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Research article

Density results on the coefficients of the triple product L-functions

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Abstract: Let H_k^* denote the set of normalized primitive holomorphic Hecke cusp forms of even integral weight k for the full modular group. Denote by $\lambda_{f\times f\times f}(n)$ the nth coefficient of the triple product L-function $L(f \times f \times f, s)$ attached to $f \in H_k^*$. Suppose $Q(\underline{x})$ is a primitive integral positive-definite binary quadratic form of fixed discriminant $D < 0$ with the class number $h(D) = 1$. In this paper, we study the distribution of $\lambda_{f\times f\times f}(n)$ on the set of all primes and its subset $\{p : p = Q(\underline{x})\}$ and obtain the analytic density and the natural density of the above sets. These results generalize previous ones.

Keywords: binary quadratic form; triple product L-function; analytic density; natural density
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1. Introduction

The density of a set of prime numbers measures its size. There are several notions of density, such as analytic density and natural density, which in general are distinct. Triple product L-functions, as vital automorphic L-functions, are investigated by some scholars (see, e.g., [6, 10]). In this paper, we focus on the distribution of coefficients of the triple product L-functions on the set of all primes and its subset, and we obtain the analytic density and the natural density of the above sets.

Let H_k^* denote the set of normalized primitive holomorphic Hecke cusp forms of even integral weight k for $SL(2,\mathbb{Z})$. $f \in H_k^*$ at the cusp ∞ has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i z},$$

where $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f(\frac{mn}{d^2}).$$

In particular, for each prime p and $r \in \mathbb{N}$, one has

$$\lambda_f^2(p^r) = 1 + \lambda_f(p^2) + \lambda_f(p^4) + \cdots + \lambda_f(p^{2r}). \tag{1.1}$$

In 1974, Deligne [3] proved the Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n),$$

where $d(n)$ is the divisor function. Deligne's result showed that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \ \ \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{1.2}$$

It can be inferred from $|\lambda_f(p)| \leqslant 2$ that there is a unique $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = e^{i\theta_f(p)} + e^{-i\theta_f(p)} = 2\cos\theta_f(p). \tag{1.3}$$

We introduce the definition of analytic density. A set S consisting of primes is said to have the analytic density $\kappa > 0$ if the following is satisfied:

$$\sum_{p \in S} \frac{1}{p^\sigma} \sim (1 + o(1))\kappa \sum_{p} \frac{1}{p^\sigma} = -(1 + o(1))\kappa \log(\sigma - 1), \quad as\ \sigma \to 1^+.$$

In this paper, let $Q(\underline{x})$ ($\underline{x} \in \mathbb{Z}^2$) be a primitive integral positive-definite binary quadratic form of fixed discriminant $D < 0$ with the class number $h(D) = 1$. Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. In [13], Vaishya investigated deterministic comparison of Fourier coefficients $\lambda_f^j(p^m)$ ($j = 1, 2$) at the primes represented by a binary quadratic form $Q(\underline{x})$. Denote by $\lambda_{f\times f\times f}(n)$ the nth coefficient of the Dirichlet expansion of the triple product L-function $L(f \times f \times f, s)$ attached to $f \in H_{k_1}^*$. In [4], Hua studied the analytic density of the sets $\{p : \lambda^j_{f\times f\times f}(p) < \lambda^j_{g\times g\times g}(p)\}$ and their respective subsets $\{p : p = Q(\underline{x})$ for some $\underline{x} \in \mathbb{Z}^2$ and $\lambda^j_{f\times f\times f}(p) < \lambda^j_{g\times g\times g}(p)\}$, where $j = 1, 2$.

The first aim of this paper is to prove the following results on analytic density.

Theorem 1.1. (1) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the two sets*

$$X_2 = \{p : \lambda_{f\times f\times f}(p^2) < \lambda_{g\times g\times g}(p^2)\}$$

and

$$Y_2 = \{p : \lambda^2_{f\times f\times f}(p^2) < \lambda^2_{g\times g\times g}(p^2)\}$$

have analytic densities at least $\frac{41}{2704}$ and $\frac{8231}{2056392}$, respectively.

(2) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the two sets*

$$X_3 = \{p : \lambda_{f\times f\times f}(p^3) < \lambda_{g\times g\times g}(p^3)\}$$

and

$$Y_3 = \{p : \lambda^2_{f\times f\times f}(p^3) < \lambda^2_{g\times g\times g}(p^3)\}$$

have analytic densities at least $\frac{211}{57600}$ and $\frac{1051157}{572166400}$, respectively.

Theorem 1.2. (1) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the two sets*

$$A_2 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } \lambda_{f\times f\times f}(p^2) < \lambda_{g\times g\times g}(p^2)\}$$

and

$$B_2 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } \lambda^2_{f\times f\times f}(p^2) < \lambda^2_{g\times g\times g}(p^2)\}$$

have analytic densities at least $\frac{41}{5408}$ and $\frac{8231}{4112784}$, respectively.

(2) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the two sets*

$$A_3 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } \lambda_{f\times f\times f}(p^3) < \lambda_{g\times g\times g}(p^3)\}$$

and

$$B_3 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } \lambda^2_{f\times f\times f}(p^3) < \lambda^2_{g\times g\times g}(p^3)\}$$

have analytic densities at least $\frac{211}{115200}$ and $\frac{1051157}{1144332800}$, respectively.

Some authors also considered the density for linear combinations of two Fourier coefficients corresponding to two distinct automorphic representations. In [15], for any given integer $j \geqslant 1$, Zou et al. arrived at a lower bound for the analytic density of the set

$$\{p : a < c_1\lambda_f(p^j) + c_2\lambda_g(p^j) < b\},$$

where $a, b, c_1, c_2 \in \mathbb{R}$, $a < b$. Hua [4] derived a lower bound for the analytic density of the set

$$\{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } a < c_1\lambda_f(p^j) + c_2\lambda_g(p^j) < b\}.$$

For the triple product L-function $L(f \times f \times f, s)$, we establish the following results.

Theorem 1.3. *Let $a, b, c_1, c_2 \in \mathbb{R}$, $a < b$.*

(1) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$F_1 = \{p : a < c_1\lambda_{f\times f\times f}(p) + c_2\lambda_{g\times g\times g}(p) < b\}$$

has an analytic density at least $f(1, a, b, c_1, c_2)$, where

$$\begin{aligned} f(1, a, b, c_1, c_2) = &\frac{(a^2 + b^2 + 4ab)(5c_1^2 + 5c_2^2) + 132(c_1^4 + c_2^4) + 150c_1^2c_2^2 + a^2b^2}{2(64(|c_1| + |c_2|)^2 + 8(|a| + |b|)(|c_1| + |c_2|) + |ab|)^2} \\ &- \frac{5c_1^2 + 5c_2^2 + ab}{2(64(|c_1| + |c_2|)^2 + 8(|a| + |b|)(|c_1| + |c_2|) + |ab|)}. \end{aligned} \tag{1.4}$$

(2) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$F_2 = \{p : a < c_1\lambda_{f\times f\times f}(p^2) + c_2\lambda_{g\times g\times g}(p^2) < b\}$$

has an analytic density at least $f(2, a, b, c_1, c_2)$, where

$$f(2, a, b, c_1, c_2)$$

$$= \frac{-2ab(a+b)(c_1+c_2)+(a^2+b^2+4ab)(42c_1^2+42c_2^2+2c_1c_2)-1490(a+b)(c_1^3+c_2^3)}{2(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$+ \frac{252ab(c_1^2c_2+c_1c_2^2)+18226(c_1^4+c_2^4)+2980(c_1^3c_2+c_1c_2^3)+10584c_1^2c_2^2+a^2b^2}{2(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$- \frac{42c_1^2+42c_2^2+2c_1c_2-(a+b)(c_1+c_2)+ab}{2(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)}. \tag{1.5}$$

(3) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$F_3 = \{p : a < c_1\lambda_{f\times f\times f}(p^3) + c_2\lambda_{g\times g\times g}(p^3) < b\}$$

has an analytic density at least $f(3,a,b,c_1,c_2)$, where

$$f(3,a,b,c_1,c_2) = \frac{(a^2+b^2+4ab)(211c_1^2+211c_2^2)+1095678(c_1^4+c_2^4)+267126c_1^2c_2^2+a^2b^2}{2(14400(|c_1|+|c_2|)^2+120(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$- \frac{211c_1^2+211c_2^2+ab}{2(14400(|c_1|+|c_2|)^2+120(|a|+|b|)(|c_1|+|c_2|)+|ab|)}. \tag{1.6}$$

Theorem 1.4. *Let $a, b, c_1, c_2 \in \mathbb{R}$, $a < b$.*
(1) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$E_1 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } a < c_1\lambda_{f\times f\times f}(p) + c_2\lambda_{g\times g\times g}(p) < b\}$$

has an analytic density at least $e(1,a,b,c_1,c_2)$, where

$$e(1,a,b,c_1,c_2) = \frac{(a^2+b^2+4ab)(5c_1^2+5c_2^2)+132(c_1^4+c_2^4)+150c_1^2c_2^2+a^2b^2}{4(64(|c_1|+|c_2|)^2+8(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$- \frac{5c_1^2+5c_2^2+ab}{4(64(|c_1|+|c_2|)^2+8(|a|+|b|)(|c_1|+|c_2|)+|ab|)}. \tag{1.7}$$

(2) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$E_2 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } a < c_1\lambda_{f\times f\times f}(p^2) + c_2\lambda_{g\times g\times g}(p^2) < b\}$$

has an analytic density at least $e(2,a,b,c_1,c_2)$, where

$$e(2,a,b,c_1,c_2) = \frac{-2ab(a+b)(c_1+c_2)+(a^2+b^2+4ab)(42c_1^2+42c_2^2+2c_1c_2)-1490(a+b)(c_1^3+c_2^3)}{4(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$+ \frac{252ab(c_1^2c_2+c_1c_2^2)+18226(c_1^4+c_2^4)+2980(c_1^3c_2+c_1c_2^3)+10584c_1^2c_2^2+a^2b^2}{4(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)^2}$$
$$- \frac{42c_1^2+42c_2^2+2c_1c_2-(a+b)(c_1+c_2)+ab}{4(1296(|c_1|+|c_2|)^2+36(|a|+|b|)(|c_1|+|c_2|)+|ab|)}. \tag{1.8}$$

(3) *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then the set*

$$E_3 = \{p : p = Q(\underline{x}) \text{ for some } \underline{x} \in \mathbb{Z}^2 \text{ and } a < c_1\lambda_{f\times f\times f}(p^3) + c_2\lambda_{g\times g\times g}(p^3) < b\}$$

has an analytic density at least $e(3, a, b, c_1, c_2)$, where

$$\begin{aligned} e(3, a, b, c_1, c_2) = &\frac{(a^2 + b^2 + 4ab)(211c_1^2 + 211c_2^2) + 1095678(c_1^4 + c_2^4) + 267126c_1^2c_2^2 + a^2b^2}{4(14400(|c_1| + |c_2|)^2 + 120(|a| + |b|)(|c_1| + |c_2|) + |ab|)^2} \\ &- \frac{211c_1^2 + 211c_2^2 + ab}{4(14400(|c_1| + |c_2|)^2 + 120(|a| + |b|)(|c_1| + |c_2|) + |ab|)}. \end{aligned} \tag{1.9}$$

In the set $S \subseteq \mathbb{P}$, where $\mathbb{P}$ is the set of all prime numbers, the natural density of the set S equals $d(S)$ if and only if $\lim_{x\to\infty} \frac{\#\{p \le x | p \in S\}}{\#\{p \le x | p \in \mathbb{P}\}} = d(S)$. Meher et al. [11] studied the distribution of the signs of sequence $\{\lambda_f(p^j)\}$. For $j = 1, 2$, Chiriac [1] proved that the analytic density of the set $\{p : \lambda_f^j(p) < \lambda_g^j(p)\}$ is at least $\frac{1}{16}$. In [2], he applied the pair-Sato–Tate conjecture to evaluate the natural density of the set $\{p : \lambda_f(p) < \lambda_g(p)\}$, which equals $\frac{1}{2}$. Here we prove the following.

Theorem 1.5. *Let $f, g \in H_k^*$ be two distinct nonzero cusp forms, and $j \ge 1$ is an integer. Then the natural density of the set $\{p : \lambda_{f\times f\times f}^j(p) \lesseqgtr \lambda_{g\times g\times g}^j(p)\}$ is $\frac{1}{2}$.*

The classical Landau lemma indicates that sequences $\{\lambda_f(n)\}_{n\ge1}$ have infinite sign transformations (see [11]). For any even positive integer j, Zou et al. [15] determined that the natural density of the set $\{p : \lambda_f(p)\lambda_f(p^j) < 0\}$ is $\frac{1}{2}$. Extending the method in [15], we formulate Theorems 1.6 and 1.7.

Theorem 1.6. *Let $f \in H_k^*$ be a nonzero cusp form.*
(1) *The sets*

$$P_i = \{p : \lambda_{f\times f\times f}(p^i) > 0\} \quad \text{and} \quad P_i' = \{p : \lambda_{f\times f\times f}(p^i) < 0\}$$

have a natural density $\frac{1}{2}$, where $i = 1, 3$.
(2) *The sets*

$$P_2 = \{p : \lambda_{f\times f\times f}(p^2) > 0\} \quad \text{and} \quad P_2' = \{p : \lambda_{f\times f\times f}(p^2) < 0\}$$

have natural densities $\frac{2\alpha}{\pi} + \frac{sin2\alpha}{\pi}$ and $1 - \frac{2\alpha}{\pi} - \frac{sin2\alpha}{\pi}$, respectively, where $\alpha \approx 0.5236$.

Theorem 1.7. *Let $f \in H_k^*$ be a nonzero cusp form.*
(1) *The sets*

$$\bar{P}_2 = \{p : \lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^2) > 0\}$$

and

$$\bar{P}_2' = \{p : \lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^2) < 0\}$$

have a natural density $\frac{1}{2}$.
(2) *The sets*

$$\bar{P}_3 = \{p : \lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^3) > 0\}$$

and

$$\bar{P}_3' = \{p : \lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^3) < 0\}$$

have natural densities $\frac{2\alpha}{\pi} + \frac{sin2\alpha}{\pi}$ *and* $1 - \frac{2\alpha}{\pi} - \frac{sin2\alpha}{\pi}$, *respectively, where* $\alpha \approx 0.7045$.

Remark 1.1. *When the natural density of a set exists, its analytic density also exists, and they are equal* (*see [12]*)*. Hence, the above three theorems are also applicable to the analytic density.*

2. Preliminaries

In this section, we introduce some facts and lemmas that will be useful in the proof of the main results in this paper.

Let $f \in H_k^*$ be a Hecke eigenform. The jth symmetric power L-function associated to f is defined as

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}, \quad Re\ (s) > 1,$$

which can be expanded into a Dirichlet series,

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p (1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}} + \cdots), \quad Re\ (s) > 1.$$

Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. At prime values, it is given by

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m = U_j(\lambda_f(p)/2),$$

where $U_j(x)$ represents the jth Chebyshev polynomial of the second kind.

Let $f \in H_{k_1}^*, g \in H_{k_2}^*$ be two distinct Hecke eigenforms. For $i, j \geq 1$, the Rankin–Selberg L-function related to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined as

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \prod_p \prod_{m=0}^{i} \prod_{n=0}^{j} (1 - \alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-n}\beta_g(p)^n p^{-s})^{-1}, \quad Re\ (s) > 1.$$

Similarly, we can also rewrite the aforementioned expression as

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(n)}{n^s} = \prod_p (1 + \sum_{\nu=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(p^\nu)}{p^{\nu s}}), \quad Re\ (s) > 1,$$

where $\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(n)$ is a real multiplicative function and satisfies

$$\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(p) = \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p). \tag{2.1}$$

Define the triple product L-function $L(f \times f \times f, s)$ attached to f as

$$L(f \times f \times f, s) = \prod_p (1 - \frac{\alpha_f(p)^3}{p^s})^{-1}(1 - \frac{\alpha_f(p)}{p^s})^{-3}(1 - \frac{\alpha_f(p)^{-1}}{p^s})^{-3}(1 - \frac{\alpha_f(p)^{-3}}{p^s})^{-1}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda_{f\times f\times f}(n)}{n^s}, \quad Re\ (s) > 1.$$

By [9, (2.5)], we know that

$$L(f \times f \times f, s) = L(f, s)^2 L(\mathrm{sym}^3 f, s).$$

Define $r_Q(n)$ as

$$r_Q(n) := \#\{\underline{x} \in \mathbb{Z}^2 : n = Q(\underline{x})\}.$$

The generating function $\theta_Q(\tau)$ associated with $Q(\underline{x})$ is given by a specific formula,

$$\theta_Q(\tau) = \sum_{\underline{x}\in\mathbb{Z}^2} q^{Q(\underline{x})} = \sum_{n=0}^{\infty} r_Q(n) q^n, q = e(\tau), \quad Im(\tau) > 0.$$

This function $\theta_Q(\tau)$ belongs to the space $M_1(\Gamma_0(|D|), \chi_D)$ (see, e.g., [5, Theorem 10.9]), where χ_D is the Dirichlet character modulo $|D|$ defined by the Jacobi symbol $\chi_D(d) = \left(\frac{D}{d}\right)$. According to Weil's bound, we have bound $r_Q(n) \ll n^{\varepsilon}$.

The character sum $r(n; D)$ is explicitly given in terms of the Jacobi symbol χ_D (see [5, (11.9), (11.10)]), with specific values

$$r(n; D) = \omega_D \sum_{d|n} \chi_D(d), \ where\ \omega_D = \begin{cases} 6, & if\ D = -3, \\ 4, & if\ D = -4, \\ 2, & if\ D < -4. \end{cases}$$

It is known that $r(n; D)$ counts the number of representations of a positive integer n by all the reduced forms of fixed discriminant D. For the quadratic form $Q(\underline{x})$ of discriminant $D < 0$ with class number $h(D) = 1$, we have

$$r_Q(n) = r(n; D) = \omega_D \sum_{d|n} \chi_D(d).$$

In particular, $r_Q(p) = \omega_D(1+\chi_D(p))$. Hence, the prime p is represented by $Q(\underline{x})$ if and only if $\chi_D(p) = 1$. Motivated by this fact, we define the characteristic function

$$1_Q(p) = \begin{cases} \dfrac{r_Q(p)}{2\omega_D}, & if\ p = Q(\underline{x})\ for\ some\ \underline{x} \in \mathbb{Z}^2, \\ 0, & otherwise. \end{cases} \tag{2.2}$$

This implies that $\dfrac{r_Q(p)}{\omega_D}$ is 2 or 0 according to whether p is represented by $Q(\underline{x})$ or not.

Next, we will introduce and prove some lemmas.

Lemma 2.1. *[13, Lemma 3.1] Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. When $\sigma \to 1^+$, for each $i,\ j \geq 0$, we have*

$$\sum_p \frac{\lambda_f^2(p^j)}{p^\sigma} = \sum_p \frac{1}{p^\sigma} + O(1) \ \ \text{and} \ \ \sum_p \frac{\lambda_f(p^i)\lambda_g(p^j)}{p^\sigma} = O(1).$$

Furthermore,

$$\sum_p \frac{\lambda_f^2(p^j)\chi_D(p)}{p^\sigma} = O(1), \quad \sum_p \frac{\lambda_f(p^i)\lambda_g(p^j)\chi_D(p)}{p^\sigma} = O(1).$$

In addition,

$$\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)}{p^\sigma} = O(1), \ \text{where} \ \ i \neq j.$$

Lemma 2.2. *Let $f \in H^*_k$ be a Hecke eigenform. When $\sigma \to 1^+$, for any $j,h \geq 1,\ 0 \leq i \leq j$ and $0 \leq k \leq h$, we have*

$$(1) \sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)\lambda_f(p^h)}{p^\sigma}$$

$$= \begin{cases} O(1), & k+h-i-j \text{ is odd or } h-k > i+j, \\ (1+i)\sum_p \frac{1}{p^\sigma} + O(1), & k+h-i-j \text{ is even and } h-k \leq j-i, \\ (1+i-\frac{h-k+i-j}{2})\sum_p \frac{1}{p^\sigma} + O(1), & k+h-i-j \text{ is even and } j-i < h-k \leq i+j. \end{cases}$$

$$(2) \sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)}{p^\sigma} = \begin{cases} \sum_p \frac{1}{p^\sigma} + O(1), & i+j-k \text{ is even and } j-i \leq k \leq i+j, \\ O(1), & \text{otherwise}. \end{cases}$$

Proof. From [15], we know that

$$\lambda_f(p^i)\lambda_f(p^j) = \sum_{l=0}^{i} \lambda_{\mathrm{sym}^{i+j-2l}f}(p). \tag{2.3}$$

Furthermore, (2.1) implies

$$\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)\lambda_f(p^h) = \sum_{l_1=0}^{i}\sum_{l_2=0}^{k} \lambda_{\mathrm{sym}^{i+j-2l_1}\times\mathrm{sym}^{k+h-2l_2}f}(p).$$

The Rankin–Selberg functions related to the above are entire functions if and only if $i+j-2l_1 \neq k+h-2l_2$ holds. Let $2(l_2-l_1) = k+h-i-j = b \geq 0$, and set

$$n_{l_1,l_2} = \#\{(l_2,l_1) : 2(l_2-l_1) = k+h-i-j = b,\ l_1 = 0,1,...,i \text{ and } l_2 = 0,1,...,k\}.$$

Noting that values of $l_2 - l_1$ that are at most k, when $b > 2k$, $n_{l_1,l_2} = 0$. In other words, $i+j-2l_1 \neq k+h-2l_2$. Obviously, in the cases where b is odd or $b > 2k$ (namely, $h-k > i+j$), we infer

$$\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)\lambda_f(p^h)}{p^\sigma} = O(1).$$

If b is even, we deal with it in two cases. When $b \leq 2(k-i)$ (namely, $h-k \leq j-i$), we have

$$\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)\lambda_f(p^h)}{p^\sigma} = (1+i)\sum_p \frac{1}{p^\sigma} + O(1).$$

When $2(k-i) < b \leq 2k$ (namely, $j-i \leq h-k \leq i+j$), we conclude that

$$\begin{aligned}\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k)\lambda_f(p^h)}{p^\sigma} &= (1+i-(\frac{b}{2}-(k-i)))\sum_p \frac{1}{p^\sigma} + O(1)\\ &= (1+i-\frac{h-k+i-j}{2})\sum_p \frac{1}{p^\sigma} + O(1).\end{aligned}$$

By referring to (2.1) and (2.3), we arrive at

$$\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^k) = \sum_{l=0}^{i} \lambda_{\mathrm{sym}^{i+j-2l}f\times\mathrm{sym}^k f}(p). \tag{2.4}$$

Following the proof as in (1), if $i+j-k$ is even and $j-i \leq k \leq i+j$, then the constant term in (2.4) is 1. Thus, the second assertion holds.

Lemma 2.3. *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. When $\sigma \to 1^+$, for any $j,h \geq 1,\ 0 \leq i \leq j$ and $0 \leq k \leq h$, we have*

$$(1)\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^k)}{p^\sigma} = O(1).$$

$$(2)\sum_p \frac{\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^k)\lambda_g(p^h)}{p^\sigma} = \begin{cases}\sum_p \frac{1}{p^\sigma} + O(1), & i=j,\ k=h,\\ O(1), & \textit{otherwise}.\end{cases}$$

Proof. (1) $\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^k) = \sum_{l=0}^{i} \lambda_{\mathrm{sym}^{i+j-2l}f\times\mathrm{sym}^k g}(p)$ is given similar to (2.4). The relation shows that the associated Dirichlet series can be decomposed into $L(\mathrm{sym}^{i+j}f\times\mathrm{sym}^k g,s)L(\mathrm{sym}^{i+j-2}f\times\mathrm{sym}^k g,s)\cdot\cdot\cdot L(\mathrm{sym}^{j-i}f\times\mathrm{sym}^k g,s)G(s)$, where $G(s)$ can be given in terms of a Euler product that converges absolutely for $Re\ (s) > \frac{1}{2}$ and $G(s) \neq 0$ at $s=1$. Therefore, (1) follows from the fact that the Rankin–Selberg L-function $L(\mathrm{sym}^i f\times\mathrm{sym}^j g,s)$ is an entire function except for the case when $i=j$ and $f=g$, in which case it has simple poles at $s=0,1$.
(2) The relation

$$\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^k)\lambda_g(p^h) = \sum_{l_1=0}^{i}\sum_{l_2=0}^{k} \lambda_{\mathrm{sym}^{i+j-2l_1}f}(p)\lambda_{\mathrm{sym}^{k+h-2l_2}g}(p)$$

implies the second assertion by a similar method.

Lemma 2.4. *Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Then we have*
(1)
$$\sum_p \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma} = O(1),$$
$$\sum_p \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} = 82 \sum_p \frac{1}{p^\sigma} + O(1).$$
(2)
$$\sum_p \frac{(\lambda_{f\times f\times f}(p^3) - \lambda_{g\times g\times g}(p^3))^2}{p^\sigma} = 422 \sum_p \frac{1}{p^\sigma} + O(1).$$

Proof.
$$\begin{aligned}&\sum_p \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma}\\ &= \sum_p \frac{\lambda^2_{f\times f\times f}(p^2)}{p^\sigma} + \sum_p \frac{\lambda^2_{g\times g\times g}(p^2)}{p^\sigma} - 2\sum_p \frac{\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2)}{p^\sigma}.\end{aligned} \tag{2.5}$$

For $j = 2m$, we learn from Lau and Lü [8] that
$$\lambda_f^j(p) = A_m + \sum_{1\le r\le m-1} C_m(r)\lambda_{\mathrm{sym}^{2r}f}(p) + \lambda_{\mathrm{sym}^{2m}f}(p),$$
where
$$A_m = \frac{(2m)!}{m!(m+1)!},\quad C_m(r) = \frac{(2m)!(2r+1)}{(m-r)!(m+r+1)!},\quad m \geqslant 1.$$
In particular, we have $\lambda_f^6(p) = 5 + 9\lambda_f(p^2) + 5\lambda_f(p^4) + \lambda_f(p^6)$, $\lambda_f^4(p) = 2 + 3\lambda_f(p^2) + \lambda_f(p^4)$ and $\lambda_f^2(p) = 1 + \lambda_f(p^2)$.

By the definition of the triple product L-functions, we have
$$\begin{aligned}\lambda_{f\times f\times f}(p^2) &= \alpha_f(p)^6 + 3\alpha_f(p)^4 + 9\alpha_f(p)^2 + 10 + 9\beta_f(p)^2 + 3\beta_f(p)^4 + \beta_f(p)^6\\ &= \lambda_f(p)^6 - 3\lambda_f(p)^4 + 6\lambda_f(p)^2 - 4\\ &= 1 + 6\lambda_f(p^2) + 2\lambda_f(p^4) + \lambda_f(p^6).\end{aligned} \tag{2.6}$$

Thus,
$$\begin{aligned}\lambda^2_{f\times f\times f}(p^2) = {}& 1 + 12\lambda_f(p^2) + 4\lambda_f(p^4) + 2\lambda_f(p^6) + 36\lambda_f^2(p^2) + 4\lambda_f^2(p^4)\\ &+ \lambda_f^2(p^6) + 24\lambda_f(p^2)\lambda_f(p^4) + 12\lambda_f(p^2)\lambda_f(p^6) + 4\lambda_f(p^4)\lambda_f(p^6),\end{aligned} \tag{2.7}$$

$$\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2) = 1 + \sum_{i=2,4,6} k_i\lambda_f(p^i) + \sum_{j=2,4,6} k_j\lambda_g(p^j) + \sum_{i,j=2,4,6} k_{ij}\lambda_f(p^i)\lambda_g(p^j), \tag{2.8}$$

where $k_2 = 6, k_4 = 2, k_6 = 1, k_{22} = 36, k_{24} = 12, k_{26} = 6, k_{42} = 12, k_{44} = 4, k_{46} = 2, k_{62} = 6, k_{64} = 2, k_{66} = 1$.
By Lemma 2.1 and (2.5), we deduce that

$$\sum_p \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma} = O(1) \ \ and \ \ \sum_p \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} = 82\sum_p \frac{1}{p^\sigma} + O(1).$$

Now we study $\lambda_{f\times f\times f}(p^3)$. For $j = 2m+1$, from [8] we obtain

$$\lambda_f^j(p) = B_m\lambda_f(p) + \sum_{1\le r\le m-1} D_m(r)\lambda_{\mathrm{sym}^{2r+1}f}(p) + \lambda_{\mathrm{sym}^{2m+1}f}(p), \tag{2.9}$$

where

$$B_m = \frac{2(2m+1)!}{m!(m+2)!}, \quad D_m(r) = \frac{(2m+1)!(2r+2)}{(m-r)!(m+r+2)!}, \quad m \geqslant 1.$$

According to the definition of the triple product L-functions and (2.9), we have

$$\begin{aligned}\lambda_{f\times f\times f}(p^3) &= \alpha_f(p)^9 + 3\alpha_f(p)^7 + 9\alpha_f(p)^5 + 20\alpha_f(p)^3 + 27\alpha_f(p)\\ &\quad + 27\beta_f(p) + 20\beta_f(p)^3 + 9\beta_f(p)^5 + 3\beta_f(p)^7 + \beta_f(p)^9\\ &= \lambda_f(p)^9 - 6\lambda_f(p)^7 + 15\lambda_f(p)^5 - 13\lambda_f^3(p)\\ &= 7\lambda_f(p) + 11\lambda_f(p^3) + 6\lambda_f(p^5) + 2\lambda_f(p^7) + \lambda_f(p^9).\end{aligned} \tag{2.10}$$

Therefore,

$$\begin{aligned}\lambda^2_{f\times f\times f}(p^3) &= 49\lambda_f^2(p) + 154\lambda_f(p)\lambda_f(p^3) + 84\lambda_f(p)\lambda_f(p^5) + 28\lambda_f(p)\lambda_f(p^7) + 14\lambda_f(p)\lambda_f(p^9)\\ &\quad + 121\lambda_f^2(p^3) + 132\lambda_f(p^3)\lambda_f(p^5) + 44\lambda_f(p^3)\lambda_f(p^7) + 22\lambda_f(p^3)\lambda_f(p^9) + 36\lambda_f^2(p^5)\\ &\quad + 24\lambda_f(p^5)\lambda_f(p^7) + 12\lambda_f(p^5)\lambda_f(p^9) + 4\lambda_f^2(p^7) + 4\lambda_f(p^7)\lambda_f(p^9) + \lambda_f^2(p^9),\end{aligned} \tag{2.11}$$

$$\lambda_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3) = \sum_{i,j=1,3,5,7,9} k_{ij}\lambda_f(p^i)\lambda_g(p^j),$$

where $k_{11} = 49, k_{13} = 77, k_{15} = 42, k_{17} = 14, ..., k_{91} = 7, k_{93} = 11, k_{95} = 6, k_{97} = 2, k_{99} = 1$.
By analogy with (2.5), we finally conclude from Lemma 2.1 that (2) holds.

Lemma 2.5. *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then we have*

(1) $$\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} = 41\sum_p \frac{1}{p^\sigma} + O(1).$$

(2) $$\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^3) - \lambda_{g\times g\times g}(p^3))^2}{p^\sigma} = 211\sum_p \frac{1}{p^\sigma} + O(1).$$

Proof. To avoid repetition, we only prove the first case.

In this lemma, because the summation extends over the primes that can be expressed in the form $Q(\underline{x})$, we introduce the characteristic function $1_{Q(p)}$ (2.2) and obtain

$$\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2}{p^\sigma} = \sum_p \frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2 1_Q(p)}{p^\sigma}$$
$$= \frac{1}{2\omega_D}\sum_p \frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2 r_Q(p)}{p^\sigma} = \frac{1}{2}\sum_p \frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2(1+\chi_D(p))}{p^\sigma}$$
$$= \frac{1}{2}(\sum_p \frac{\lambda^2_{f\times f\times f}(p^2)}{p^\sigma} + \sum_p \frac{\lambda^2_{g\times g\times g}(p^2)}{p^\sigma} - 2\sum_p \frac{\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2)}{p^\sigma})$$
$$+ \frac{1}{2}(\sum_p \frac{\lambda^2_{f\times f\times f}(p^2)\chi_D(p)}{p^\sigma} + \sum_p \frac{\lambda^2_{g\times g\times g}(p^2)\chi_D(p)}{p^\sigma} - 2\sum_p \frac{\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2)\chi_D(p)}{p^\sigma}).$$

By (2.7), (2.8) and Lemma 2.1, we prove (1).

Lemma 2.6. *Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Then we have*
(1)

$$\sum_p \frac{\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2)}{p^\sigma} = O(1)\ and\ \sum_p \frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} = 32924\sum_p \frac{1}{p^\sigma} + O(1).$$

(2)

$$\sum_p \frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma} = 2102314\sum_p \frac{1}{p^\sigma} + O(1).$$

Proof. Substituting (1.1) into (2.7), we have

$$\begin{aligned}\lambda^2_{f\times f\times f}(p^2) &= 42 + 53\lambda_f(p^2) + 45\lambda_f(p^4) + 7\lambda_f(p^6) + 5\lambda_f(p^8) + \lambda_f(p^{10}) + \lambda_f(p^{12})\\ &\quad + 24\lambda_f(p^2)\lambda_f(p^4) + 12\lambda_f(p^2)\lambda_f(p^6) + 4\lambda_f(p^4)\lambda_f(p^6). \end{aligned} \tag{2.12}$$

Hence,

$$\begin{aligned}\lambda^4_{f\times f\times f}(p^2) &= 2500 + 6161\lambda^2_f(p^2) + 5457\lambda^2_f(p^4) + 1001\lambda^2_f(p^6) + 41\lambda^2_f(p^8) + \lambda^2_f(p^{10}) + \lambda^2_f(p^{12})\\ &\quad + \sum_{\substack{i=2,4,6,\\8,10,12}} k_i\lambda_f(p^i) + \sum_{\substack{i,j=2,4,\\6,8,10,12\\ i\neq j}} k_{ij}\lambda_f(p^i)\lambda_f(p^j) + \sum_{\substack{i,j,h=2,\\4,6,8,10,12\\ i\neq j\neq h}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^h),\end{aligned}$$

where $k_2 = 8540, k_4 = 7948, k_6 = 3740, k_8 = 1172, ..., k_{248} = 336, k_{268} = 312, k_{468} = 40$.

$$\begin{aligned}\lambda^2_{f\times f\times f}(p^2)\lambda^2_{g\times g\times g}(p^2) &= 1764 + \sum_{\substack{i=2,4,6,\\8,10,12}} k_i\lambda_f(p^i) + \sum_{\substack{j=2,4,6,\\8,10,12}} k_j\lambda_g(p^j) + \sum_{\substack{i,j=2,4,\\6,8,10,12}} k'_{ij}\lambda_f(p^i)\lambda_g(p^j)\\ &\quad + \sum_{\substack{i,j=2,4,\\6,8,10,12\\ i\neq j}} k_{ij}\lambda_f(p^i)\lambda_f(p^j) + \sum_{\substack{i,j=2,4,\\6,8,10,12\\ i\neq j}} k_{ij}\lambda_g(p^i)\lambda_g(p^j) + \sum_{\substack{i,j,h=2,4,\\6,8,10,12\\ j\neq h}} k_{ijh}\lambda_f(p^i)\lambda_g(p^j)\lambda_g(p^h)\end{aligned}$$

$$+\sum_{\substack{i,j,h=2,4,\\6,8,10,12\\i\neq j}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h)+\sum_{\substack{i,j,h,l=2,4,\\6,8,10,12\\i\neq j,h\neq l}} k_{ijhl}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h)\lambda_g(p^l),$$

where $k_2 = 2226, k_4 = 1890, k_6 = 294, k_8 = 210, ..., k_{24} = 1008, k_{26} = 504, k_{46} = 168, ..., k_{4624} = 96, k_{4626} = 48, k_{4646} = 16$.
To see that (1) holds, we use Lemmas 2.1 and 2.3.

Now we study the case when $\lambda_{f\times f\times f}(p^3)$. Substituting (1.1) into (2.11), we have

$$\begin{aligned}\lambda^2_{f\times f\times f}(p^3) &= 211 + 211\lambda_f(p^2) + 162\lambda_f(p^4) + 162\lambda_f(p^6) + 41\lambda_f(p^8) + 41\lambda_f(p^{10}) + 5\lambda_f(p^{12})\\ &+ 5\lambda_f(p^{14}) + \lambda_f(p^{16}) + \lambda_f(p^{18}) + 154\lambda_f(p)\lambda_f(p^3) + 84\lambda_f(p)\lambda_f(p^5) + 28\lambda_f(p)\lambda_f(p^7)\\ &+ 14\lambda_f(p)\lambda_f(p^9) + 132\lambda_f(p^3)\lambda_f(p^5) + 44\lambda_f(p^3)\lambda_f(p^7) + 22\lambda_f(p^3)\lambda_f(p^9)\\ &+ 24\lambda_f(p^5)\lambda_f(p^7) + 12\lambda_f(p^5)\lambda_f(p^9) + 4\lambda_f(p^7)\lambda_f(p^9). \end{aligned}\tag{2.13}$$

Hence,

$$\begin{aligned}\lambda^4_{f\times f\times f}(p^3) &= 96853 + 96853\lambda^2_f(p^2) + 46824\lambda^2_f(p^4) + 46824\lambda^2_f(p^6) + 2417\lambda^2_f(p^8)\\ &+ 2417\lambda^2_f(p^{10}) + 41\lambda^2_f(p^{12}) + 41\lambda^2_f(p^{14}) + \lambda^2_f(p^{16}) + \lambda^2_f(p^{18})\\ &+\sum_{\substack{i=2,4,6,8,10,\\12,14,16,18}} k_i\lambda_f(p^i)+\sum_{\substack{i,j=2,4,6,8,\\10,12,14,16,18\\i\neq j}} k_{ij}\lambda_f(p^i)\lambda_f(p^j)+\sum_{\substack{i,j=1,\\3,5,7,9\\i\neq j}} k_{ij}\lambda_f(p^i)\lambda_f(p^j)\\ &+\sum_{\substack{i=2,4,6,8,10,\\12,14,16,18\\j,h=1,3,5,7,9\\j\neq h}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^h)+\sum_{\substack{i,j,h,l=\\1,3,5,7,9\\i\neq j\neq h\neq l}} k_{ijhl}\lambda_f(p^i)\lambda_f(p^j)\lambda_f(p^h)\lambda_f(p^l),\end{aligned}$$

where $k_2 = 89042, k_4 = 68364, k_6 = 68364, k_8 = 17302, ..., k_{1579} = 2016, k_{3579} = 3168$.

$$\begin{aligned}\lambda^2_{f\times f\times f}(p^3)\lambda^2_{g\times g\times g}(p^3) &= 44521 +\sum_{\substack{i=2,4,6,8,10,\\12,14,16,18}} k_i\lambda_f(p^i)+\sum_{\substack{j=2,4,6,8,10,\\12,14,16,18}} k_j\lambda_g(p^j)+\sum_{\substack{i,j=1,\\3,5,7,9\\i\neq j}} k_{ij}\lambda_f(p^i)\lambda_f(p^j)\\ &+\sum_{\substack{i,j=2,4,\\6,8,10,12,\\14,16,18}} k_{ij}\lambda_f(p^i)\lambda_g(p^j)+\sum_{\substack{i,j=1,\\3,5,7,9\\i\neq j}} k_{ij}\lambda_g(p^i)\lambda_g(p^j)+\sum_{\substack{i=2,4,6,8,10,\\12,14,16,18\\j,h=1,3,5,7,9\\j\neq h}} k_{ijh}\lambda_f(p^i)\lambda_g(p^j)\lambda_g(p^h)\\ &+\sum_{\substack{h=2,4,6,8,10,\\12,14,16,18\\i,j=1,3,5,7,9\\i\neq j}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h)+\sum_{\substack{i,j=1,3,5,7,9\\h,l=1,3,5,7,9\\i\neq j,h\neq l}} k_{ijhl}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h)\lambda_g(p^l),\end{aligned}$$

where $k_2 = 44521, k_4 = 34182, k_6 = 34182, k_8 = 8651, ..., k_{7959} = 48, k_{7979} = 16$.
From the calculations above, we show that (2) holds.

Lemma 2.7. *Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Then we have*
(1)

$$\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} = 16462\sum_p \frac{1}{p^\sigma} + O(1).$$

(2)

$$\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma} = 1051157\sum_p \frac{1}{p^\sigma} + O(1).$$

Proof. Lemma 2.7 readily follows from the preceding analysis in Lemmas 2.5 and 2.6.

3. Proof of Theorems 1.1 and 1.2

In order to demonstrate Theorems 1.1 and 1.2, we introduce the lemma below.

3.1. Lemma

Lemma 3.1. *Let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Then*

$$(1)\ |\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2)| \le 52 \ \ and\ \ |\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3)| \le 240.$$
$$(2)\ |\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2)| \le 2028 \ \ and\ \ |\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3)| \le 23920.$$

Proof. From [7, Lemma 3.1], for any $j \ge 1$, we find that

$$|\lambda_f(p^j)-\lambda_g(p^j)| \le 4[\frac{j+1}{2}], \tag{3.1}$$

where $[t]$ denotes the integral part of t. Hence, (2.6), (2.10), and (3.1) yield

$$\begin{aligned}&|\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2)|\\ &\le |\lambda_f(p^6)-\lambda_g(p^6)| + 2|\lambda_f(p^4)-\lambda_g(p^4)| + 6|\lambda_f(p^2)-\lambda_g(p^2)| \le 52\end{aligned}$$

and $|\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3)| \le 240$.

From [15, Lemma 6], for any $j \ge 1,\ 0 \le i \le j$, we have

$$|\lambda_f(p^i)\lambda_f(p^j)-\lambda_g(p^i)\lambda_g(p^j)| \le 2(ij+j+\delta), \tag{3.2}$$

where $\delta = \begin{cases} i+1, & 2\nmid(i+j),\\ 0, & 2\mid(i+j).\end{cases}$ By (2.12), (2.13), (3.1), and (3.2), we arrive at $|\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2)| \le 2028$ and $|\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3)| \le 23920$.

3.2. Proof of Theorem 1.1

We consider the set $X_2 = \{p : \lambda_{f\times f\times f}(p^2) < \lambda_{g\times g\times g}(p^2)\}$. In order to establish the lower bound for the analytic density of the set X_2, we analyze the following summation and obtain

$$\begin{aligned}
&\sum_{p\in X_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \\
&= \sum_{p\in X_2} \frac{|\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)||\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)|}{p^\sigma} \\
&\leq 2704 \sum_{p\in X_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \qquad (3.3)
\end{aligned}$$

If $p \notin X_2$, then $\lambda_{f\times f\times f}(p^2) \geq \lambda_{g\times g\times g}(p^2)$. Applying the first assertion in Lemma 2.4 yields:

$$\begin{aligned}
&\sum_{p\notin X_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \leq 52 \sum_{p\notin X_2} \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma} \\
&= 52(\sum_{p} \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma} - \sum_{p\in X_2} \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma}) \\
&\leq 52(O(1) + \sum_{p\in X_2} \frac{\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)}{p^\sigma}) \\
&\leq 2704 \sum_{p\in X_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \qquad (3.4)
\end{aligned}$$

Now we combine the estimates from (3.3) and (3.4) to get

$$\begin{aligned}
&\sum_{p} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \\
&= \sum_{p\in X_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} + \sum_{p\notin X_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \\
&\leq 5408 \sum_{p\in X_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \qquad (3.5)
\end{aligned}$$

Lemma 2.4 and (3.5) lead to

$$82 \sum_{p} \frac{1}{p^\sigma} + O(1) = \sum_{p} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \leq 5408 \sum_{p\in X_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

This shows that the analytic density of the set X_2 is at least $\frac{41}{2704}$.

We discuss the set $Y_2 = \{p : \lambda^2_{f\times f\times f}(p^2) < \lambda^2_{g\times g\times g}(p^2)\}$. This proof parallels the approach taken in that X_2, with the primary distinction lying in the substitution of Lemma 2.4 for Lemma 2.6.

$$\sum_{p\in Y_2} \frac{(\lambda^2_{f\times f\times f}(p^2) - \lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} \leq 4112784 \sum_{p\in Y_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

$$\sum_{p\notin Y_2}\frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma}\leq 4112784\sum_{p\in Y_2}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

Now, by combining the estimates above, we derive that

$$32924\sum_{p}\frac{1}{p^\sigma}+O(1)=\sum_{p}\frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma}\leq 8225568\sum_{p\in Y_2}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

The preceding discussion confirms that the analytic density of the set Y_2 is at least $\frac{8231}{2056392}$.

Similarly, we consider the sets X_3 and Y_3.

$$\sum_{p\in X_3}\frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 57600\sum_{p\in X_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$
$$\sum_{p\notin X_3}\frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 57600\sum_{p\in X_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

Combining these two estimates and Lemma 2.4 implies that

$$422\sum_{p}\frac{1}{p^\sigma}+O(1)=\sum_{p}\frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 115200\sum_{p\in X_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

This shows that the analytic density of the set X_3 is at least $\frac{211}{57600}$.

$$\sum_{p\in Y_3}\frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 57216640\sum_{p\in Y_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$
$$\sum_{p\notin Y_3}\frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 57216640\sum_{p\in Y_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

According to Lemma 2.6, as $\sigma\to 1^+$, we have

$$2102314\sum_{p}\frac{1}{p^\sigma}+O(1)=\sum_{p}\frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma}\leq 114433280\sum_{p\in Y_3}\frac{1}{p^\sigma}+O(1).$$

This directly yields that the analytic density of the set Y_3 is at least $\frac{1051157}{572166400}$.

3.3. Proof of Theorem 1.2

We focus on the set $A_2=\{p:p=Q(\underline{x})$ for some $\underline{x}\in\mathbb{Z}^2$ and $\lambda_{f\times f\times f}(p^2)<\lambda_{g\times g\times g}(p^2)\}$. By the characteristic function (2.2) and Lemma 3.1, we have

$$\sum_{\substack{p\in A_2,\,p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}}\frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2}{p^\sigma}=\frac{1}{2\omega_D}\sum_{p\in A_2}\frac{(\lambda_{f\times f\times f}(p^2)-\lambda_{g\times g\times g}(p^2))^2 r_Q(p)}{p^\sigma}$$

$$= \frac{1}{2\omega_D} \sum_{p \in A_2} \frac{|\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)||\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2)|r_Q(p)}{p^\sigma}$$

$$\leq 2704 \sum_{p \in A_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

If $p \notin A_2$, then $\lambda_{f\times f\times f}(p^2) \geq \lambda_{g\times g\times g}(p^2)$ or p cannot be represented by $Q(\underline{x})$ for any $\underline{x} \in \mathbb{Z}^2$. In the second case, $r_Q(n) = 0$. Moreover, $|\frac{r_Q(p)}{\omega_D}| \leq 2$. This results in the following estimate by Lemma 2.4 (1):

$$\sum_{\substack{p \notin A_2,\, p=Q(\underline{x}) \\ for\ some\ \underline{x} \in \mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} = \frac{1}{2\omega_D} \sum_{p \notin A_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2 r_Q(p)}{p^\sigma}$$

$$\leq \frac{26}{\omega_D}\Big(\sum_{p} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))r_Q(p)}{p^\sigma} - \sum_{p \in A_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))r_Q(p)}{p^\sigma}\Big)$$

$$\leq 52\Big(O(1) + \frac{1}{2\omega_D} \sum_{p \in A_2} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))r_Q(p)}{p^\sigma}\Big)$$

$$\leq 2704 \sum_{p \in A_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

The two estimates above imply

$$\sum_{\substack{p=Q(\underline{x}) \\ for\ some\ \underline{x} \in \mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma}$$

$$= \sum_{\substack{p \in A_2,\, p=Q(\underline{x}) \\ for\ some\ \underline{x} \in \mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} + \sum_{\substack{p \notin A_2,\, p=Q(\underline{x}) \\ for\ some\ \underline{x} \in \mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma}$$

$$\leq 5408 \sum_{p \in A_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \tag{3.6}$$

On combining Lemma 2.5 and (3.6), we infer that

$$41 \sum_{p} \frac{1}{p^\sigma} + O(1) = \sum_{\substack{p=Q(\underline{x}) \\ for\ some\ x \in \mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^2) - \lambda_{g\times g\times g}(p^2))^2}{p^\sigma} \leq 5408 \sum_{p \in A_2} \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

This shows that the analytic density of the set A_2 is at least $\frac{41}{5408}$.

We deal with the set $B_2 = \{p : p = Q(\underline{x})$ for some $\underline{x} \in \mathbb{Z}^2$ and $\lambda^2_{f\times f\times f}(p^2) < \lambda^2_{g\times g\times g}(p^2)\}$. This proof follows the same line as that of A_2, and the main difference is the use of Lemmas 2.6 and 2.7 instead of Lemmas 2.4 and 2.5.

$$\sum_{\substack{p\in B_2,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} \le 4112784 \sum_{p\in B_2}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

$$\sum_{\substack{p\notin B_2,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} \le 4112784 \sum_{p\in B_2}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

Now we combine the two estimates and utilize Lemma 2.7 to get

$$16462\sum_{p}\frac{1}{p^\sigma}+O(1)=\sum_{\substack{p=Q(\underline{x})\ for\\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^2)-\lambda^2_{g\times g\times g}(p^2))^2}{p^\sigma} \le 8225568 \sum_{p\in B_2}\frac{1}{p^\sigma}+O(1),$$

$as\ \sigma\to 1^+.$

It is evident that the analytic density of the set B_2 is at least $\frac{8231}{4112784}$.

Similarly, we turn to the sets A_3 and B_3.

$$\sum_{\substack{p\in A_3,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma} \le 57600 \sum_{p\in A_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

$$\sum_{\substack{p\notin A_3,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma} \le 57600 \sum_{p\in A_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

Hence, from Lemma 2.5, we find that

$$211\sum_{p}\frac{1}{p^\sigma}+O(1)=\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda_{f\times f\times f}(p^3)-\lambda_{g\times g\times g}(p^3))^2}{p^\sigma}$$
$$\le 115200\sum_{p\in A_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

We thus demonstrate that the analytic density of the set A_3 is at least $\frac{211}{115200}$.

$$\sum_{\substack{p\in B_3,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma} \le 57216640 \sum_{p\in B_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

$$\sum_{\substack{p\notin B_3,\, p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}} \frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^\sigma} \le 57216640 \sum_{p\in B_3}\frac{1}{p^\sigma}+O(1),\ as\ \sigma\to 1^+.$$

In view of Lemma 2.7, we deduce that

$$1051157\sum_{p}\frac{1}{p^{\sigma}}+O(1)=\sum_{\substack{p=Q(\underline{x})\\ for\ some\ \underline{x}\in\mathbb{Z}^2}}\frac{(\lambda^2_{f\times f\times f}(p^3)-\lambda^2_{g\times g\times g}(p^3))^2}{p^{\sigma}}$$
$$\leq 114433280\sum_{p\in B_3}\frac{1}{p^{\sigma}}+O(1),\ as\ \sigma\to 1^+.$$

This shows that the analytic density of the set B_3 is at least $\frac{1051157}{1144332800}$.

4. Proof of Theorems 1.3 and 1.4

To prove Theorems 1.3 and 1.4, we provide the following lemmas.

4.1. Some lemmas

Lemma 4.1. *[15, Lemma 4] Let $K(p)$ be a set of real numbers, and $|K(p)|\leq B$ with a absolute bound B. Suppose that there exist two absolute constants m, M such that*

$$\sum_{p}\frac{K^2(p)}{p^{\sigma}}=m\sum_{p}\frac{1}{p^{\sigma}}+O(1)$$

and

$$\sum_{p}\frac{K(p)}{p^{\sigma}}=M\sum_{p}\frac{1}{p^{\sigma}}+O(1),$$

as $\sigma\to 1^+$. Then the set $\{p: K(p)<0\}$ has an analytic density at least $\frac{m-MB}{2B^2}$.

Lemma 4.2. *[4, Lemma 2.3] Let $K(p)$ be a set of real numbers, and $|K(p)|\leq B$ with a absolute bound B. Suppose that there exist two absolute constants c, C such that*

$$\sum_{p}\frac{K^2(p)1_Q(p)}{p^{\sigma}}=c\sum_{p}\frac{1}{p^{\sigma}}+O(1)$$

and

$$\sum_{p}\frac{K(p)1_Q(p)}{p^{\sigma}}=C\sum_{p}\frac{1}{p^{\sigma}}+O(1),$$

as $\sigma\to 1^+$. Then the set $\{p: p=Q(\underline{x})\ for\ some\ \underline{x}\in\mathbb{Z}^2\ and\ K(p)<0\}$ has an analytic density at least $\frac{c-CB}{2B^2}$.

4.2. Proof of Theorem 1.3

Set

$$S_i(p)=(c_1\lambda_{f\times f\times f}(p^i)+c_2\lambda_{g\times g\times g}(p^i)-a)(c_1\lambda_{f\times f\times f}(p^i)+c_2\lambda_{g\times g\times g}(p^i)-b)$$

$$= c_1^2\lambda_{f\times f\times f}^2(p^i) + c_2^2\lambda_{g\times g\times g}^2(p^i) + 2c_1c_2\lambda_{f\times f\times f}(p^i)\lambda_{g\times g\times g}(p^i)$$
$$- (a+b)(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i)) + ab, \quad i = 1,2,3. \tag{4.1}$$

Therefore,

$$\sum_p \frac{S_i(p)}{p^\sigma} = c_1^2\sum_p \frac{\lambda_{f\times f\times f}^2(p^i)}{p^\sigma} + c_2^2\sum_p \frac{\lambda_{f\times f\times f}^2(p^i)}{p^\sigma} + 2c_1c_2\sum_p \frac{\lambda_{f\times f\times f}(p^i)\lambda_{g\times g\times g}(p^i)}{p^\sigma}$$
$$- (a+b)\sum_p \frac{c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i)}{p^\sigma} + ab\sum_p \frac{1}{p^\sigma}$$

and

$$\sum_p \frac{S_i^2(p)}{p^\sigma} = a^2b^2\sum_p \frac{1}{p^\sigma}$$
$$- 2ab(a+b)\sum_p \frac{c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i)}{p^\sigma} - 2(a+b)\sum_p \frac{(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^3}{p^\sigma}$$
$$+ ((a+b)^2 + 2ab)\sum_p \frac{(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^2}{p^\sigma} + \sum_p \frac{(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^4}{p^\sigma},$$

where

$$\begin{aligned}
(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^2 &= c_1^2\lambda_{f\times f\times f}^2(p^i) + 2c_1c_2\lambda_{f\times f\times f}(p^i)\lambda_{g\times g\times g}(p^i) + c_2^2\lambda_{g\times g\times g}^2(p^i),\\
(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^3 &= c_1^3\lambda_{f\times f\times f}^3(p^i) + 3c_1^2c_2\lambda_{f\times f\times f}^2(p^i)\lambda_{g\times g\times g}(p^i)\\
&\quad + 3c_1c_2^2\lambda_{f\times f\times f}(p^i)\lambda_{g\times g\times g}^2(p^i) + c_2^3\lambda_{g\times g\times g}^2(p^i),\\
(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i))^4 &= c_1^4\lambda_{f\times f\times f}^4(p^i) + 4c_1^3c_2\lambda_{f\times f\times f}^3(p^i)\lambda_{g\times g\times g}(p^i)\\
&\quad + 6c_1^2c_2^2\lambda_{f\times f\times f}^2(p^i)\lambda_{g\times g\times g}^2(p^i)\\
&\quad + 4c_1c_2^3\lambda_{f\times f\times f}(p^i)\lambda_{g\times g\times g}^3(p^i) + c_2^4\lambda_{g\times g\times g}^4(p^i).
\end{aligned}$$

Because the proofs for these three cases are similar, to avoid redundancy, we only prove the third case. We derive by (1.2) and (2.10) that

$$|\lambda_{f\times f\times f}(p^3)| \le |\alpha_f(p)|^9 + 3|\alpha_f(p)|^7 + 9|\alpha_f(p)|^5 + 20|\alpha_f(p)|^3 + 27|\alpha_f(p)|$$
$$+ 27|\beta_f(p)| + 20|\beta_f(p)|^3 + 9|\beta_f(p)|^5 + 3|\beta_f(p)|^7 + |\beta_f(p)|^9 = 120.$$

Thus,

$$|\lambda_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3)| \le 14400, \quad |\lambda_{f\times f\times f}^2(p^3)| \le 14400.$$

Following the formula of (4.1), we find that

$$|S_3(p)| \le 14400(|c_1| + |c_2|)^2 + 120(|a| + |b|)(|c_1| + |c_2|) + |ab|.$$

Lemma 2.1 gives

$$\sum_p \frac{S_3(p)}{p^\sigma} = (211c_1^2 + 211c_2^2 + ab)\sum_p \frac{1}{p^\sigma} + O(1),\ as\ \sigma \to 1^+. \tag{4.2}$$

Now we calculate $\lambda^3_{f\times f\times f}(p^3)$, $\lambda^2_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3)$ and $\lambda^3_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3)$.

$$\begin{aligned}\lambda^3_{f\times f\times f}(p^3) &= \lambda^2_{f\times f\times f}(p^3)\lambda_{f\times f\times f}(p^3)\\ &= \sum_{i=1,3,5,7,9} k_i\lambda_f(p^i) + \sum_{\substack{i=1,3,5,7,9\\ j=2,4,6,8,10,\\ 12,14,16,18}} k_{ij}\lambda_f(p^i)\lambda_f(p^j),\end{aligned}$$

$$\begin{aligned}&\lambda^2_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3)\\ &= \sum_{i=1,3,5,7,9} k_i\lambda_g(p^i) + \sum_{\substack{i=2,4,6,8,\\ 10,12,14\\ j=1,3,5,7,9}} k_{ij}\lambda_f(p^i)\lambda_g(p^j) + \sum_{\substack{i,j=1,3,5,7,9\\ i\neq j\\ h=1,3,5,7,9}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h),\end{aligned}$$

$$\lambda^3_{f\times f\times f}(p^3)\lambda_{g\times g\times g}(p^3) = \sum_{\substack{i=1,3,5,7,9\\ j=1,3,5,7,9}} k_{ij}\lambda_f(p^i)\lambda_g(p^j) + \sum_{\substack{i=2,4,6,8,10,\\ 12,14,16,18\\ j=1,3,5,7,9\\ h=1,3,5,7,9}} k_{ijh}\lambda_f(p^i)\lambda_f(p^j)\lambda_g(p^h).$$

It follows from Lemmas 2.1 and 2.2 in combination with Lemma 2.3 that

$$\begin{aligned}\sum_p \frac{S_3^2(p)}{p^\sigma} &= ((a^2 + b^2 + 4ab)(211c_1^2 + 211c_2^2) + 1095678(c_1^4 + c_2^4)\\ &\quad + 267126c_1^2c_2^2 + a^2b^2)\sum_p \frac{1}{p^\sigma} + O(1),\ as\ \sigma \to 1^+.\end{aligned} \tag{4.3}$$

According to Lemma 4.1, we obtain that the set F_3 has an analytic density at least $f(3, a, b, c_1, c_2)$, where $f(3, a, b, c_1, c_2)$ is defined as in (1.6).
Next, we present the results of the first and second cases.
Case 1:

$$|S_1(p)| \leq 64(|c_1| + |c_2|)^2 + 8(|a| + |b|)(|c_1| + |c_2|) + |ab|.$$

$$\sum_p \frac{S_1(p)}{p^\sigma} = (5c_1^2 + 5c_2^2 + ab)\sum_p \frac{1}{p^\sigma} + O(1),\ as\ \sigma \to 1^+. \tag{4.4}$$

$$\sum_p \frac{S_1^2(p)}{p^\sigma} = ((a^2 + b^2 + 4ab)(5c_1^2 + 5c_2^2) + 132(c_1^4 + c_2^4) + 150c_1^2c_2^2 + a^2b^2)\sum_p \frac{1}{p^\sigma} + O(1), \tag{4.5}$$

as $\sigma \to 1^+$. According to Lemma 4.1, we obtain that the set F_1 has an analytic density at least $f(1, a, b, c_1, c_2)$, where $f(1, a, b, c_1, c_2)$ is defined as in (1.4).

Case 2:

$$|S_2(p)| \le 1296(|c_1| + |c_2|)^2 + 36(|a| + |b|)(|c_1| + |c_2|) + |ab|.$$

$$\sum_p \frac{S_2(p)}{p^\sigma} = (42c_1^2 + 42c_2^2 + 2c_1c_2 - (c_1 + c_2)(a + b) + ab)\sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \tag{4.6}$$

$$\begin{aligned}\sum_p \frac{S_2^2(p)}{p^\sigma} &= (-2ab(a + b)(c_1 + c_2) + (a^2 + b^2 + 4ab)(42c_1^2 + 42c_2^2 + 2c_1c_2) \\ &\quad - (a + b)(1490(c_1^3 + c_2^3) + 252(c_1^2c_2 + c_1c_2^2)) + 18226(c_1^4 + c_2^4) + 2980(c_1^3c_2 + c_1c_2^3) \\ &\quad + 10584c_1^2c_2^2 + a^2b^2)\sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+. \end{aligned}\tag{4.7}$$

In light of Lemma 4.1, we obtain that the set F_2 has an analytic density at least $f(2, a, b, c_1, c_2)$, where $f(2, a, b, c_1, c_2)$ is defined as in (1.5).

4.3. *Proof of Theorem 1.4*

Set

$$S_i(p) = (c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i) - a)(c_1\lambda_{f\times f\times f}(p^i) + c_2\lambda_{g\times g\times g}(p^i) - b), \ \ i = 1, 2, 3.$$

$$\sum_p \frac{S_i(p)1_Q(p)}{p^\sigma} = \frac{1}{2\omega_D}\sum_p \frac{S_i(p)r_Q(p)}{p^\sigma} = \frac{1}{2}\sum_p \frac{S_i(p)(1 + \chi_D(p))}{p^\sigma}.$$

$$\sum_p \frac{S_i^2(p)1_Q(p)}{p^\sigma} = \frac{1}{2\omega_D}\sum_p \frac{S_i^2(p)r_Q(p)}{p^\sigma} = \frac{1}{2}\sum_p \frac{S_i^2(p)(1 + \chi_D(p))}{p^\sigma}.$$

We invoke (4.4) and (4.5) to compute

$$\sum_p \frac{S_1(p)1_Q(p)}{p^\sigma} = (\frac{5}{2}c_1^2 + \frac{5}{2}c_2^2 + \frac{1}{2}ab)\sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+,$$

$$\begin{aligned}\sum_p \frac{S_1^2(p)1_Q(p)}{p^\sigma} &= ((a^2 + b^2 + 4ab)(\frac{5}{2}c_1^2 + \frac{5}{2}c_2^2) + 66(c_1^4 + c_2^4) \\ &\quad + 75c_1^2c_2^2 + \frac{1}{2}a^2b^2)\sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.\end{aligned}$$

With reference to Lemma 4.2, we obtain that the set E_1 has an analytic density at least $e(1, a, b, c_1, c_2)$, where $e(1, a, b, c_1, c_2)$ is defined as in (1.7).

For E_2, applying (4.6) and (4.7), we have

$$\sum_p \frac{S_2(p)1_Q(p)}{p^\sigma} = (21c_1^2 + 21c_2^2 + c_1c_2 - \frac{1}{2}(c_1 + c_2)(a + b) + \frac{1}{2}ab) \sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

$$\begin{aligned}\sum_p \frac{S_2^2(p)1_Q(p)}{p^\sigma} &= (-ab(a + b)(c_1 + c_2) + (a^2 + b^2 + 4ab)(21c_1^2 + 21c_2^2 + c_1c_2) \\ &\quad - (a + b)(745(c_1^3 + c_2^3) + 126(c_1^2c_2 + c_1c_2^2)) + 9113(c_1^4 + c_2^4) + 1490(c_1^3c_2 + c_1c_2^3) \\ &\quad + 5292c_1^2c_1^2 + \frac{1}{2}a^2b^2) \sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.\end{aligned}$$

Following Lemma 4.2, we derive that the set E_2 has an analytic density at least $e(2, a, b, c_1, c_2)$, where $e(2, a, b, c_1, c_2)$ is defined as in (1.8).

For E_3, employing (4.2) and (4.3), we have

$$\sum_p \frac{S_3(p)1_Q(p)}{p^\sigma} = \frac{1}{2}(211c_1^2 + 211c_2^2 + ab) \sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.$$

$$\begin{aligned}\sum_p \frac{S_3^2(p)1_Q(p)}{p^\sigma} &= \frac{1}{2}((a^2 + b^2 + 4ab)(211c_1^2 + 211c_2^2) + 1095678(c_1^4 + c_2^4) \\ &\quad + 267126c_1^2c_1^2 + a^2b^2) \sum_p \frac{1}{p^\sigma} + O(1), \ as\ \sigma \to 1^+.\end{aligned}$$

Lemma 4.2 shows that the set E_3 has an analytic density at least $e(3, a, b, c_1, c_2)$, where $e(3, a, b, c_1, c_2)$ is defined as in (1.9).

5. Proof of Theorem 1.5

5.1. Lemma

We define the Pair-Sato–Tate measure and state the Pair-Sato–Tate conjecture, which will be used to prove Theorem 1.5 (see [14, Theorem 1.1]).

Definition 5.1. *The Pair-Sato–Tate measure μ_{ST}^2 is a probability measure on the interval $[0, \pi]$ and* $\mu_{ST}^2 = \frac{4}{\pi^2} \sin^2 \theta_f \mathrm{d}\theta_f \sin^2 \theta_g \mathrm{d}\theta_g$.

Lemma 5.2. (*Pair-Sato–Tate conjecture*) *Let $f, g \in H_k^*$ be two nonzero cusp forms and $\theta_f(p), \theta_g(p)$ be Frobenius angles at p of f and g, respectively. The sequence $(\theta_f(p), \theta_g(p))$ based on the Pair-Sato–Tate measure μ_{ST}^2 is uniformly distributed on $[0, \pi]^2$. In particular, for any two subintervals $I_1 \subseteq [0, \pi]$ and $I_2 \subseteq [0, \pi]$, we have*

$$\lim_{x \to \infty} \frac{\#\{p \le x | (\theta_f(p), \theta_g(p)) \in I_1 \times I_2\}}{\#\{p \le x | p \in \mathbb{P}\}}$$

$$= \mu_{ST}^2(I_1 \times I_2)$$
$$= \frac{4}{\pi^2}\int_{I_1} \sin^2\theta_f \mathrm{d}\theta_f \int_{I_2} \sin^2\theta_g \mathrm{d}\theta_g.$$

5.2. Proof of Theorem 1.5

By (1.2) and (1.3), we have

$$\lambda_{f\times f\times f}(p) = \lambda_f^3(p) = 8\cos^3\theta_f(p). \tag{5.1}$$

Expressing $\lambda_{f\times f\times f}^j(p) > \lambda_{g\times g\times g}^j(p)$ in trigonometric form gives

$$\cos^{3j}\theta_f(p) > \cos^{3j}\theta_g(p). \tag{5.2}$$

The following will analyze the parity of j to prove Theorem 1.5.
Case 1: When j is even, function $\cos^{3j}\theta$ is decreasing on interval $[0, \frac{\pi}{2}]$ and increasing on interval $[\frac{\pi}{2}, \pi]$, and its graph is symmetrical about line $\theta = \frac{\pi}{2}$. The ranges of $\theta_f(p)$ and $\theta_g(p)$ that satisfy (5.2) are shown in the shaded area of the Figure 1.

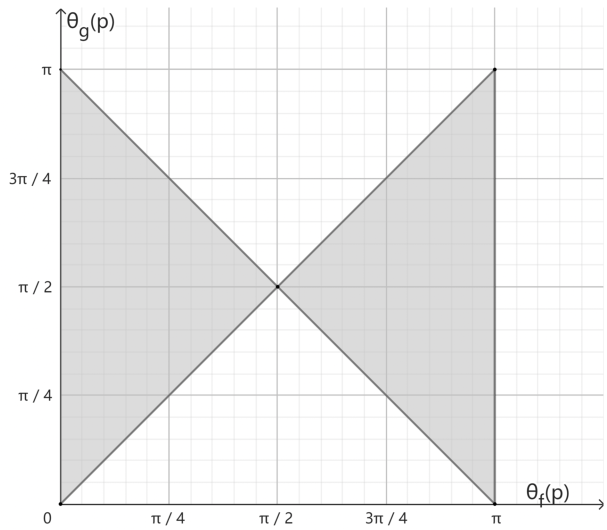

Figure 1. Range map of $\theta_f(p)$ and $\theta_g(p)$.

Decompose the shaded area of Figure 1 into two regions:

$$D_1 = \{(\theta_f(p), \theta_g(p)) : 0 \le \theta_f(p) \le \frac{\pi}{2}, \theta_f(p) \le \theta_g(p) \le \pi - \theta_f(p)\},$$
$$D_2 = \{(\theta_f(p), \theta_g(p)) : \frac{\pi}{2} \le \theta_f(p) \le \pi, \pi - \theta_f(p) \le \theta_g(p) \le \theta_f(p)\}.$$

Based on Lemma 5.2, we first perform the calculation for region D_1. Applying formula $\int \sin^2 x\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C$ yields

$$\int_0^{\frac{\pi}{2}} \sin^2\theta_f \mathrm{d}\theta_f \int_{\theta_f}^{\pi-\theta_f} \sin^2\theta_g \mathrm{d}\theta_g = \int_0^{\frac{\pi}{2}} \sin^2\theta_f (\frac{\pi}{2} - \theta_f + \frac{sin2\theta_f}{2})\mathrm{d}\theta_f$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta_f \mathrm{d}\theta_f - \int_0^{\frac{\pi}{2}} \theta_f \sin^2\theta_f \mathrm{d}\theta_f + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta_f \sin 2\theta_f \mathrm{d}\theta_f.$$

Each integral in the above expression can be calculated using integration by parts.

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta_f \mathrm{d}\theta_f = \frac{\pi^2}{8},\ \ \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta_f \sin 2\theta_f \mathrm{d}\theta_f = \frac{1}{4}.$$

$$\int_0^{\frac{\pi}{2}} \theta_f \sin^2\theta_f \mathrm{d}\theta_f = \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta_f(1-\cos 2\theta_f)\mathrm{d}\theta_f = \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta_f \mathrm{d}\theta_f - \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta_f \cos 2\theta_f \mathrm{d}\theta_f = \frac{\pi^2}{16} + \frac{1}{4}.$$

In summary,

$$\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \sin^2\theta_f \mathrm{d}\theta_f \int_{\theta_f}^{\pi-\theta_f} \sin^2\theta_g \mathrm{d}\theta_g = \frac{1}{4}.$$

Similarly, the integral result for region D_2 is $\frac{1}{4}$. This indicates that the natural density of set $\{p : \lambda^j_{f\times f\times f}(p) > \lambda^j_{g\times g\times g}(p)\}$ is $\frac{1}{2}$. In view of symmetry, the natural density of set $\{p : \lambda^j_{f\times f\times f}(p) < \lambda^j_{g\times g\times g}(p)\}$ is also $\frac{1}{2}$.

Case 2: When j is odd, the function $\cos^{3j}\theta$ is decreasing on $[0, \pi]$, and its graph is centrally symmetric about point $(\frac{\pi}{2}, 0)$. The ranges of $\theta_f(p)$ and $\theta_g(p)$ that satisfy (5.2) are indicated by the shaded area in the Figure 2.

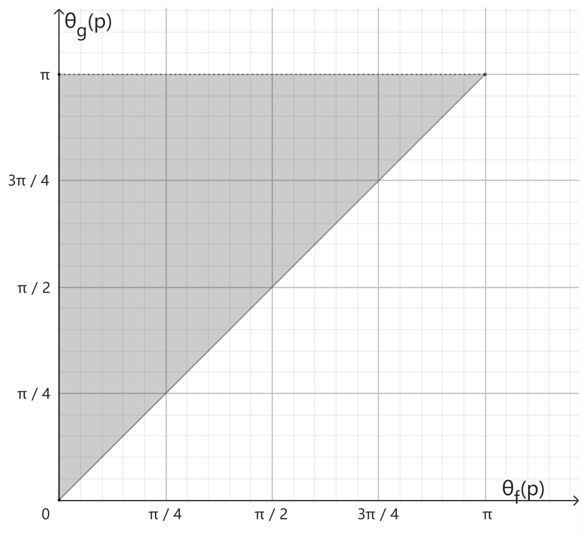

Figure 2. Range map of $\theta_f(p)$ and $\theta_g(p)$.

The shaded area in Figure 2 is defined by

$$D_3 = \{(\theta_f(p), \theta_g(p)) : 0 \le \theta_f(p) \le \pi, \theta_f(p) \le \theta_g(p) \le \pi\}.$$

Similarly, by performing calculations on region D_3, we obtain

$$\frac{4}{\pi^2}\int_0^{\pi}\sin^2\theta_f \mathrm{d}\theta_f\int_{\theta_f}^{\pi}\sin^2\theta_g \mathrm{d}\theta_g = \frac{1}{2}.$$

This indicates that the natural density of the set $\{p : \lambda^j_{f\times f\times f}(p) \leqslant \lambda^j_{g\times g\times g}(p)\}$ is $\frac{1}{2}$.

6. Proof of Theorems 1.6 and 1.7

6.1. Lemma

The definition of the Sato–Tate measure is given, followed by a statement of the Sato–Tate conjecture (see [11, Theorem 2.3]), which will be used to prove Theorems 1.6 and 1.7.

Definition 6.1. *The Sato–Tate measure μ_{ST} is a probability measure on the interval $[0,\pi]$ and $\mu_{ST} = \frac{2}{\pi}\sin^2\theta\mathrm{d}\theta$.*

Lemma 6.2. (*Sato–Tate conjecture*) *Let $f \in H_k^*$ be a nonzero cusp form. The sequence $\{\theta_p\}$ based on the Sato–Tate measure μ_{ST} is uniformly distributed on $[0,\pi]$. In particular, for any subinterval $I \subseteq [0,\pi]$, we have*

$$\lim_{x\to\infty}\frac{\#\{p \le x|\theta_p \in I\}}{\#\{p \le x|p \in \mathbb{P}\}} = \mu_{ST}(I) = \frac{2}{\pi}\int_I \sin^2\theta\mathrm{d}\theta.$$

6.2. Proof of Theorem 1.6

For simplicity, $\theta_f(p)$ will be replaced by θ in the next context.

Define the characteristic function $\varepsilon(\theta) = \mathrm{sgn}(\lambda_{f\times f\times f}(p)) = \mathrm{sgn}(8\cos^3\theta)$ with the help of (5.1). Calculate the natural density $d(P_1)$ and $d(P_1')$ of sets $P_1 = \{p : \lambda_{f\times f\times f}(p) > 0\}$ and $P_1' = \{p : \lambda_{f\times f\times f}(p) < 0\}$. According to the definition of natural density and Lemma 6.2, we know

$$d(P_1) = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin^2\theta\mathrm{d}\theta = \frac{1}{2}$$

and

$$d(P_1') = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=-1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin^2\theta\mathrm{d}\theta = \frac{1}{2}.$$

For P_3 and P_3',

$$\begin{aligned}\lambda_{f\times f\times f}(p^3) &= \lambda_f(p)^9 - 6\lambda_f(p)^7 + 15\lambda_f(p)^5 - 13\lambda_f(p)^3\\ &= 512\cos^9\theta - 768\cos^7\theta + 480\cos^5\theta - 104\cos^3\theta.\end{aligned}$$

The definition of natural density and Lemma 6.2 imply

$$d(P_3) = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta \;\; and \;\; d(P_3') = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=-1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta,$$

where the characteristic function $\varepsilon(\theta) = \mathrm{sgn}(512\cos^9\theta - 768\cos^7\theta + 480\cos^5\theta - 104\cos^3\theta)$. Then,

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}\varepsilon(\theta)\sin^2\theta\mathrm{d}\theta &= \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta + \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=-1\}\cap[0,\pi]}(-1)\sin^2\theta\mathrm{d}\theta\\ &= d(P_3) - d(P_3').\end{aligned}$$

Performing a transformation on the integral $t = \theta - \frac{\pi}{2}$ leads to

$$\frac{2}{\pi}\int_0^{\pi}\varepsilon(\theta)\sin^2\theta\mathrm{d}\theta = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon_3(t)\cos^2 t\mathrm{d}t = T_3,$$

where $\varepsilon_3(t) := -\mathrm{sgn}(512\sin^9 t - 768\sin^7 t + 480\sin^5 t - 104\sin^3 t)$ is an odd function in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $T_3 = 0$. In other words,

$$d(P_3) - d(P_3') = 0.$$

$d(P_3) + d(P_3') = 1$ implies $d(P_3) = d(P_3') = \frac{1}{2}$.

Now we consider P_2 and P_2'.

$$\begin{aligned}\lambda_{f\times f\times f}(p^2) &= \lambda_f(p)^6 - 3\lambda_f(p)^4 + 6\lambda_f(p)^2 - 4\\ &= 64\cos^6\theta - 48\cos^4\theta + 24\cos^2\theta - 4.\end{aligned}$$

In view of Lemma 6.2, one has

$$d(P_2) = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]}\sin^2\theta\mathrm{d}\theta,$$

where

$$\begin{aligned}\varepsilon(\theta) &= \mathrm{sgn}(\lambda_{f\times f\times f}(p^2)) = \mathrm{sgn}(64\cos^6\theta - 48\cos^4\theta + 24\cos^2\theta - 4)\\ &= \mathrm{sgn}(64\sin^6 t - 48\sin^4 t + 24\sin^2 t - 4) =: \varepsilon_2(t).\end{aligned} \tag{6.1}$$

Noting (6.1), we learn that $\varepsilon_2(t)$ is an even function in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then

$$\frac{2}{\pi}\int_0^{\pi}\varepsilon(\theta)\sin^2\theta\mathrm{d}\theta = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon_2(t)\cos^2 t\mathrm{d}t = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\varepsilon_2(t)\cos^2 t\mathrm{d}t.$$

It is easy to see that

$$d(P_2) = \frac{4}{\pi}\int_{\{t:\varepsilon_2(t)=1\}\cap[0,\frac{\pi}{2}]}\cos^2 t\mathrm{d}t. \tag{6.2}$$

Denote by α the zero point of the function $h(t) = 64\sin^6 t - 48\sin^4 t + 24\sin^2 t - 4$ in $[0, \frac{\pi}{2}]$. Substituting x for $\sin^2 t$, we write $f(x) := 64x^3 - 48x^2 + 24x - 4,\ x \in [0, 1]$. Its derivative $f'(x) = 192(x - \frac{1}{4})^2 + 12 >$

0 implies $f(x)$ is monotonically increasing in the interval $[0,1]$. Therefore, $h(t)$ is monotonically increasing in $[0,\frac{\pi}{2}]$. In view of $h(0) < 0$ and $h(\frac{\pi}{2}) > 0$, $h(t)$ has a unique zero α in $[0,\frac{\pi}{2}]$. Using MATLAB, we know that α approximately equal to 0.5236 and

$$\varepsilon_2(t) = 1,\ t \in [0,\frac{\pi}{2}] \Leftrightarrow t \in (\alpha,\frac{\pi}{2}].$$

Using (6.2), we have

$$d(P_2) = \frac{4}{\pi}\int_{\alpha}^{\frac{\pi}{2}} \cos^2 t\mathrm{d}t = 1 - \frac{2\alpha}{\pi} - \frac{\sin 2\alpha}{\pi}.$$

$d(P_2) + d(P_2') = 1$ implies $d(P_2') = \frac{2\alpha}{\pi} + \frac{\sin 2\alpha}{\pi}$.

6.3. Proof of Theorem 1.7

The proof of Theorem 1.7 is similar to that of Theorem 1.6, both being proved using Lemma 6.2.

$$\begin{aligned}\lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^2) &= \lambda_f(p)^9 - 3\lambda_f(p)^7 + 6\lambda_f(p)^5 - 4\lambda_f(p)^3\\ &= 512\cos^9\theta - 384\cos^7\theta + 192\cos^5\theta - 32\cos^3\theta.\end{aligned}$$

Recalling the definition of natural density and Lemma 6.2, we deduce

$$d(\bar{P}_2) = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]} \sin^2\theta\mathrm{d}\theta \quad and \quad d(\bar{P}_2') = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=-1\}\cap[0,\pi]} \sin^2\theta\mathrm{d}\theta,$$

where

$$\begin{aligned}\varepsilon(\theta) = \mathrm{sgn}(\lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^2)) &= \mathrm{sgn}(512\cos^9\theta - 384\cos^7\theta + 192\cos^5\theta - 32\cos^3\theta)\\ &= -\mathrm{sgn}(512\sin^9 t - 384\sin^7 t + 192\sin^5 t - 32\sin^3 t)\\ &=: \varepsilon_4(t).\end{aligned}$$

It is obvious to see that $\varepsilon_4(t)$ is an odd function. Then $\frac{2}{\pi}\int_0^{\pi}\varepsilon(\theta)\sin^2\theta\mathrm{d}\theta = 0$. In other words,

$$d(\bar{P}_2) - d(\bar{P}_2') = 0.$$

$d(\bar{P}_2) + d(\bar{P}_2') = 1$ implies $d(\bar{P}_2) = d(\bar{P}_2') = \frac{1}{2}$.

Now we consider $\bar{P}_3$ and $\bar{P}_3'$.

$$\begin{aligned}\lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^3) &= \lambda_f(p)^{12} - 6\lambda_f(p)^{10} + 15\lambda_f(p)^8 - 13\lambda_f(p)^6\\ &= 4096\cos^{12}\theta - 6144\cos^{10}\theta + 3840\cos^8\theta - 832\cos^6\theta.\end{aligned}$$

From Lemma 6.2, we find that

$$d(\bar{P}_3) = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=1\}\cap[0,\pi]} \sin^2\theta\mathrm{d}\theta \quad and \quad d(\bar{P}_3') = \frac{2}{\pi}\int_{\{\theta:\varepsilon(\theta)=-1\}\cap[0,\pi]} \sin^2\theta\mathrm{d}\theta,$$

where

$$\begin{aligned}\varepsilon(\theta) = \operatorname{sgn}(\lambda_{f\times f\times f}(p)\lambda_{f\times f\times f}(p^3)) &= \operatorname{sgn}(4096\cos^{12}\theta - 6144\cos^{10}\theta + 3840\cos^8\theta - 832\cos^6\theta)\\ &= \operatorname{sgn}(4096\sin^{12}t - 6144\sin^{10}t + 3840\sin^8 t - 832\sin^6 t)\\ &=: \varepsilon_5(t).\end{aligned}$$

An analysis similar to that in Theorem 1.6 shows that $h(t) = 4096\sin^{12}t - 6144\sin^{10}t + 3840\sin^8 t - 832\sin^6 t$ has a unique zero α in $(0, \frac{\pi}{2}]$. Using MATLAB, we know $\alpha \approx 0.7045$.

$$\varepsilon_5(t) = 1,\ t \in [0, \frac{\pi}{2}] \Leftrightarrow t \in (\alpha, \frac{\pi}{2}]. \tag{6.3}$$

It follows from (6.3) that

$$d(\bar{P}_3) = \frac{4}{\pi}\int_{\alpha}^{\frac{\pi}{2}} \cos^2 t\mathrm{d}t = 1 - \frac{2\alpha}{\pi} - \frac{\sin 2\alpha}{\pi}.$$

$d(\bar{P}_3) + d(\bar{P}'_3) = 1$ implies $d(\bar{P}'_3) = \frac{2\alpha}{\pi} + \frac{\sin 2\alpha}{\pi}$.

7. Conclusions

This paper presents a total of seven theorems, including the analytic and natural density of the given sets. The proofs of the first four theorems rely on the relationships between the triple product L-functions, symmetric power L-functions, and Rankin–Selberg L-functions as well as the tools in analytic number theory in connection with these automorphic L-functions. Based on the now-proven Sato–Tate conjecture (or pair-Sato–Tate conjecture), we introduce the characteristic functions to establish the natural density of the sets in the last three theorems.

Author contributions

Ying Han: writing—original draft, methodology, validation, formal analysis; Huixue Lao: writing—review and editing, resources, methodology, supervision, validation, formal analysis, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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