_2}z_1^{v_1+p_1}z_2^{v_2}\bar{z}_1^{\,l_1}\bar{z}_2^{\,l_2} + W_0(z), \tag{3.10}$$

($\widetilde{\varphi}(z)$ and $\alpha_{v_1(v_2-p_2)s_1l_1s_2l_2}$ are the same as in (i)) on the condition that

$$\begin{aligned} &\frac{2}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\ &=\Re\Big\{\frac{1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar{\zeta}_2}[\bar{\zeta}_2^{p_2}f_{1k}(\zeta)]\frac{\bar{z}_2}{\bar{z}_2\zeta_2-1}\Big\{z_1^{-p_1}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}+g_1(\zeta_1,z_1)-g_1(\zeta_1,0)\Big\}d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &+\frac{z_1^{-p_1}\bar{z}_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}f_{2k}(z_1,\zeta_2)\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_2} \\ &+\frac{\bar{z}_2^{p_2+1}}{\pi^2[(k-1)!]^2}\int_{D^2}\partial^k_{\bar{\zeta}_2}f_{1k}(\zeta)\Big[z_1^{-p_1}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}-g_1(\zeta_1,z_1)+g_1(\zeta_1,0)\Big]\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &-\frac{\bar{z}_2^{p_2+1}}{\pi^2(k-1)!}\int_{D^2}\frac{z_1\zeta_1^{-p_1-1}}{\zeta_1-z_1}\partial_{\bar{\zeta}_1}f_{2k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar{z}_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &+\frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}[(T_{0,k-1})_1\partial_{\zeta_2}f_{1k}+(T_{-1,k})_2\partial_{\bar{\zeta}_1}f_{2k}-(T_{-1,k})_2(T_{0,k-1})_1\partial^k_{\bar{\zeta}_2}f_{1k}](\zeta)\frac{\bar{z}_1 z_2 d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_1\zeta_1)(1-z_2\bar{\zeta}_2)}\Big\} \end{aligned} \tag{3.11}$$

$$+\frac{2}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2},$$

where

$$g_1(\zeta_1', z_1) = \frac{1}{k(-p_1-1)!}\Big\{\partial_{\bar\zeta_1}^{-p_1-1}\Big[\frac{(\overline{\zeta_1-\zeta_1'})^k}{(\bar\zeta_1\zeta_1'-1)(1-z_1\bar\zeta_1)}\Big]\Big\}_{\zeta_1=0}, \tag{3.12}$$

and for $l=0,1,\cdots,-p_1-1$,

$$\int_{\partial D_1}\Re\{\zeta_1^{-p_1}\zeta_2^{-p_2}[f_{1(k-1)}-(T_{0,1})_1f_{1k}-(T_{0,k})_2\partial_{\bar\zeta_1}^{k-1}f_{2k}+(T_{0,k})_2(T_{0,1})_1\partial_{\bar\zeta_2}^{k}f_{1k}](\zeta)\}\frac{d\zeta_1}{\zeta_1^{l+1}}=0. \tag{3.13}$$

(iii) In the case of $p_1, p_2 < 0$,

$$W(z)=z_1^{p_1}z_2^{p_2}\widetilde{\varphi}(z)|_{p_1=p_2=0}+\sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{v_1v_2s_1l_1s_2l_2}z_1^{v_1+p_1}z_2^{v_2+p_2}\bar z_1^{\,l_1}\bar z_2^{\,l_2}+W_0(z) \tag{3.14}$$

($\widetilde{\varphi}(z)$ and $\alpha_{v_1v_2s_1l_1s_2l_2}$ are the same as in (i)) on the condition that

$$\begin{aligned}&\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar z_2}{(\zeta_1-z_1)(\overline{\zeta_2-z_2})}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=-2\Re\Big\{\int_{D^2}\zeta_1^{-p_1-1}\zeta_2^{-p_2-1}\partial_{\bar\zeta_2}\partial_{\bar\zeta_1}W_0(\zeta)\Big[\frac{\zeta_2 z_1\bar z_2}{(\zeta_1-z_1)(1-\bar z_2\zeta_2)}+\frac{\zeta_1\bar z_1 z_2}{(1-\bar z_1\zeta_1)(\zeta_2-z_2)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&+\pi z_1^{-p_1}\bar z_2\int_{D_2}\zeta_2^{-p_2}\frac{\partial_{\bar\zeta_2}W_0(z_1,\zeta_2)}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}+\pi\bar z_1 z_2^{-p_2}\int_{D_1}\zeta_1^{-p_1}\frac{\partial_{\bar\zeta_1}W_0(\zeta_1,z_2)}{1-\bar z_1\zeta_1}d\sigma_{\zeta_1}\Big\}\\
&=\Re\Big\{\int_{D^2}\zeta_1^{-p_1-1}\zeta_2^{-p_2-1}[(T_{0k-1})_1\partial_{\bar\zeta_2}f_{1k}+(T_{0,k-1})_2\partial_{\bar\zeta_1}f_{2k}-(T_{0,k-1})_2(T_{0,k-1})_1\partial_{\bar\zeta_2}^{k}f_{1k}](\zeta)\Re\frac{-4z_1\bar z_2}{(\zeta_1-z_1)(\overline{\zeta_2-z_2})}\cdot d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\
&+\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\end{aligned} \tag{3.15}$$

and

$$\begin{cases}\int_{\partial D_1}\Re\{\zeta_1^{-p_1}\zeta_2^{-p_2}[f_{1(k-1)}-(T_{0,1})_1f_{1k}-(T_{0,k})_2\partial_{\bar\zeta_1}^{k-1}f_{2k}+(T_{0,k})_2(T_{0,1})_1\partial_{\bar\zeta_2}^{k}f_{1k}](\zeta)\}\frac{d\zeta_1}{\zeta_1^{l+1}}=0,\\ \int_{\partial D_2}\Re\{\zeta_1^{-p_1}\zeta_2^{-p_2}[f_{2(k-1)}-(T_{0,k})_1\partial_{\bar\zeta_2}^{k-1}f_{1k}-(T_{0,1})_2f_{2k}+(T_{0,1})_2(T_{0,k})_1\partial_{\bar\zeta_2}^{k}f_{1k}](\zeta)\}\frac{d\zeta_2}{\zeta_2^{r+1}}=0\end{cases} \tag{3.16}$$

for $l=0,1,\cdots,-p_1-1$ and $r=0,1,\cdots,-p_2-1$.

Proof. For

$$T_{0,k}f(z)=\frac{-1}{\pi}\int_D\frac{1}{(k-1)!}\frac{(\overline{z-\zeta})^{k-1}}{\zeta-z}f(\zeta)d\sigma_\zeta, \tag{3.17}$$

by Lemma 2.3,

$$\partial_{\bar z}^{k}(T_{0,k}f(z))=T_{0,0}f(z)=f(z). \tag{3.18}$$

Applying (3.18) and $\partial_{\bar{z}_2}^{k_2} f_{1k_1} = \partial_{\bar{z}_1}^{k_1} f_{2k_2}$, we get that

$$\begin{aligned}\partial_{\bar{z}_1}^{k} W_0(z) &= f_{1k}(z) + \frac{-1}{\pi}\int_{D_2}\frac{1}{(k-1)!}\frac{(\overline{z_2-\zeta_2})^{k-1}}{\zeta_2-z_2}\partial_{\bar{z}_1}^{k} f_{2k}(z_1,\zeta_2)d\sigma_{\zeta_2}\\ &-\frac{-1}{\pi}\int_{D_2}\frac{1}{(k-1)!}\frac{(\overline{z_2-\zeta_2})^{k-1}}{\zeta_2-z_2}\partial_{\bar{z}_1}^{k}\Big[\frac{-1}{\pi(k-1)!}\int_{D_1}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}\partial_{\bar{\zeta}_2}^{k} f_{1k}(\zeta_1,\zeta_2)d\sigma_{\zeta_1}\Big]d\sigma_{\zeta_2}\\ &= f_{1k}(z).\end{aligned}\tag{3.19}$$

Similarly, $\partial_{\bar{z}_2}^{k} W_0(z) = f_{2k}(z)$, and thus $W_0(z)$ is a particular solution to $\partial_{\bar{z}_l}^{k} W(z) = f_{lk}(l = 1,2)$. The general solution is $W(z) = \varphi(z) + W_0(z)$, where $\varphi(z)$ is k-holomorphic on D^2, and

$$\Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)] = \Re[\{\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}[W(\zeta) - W_0(\zeta)]\} = \gamma(\zeta) - \Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2} W_0(\zeta)] \doteq \gamma_0(\zeta).$$

(i) In the case of $p_1, p_2 \geq 0$, let $z_1^{-p_1} z_2^{-p_2}\varphi(z) = \widetilde{\varphi}(z)$, and then $\varphi(z)$ is k-holomorphic if $\widetilde{\varphi}(z)$ is k-holomorphic. By Lemma 2.2, $z_1^{p_1} z_2^{p_2}\widetilde{\varphi}(z)$ is a particular solution to $\Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)] = \gamma_0(\zeta)$, where

$$\begin{aligned}\widetilde{\varphi}(z) =& \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma_0(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - 1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\ &+\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}\partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2}(\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta))\Big]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - 1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\ &+\frac{1}{(2\pi)^2 i}\int_{\partial_0 D^2}\Im\Big[\sum_{k_1,k_2=0}^{k-1}\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}-(-\bar{\zeta}_1)^{k_1}(-\bar{\zeta}_2)^{k_2}}{k_1!k_2!}\partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2}(\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta))\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2},\end{aligned}\tag{3.20}$$

on the condition of (2.4) in which γ is replaced by $\gamma_0 = \gamma(\zeta) - \Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2} W_0(\zeta)]$ and $\partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2} W(\zeta)$ is replaced by $\partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2}[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}(W - W_0)(\zeta)]$, that is,

$$\begin{aligned}&\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} - \Re\int_{\partial_0 D^2}[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2} W_0(\zeta)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{W_0(\zeta)}]\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{2\zeta_1\zeta_2}\\ &= \int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} - \frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}A(\zeta) &= \partial_{\bar{\zeta}_1}^{k_1}\partial_{\bar{\zeta}_2}^{k_2}[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}(W - W_0)(\zeta)] = \partial_{\bar{\zeta}_1}^{k_1}\{\bar{\zeta}_1^{\,p_1}[\sum_{m_1=0}^{k_2} C_{k_2}^{m_1}\partial_{\bar{\zeta}_2}^{m_1}(W - W_0)\partial_{\bar{\zeta}_2}^{k_2-m_1}\bar{\zeta}_2^{\,p_2}]\}\\ &= \partial_{\bar{\zeta}_1}^{k_1}\{\bar{\zeta}_1^{\,p_1}[\sum_{m_1=0}^{k_2} C_{k_2}^{m_1}(f_{2m_1}-\partial_{\bar{\zeta}_2}^{m_1} W_0)\partial_{\bar{\zeta}_2}^{k_2-m_1}\bar{\zeta}_2^{\,p_2}]\} = \sum_{m_1=0}^{k_2} C_{k_2}^{m_1}\{\partial_{\bar{\zeta}_1}^{k_1}[\bar{\zeta}_1^{\,p_1}(f_{2m_1}-\partial_{\bar{\zeta}_2}^{m_1} W_0)]\}\partial_{\bar{\zeta}_2}^{k_2-m_1}\bar{\zeta}_2^{\,p_2}\\ &= \sum_{m_1=0}^{k_2} C_{k_2}^{m_1}\{\sum_{m_2=0}^{k_1} C_{k_1}^{m_2}(\partial_{\bar{\zeta}_1}^{m_2} f_{2m_1} - \partial_{\bar{\zeta}_1}^{m_2}\partial_{\bar{\zeta}_2}^{m_1} W_0)\partial_{\bar{\zeta}_1}^{k_1-m_2}\bar{\zeta}_1^{p_1}\}\partial_{\bar{\zeta}_2}^{k_2-m_1}\bar{\zeta}_2^{\,p_2}\end{aligned}\tag{3.22}$$

$$
\begin{aligned}
&= \sum_{m_1=0}^{k_2}\sum_{m_2=0}^{k_1} C_{k_2}^{m_1} C_{k_1}^{m_2} \partial_{\bar\zeta_2}^{k_2-m_1}\bar\zeta_2^{p_2}\partial_{\bar\zeta_1}^{k_1-m_2}\bar\zeta_1^{p_1}\cdot \partial_{\bar\zeta_1}^{m_2}[f_{2m_1}(\zeta)-\partial_{\bar\zeta_2}^{m_1}W_0(\zeta)]\\
&= \sum_{m_1=0}^{k_2}\sum_{m_2=0}^{k_1} C_{k_2}^{m_1} C_{k_1}^{m_2} \partial_{\bar\zeta_2}^{k_2-m_1}\bar\zeta_2^{p_2}\partial_{\bar\zeta_1}^{k_1-m_2}\bar\zeta_1^{p_1}\\
&\quad \cdot \partial_{\bar\zeta_1}^{m_2}\{f_{2m_1}(\zeta)-[(T_{0,k})_1\partial_{\bar\zeta_2}^{m_1}f_{1k}(\zeta)+(T_{0,k-m_1})_2f_{2k}(\zeta)-(T_{0,k-m_1})_2(T_{0,k})_1\partial_{\bar\zeta_2}^{k}f_{1k}(\zeta)]\}\\
&= \sum_{m_1=0}^{k_2}\sum_{m_2=0}^{k_1} C_{k_2}^{m_1} C_{k_1}^{m_2} \partial_{\bar\zeta_2}^{k_2-m_1}\bar\zeta_2^{p_2}\partial_{\bar\zeta_1}^{k_1-m_2}\bar\zeta_1^{p_1}\cdot[\partial_{\bar\zeta_1}^{m_2}f_{2m_1}(\zeta)-(T_{0,k-m_2})_1\partial_{\bar\zeta_2}^{m_1}f_{1k}(\zeta)\\
&\quad -(T_{0,k-m_1})_2\partial_{\bar\zeta_1}^{m_2}f_{2k}(\zeta)+(T_{0,k-m_1})_2(T_{0,k-m_2})_1\partial_{\bar\zeta_2}^{k}f_{1k}(\zeta)].
\end{aligned}
$$

① First, we simplify (3.20), in which

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma_0(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&-\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}[\bar\zeta_1^{p_1}\bar\zeta_2^{p_2}W_0(\zeta)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{W_0(\zeta)}]\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}
\tag{3.23}
$$

In addition, (3.17) leads to

$$
\frac{1}{2\pi i}\int_{\partial D}\zeta^{-p}T_{0,k}f_k(\zeta)\frac{d\zeta}{\zeta-z}=\frac{1}{(k-1)!}\frac{-1}{\pi}\int_D f_k(\zeta')\Big[\frac{-1}{2\pi i}\int_{\partial D}\bar\zeta^{p}\frac{(\overline{\zeta-\zeta'})^{k-1}}{\bar\zeta\zeta'-1}\frac{d\bar\zeta}{z\bar\zeta-1}\Big]d\sigma_{\zeta'}=0,
\tag{3.24}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D}\zeta^{p}\overline{T_{0,k}f_k(\zeta)}\frac{d\zeta}{\zeta-z}=\frac{1}{(k-1)!}\frac{-1}{\pi}\int_D\overline{f_k(\zeta')}\Big[\frac{1}{2\pi i}\int_{\partial D}\zeta^{p}\frac{(\zeta-\zeta')^{k-1}}{\bar\zeta'-\bar\zeta}\frac{d\zeta}{\zeta-z}\Big]d\sigma_{\zeta'}\\
&=\frac{1}{(k-1)!}\frac{-1}{\pi}\int_D\overline{f_k(\zeta')}\frac{z^{p+1}(z-\zeta')^{k-1}}{z\bar\zeta'-1}d\sigma_{\zeta'}=\frac{z^p}{(k-1)!}\frac{-1}{\pi}\int_D\overline{f_k(\zeta)}\frac{z(z-\zeta)^{k-1}}{z\bar\zeta-1}d\sigma_{\zeta}.
\end{aligned}
\tag{3.25}
$$

According to (3.24), we have that

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar\zeta_1^{p_1}\bar\zeta_2^{p_2}W_0(\zeta)\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar\zeta_1^{p_1}\bar\zeta_2^{p_2}[(T_{0,k})_1f_{1k}+(T_{0,k})_2f_{2k}-(T_{0,k})_2(T_{0,k})_1\partial_{\bar\zeta_2}^{k}f_{1k}](\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&=\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\bar\zeta_1^{p_1}(T_{0,k})_1f_{1k}(\zeta)\frac{d\zeta_1}{\zeta_1-z_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\bar\zeta_1^{p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}(T_{0,k})_2f_{2k}(\zeta)\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}\\
&-\frac{1}{2\pi i}\int_{\partial D_1}\bar\zeta_1^{p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}(T_{0,k})_2(T_{0,k})_1\partial_{\bar\zeta_2}^{k}f_{1k}(\zeta)\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}\\
&=0,
\end{aligned}
\tag{3.26}
$$

and it follows that

$$\begin{cases} \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2} W_0(\zeta)\frac{1}{2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} = 0, \\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \zeta_1^{p_1}\zeta_2^{p_2} \overline{W_0(\zeta)}\frac{1}{2}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} = 0. \end{cases} \tag{3.27}$$

Moreover, according to (3.25), we have that

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\zeta_1^{p_1}\zeta_2^{p_2}\overline{W_0(\zeta)}\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\zeta_1^{p_1}\zeta_2^{p_2}[\overline{(T_{0,k})_1 f_{1k}} + \overline{(T_{0,k})_2 f_{2k}} - \overline{(T_{0,k})_2(T_{0,k})_1\partial^k_{\bar{\zeta}_2} f_{1k}}](\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)} \\
&= \frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\overline{(T_{0,k})_1 f_{1k}(\zeta)}\frac{d\zeta_1}{\zeta_1-z_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2} \\
&+ \frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\overline{(T_{0,k})_2 f_{2k}(\zeta)}\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1} \\
&- \frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\overline{(T_{0,k})_2(T_{0,k})_1\partial^k_{\bar{\zeta}_2} f_{1k}(\zeta)}\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1} \\
&= \frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\Big[\frac{z_1^{p_1}}{(k-1)!}\frac{-1}{\pi}\int_{D_1}\overline{f_{1k}(\zeta)}\frac{z_1(z_1-\zeta_1)^{k-1}}{z_1\bar{\zeta}_1-1}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2} \\
&+ \frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{z_2^{p_2}}{(k-1)!}\frac{-1}{\pi}\int_{D_2}\overline{f_{2k}(\zeta)}\frac{z_2(z_2-\zeta_2)^{k-1}}{z_2\bar{\zeta}_2-1}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1} \\
&- \frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{z_2^{p_2}}{(k-1)!}\frac{-1}{\pi}\int_{D_2}\overline{(T_{0,k})_1\partial^k_{\bar{\zeta}_2} f_{1k}(\zeta)}\frac{z_2(z_2-\zeta_2)^{k-1}}{z_2\bar{\zeta}_2-1}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}.
\end{aligned} \tag{3.28}$$

Applying the Cauchy-Pompeiu formula (2.1) on D_2, we get that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\Big[\frac{z_1^{p_1}}{(k-1)!}\frac{-1}{\pi}\int_{D_1}\overline{f_{1k}(\zeta)}\frac{z_1(z_1-\zeta_1)^{k-1}}{z_1\bar{\zeta}_1-1}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2} \\
&= \frac{z_1^{p_1}}{(k-1)!}\frac{1}{\pi}\int_{D_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{p_2}\overline{f_{1k}(\zeta)}\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}d\sigma_{\zeta_1} \\
&= \frac{z_1^{p_1}}{(k-1)!}\frac{1}{\pi}\Big[z_2^{p_2}\overline{f_{1k}(\zeta_1,z_2)} + \frac{1}{\pi}\int_{D_2}\zeta_2^{p_2}\overline{\partial_{\zeta_2} f_{1k}(\zeta)}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\Big]\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}d\sigma_{\zeta_1} \\
&= \frac{z_1^{p_1}z_2^{p_2}}{\pi(k-1)!}\int_{D_1}\overline{f_{1k}(\zeta_1,z_2)}\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}d\sigma_{\zeta_1} \\
&+ \frac{z_1^{p_1}}{\pi^2(k-1)!}\int_{D_1}\int_{D_2}\zeta_2^{p_2}\overline{\partial_{\zeta_2} f_{1k}(\zeta)}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}d\sigma_{\zeta_1}.
\end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{z_2^{p_2}}{(k-1)!}\frac{-1}{\pi}\int_{D_2}\overline{f_{2k}(\zeta)}\frac{z_2(z_2-\zeta_2)^{k-1}}{z_2\bar{\zeta}_2-1}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1} \\
&= \frac{z_1^{p_1}z_2^{p_2}}{\pi(k-1)!}\int_{D_2}\overline{f_{2k}(z_1,\zeta_2)}\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2} \\
&+ \frac{z_2^{p_2}}{\pi^2(k-1)!}\int_{D_1}\int_{D_2}\zeta_1^{p_1}\overline{\partial_{\zeta_1} f_{2k}(\zeta)}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2}.
\end{aligned} \tag{3.30}$$

By (3.25), we obtain that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\Big[\frac{z_2^{p_2}}{(k-1)!}\frac{-1}{\pi}\int_{D_2}\overline{(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}(\zeta)\frac{z_2(z_2-\zeta_2)^{k-1}}{z_2\bar{\zeta}_2-1}}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}\\
&=\frac{z_2^{p_2}}{\pi(k-1)!}\int_{D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{p_1}\overline{(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}(\zeta)}\frac{d\zeta_1}{\zeta_1-z_1}\Big]\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2}\\
&=\frac{z_2^{p_2}}{\pi(k-1)!}\int_{D_2}\Big[\frac{z_1^{p_1}}{(k-1)!}\frac{-1}{\pi}\int_{D_1}\overline{\partial_{\bar{\zeta}_2}^k f_{1k}(\zeta)\frac{z_1(z_1-\zeta_1)^{k-1}}{z_1\bar{\zeta}_1-1}}d\sigma_{\zeta_1}\Big]\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2}\\
&=\frac{z_1^{p_1}z_2^{p_2}}{\pi^2[(k-1)!]^2}\int_{D_1}\int_{D_2}\overline{\partial_{\bar{\zeta}_2}^k f_{1k}(\zeta)}\frac{z_1(z_1-\zeta_1)^{k-1}}{1-z_1\bar{\zeta}_1}\frac{z_2(z_2-\zeta_2)^{k-1}}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}.
\end{aligned}\tag{3.31}$$

Plugging (3.23)–(3.31) into (3.20), we get (3.6).

② Second, we simplify (3.21), in which

$$\begin{aligned}
&\int_{\partial_0 D^2}[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{W_0(\zeta)}]\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{2\zeta_1\zeta_2}\\
&=\int_{\partial_0 D^2}\{\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}[(T_{0,k})_1 f_{1k}+(T_{0,k})_2 f_{2k}-(T_{0,k})_2(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}](\zeta)\\
&+\zeta_1^{p_1}\zeta_2^{p_2}[\overline{(T_{0,k})_1 f_{1k}(\zeta)}+\overline{(T_{0,k})_2 f_{2k}(\zeta)}-\overline{(T_{0,k})_2(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}(\zeta)}]\}\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{2\zeta_1\zeta_2}.
\end{aligned}\tag{3.32}$$

Applying (3.24), (3.17), and (2.2), the terms on the right side of Eq (3.32) can be simplified. Substituting the simplified results into (3.21) yields (3.5).

③ If $\varphi_3(z)$ is k-holomorphic on D^2 with $\Re[\zeta_1^{-p_1}\zeta_2^{-p_2}\varphi_3(\zeta)]=0$, then $z_1^{p_1}z_2^{p_2}\widetilde{\varphi}(z)+\varphi_3(z)$ is the general solution to $\Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)]=\gamma_0(\zeta)$. In the following, we seek $\varphi_3(z)$. As $\varphi_3(z)$ is k-holomorphic on D^2, then

$$\varphi_3(z)=\sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{(v_1-p_1)(v_2-p_2)s_1l_1s_2l_2}z_1^{v_1}z_2^{v_2}\bar{z}_1^{l_1}\bar{z}_2^{l_2},$$

where $\alpha_{(v_1-p_1)(v_2-p_2)s_1l_1s_2l_2}$ are arbitrary complex constants. Similarly, for the k-holomorphic function $\widetilde{\varphi_3}(z)$ on D, it can be expressed as

$$\widetilde{\varphi_3}(z)=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=0}^{+\infty}\alpha_{sl(v-p)}z^v\bar{z}^l,$$

where $\alpha_{sl(v-p)}$ are arbitrary complex constants. Denote $\alpha_{slv}+\overline{\alpha_{sl(2l-v)}}\doteq A_{slv}$, $\alpha_{slv}\doteq B_{slv}$, $\overline{\alpha_{sl(2l-v)}}\doteq C_{slv}$. As $\sum_{l=0}^{k-1}\sum_{s=l}^{k-1}C_{slv}=\sum_{s=0}^{k-1}\sum_{l=0}^{s}C_{slv}$, then

$$\begin{aligned}
0=\Re\{\bar{\zeta}^p\widetilde{\varphi_3}(\zeta)\}&=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-p}^{+\infty}(\alpha_{slv}\zeta^{v-l}+\overline{\alpha_{slv}}\zeta^{l-v})\\
&=\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-p}^{p+2l}A_{slv}\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=p+2l+1}^{+\infty}B_{slv}\zeta^{v-l}+\sum_{s=0}^{k-1}\sum_{l=0}^{s}\sum_{v=-\infty}^{-p-1}C_{slv}\zeta^{v-l}
\end{aligned}\tag{3.33}$$

$$
\begin{aligned}
&= \Big\{ \sum_{v=-p}^{p} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} A_{sl(v+l)}\Big] + \zeta^{-p-1} \sum_{l=1}^{k-1} \sum_{s=l}^{k-1} A_{sl(-p-1+l)} + \cdots + \zeta^{-p-k+1} A_{(k-1)(k-l)(-p)} \\
&+ \zeta^{p+1} \sum_{l=1}^{k-1} \sum_{s=l}^{k-1} A_{sl(p+1+l)} + \zeta^{p+2} \sum_{l=2}^{k-1} \sum_{s=l}^{k-1} A_{sl(p+2+l)} + \cdots + \zeta^{p+k-1} A_{(k-1)(k-1)(p+2(k-1))}\Big\} \\
&+ \Big\{\zeta^{p+1} \sum_{s=0}^{k-1} B_{s0(p+1)} + \zeta^{p+2}\Big[\sum_{s=0}^{k-1} B_{s0(p+2)} + \sum_{s=1}^{k-1} B_{s1(p+3)}\Big] \\
&+ \cdots + \zeta^{p+k-1} \sum_{l=0}^{k-2} \sum_{s=l}^{k-1} B_{sl(p+k-1+l)} + \sum_{v=p+k}^{+\infty} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} B_{sl(v+l)}\Big]\Big\} \\
&+ \Big\{\zeta^{-p-1} \sum_{s=0}^{k-1} C_{s0(-p-1)} + \zeta^{-p-2}\Big[\sum_{s=0}^{k-1} C_{s0(-p-2)} + \sum_{s=1}^{k-1} C_{s1(-p-1)}\Big] \\
&+ \cdots + \zeta^{-p-k+1} \sum_{l=0}^{k-2} \sum_{s=l}^{k-1} C_{sl(-p-k+1+l)} + \sum_{v=-\infty}^{-p-k} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} C_{sl(v+l)}\Big]\Big\} \\
&= \sum_{v=-\infty}^{-p-k} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} C_{sl(v+l)}\Big] + \zeta^{-p-k+1}\Big[\sum_{l=0}^{k-2} \sum_{s=l}^{k-1} C_{sl(-p-k+1+l)} + A_{(k-1)(k-l)(-p)}\Big] + \cdots \\
&+ \zeta^{-p-1}\Big[\sum_{s=0}^{k-1} C_{s0(-p-1)} + \sum_{l=1}^{k-1} \sum_{s=l}^{k-1} A_{sl(-p-1+l)}\Big] + \sum_{v=-p}^{p} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} A_{sl(v+l)}\Big] \\
&+ \zeta^{p+1}\Big[\sum_{s=0}^{k-1} B_{s0(p+1)} + \sum_{l=1}^{k-1} \sum_{s=l}^{k-1} A_{sl(p+1+l)}\Big] + \zeta^{p+k-1}\Big[\sum_{l=0}^{k-2} \sum_{s=l}^{k-1} B_{sl(p+k-1+l)} \\
&+ A_{(k-1)(k-1)(p+2(k-1))}\Big] + \sum_{v=p+k}^{+\infty} \zeta^{v}\Big[\sum_{l=0}^{k-1} \sum_{s=l}^{k-1} B_{sl(v+l)}\Big],
\end{aligned}
$$

which is equivalent to

$$
\begin{cases}
\displaystyle\sum_{s=0}^{k-1} \sum_{l=0}^{s} \alpha_{sl(v+l)} = 0 \ (v \geq p+k), \ \ \sum_{s=0}^{k-1} \sum_{l=0}^{s} [\alpha_{sl(v+l)} + \overline{\alpha_{sl(l-v)}}] = 0 \ (-p \leq v \leq p), \\
\displaystyle\sum_{s=0}^{k-1} \sum_{l=0}^{s} \alpha_{sl(p+1+t+l)} + \sum_{s=t+1}^{k-1} \sum_{l=t+1}^{s} \overline{\alpha_{sl(-p-1-t+l)}} = 0 \ \ (t = 0, 1, \cdots, k-2, \ k \geq 2),
\end{cases}
$$

and in the case of $k = 1$, the last equation in the above system is non-existent.

Therefore, for $\forall \zeta_1 \in \partial D$ and $\zeta_2 \in \partial D$ fixed,

$$
\begin{aligned}
0 = \Re[\zeta_1^{-p_1} \zeta_2^{-p_2} \varphi_3(\zeta)] = \sum_{s_1=0}^{k-1} \sum_{l_1=0}^{s_1} \sum_{v_1=-p_1}^{+\infty} \Big[\Big(\sum_{s_2=0}^{k-1} \sum_{l_2=0}^{s_2} \sum_{v_2=-p_2}^{+\infty} \alpha_{v_1 v_2 s_1 l_1 s_2 l_2} \zeta_2^{v_2-l_2}\Big) \zeta_1^{v_1-l_1} \\
+ \Big(\sum_{s_2=0}^{k-1} \sum_{l_2=0}^{s_2} \sum_{v_2=-p_2}^{+\infty} \overline{\alpha_{v_1 v_2 s_1 l_1 s_2 l_2}}\, \overline{\zeta}_2^{v_2-l_2}\Big) \zeta_1^{l_1-v_1}\Big]
\end{aligned}
$$

is equivalent to

$$\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\Big[\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2}^{+\infty}\alpha_{(v_1+l_1)v_2s_1l_1s_2l_2}\zeta_2^{v_2-l_2}\Big]=0\ \ (v_1\geq p_1+k), \tag{3.34}$$

$$\sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\Big\{\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2}^{+\infty}\Big[\alpha_{(v_1+l_1)v_2s_1l_1s_2l_2}\zeta_2^{v_2-l_2}+\overline{\alpha_{(l_1-v_1)v_2s_1l_1s_2l_2}}\,\bar{\zeta}_2^{v_2-l_2}\Big]\Big\}=0\ \ (-p_1\leq v_1\leq p_1), \tag{3.35}$$

$$\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2}^{+\infty}\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(p_1+1+t+l_1)v_2s_1l_1s_2l_2}\zeta_2^{v_2-l_2}+\sum_{s_1=t+1}^{k-1}\sum_{l_1=t+1}^{s_1}\overline{\alpha_{(l_1-p_1-1-t)v_2s_1l_1s_2l_2}}\,\bar{\zeta}_2^{v_2-l_2}\Big]=0, \tag{3.36}$$

where $t=0,1,\cdots,k-2$ $(k\geq 2)$, and (3.36) will disappear if $k=1$.

Furthermore, for $\forall\zeta_2\in\partial D$, $v_1\geq p_1+k$, and $\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\alpha_{s_2,l_2}=\sum_{l_2=0}^{k-1}\sum_{s_2=l_2}^{k-1}\alpha_{s_2,l_2}$, (3.34) is equivalent to

$$\begin{aligned}
0=&\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2-l_2}^{+\infty}\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2}\Big]\zeta_2^{v_2}\\
=&\Big[\sum_{l_2=1}^{k-1}\sum_{s_2=l_2}^{k-1}\sum_{v_2=-p_2-l_2}^{-p_2-1}+\sum_{v_2=-p_2}^{+\infty}\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\Big]\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2}\Big]\zeta_2^{v_2}\\
=&\sum_{l_2=1}^{k-1}\sum_{s_2=l_2}^{k-1}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(-p_2-1+l_2)s_1l_1s_2l_2}\zeta_2^{-p_2-1}+\sum_{l_2=2}^{k-1}\sum_{s_2=l_2}^{k-1}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(-p_2-2+l_2)s_1l_1s_2l_2}\zeta_2^{-p_2-2}\\
&+\cdots+\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(-p_2)s_1l_1s_2l_2}\zeta_2^{-p_2-(k-1)}+\sum_{v_2=-p_2}^{+\infty}\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)(v_2+l_2)s_1l_1s_2l_2}\Big]\zeta_2^{v_2}.
\end{aligned}$$

So we get (3.7) in which the first equation will disappear for $k=1$.

For $\forall\zeta_2\in\partial D$, similar to the discussion of (3.33), (3.35) is equivalent to

$$\begin{aligned}
0=&\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2}^{+\infty}\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)v_2s_1l_1s_2l_2}\zeta_2^{v_2-l_2}+\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\overline{\alpha_{(l_1-v_1)v_2s_1l_1s_2l_2}}\zeta_2^{l_2-v_2}\Big]\\
=&\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-p_2}^{p_2+2l_2}\Big[\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)v_2s_1l_1s_2l_2}+\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\overline{\alpha_{(l_1-v_1)(2l_2-v_2)s_1l_1s_2l_2}}\Big]\zeta_2^{v_2-l_2}\\
&+\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=p_2+2l_2+1}^{+\infty}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\alpha_{(v_1+l_1)v_2s_1l_1s_2l_2}\zeta_2^{v_2-l_2}\\
&+\sum_{s_2=0}^{k-1}\sum_{l_2=0}^{s_2}\sum_{v_2=-\infty}^{-p_2-1}\sum_{s_1=0}^{k-1}\sum_{l_1=0}^{s_1}\overline{\alpha_{(l_1-v_1)(2l_2-v_2)s_1l_1s_2l_2}}\zeta_2^{v_2-l_2},
\end{aligned}$$

which means (3.8) and the last equation in (3.8) will disappear if $k=1$. Similarly, (3.36) is equivalent to (3.9), and (3.9) will disappear for $k=1$.

Thus the general solution to $\Re[\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)]=\gamma_0(\zeta)$ is $\varphi(z)=z_1^{p_1}z_2^{p_2}\widetilde{\varphi}(z)+\varphi_3(z)$ in which the coefficients are arbitrary complex constants satisfying (3.7)–(3.9).

By ①, ②, and ③, (3.4) is the solution of Problem RH on the condition of (3.5).

(ii) In the case of $p_1 < 0$ and $p_2 \geq 0$, the boundary condition is $\Re\{\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)\} = \gamma_0(\zeta)$. Let $z_1^{-p_1}\varphi(z) = \varphi_1(z)$, and then

$$\Re\{\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}\varphi(\zeta)\} = \gamma_0(\zeta) \Leftrightarrow \Re\{\bar{\zeta}_2^{p_2}\varphi_1(\zeta)\} = \gamma_0(\zeta).$$

Since $p_1 < 0$ and $\varphi(z)$ is k-holomorphic, then $\varphi_1(z)$ is k-holomorphic, and therefore $\varphi_1(z)$ is the solution of the boundary value problem for k-holomorphic functions satisfying $\Re\{\bar{\zeta}_2^{p_2}\varphi_1(\zeta)\} = \gamma_0(\zeta)$. To get $\varphi(z)$, we need to find the solution to $\Re\{\bar{\zeta}_2^{p_2}\varphi_1(\zeta)\} = \gamma_0(\zeta)$, where $\varphi_1(z)$ is k-holomorphic, and ensure that $\varphi(z)$ is k-holomorphic at the same time.

① First, we seek the solution to $\Re\{\bar{\zeta}_2^{p_2}\varphi_1(\zeta)\} = \gamma_0(\zeta)$, which is just the case of $p_1 = 0$ and $p_2 \geq 0$ in (i). Therefore,

$$\varphi_1(z) = z_2^{p_2}\widetilde{\varphi}(z)|_{p_1=0} + \sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{v_1(v_2-p_2)s_1l_1s_2l_2}z_1^{v_1}z_2^{v_2}\bar{z}_1 \tag{3.37}$$

and the condition (3.21) leads to

$$\begin{aligned}
&\frac{2}{(2\pi i)^2}\int_{\partial_0 D^2}\gamma(\zeta)\Re\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&= \Re\Big\{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\Big[\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}} + \frac{\bar{z}_1 z_2}{\overline{(\zeta_1-z_1)}(\zeta_2-z_2)}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\Big\}\\
&+\frac{2}{(2\pi i)^2}\int_{\partial_0 D^2}\sum_{\substack{k_1,k_2=0\\ k_1^2+k_2^2\neq 0}}^{k-1}\Re\Big[\frac{(\overline{z_1-\zeta_1})^{k_1}(\overline{z_2-\zeta_2})^{k_2}}{k_1!k_2!}A(\zeta)\Big]\Re\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{1}{2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}.
\end{aligned}\tag{3.38}$$

By (2.1), (2.2), (3.1), and Lemma 2.3, we get that

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{z_1\bar{z}_2}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&= \frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2^{p_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{-p_1-1}W_0(\zeta)\frac{z_1}{\zeta_1-z_1}d\zeta_1\Big]\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\zeta_2\\
&= \frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2^{p_2}\Big[z_1^{-p_1}W_0(z_1,\zeta_2) + \frac{1}{\pi}\int_{D_1}\zeta_1^{-p_1-1}\partial_{\bar{\zeta}_1}W_0(\zeta)\frac{z_1}{\zeta_1-z_1}d\sigma_{\zeta_1}\Big]\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\zeta_2\\
&=\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar{\zeta}_2}[\bar{\zeta}_2^{p_2}W_0(z_1,\zeta_2)]\frac{\bar{z}_2 d\sigma_{\zeta_2}}{1-\bar{z}_2\zeta_2}+\frac{1}{\pi^2}\int_{D^2}\partial_{\bar{\zeta}_2}[\bar{\zeta}_2^{p_2}\partial_{\bar{\zeta}_1}W_0(\zeta)]\zeta_1^{-p_1-1}\frac{z_1}{\zeta_1-z_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar{\zeta}_2}\{\bar{\zeta}_2^{p_2}[(T_{0,k})_1 f_{1k} + (T_{0,k})_2 f_{2k} - (T_{0,k})_2(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}](z_1,\zeta_2)\}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_2}\\
&+\frac{1}{\pi^2}\int_{D^2}\partial_{\bar{\zeta}_2}\{\bar{\zeta}_2^{p_2}[(T_{0,k-1})_1 f_{1k}+(T_{0,k})_2\partial_{\bar{\zeta}_1}f_{2k}-(T_{0,k})_2(T_{0,k-1})_1\partial_{\bar{\zeta}_2}^k f_{1k}](\zeta)\}\zeta_1^{-p_1-1}\frac{z_1 d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar{z}_2 d\sigma_{\zeta_2}}{1-\bar{z}_2\zeta_2}\\
&=\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar{\zeta}_2}\{\bar{\zeta}_2^{p_2}[(T_{0,k})_1 f_{1k} + (T_{0,k})_2 f_{2k} - (T_{0,k})_2(T_{0,k})_1\partial_{\bar{\zeta}_2}^k f_{1k}](z_1,\zeta_2)\}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_2}\\
&+\frac{1}{\pi^2}\int_{D^2}\partial_{\bar{\zeta}_2}\{\bar{\zeta}_2^{p_2}[(T_{0,k-1})_1 f_{1k}+(T_{0,k})_2\partial_{\bar{\zeta}_1}f_{2k}-(T_{0,k})_2(T_{0,k-1})_1\partial_{\bar{\zeta}_2}^k f_{1k}](\zeta)\}\zeta_1^{-p_1}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar{z}_2 d\sigma_{\zeta_2}}{1-\bar{z}_2\zeta_2}
\end{aligned}\tag{3.39}$$

$$-\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}\{\bar\zeta_2^{p_2}[(T_{0,k-1})_1 f_{1k}+(T_{0,k})_2\partial_{\bar\zeta_1}f_{2k}-(T_{0,k})_2(T_{0,k-1})_1\partial^k_{\bar\zeta_2}f_{1k}](\zeta)\}\zeta_1^{-p_1-1}d\sigma_{\zeta_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2},$$

where

$$\begin{aligned}
&\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_1 f_{1k}(z_1,\zeta_2)]\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&=\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar\zeta_2}\Big[\frac{-1}{\pi}\int_{D_1}\frac{\bar\zeta_2^{p_2}}{(k-1)!}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}f_{1k}(\zeta)d\sigma_{\zeta_1}\Big]\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&=\frac{-z_1^{-p_1}}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}f_{1k}(\zeta)]\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2},
\end{aligned}\tag{3.40}$$

$$\begin{aligned}
&\frac{z_1^{-p_1}}{\pi}\int_{D_2}\partial_{\bar\zeta_2}\{\bar\zeta_2^{p_2}[(T_{0,k})_2 f_{2k}-(T_{0,k})_2(T_{0,k})_1\partial^k_{\bar\zeta_2}f_{1k}](z_1,\zeta_2)\}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&=\frac{z_1^{-p_1}}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}(T_{0,k})_2[f_{2k}-(T_{0,k})_1\partial^k_{\bar\zeta_2}f_{1k}](z_1,\zeta_2)\frac{\bar z_2}{1-\bar z_2\zeta_2}d\zeta_2\\
&=\frac{z_1^{-p_1}}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2-1}(T_{0,k})_2[f_{2k}-(T_{0,k})_1\partial^k_{\bar\zeta_2}f_{1k}](z_1,\zeta_2)\frac{\bar z_2}{\bar\zeta_2-\bar z_2}d\bar\zeta_2\\
&=\frac{z_1^{-p_1}}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2-1}\Big\{\frac{-1}{\pi}\int_{D_2}\frac{1}{(k-1)!}\frac{(\overline{\zeta_2-\zeta_2'})^{k-1}}{\zeta_2'-\zeta_2}[f_{2k}-(T_{0,k})_1\partial^k_{\bar\zeta_2'}f_{1k}](z_1,\zeta_2')d\sigma_{\zeta_2'}\Big\}\frac{\bar z_2}{\bar\zeta_2-\bar z_2}d\bar\zeta_2\\
&=\frac{-z_1^{-p_1}}{\pi}\int_{D_2}\frac{1}{(k-1)!}[f_{2k}-(T_{0,k})_1\partial^k_{\bar\zeta_2'}f_{1k}](z_1,\zeta_2')\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2-1}\frac{(\overline{\zeta_2-\zeta_2'})^{k-1}}{\zeta_2'-\zeta_2}\frac{\bar z_2}{\bar\zeta_2-\bar z_2}d\bar\zeta_2\Big]d\sigma_{\zeta_2'}\\
&=\frac{z_1^{-p_1}\bar z_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}[f_{2k}-(T_{0,k})_1\partial^k_{\bar\zeta_2}f_{1k}](z_1,\zeta_2)\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&=\frac{z_1^{-p_1}\bar z_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}f_{2k}(z_1,\zeta_2)\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&-\frac{z_1^{-p_1}\bar z_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}\Big[\frac{-1}{\pi}\int_{D_1}\frac{1}{(k-1)!}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}\partial^k_{\bar\zeta_2}f_{1k}(\zeta)d\sigma_{\zeta_1}\Big]\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&=\frac{z_1^{-p_1}\bar z_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}f_{2k}(z_1,\zeta_2)\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\\
&+\frac{z_1^{-p_1}\bar z_2^{p_2+1}}{\pi^2[(k-1)!]^2}\int_{D^2}\partial^k_{\bar\zeta_2}f_{1k}(\zeta)\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar z_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2},
\end{aligned}\tag{3.41}$$

$$\begin{aligned}
&\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k-1})_1 f_{1k}(\zeta)]\zeta_1^{-p_1}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2}\\
&=\frac{1}{\pi^2}\int_{D^2}\Big\{\frac{-1}{\pi(k-1)!}\int_{D_1}\frac{(\overline{\zeta_1-\zeta_1'})^{k-1}}{\zeta_1'-\zeta_1}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}f_{1k}(\zeta_1',\zeta_2)]d\sigma_{\zeta_1'}\Big\}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\frac{1}{\pi}\int_{D_2}\Big\{\frac{-1}{\pi(k-1)!}\int_{D_1}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}f_{1k}(\zeta_1',\zeta_2)]\Big[\frac{1}{\pi}\int_{D_1}\frac{(\overline{\zeta_1-\zeta_1'})^{k-1}}{\zeta_1'-\zeta_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}d\sigma_{\zeta_1}\Big]d\sigma_{\zeta_1'}\Big\}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}
\end{aligned}\tag{3.42}$$

$$= \frac{1}{\pi}\int_{D_2}\Big\{\frac{-1}{\pi(k-1)!}\int_{D_1}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2} f_{1k}(\zeta_1', \zeta_2)]\Big[\frac{1}{2k\pi i}\int_{\partial D_1}\frac{(\overline{\zeta_1-\zeta_1'})^k}{\bar\zeta_1\zeta_1'-1}\frac{d\bar\zeta_1}{(1-z_1\bar\zeta_1)\bar\zeta_1^{-p_1}}\Big]d\sigma_{\zeta_1'}\Big\}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}$$
$$= \frac{-1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2} f_{1k}(\zeta)]g_1(\zeta_1, z_1)\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2},$$

$$\begin{aligned}
&\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_2\partial_{\bar\zeta_1}f_{2k}(\zeta)]\zeta_1^{-p_1}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2}\\
&=\frac{1}{\pi}\int_{D_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\Big\{\frac{1}{\pi}\int_{D_2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_2\partial_{\bar\zeta_1}f_{2k}(\zeta)]\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_2}\Big\}d\sigma_{\zeta_1}\\
&=\frac{1}{\pi}\int_{D_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\Big\{\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}(T_{0,k})_2\partial_{\bar\zeta_1}f_{2k}(\zeta)\frac{\bar z_2}{1-\bar z_2\zeta_2}d\zeta_2\Big\}d\sigma_{\zeta_1}\\
&=\frac{1}{\pi}\int_{D_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\Big\{\frac{-1}{\pi(k-1)!}\int_{D_2}\partial_{\bar\zeta_1}f_{2k}(\zeta_1,\zeta_2')\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}\frac{(\overline{\zeta_2-\zeta_2'})^{k-1}}{\zeta_2'-\zeta_2}\frac{\bar z_2}{1-\bar z_2\zeta_2}d\zeta_2\Big]d\sigma_{\zeta_2'}\Big\}d\sigma_{\zeta_1}\\
&=\frac{1}{\pi}\int_{D_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\Big\{\frac{-1}{\pi(k-1)!}\int_{D_2}\partial_{\bar\zeta_1}f_{2k}(\zeta_1,\zeta_2')\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar\zeta_2^{p_2}\frac{(\overline{\zeta_2-\zeta_2'})^{k-1}}{\bar\zeta_2\zeta_2'-1}\frac{\bar z_2}{\bar\zeta_2-\bar z_2}d\bar\zeta_2\Big]d\sigma_{\zeta_2'}\Big\}d\sigma_{\zeta_1}\\
&=\frac{1}{\pi}\int_{D_1}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\Big\{\frac{-\bar z_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}\partial_{\bar\zeta_1}f_{2k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_2}\Big\}d\sigma_{\zeta_1}\\
&=\frac{-\bar z_2^{p_2+1}}{\pi^2(k-1)!}\int_{D^2}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\partial_{\bar\zeta_1}f_{2k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2},
\end{aligned}\tag{3.43}$$

and similarly,

$$\begin{aligned}
&\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_2(T_{0,k-1})_1\partial_{\bar\zeta_2}^k f_{1k}(\zeta)]\zeta_1^{-p_1}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2}\\
&=\frac{-\bar z_2^{p_2+1}}{\pi^2(k-1)!}\int_{D^2}\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}(T_{0,k-1})_1\partial_{\bar\zeta_2}^k f_{1k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\frac{-\bar z_2^{p_2+1}}{(k-1)!}\frac{1}{\pi^2}\int_{D^2}(T_{0,k-1})_1\partial_{\bar\zeta_2}^k f_{1k}(\zeta)\frac{\zeta_1^{-p_1}}{\zeta_1-z_1}\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\frac{-\bar z_2^{p_2+1}}{(k-1)!}\frac{-1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar\zeta_2}^k[f_{1k}(\zeta)]g_1(\zeta_1,z_1)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2},
\end{aligned}\tag{3.44}$$

in which the last equation is similar to (3.42), and $g_1(\zeta_1', z_1)$ is the function in (3.12).

Since the third term at the right end of (3.44) is the opposite of its second term for $z_1 = 0$, then from (3.42)–(3.44), we get that

$$\begin{cases}
\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k-1})_1 f_{1k}(\zeta)]\zeta_1^{-p_1-1}d\sigma_{\zeta_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2} = \frac{-1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2} f_{1k}(\zeta)]g_1(\zeta_1,0)\frac{\bar z_2}{1-\bar z_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2},\\
\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_2\partial_{\bar\zeta_1}f_{2k}(\zeta)]\zeta_1^{-p_1-1}d\sigma_{\zeta_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2} = \frac{-\bar z_2^{p_2+1}}{\pi^2(k-1)!}\int_{D^2}\zeta_1^{-p_1-1}\partial_{\bar\zeta_1}f_{2k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2},\\
\frac{1}{\pi^2}\int_{D^2}\partial_{\bar\zeta_2}[\bar\zeta_2^{p_2}(T_{0,k})_2(T_{0,k-1})_1\partial_{\bar\zeta_2}^k f_{1k}(\zeta)]\zeta_1^{-p_1}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{\bar z_2 d\sigma_{\zeta_2}}{1-\bar z_2\zeta_2}\\
=\frac{-\bar z_2^{p_2+1}}{(k-1)!}\frac{-1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar\zeta_2}^k[f_{1k}(\zeta)]g_1(\zeta_1,z_1)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar z_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2}.
\end{cases}\tag{3.45}$$

Plugging (3.40)–(3.45) into (3.39), we obtain that

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{z_1\bar{z}_2}{(\zeta_1-z_1)(\overline{\zeta_2-z_2})}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=\frac{1}{\pi^2(k-1)!}\int_{D^2}\partial_{\bar{\zeta}_2}[\bar{\zeta}_2^{p_2}f_{1k}(\zeta)]\frac{\bar{z}_2}{\bar{z}_2\zeta_2-1}\Big\{z_1^{-p_1}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}+g_1(\zeta_1,z_1)-g_1(\zeta_1,0)\Big\}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&\quad+\frac{z_1^{-p_1}\bar{z}_2^{p_2+1}}{\pi(k-1)!}\int_{D_2}f_{2k}(z_1,\zeta_2)\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_2}\\
&\quad+\frac{\bar{z}_2^{p_2+1}}{\pi^2[(k-1)!]^2}\int_{D^2}\partial^k_{\bar{\zeta}_2}f_{1k}(\zeta)\Big[z_1^{-p_1}\frac{(\overline{z_1-\zeta_1})^{k-1}}{\zeta_1-z_1}-g_1(\zeta_1,z_1)+g_1(\zeta_1,0)\Big]\frac{(\overline{z_2-\zeta_2})^{k-1}}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&\quad-\frac{\bar{z}_2^{p_2+1}}{\pi^2(k-1)!}\int_{D^2}\frac{z_1\zeta_1^{-p_1-1}}{\zeta_1-z_1}\partial_{\bar{\zeta}_1}f_{2k}(\zeta)\frac{(\overline{z_2-\zeta_2})^{k-1}}{\bar{z}_2\zeta_2-1}d\sigma_{\zeta_1}d\sigma_{\zeta_2}.
\end{aligned}\tag{3.46}$$

On the other hand, by Lemma 2.3 and (2.2), we get that

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{\bar{z}_1 z_2}{(\overline{\zeta_1-z_1})(\zeta_2-z_2)}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&=\frac{-1}{2\pi i}\int_{\partial D_1}\zeta_1^{-p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{z_2}{1-z_2\bar{\zeta}_2}d\bar{\zeta}_2\Big]\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}d\zeta_1\\
&=\frac{-1}{\pi}\int_{D_1}\zeta_1^{-p_1}\partial_{\bar{\zeta}_1}\Big[\frac{-1}{\pi}\int_{D_2}\bar{\zeta}_2^{p_2}\partial_{\zeta_2}W_0(\zeta)\frac{z_2}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_2}\Big]\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}d\sigma_{\zeta_1}\\
&=\frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}\partial_{\bar{\zeta}_1}[\partial_{\zeta_2}W_0(\zeta)]\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{z_2}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}\partial_{\zeta_2}[(T_{0,k-1})_1f_{1k}+(T_{0,k})_2\partial_{\bar{\zeta}_1}f_{2k}-(T_{0,k})_2(T_{0,k-1})_1\partial^k_{\bar{\zeta}_2}f_{1k}](\zeta)\frac{\bar{z}_1 z_2 d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_1\zeta_1)(1-z_2\bar{\zeta}_2)}\\
&=\frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1}\bar{\zeta}_2^{p_2}[(T_{0,k-1})_1\partial_{\zeta_2}f_{1k}+(T_{-1,k})_2\partial_{\bar{\zeta}_1}f_{2k}-(T_{-1,k})_2(T_{0,k-1})_1\partial^k_{\bar{\zeta}_2}f_{1k}](\zeta)\frac{\bar{z}_1 z_2 d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_1\zeta_1)(1-z_2\bar{\zeta}_2)}.
\end{aligned}\tag{3.47}$$

Plugging (3.46) and (3.47) into (3.38), we get (3.11). Hence, $\varphi_1(z)$ in (3.37) is the solution to $\Re\{\bar{\zeta}_2{}^{p_2}\varphi_1(\zeta)\}=\gamma_0(\zeta)$ on the condition of (3.11).

② Second, we seek the condition to ensure that $\varphi(z)$ is k-holomorphic for $\varphi_1(z)=z_1^{-p_1}\varphi(z)$. We need only to ensure that $\varphi(z)$ is k-holomorphic for z_1 since the k-holomorphism for z_2 is the same for $\varphi(z)$ and $\varphi_1(z)$. As the k-holomorphism of $\varphi_1(z)=z_1^{-p_1}\varphi(z)$ for z_1 is equivalent to the holomorphism of $\partial^{k-1}_{\bar{z}_1}\varphi_1(z)=z_1^{-p_1}\partial^{k-1}_{\bar{z}_1}\varphi(z)$ for z_1, then, applying the properties of the Schwarz operator for holomorphic functions, we get that

$$\begin{aligned}
&\zeta_2^{-p_2}z_1^{-p_1}\partial^{k-1}_{\bar{z}_1}\varphi(z_1,\zeta_2)\\
&=\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_1^{-p_1}\zeta_2^{-p_2}\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)}]\Big[\frac{2\zeta_1}{\zeta_1-z_1}-1\Big]\frac{d\zeta_1}{2\zeta_1}\\
&=\sum_{l=0}^{\infty}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_1^{-p_1}\zeta_2^{-p_2}\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)}]\frac{d\zeta_1}{\zeta_1^{l+1}}\Big\}z_1^l\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_1^{-p_1}\zeta_2^{-p_2}\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)+\zeta_1^{p_1}\zeta_2^{p_2}\overline{\partial^{k-1}_{\bar{\zeta}_1}\varphi(\zeta_1,\zeta_2)}]\frac{d\zeta_1}{2\zeta_1}.
\end{aligned}$$

Since $p_1 < 0$, then $\varphi(z_1, \zeta_2)$ is k-holomorphic for z_1 (i.e., $\partial_{\bar{z}_1}^{k-1}\varphi(z_1, \zeta_2)$ is holomorphic for z_1) if and only if $z_1^{-p_1}\zeta_2^{-p_2}\partial_{\bar{z}_1}^{k-1}\varphi(z_1, \zeta_2)$ has a zero of order at least $-p_1$ at $z_1 = 0$. Therefore,

$$\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_1^{-p_1}\zeta_2^{-p_2}\partial_{\bar{\zeta}_1}^{k-1}\varphi(\zeta_1, \zeta_2) + \zeta_1^{p_1}\zeta_2^{p_2}\overline{\partial_{\bar{\zeta}_1}^{k-1}\varphi(\zeta_1, \zeta_2)}]\frac{d\zeta_1}{\zeta_1^{l+1}} = 0 \ \ (l = 0, 1, \cdots, -p_1 - 1),$$

that is, (3.13).

Hence, by ① and ②, we get that (3.10) is the solution of Problem RH on the conditions of (3.11) and (3.13).

(iii) In the case of $p_1, p_2 < 0$, the boundary condition is $\Re[\zeta_1^{-p_1}\zeta_2^{-p_2}\varphi(\zeta)] = \gamma_0(\zeta)$. Let $z_1^{-p_1}z_2^{-p_2}\varphi(z) = \varphi_2(z)$, and then

$$\Re[\zeta_1^{-p_1}\zeta_2^{-p_2}\varphi(\zeta)] = \gamma_0(\zeta) \Leftrightarrow \Re[\varphi_2(\zeta)] = \gamma_0(\zeta).$$

Similar to (ii), we need to find the solution to $\Re[\varphi_2(\zeta)] = \gamma_0(\zeta)$, where $\varphi_2(z)$ is k-holomorphic, and ensure that $\varphi(z)$ is k-holomorphic at the same time.

① First, $\Re[\varphi_2(\zeta)] = \gamma_0(\zeta)$ is just the case of $p_1 = p_2 = 0$ in (i). Therefore

$$\varphi_2(z) = \widetilde{\varphi}(z)|_{p_1=p_2=0} + \sum_{s_1,s_2=0}^{k-1}\sum_{l_1=0}^{s_1}\sum_{l_2=0}^{s_2}\sum_{v_1,v_2=0}^{+\infty}\alpha_{v_1v_2s_1l_1s_2l_2}z_1^{v_1}z_2^{v_2}\bar{z}_1^{l_1}\bar{z}_2^{l_2} \tag{3.48}$$

on the condition of (3.38). Moreover, by (2.1) and (2.2), we get that

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{z_1\bar{z}_2}{(\zeta_1 - z_1)(\overline{\zeta_2 - z_2})}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&= \frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{-p_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{-p_1-1}W_0(\zeta)\frac{z_1}{\zeta_1 - z_1}d\zeta_1\Big]\frac{\bar{z}_2}{\overline{\zeta_2 - z_2}}\frac{d\zeta_2}{\zeta_2}\\
&= \frac{1}{\pi}\int_{D_2}\zeta_2^{-p_2}\partial_{\bar{\zeta}_2}\Big[\frac{1}{\pi}\int_{D_1}\zeta_1^{-p_1-1}\partial_{\bar{\zeta}_1}W_0(\zeta)\frac{z_1}{\zeta_1 - z_1}d\sigma_{\zeta_1} + z_1^{-p_1}W_0(z_1, \zeta_2)\Big]\frac{\bar{z}_2}{1 - \bar{z}_2\zeta_2}d\sigma_{\zeta_2}\\
&= \frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1-1}\zeta_2^{-p_2}\partial_{\bar{\zeta}_2}[\partial_{\bar{\zeta}_1}W_0(\zeta)]\frac{z_1\bar{z}_2}{(\zeta_1 - z_1)(1 - \bar{z}_2\zeta_2)}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&+\frac{1}{\pi}\int_{D_2}z_1^{-p_1}\zeta_2^{-p_2}\partial_{\bar{\zeta}_2}W_0(z_1, \zeta_2)\frac{\bar{z}_2}{1 - \bar{z}_2\zeta_2}d\sigma_{\zeta_2},
\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\bar{\zeta}_1^{p_1}\bar{\zeta}_2^{p_2}W_0(\zeta)\frac{\bar{z}_1 z_2}{(\overline{\zeta_1 - z_1})(\zeta_2 - z_2)}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&= \frac{1}{2\pi i}\int_{\partial D_1}\zeta_1^{-p_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\zeta_2^{-p_2-1}W_0(\zeta)\frac{z_2}{\zeta_2 - z_2}d\zeta_2\Big]\frac{\bar{z}_1}{\overline{\zeta_1 - z_1}}\frac{d\zeta_1}{\zeta_1}\\
&= \frac{1}{\pi}\int_{D_1}\zeta_1^{-p_1}\partial_{\bar{\zeta}_1}\Big[\frac{1}{\pi}\int_{D_2}\zeta_2^{-p_2-1}\partial_{\bar{\zeta}_2}W_0(\zeta)\frac{z_2}{\zeta_2 - z_2}d\sigma_{\zeta_2} + z_2^{-p_2}W_0(\zeta_1, z_2)\Big]\frac{\bar{z}_1}{1 - \bar{z}_1\zeta_1}d\sigma_{\zeta_1}\\
&= \frac{1}{\pi^2}\int_{D^2}\zeta_1^{-p_1}\zeta_2^{-p_2-1}\partial_{\bar{\zeta}_2}[\partial_{\bar{\zeta}_1}W_0(\zeta)]\frac{\bar{z}_1 z_2}{(1 - \bar{z}_1\zeta_1)(\zeta_2 - z_2)}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&+ \frac{1}{\pi}\int_{D_1}\zeta_1^{-p_1}z_2^{-p_2}\partial_{\bar{\zeta}_1}W_0(\zeta_1, z_2)\frac{\bar{z}_1}{1 - \bar{z}_1\zeta_1}d\sigma_{\zeta_1}.
\end{aligned} \tag{3.50}$$

Plugging (3.49) and (3.50) into (3.38), we get (3.15). Therefore, $\varphi_2(z)$ in (3.48) is the solution to $\Re[\varphi_2(\zeta)] = \gamma_0(\zeta)$ on the condition of (3.15).

② Second, we seek the condition to ensure that $\varphi(z)$ is k-holomorphic from $\varphi_2(z) = z_1^{-p_1} z_2^{-p_2}\varphi(z)$. Similar to the discussion in ② of (ii), $\varphi(z)$ is k-holomorphic for z_1 and z_2 if and only if (3.16).

Hence, by ① and ②, we obtain that (3.14) is the solution of Problem RH on the conditions of (3.15) and (3.16). □

Remark 3.2. *Setting $k = 1$ in Theorem 3.1, we obtain the corresponding conclusion of the Riemann-Hilbert problem for $\Re[\bar{\zeta}^p W(\zeta)] = \gamma(\zeta)$ $(\zeta \in \partial_0 D^2)$ and $\frac{\partial W(z)}{\partial \bar{z}_l} = f_l(z)(z \in D^2)$, which is more concise than that of Theorem 3.1.*

Corollary 3.3. *Let $\gamma, f_1, f_2 \in C(\partial_0 D^2)$ with $\partial_{\bar{z}_1} f_2 = \partial_{\bar{z}_2} f_1$, and let*

$$T_i f(z_i) = \frac{-1}{\pi}\int_{D_i} \frac{f(\zeta_i)}{\zeta_i - z_i} d\sigma_{\zeta_i} \ \ (i = 1, 2, \ \ D_1 = D_2 = D).$$

Then, the Riemann-Hilbert problem

$$\Re[\bar{\zeta}^p W(\zeta)] = \gamma(\zeta)\ (\zeta \in \partial_0 D^2), \quad \partial_{\bar{z}_l} W(z) = f_l(z)\ (l = 1, 2,\ z \in D^2)$$

is solvable, and the solution is as follows:

(i) In the case of $p_1, p_2 \geq 0$,

$$W(z) = z_1^{p_1} z_2^{p_2}\widetilde{\varphi}(z) + \sum_{v_1=0}^{2p_1}\sum_{v_2=0}^{2p_2} \alpha_{(v_1-p_1)(v_2-p_2)} z_1^{v_1} z_2^{v_2} + [T_1 f_1 + T_2 f_2 - T_2 T_1 \partial_{\bar{\zeta}_2} f_1](z),$$

where $\alpha_{v_1 v_2}$ are arbitrary complex constants satisfying

$$\alpha_{v_1 v_2} + \overline{\alpha_{(-v_1)(-v_2)}} = 0 \ \ (-p_1 \leq v_1 \leq p_1, \ \ -p_2 \leq v_2 \leq p_2),$$

and

$$\begin{aligned}
\widetilde{\varphi}(z) =& \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} \gamma(\zeta)\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&- \frac{z_1^{p_1} z_2^{p_2}}{\pi}\Big[\int_{D_1} \frac{z_1\overline{f_1(\zeta_1, z_2)}}{1 - z_1\bar{\zeta}_1} d\sigma_{\zeta_1} + \int_{D_2} \frac{z_2\overline{f_2(z_1, \zeta_2)}}{1 - z_2\bar{\zeta}_2} d\sigma_{\zeta_2}\Big] \\
&- \frac{1}{\pi^2}\int_{D_1}\int_{D_2}\Big[\frac{z_1^{p_1+1}\zeta_2^{p_2}\overline{\partial_{\zeta_2} f_1(\zeta)}}{(1 - z_1\bar{\zeta}_1)(\zeta_2 - z_2)} + \frac{\zeta_1^{p_1} z_2^{p_2+1}\overline{\partial_{\zeta_1} f_2(\zeta)}}{(1 - z_2\bar{\zeta}_2)(\zeta_1 - z_1)} - \overline{\partial_{\bar{\zeta}_2} f_1(\zeta)}\frac{z_1^{p_1+1}}{1 - z_1\bar{\zeta}_1}\frac{z_2^{p_2+1}}{1 - z_2\bar{\zeta}_2}\Big] d\sigma_{\zeta_1} d\sigma_{\zeta_2},
\end{aligned}$$

on the condition that

$$\begin{aligned}
&\int_{\partial_0 D^2} \gamma(\zeta)\Re\frac{z_1\bar{z}_2}{(\zeta_1 - z_1)\overline{(\zeta_2 - z_2)}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&= -2\Re\Big\{\int_{D^2}\Big[\bar{\zeta}_1^{p_1}\bar{z}_2^{p_2}\partial_{\zeta_1} f_2(\zeta) + \zeta_2^{p_2} z_1^{p_1}\overline{\partial_{\zeta_2} f_1(\zeta)}\Big]\frac{z_1}{1 - z_1\bar{\zeta}_1}\frac{\bar{z}_2}{1 - \bar{z}_2\zeta_2} d\sigma_{\zeta_1} d\sigma_{\zeta_2}\Big\}.
\end{aligned}$$

(ii) In the case of $p_1 < 0$ *and* $p_2 \geq 0$,

$$W(z) = z_1^{p_1}\Big[z_2^{p_2}\widetilde{\varphi}(z)|_{p_1=0} + \sum_{v_2=0}^{2p_2} \alpha_{0(v_2-p_2)} z_2^{v_2}\Big] + [T_1 f_1 + T_2 f_2 - T_2 T_1 \partial_{\bar{\zeta}_2} f_1](z)$$

($\widetilde{\varphi}(z)$ *and* $\alpha_{v_1 v_2}$ *are the same as in (i)) on the condition that*

$$\begin{aligned}
&\frac{2}{(2\pi i)^2}\int_{\partial_0 D^2} \gamma(\zeta)\Re \frac{z_1\bar{z}_2}{(\zeta_1 - z_1)\overline{(\zeta_2 - z_2)}} \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&= \Re\Big\{\frac{1}{\pi^2}\int_{D^2} \partial_{\bar{\zeta}_2}[\bar{\zeta}_2^{p_2} f_1(\zeta)]\frac{\bar{z}_2}{\bar{z}_2\zeta_2 - 1}\Big\{\frac{z_1^{-p_1}}{\zeta_1 - z_1} + g_1(\zeta_1, z_1) - g_1(\zeta_1, 0)\Big\} d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\
&+ \frac{z_1^{-p_1}\bar{z}_2^{p_2+1}}{\pi}\int_{D_2} \frac{f_2(z_1,\zeta_2)}{1 - \bar{z}_2\zeta_2} d\sigma_{\zeta_2} + \frac{\bar{z}_2^{p_2+1}}{\pi^2}\int_{D^2}\Big[\frac{z_1^{-p_1}}{\zeta_1 - z_1} - g_1(\zeta_1, z_1) + g_1(\zeta_1, 0)\Big]\frac{\partial_{\bar{\zeta}_2} f_1(\zeta)}{1 - \bar{z}_2\zeta_2} d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\
&- \frac{\bar{z}_2^{p_2+1}}{\pi^2}\int_{D^2} \frac{z_1\zeta_1^{-p_1-1}}{\zeta_1 - z_1}\frac{\partial_{\bar{\zeta}_1} f_2(\zeta)}{\bar{z}_2\zeta_2 - 1} d\sigma_{\zeta_1} d\sigma_{\zeta_2} + \frac{1}{\pi^2}\int_{D^2} \zeta_1^{-p_1}\bar{\zeta}_2^{p_2}\partial_{\bar{\zeta}_2} f_1(\zeta)\frac{\bar{z}_1 z_2 d\sigma_{\zeta_1} d\sigma_{\zeta_2}}{(1 - \bar{z}_1\zeta_1)(1 - z_2\bar{\zeta}_2)}\Big\}
\end{aligned}$$

and

$$\int_{\partial D_1} \{\gamma(\zeta) - \Re[\zeta_1^{-p_1}\zeta_2^{-p_2}(T_1 f_1 + T_2 f_2 - T_2 T_1 \partial_{\bar{\zeta}_2} f_1)(\zeta)]\}\frac{d\zeta_1}{\zeta_1^{l+1}} = 0 \ \ (l = 0, 1, \cdots, -p_1 - 1),$$

where

$$g_1(\zeta_1', z_1) = \frac{1}{(-p_1 - 1)!}\Big\{\partial_{\bar{\zeta}_1}^{-p_1-1}\Big[\frac{1}{(\bar{\zeta}_1\zeta_1' - 1)(1 - z_1\bar{\zeta}_1)}\Big]\Big\}_{\zeta_1=0}.$$

(iii) In the case of $p_1, p_2 < 0$,

$$W(z) = z_1^{p_1} z_2^{p_2}\Big[\widetilde{\varphi}(z)|_{p_1=p_2=0} + \alpha_{00}\Big] + [T_1 f_1 + T_2 f_2 - T_2 T_1 \partial_{\bar{\zeta}_2} f_1](z).$$

($\widetilde{\varphi}(z)$ *is the same as in (i) and* $\Re\alpha_{00} = 0$) *on the condition that*

$$\begin{aligned}
&\int_{\partial_0 D^2} \gamma(\zeta)\Re \frac{z_1\bar{z}_2}{(\zeta_1 - z_1)\overline{(\zeta_2 - z_2)}} \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
&= -2\Re\Big\{\int_{D^2} \zeta_1^{-p_1-1}\zeta_2^{-p_2-1}\partial_{\bar{\zeta}_1} f_2(\zeta)\Big[\frac{\zeta_2 z_1\bar{z}_2}{(\zeta_1 - z_1)(1 - \bar{z}_2\zeta_2)} + \frac{\zeta_1\bar{z}_1 z_2}{(1 - \bar{z}_1\zeta_1)(\zeta_2 - z_2)}\Big] d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\
&+ \pi z_1^{-p_1}\bar{z}_2\int_{D_2} \zeta_2^{-p_2}\frac{f_2(z_1, \zeta_2)}{1 - \bar{z}_2\zeta_2} d\sigma_{\zeta_2} + \pi\bar{z}_1 z_2^{-p_2}\int_{D_1} \zeta_1^{-p_1}\frac{f_1(\zeta_1, z_2)}{1 - \bar{z}_1\zeta_1} d\sigma_{\zeta_1}\Big\}.
\end{aligned}$$

For $j = 1, 2$,

$$\int_{\partial D_j} \{\gamma(\zeta) - \Re[\zeta_1^{-p_1}\zeta_2^{-p_2}(T_1 f_1 + T_2 f_2 - T_2 T_1 \partial_{\bar{\zeta}_2} f_1)(\zeta)]\}\frac{d\zeta_j}{\zeta_j^{r_j+1}} = 0 \ \ (r_j = 0, 1, \cdots, -p_j - 1).$$

The following is a simple example of the corresponding Riemann-Hilbert problem. Through simple calculations, it can be verified that the solvable conditions in Corollary 3.3 are satisfied.

Example 3.4. *Let $f_1 = f_2 = 0$ and*

$$\gamma(\zeta) = \begin{cases} 1, & p_1, p_2 \geq 0, \\ \zeta_1^{-p_1}, & p_1 < 0, p_2 \geq 0, \\ \zeta_1^{-p_1}\zeta_2^{-p_2}, & p_1, p_2 < 0 \end{cases}$$

for $\zeta \in \partial_0 D^2$ and $p_1, p_2 \in \mathbb{Z}$. Then the solution to the Riemann-Hilbert problem

$$\Re[\bar{\zeta}^p W(\zeta)] = \gamma(\zeta)\ (\zeta \in \partial_0 D^2,\ p = (p_1, p_2)), \quad \partial_{\bar{z}_l} W(z) = f_l(z)\ (l = 1, 2,\ z \in D^2)$$

is:

$$\begin{aligned} W(z) = &\frac{z_1^{p_1} z_2^{p_2}}{(2\pi i)^2} \int_{\partial_0 D^2} \gamma(\zeta) \Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big] \frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\ &+ \begin{cases} \sum\limits_{v_1=0}^{2p_1} \sum\limits_{v_2=0}^{2p_2} \alpha_{(v_1-p_1)(v_2-p_2)} z_1^{v_1} z_2^{v_2}, & p_1, p_2 \geq 0, \\ \sum\limits_{v_2=0}^{2p_2} \alpha_{0(v_2-p_2)} z_2^{v_2}, & p_1 < 0, p_2 \geq 0, \\ \alpha_{00}, & p_1, p_2 < 0, \end{cases} \end{aligned}$$

where $\alpha_{v_1 v_2}$ are arbitrary complex constants satisfying

$$\alpha_{v_1 v_2} + \overline{\alpha_{(-v_1)(-v_2)}} = 0 \ \ (-p_1 \leq v_1 \leq p_1, \quad -p_2 \leq v_2 \leq p_2).$$

4. Conclusions

Applying the classical Cauchy-Pompeiu formula and the Gauss theorem, we obtain the solvable condition and the specific solution of an inhomogeneous Riemann-Hilbert boundary value problem for complex partial differential operators of higher order on the bicylinder D^2 in $\mathbb{C}^2$. With the methods in this article, some other boundary value problems for complex partial differential operators in $\mathbb{C}^n$ can be studied further. The conclusions generalize the existing results of the corresponding Riemann-Hilbert boundary value problems for lower-order partial differential equations, enrich the research of complex partial differential equations in $\mathbb{C}^n$, and provide a solid basis for future research of boundary value problems in $\mathbb{C}^n$. The method proposed here can also be extended to more general domains in $\mathbb{C}^2$, although the integral representations would become more complicated. In addition, the conclusions drawn in this article are expected to contribute to some physical or engineering problems, such as string theory and quantum gravity research, high-dimensional signal and image processing, and so on.

Author contributions

Yanyan Cui: Conceptualization, Project administration, Writing original draft, Writing-review and editing; Chaojun Wang: Investigation, Writing original draft, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the NSF of China (No. 11601543), the NSF of Henan Province (No. 222300420397), and the Science and Technology Research Projects of Henan Provincial Education Department (No. 19B110016). The authors are grateful to the anonymous referees for their valuable comments and suggestions which improved the quality of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. X. Li, An application of the periodic Riemann boundary value problem to a periodic crack problem, In: *Complex Methods for Partial Differential and Integral Equations*, New York: Springer, 1999. https://doi.org/10.1007/978-1-4613-3291-6_7
2. J. W. Cohen, O. J. Boxma, *Boundary Value Problems in Queueing System Analysis*, New York: North-Holland Pub. Co., 1983.
3. M. Reed, *Abstract Non-Linear Wave Equations*, Berlin: Springer Lecture Notes in Mathematics, 1976. https://doi.org/10.1007/BFb0079271
4. Y. Li, S. F. Tian, J. J. Yang, Riemann-Hilbert problem and interactions of solitons in the n-component nonlinear Schrödinger equations, *Stud. Appl. Math.*, **148** (2022), 577–605. https://doi.org/10.1111/sapm.12450
5. Z. Y. Wang, S. F. Tian, J. Cheng, The $\bar{\partial}$-dressing method and soliton solutions for the three-component coupled Hirota equations, *J. Math. Phys.*, **62** (2021), 093510. https://doi.org/10.1063/5.0046806
6. J. J. Yang, S. F. Tian, Z. Q. Li, Riemann-Hilbert problem for the focusing nonlinear Schrödinger equation with multiple high-order poles under nonzero boundary conditions, *Phys. D*, **432** (2022), 133162. https://doi.org/10.1016/j.physd.2022.133162
7. X. Wu, S. F. Tian, On long-time asymptotics to the nonlocal short pulse equation with the Schwartz-type initial data: Without solitons, *Phys. D.*, **448** (2023), 133733. https://doi.org/10.1016/j.physd.2023.133733
8. Z. Q. Li, S. F. Tian, J. J. Yang, On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions, *Adv. Math.*, **409** (2022), 108639. https://doi.org/10.1016/j.aim.2022.108639
9. Z. Q. Li, S. F. Tian, J. J. Yang, Soliton resolution for the Wadati-Konno-Ichikawa equation with weighted sobolev initial data, *Ann. Henri Poincaré*, **23** (2022), 2611–2655. https://doi.org/10.1007/s00023-021-01143-z

10. Z. Q. Li, S. F. Tian, J. J. Yang, E. Fan, Soliton resolution for the complex short pulse equation with weighted Sobolev initial data in space-time solitonic regions, *J. Differ. Equ.*, **329** (2022), 31–88. https://doi.org/10.1016/j.jde.2022.05.003
11. O. Suzuki, Z. D. Zhang, A method of Riemann-Hilbert problem for Zhang's conjecture 1 in a ferromagnetic 3D Ising model: Trivialization of topological structure, *Mathematics*, **9** (2021), 776. https://doi.org/10.3390/math9070776
12. Z. D. Zhang, O. Suzuki, A method of the Riemann-Hilbert problem for Zhang's conjecture 2 in a ferromagnetic 3D Ising model: Topological phases, *Mathematics*, **9** (2021), 2936. https://doi.org/10.3390/math9222936
13. D. Pompeiu, Sur une classe de fonctions d'une variable complexe et sur certaine equations integrals, *Rend. Circ. Mat. Palermo.*, **35** (1913), 277–281.
14. S. Rogosin, A. O. Çelebi, *Analysis as a Life*, Birkh*ä*user: Springer, 2019. https://doi.org/10.1007/978-3-030-02650-9
15. I. N. Dorofeeva, A. B. Rasulov, A linear matching problem for a generalized Cauchy-Riemann equation with super-singular points on a half-plane, *Comput. Math. and Math. Phys.*, **62** (2022), 1859–1864. https://doi.org/10.1134/S0965542522110069
16. A. Mera, N. Tarkhanov, An elliptic equation of finite index in a domain, *Bol. Soc. Mat. Mex.*, **28** (2022), 49. https://doi.org/10.1007/s40590-022-00442-7
17. V. Gutlyanskiĭ, O. Nesmelova, V. Ryazanov, A. Yefimushkin, On boundary-value problems for semi-linear equations in the plane, *J. Math. Sci.*, **259** (2021), 53–74. https://doi.org/10.1007/s10958-021-05604-y
18. D. E. G. Valencia, R. A. Blaya, M. P. R. Alejandre, A. M. Garcia, On the plane Lamé-Navier system in fractal domains, *Complex Anal. Oper. Theory*, **15** (2021), 43. https://doi.org/10.1007/s11785-021-01088-5
19. L. K. Ha, C^k-estimates for $\overline{\partial}$-equation on certain convex domains of infinite type in $\mathbb{C}^n$, *J. Geom. Anal.*, **31** (2021), 2058–2087. https://doi.org/10.1007/S12220-019-00332-X
20. J. Bory-Reyes, D. Barseghyan, B. Schneider, Dirichlet-type problems for certain Beltrami equations. *Mediterr. J. Math.*, **18** (2021), 103. https://doi.org/10.1007/s00009-021-01760-2
21. L. Kim Ha, Hölder and L^p estimates for the $\overline{\partial}$-equation in a class of convex domains of infinite type in $\mathbb{C}^n$, *Monatsh Math.*, **190** (2019), 517–540. https://doi.org/10.1007/s00605-019-01327-0
22. M. M. Sirazhudinov, S. P. Dzhamaludinova, Estimates for the locally periodic homogenization of the Riemann-Hilbert problem for a generalized Beltrami equation, *Diff. Equat.*, **58** (2022), 771–790. https://doi.org/10.1134/S0012266122060064
23. S. Wang, F. He, On the variable exponent Riemann boundary value problem for Liapunov open curve, *J. Geom. Anal.*, **33** (2023), 62. https://doi.org/10.1007/s12220-022-01113-9
24. M. Marin, S. Pirlog, O. M. Hapenciuc, Study of plane boundary value problems in the asymmetric elasticity theory, *Contin. Mech. Therm.*, **37** (2025), 79. https://doi.org/10.1007/s00161-025-01412-w
25. S. Alexandrov, E. Lyamina, Y. R. Jeng, A new method of solving plane-strain boundary value problems for the double slip and rotation model, *J. Mech.*, **40** (2024), 53. https://doi.org/10.1093/jom/ufae004

26. H. Emkanpour, N. Taghizadeh, Three boundary value problems of the Cauchy-Riemann equation in the Eclipse domain, *Complex Var. Elliptic Equ.*, **69** (2022), 510–529. https://doi.org/10.1080/17476933.2022.2151004
27. S. Sambou, E. Bodian, W. O. Ingoba, I. Hamidine, Solution of the $\partial\overline{\partial}$-problem in a pseudoconvex domain of $\mathbb{C}^n$, *J. Anal.*, **30** (2022), 1361–1375. https://doi.org/10.1007/s41478-022-00385-2
28. D. Chakrabarti, P. S. Harrington, Exact sequences and estimates for the $\overline{\partial}$-problem, *Math. Z.*, **299** (2021), 1837–1873. https://doi.org/10.1007/S00209-021-02750-6
29. M. Fassina, S. Pinton, Existence and interior regularity theorems for $\overline{\partial}$ on q-convex domains, *Complex Anal. Oper. Theory*, **13** (2019), 2487–2494. https://doi.org/10.1007/s11785-018-0874-6
30. A. El-Gasmi, The Dirichlet problem for the complex Hessian operator in the class $N_m(\Omega, f)$, *Math. Scand.*, **127** (2021), 287–316. https://doi.org/10.7146/math.scand.a-125994
31. O. Suzuki, The problems of Riemann and Hilbert and the relations of fuchs in several complex variables, In: *Equations Différentielles et Systèmes de Pfaff dans le Champ Complexe, Berlin: Springer*, **712** (1977), 325–364. https://doi.org/10.1007/BFb0062822
32. A. Okay Çelebi, Neumann problem in polydomains, In: *Analysis, Probability, Applications, and Computation*, Switzerland: Springer, 2019. http://doi.org/10.1007/978-3-030-04459-6_12
33. A. Mohammed, The Riemann-Hilbert problem for certain poly domains and its connection to the Riemann problem, *J. Math. Anal. Appl.*, **343** (2008), 706–723. https://doi.org/10.1016/j.jmaa.2008.01.053
34. Y. Qiao, Y. Cui, Z. Li, L. Wang, k-holomorphic functions in spaces of several complex variables, *Complex Var. Ellipt. Equ.*, **65** (2020), 1826–1845. https://doi.org/10.1080/17476933.2019.1681417
35. Y. Cui, C. Wang, Y. Xie, Y. Qiao, Boundary value problems of conjugate and generalized k-holomorphic functions in $\mathbb{C}^n$, *Acta Math. Sci.*, **44** (2024), 1837–1852. https://doi.org/10.1007/s10473-024-0511-6
36. I. N. Vekua, *Generalized Analytic Functions*, Oxford: Pergamon Press, 1962. https://doi.org/10.1007/3-540-34459-4_4
37. H. Begehr, G. N. Hile, A hierarchy of integral operators, *Rocky MT. J. Math.*, **27** (1997), 669–706. https://doi.org/10.1216/rmjm/1181071888