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Research article

Analysis of fractional stochastic systems driven by fractional Brownian motion with general memory kernel

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Abstract: Fractional stochastic differential equations (FSDEs) driven by fractional Brownian motion (fBm) have attracted growing attention due to their ability to model systems exhibiting non-Markovian dynamics and long-range dependence, which naturally arise in many real-world phenomena characterized by hereditary and persistent randomness. In this work, we establish the existence and uniqueness of mild solutions using the Picard iteration technique for the case where the Hurst parameter satisfies $H \in \left(\frac{1}{2}, 1\right)$. Moreover, we establish the approximate controllability of the systems under suitable conditions. To generalize the theoretical framework, we employ the Caputo–Katugampola fractional derivative (CKFD), thereby extending the analysis to a broader class of fractional stochastic systems.

Keywords: fractional Brownian motion; mild solutions; Picard iteration approach; fixed point approach
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1. Introduction

Unlike classical derivatives, which describe the instantaneous rate of change at a specific point, fractional derivatives capture the memory and hereditary characteristics of processes, making them particularly suitable for modeling systems that exhibit long-term memory effects, such as viscoelastic materials, anomalous diffusion, and various biological phenomena. There are many different kinds of fractional operators, each with unique characteristics and uses. Examples include the Riemann-Liouville, Caputo fractional derivative (CFD), Grünwald-Letnikov, Caputo-Fabrizio, conformable, CKFD, and Caputo-Hadamard fractional derivative (CHFD) [1–3]. Selecting a suitable type of fractional derivative depends on the problem's unique features and the mathematical behavior intended to be captured.

Among the various fractional operators, the CKFD is particularly notable and is defined as follows [4]:

$$D^{\alpha,\beta} f(t) = \frac{\beta^{\alpha}}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\gamma)}{(t^{\beta}-\gamma^{\beta})^{\alpha}} d\gamma. \tag{1.1}$$

When $\beta = 1$, the CKFD reduces to the CFD, and when $\beta \to 0^+$, it becomes the CHFD [5]. The Caputo-Katugampola integral is defined as [6]:

$$\mathcal{I}^{\alpha,\beta} f(t) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\gamma^{\beta-1} f(\gamma)}{(t^{\beta}-\gamma^{\beta})^{1-\alpha}} d\gamma. \tag{1.2}$$

Remark 1.1. *The parameter β plays a fundamental role in the CKFD, as it governs the manner in which past states influence the present dynamics of the system. In particular, the memory kernel of the operator depends explicitly on β, allowing a continuous adjustment of memory effects. When $\beta = 1$, the CKFD reduces to the CFD, corresponding to a uniform power-law memory. As $\beta \to 0^+$, the operator approaches the CHFD, which is associated with logarithmic memory behavior. For values $\beta \neq 1$, the derivative exhibits intermediate memory effects, lying between the power–law and logarithmic regimes. This flexibility constitutes a significant advantage of the Caputo–Katugampola framework, as it enables more accurate modeling of real-world systems whose hereditary properties cannot be adequately described by a single pure power–law memory kernel.*

The CKFD has recently garnered significant attention from researchers. For example, the authors of [7] developed a novel approach for solving various fractional-order models using the CKFD, which they applied to obtain approximate solutions and establish well-posedness. Similarly, the authors of [8] addressed several problems by proposing a practical technique based on this operator. The existence and uniqueness (Ex-Un) of solutions for fractional models were examined in [9], where the authors also derived several stability results. Additionally, the authors of [10] introduced a new method for solving a range of related problems. The foundational work by the authors of [11] provided key inequalities and concepts for the CKFD. This theoretical framework was subsequently strengthened through proofs of Ex-Un for solutions to fractional systems [12–14]. The applicability of the CKFD has been further demonstrated both in solving specific fractional models [15] and in analyzing the asymptotic properties of stochastic systems [16].

FSDEs driven by fBm offer an effective modeling framework for dynamical systems that exhibit memory effects and long-range dependence. In contrast to classical Brownian motion, which relies on independent and stationary increments, fBm is a Gaussian process characterized by the Hurst parameter $H \in (0, 1)$, allowing the representation of persistent behavior when $H > 0.5$. When FSDEs incorporate fBm, they can model more realistic dynamics in complex systems where the influence of past events and temporal correlations cannot be ignored. Such models find extensive applications in areas including finance (to capture long-memory effects in asset prices), biology (to describe anomalous diffusion in crowded environments), and physics (to model viscoelastic materials or climate systems) [17, 18]. The mathematical treatment of FSDEs with fBm is more intricate, requiring advanced integration techniques such as the pathwise Riemann-Stieltjes integral for ($H > 0.5$) or Malliavin calculus for ($H < 0.5$), due to the non-semimartingale nature of fBm.

Controllability of FSDEs describes the capability of steering the system state from a given initial condition to a prescribed target state over a finite time interval, while accounting for both memory effects arising from fractional derivatives and uncertainty introduced by stochastic components. Exact controllability denotes the situation in which the system can be driven exactly to the desired state, whereas approximate controllability means that the system can be guided to an arbitrarily small neighborhood of the target state. Due to the intrinsic nonlocality of fractional derivatives and the presence of stochastic perturbations, attaining exact controllability is exceedingly challenging, particularly in the context of infinite-dimensional fractional stochastic systems. Therefore, approximate controllability is more commonly studied and practically achievable. The controllability of FSDEs has vital applications in various fields, such as controlling disease spread in biological systems, designing robust financial strategies under uncertainty, managing vibrations and noise in engineering systems, and optimizing decision-making in systems with memory and random disturbances.

A variety of results on FSDEs have been reported in the existing literature. For example, Kahouli et al. [19] derived stability conditions for these systems. By employing Picard's iteration method (PIM), the authors in [20] proved the existence and uniqueness of solutions for coupled FSDEs. Moreover, Raheem et al. [21] investigated the existence, uniqueness, and controllability of FSDEs through the use of sectorial operators. Further key results were established in [22, 23]. The authors of [24] proved the Ex-Un for FSDEs. The authors of [25] found results on Ex-Un and controllability for FSDEs concerning the CHFD. Moualkia and Xu [26] investigated the various useful properties of variable-order FSDEs, including the use of PIM to obtain solutions. The authors [27] investigated a financial model structured as an FSDE, providing numerical solutions and graphical interpretations under multiple scenarios. Abouagwa et al. [28] explored Ex-Un for impulsive FSDEs, while the authors of [29] addressed solution existence, uniqueness, and finite-time stability for a specific category of FSDEs. The integration of neural networks and nonparametric learning with stochastic processes have been used for describe degradation processes, where fractional derivatives appear to naturally describe memory and hereditary effects in degradation processes [30–32].

In this research work, we consider the following FSDEs with fBm:

$$\begin{cases} D^{\alpha,\beta} f(t) = \Im f(t) + Ku(t) + \eta(t, f(t)) + \varphi(t)\frac{dw^H(t)}{dt}, \ t \in I = [0, L], \\ f(0) = f_0, \end{cases} \tag{1.3}$$

where $\alpha \in \left(\frac{1}{2}, 1\right]$ and $D^{\alpha,\beta}$ denotes the CKFD. The infinitesimal generator $\Im$ yields a strongly continuous semigroup $\{S_t\}_{t\geq 0}$ on a real separable Hilbert space X. Furthermore, let w^H denote an fBm defined on another infinite-dimensional real separable Hilbert space Z, with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. Additionally, we introduce the operator $K : V_F^2(I, Y) \to V^2(I, X)$ and define $\eta : I \times X \to X$ along with $\varphi : I \to V_J^0(Z, X)$. These latter two operators will be specified in greater detail in the following section.

In this study, notable results are obtained for FSDEs within the framework of CKFD. The following are some significant elements that our study contributes:

(1) In many existing works on approximate controllability of fractional stochastic systems, the nonlinear term is assumed to be uniformly bounded. This assumption implies that the magnitude of the nonlinear term remains bounded independently of the size of the state variable.

This hypothesis has been widely adopted in the literature mainly for technical convenience. First, it significantly simplifies the analysis of Ex-Un of mild solutions for FSDEs. Second, it ensures that the nonlinear term does not dominate the control input or the linear dynamics, which makes the controllability analysis more tractable. Finally, it allows the approximate controllability of the nonlinear system to be treated as a small perturbation of the corresponding linear system.

However, the uniform boundedness assumption is highly restrictive and limits the applicability of the results. In many realistic models arising in physics, engineering, and biology, nonlinearities naturally depend on the state and typically exhibit growth with respect to x. Examples include polynomial-type nonlinearities, saturation effects, and other state-dependent feedback mechanisms. Such nonlinear terms are not uniformly bounded, and therefore cannot be covered under this assumption.

In the present paper, this restrictive hypothesis is removed. Instead, we assume a linear growth condition which allows the nonlinearity to grow with the state variable while remaining mathematically manageable. This condition is standard in the theory of stochastic and fractional differential equations and is sufficient to guarantee the Ex-Un of mild solutions. More importantly, it enables us to establish approximate controllability results for systems with unbounded nonlinearities, thereby significantly broadening the class of admissible models compared to earlier works.

(2) In many existing works, it is assumed that the fractional linear system obtained by setting $\eta = 0$ is already approximately controllable. This hypothesis allows the controllability of the nonlinear system to be deduced directly as a perturbation of a controllable linear system.

However, this assumption is rather restrictive. Approximate controllability of fractional linear systems in infinite-dimensional spaces is itself difficult to verify and depends strongly on the fractional order, the semigroup generated by $\mathfrak{I}$, and the structure of the control operator K. As a result, requiring this property a priori significantly limits the applicability of the results.

In this paper, we remove this constraint and do not assume approximate controllability of the linear system. Instead, we establish it through an operator-theoretic approach, which enables us to prove approximate controllability of the nonlinear fractional stochastic system under substantially assumptions.

(3) Using the PIM, we establish the Ex-Un of mild solutions.

(4) We generalize the results on Ex-Un and approximate controllability in the context of fractional derivatives by establishing them within the framework of the CKFD.

Section 2 presents the fundamental results that form the foundation of the findings established in this research work. In Sections 3 and 4, we then provide our main findings about FSDEs in the CKFD framework. In Section 5, an example shows how these theoretical findings can be applied. In Section 6, the paper ends with a summary.

2. Preliminaries

In this section, we present results that serve as the foundation for the findings established in this research work.

Suppose X, Y, and Z are real separable Hilbert spaces. We work on a complete probability space (Ω, F, P), furnished with a normal filtration $\{F_t\}_{t\in[0,L]}$ that adheres to the standard assumptions, and note that $F_L = F$.

Consider a control function u in $V_F^2(I, Y)$. This space is defined as the closed subspace of $V^2(I\times\Omega, Y)$ containing all F_t-adapted processes that take values in Y. We also define, for the entirety of this work, $\varpi = \sup_{t\in[0,+\infty)} |S(t)| < \infty$.

We now define the following Banach spaces. First, let $V(Z, X)$ represent the space of all bounded linear operators from the Hilbert space Z into X, endowed with the conventional operator norm, expressed as

$$V(Z, X) := \{T : Z \to X \,\big|\, T \text{ is a bounded linear operator}\}.$$

Next, consider the set $V^2(\Omega, X)$, which comprises all X-valued random variables f on Ω that are F-measurable and possess finite second moment. Formally, this space is defined as

$$V^2(\Omega, X) := \left\{f : \Omega \to X \,\big|\, f \text{ is } F\text{-measurable and } E\|f\|^2 < \infty\right\}$$

and is endowed with its canonical square norm.

Furthermore, we define $\vartheta(I, V^2(\Omega, X))$ as the collection of all functions $f : I \to V^2(\Omega, X)$ satisfying the following conditions:

(1) For every $t \in I$, $f(t)$ is adapted to the filtration F_t.

(2) The mapping f is continuous in t.

(3) The supremum $\sup_{t\in I} E\|f(t)\|^2$ is finite.

Denoting $A = \vartheta(I, V^2(\Omega, X))$, we equip this space with the norm

$$\|f\|_A = \left(\sup_{t\in I} E\|f(t)\|^2\right)^{1/2},$$

under which it becomes a Banach space.

Definition 2.1. *[33] For a Hurst parameter $H \in (0, 1)$, a one-dimensional fBm is a family of Gaussian processes $W^H = \{W^H(t),\ t \in I\}$ with mean zero and continuous sample paths. Its covariance structure is specified by:*

$$R^H(t, \gamma) = E[W^H(t)W^H(\gamma)] = \frac{1}{2}(t^{2H} + \gamma^{2H} - |t - \gamma|^{2H}), \quad \textit{for } t, \gamma \in I.$$

In the subsequent analysis, we restrict our attention to the case $H \in \left(\frac{1}{2}, 1\right)$. For such H, the process W^H can be represented in integral form as

$$W^H(t) = \int_0^t B^H(t, \gamma) dW(\gamma),$$

where $W = \{W(t),\ t \in I\}$ is a standard one-dimensional Brownian motion, and the kernel B^H is given by

$$B^H(t,\gamma) = \Phi_H \gamma^{\frac{1}{2}-H} \int_\gamma^t (\mathrm{e}-\gamma)^{H-\frac{3}{2}} \mathrm{e}^{H-\frac{1}{2}} d\mathrm{e},\ t > \gamma, \tag{2.1}$$

with the constant

$$\Phi_H = \sqrt{\frac{H(2H-1)}{Be(2-2H,\ H-\frac{1}{2})}}.$$

Now, for a deterministic function $G \in V^2(I)$, the following equality holds:

$$\int_0^L G(\gamma) dW^H(\gamma) = \int_0^L (B_L^* G)(\gamma) dW(\gamma),$$

where the adjoint operator B_p^* acts on G as

$$(B_p^* G)(\gamma) = \int_\gamma^u G(a) \frac{\partial B^H(a,\gamma)}{\partial a} dz,\ a \in [0,L].$$

We now introduce the definition of fBm within an infinite-dimensional framework, together with its corresponding stochastic integral. Let $J \in V(Z,Z)$ be a self-adjoint, nonnegative trace-class operator satisfying

$$J z_j = \upsilon_j z_j,$$

where $\{z_j\}_{j=1}^\infty$ forms a complete orthonormal system in the Hilbert space Z, and the corresponding eigenvalues $\{\upsilon_j\}_{j=1}^\infty \subseteq [0,\infty)$ fulfill

$$\mathrm{tr}(J) = \sum_{j=1}^\infty \upsilon_j < \infty.$$

A Z-valued J-cylindrical fBm on the probability space (Ω, F, P), having covariance operator J, is then defined by the series

$$w^H(t) = \sum_{j=1}^\infty J^{\frac{1}{2}} G_j W_j^H(t) = \sum_{j=1}^\infty \sqrt{\upsilon_j} G_j W_j^H(t),$$

in which each W_j^H denotes an independent one-dimensional fBm.

Consider the space $V_J^0(Z,X)$ of J-Hilbert-Schmidt operators. An operator $\Psi \in V(Z,X)$ is called a J-Hilbert-Schmidt operator if it satisfies

$$\|\Psi\|_{V_J^0(Z,X)}^2 = \sum_j^\infty \|\sqrt{\upsilon_j} \Psi G_j\|^2 < \infty.$$

Definition 2.2. *[34] Let $\psi : I \to V_J^0(Z,X)$ be a mapping representing a random operator-valued function defined on I and taking values in the space of J-Hilbert-Schmidt operators between the real separable Hilbert spaces Z and X. If ψ fulfills*

$$\sum_{j=1}^{\infty} \|B_L^*(\psi J^{\frac{1}{2}})G_j\|_{V^2(I,X)} < \infty, \tag{2.2}$$

then the stochastic integral of $\psi(\gamma)$ with respect to the cylindrical fBm $w^H(\gamma)$ is defined by the expression

$$\begin{aligned}\int_0^t \psi(\gamma)dw^H(\gamma) &= \sum_{j=1}^{\infty}\int_0^t \psi(\gamma)J^{\frac{1}{2}}G_j dW_j^H(\gamma)\\ &= \sum_{j=1}^{\infty}\int_0^t (B_L^*(\psi J^{\frac{1}{2}}G_j))(\gamma)dW(\gamma), \ \ \textit{for all } t\in I.\end{aligned}$$

Lemma 2.3. *[35] Assume that $\psi : I \to V_J^0(Z,X)$ satisfies the condition*

$$\sum_{j=1}^{\infty} \|\psi J^{\frac{1}{2}}G_j\|_{V^{\frac{1}{H}}(I,X)} < \infty. \tag{2.3}$$

Then, for any $0 \le \gamma < t \le L$, the following inequality holds:

$$E\Big\|\int_\gamma^t \psi(p)dw^H(p)\Big\|_X^2 \le \Phi_H(t-\gamma)^{2H-1}\sum_{j=1}^{\infty}\int_\gamma^t \|\psi(p)J^{\frac{1}{2}}G_j\|_X^2 dp,$$

where the positive constant Φ_H depends solely on the Hurst parameter H.

Moreover, if the series $\sum_{j=1}^{\infty}\|\psi(t)J^{\frac{1}{2}}z_j\|_X$ converges, then

$$E\Big\|\int_\gamma^t \psi(p)dw^H(p)\Big\|_X^2 \le \Phi_H(t-\gamma)^{2H-1}\int_\gamma^t \|\psi(p)\|_{V_J^0(Z,X)}dp. \tag{2.4}$$

Definition 2.4. *[36] We define the stochastic process $\{f(t)\}_{t\in I}$ to be a mild solution of system* (1.3) *if, for any admissible control $u \in V_F^2(I,Y)$, it fulfills the following integral equation:*

$$\begin{aligned} f(t) = S_{\alpha,\beta}(t)f_0 &+ \int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)[\eta(\gamma,f(\gamma))+Ku(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}}\\ &+ \int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}, \ t\in I, \end{aligned} \tag{2.5}$$

where $S_{\alpha,\beta}(t) = \int_0^\infty \Psi_\alpha(\lambda)S\big((\frac{t^\beta}{\beta})^\alpha\lambda\big)d\lambda$, $\mathcal{Y}_{\alpha,\beta}(t) = \alpha\int_0^\infty \lambda\Psi_\alpha(\lambda)S\big((\frac{t^\beta-\gamma^\beta}{\beta})^\alpha\lambda\big)d\lambda$, and

$$\Psi_\alpha(\lambda) = \frac{1}{\alpha}\lambda^{-(1+\frac{1}{\alpha})}\overline{\omega}_\alpha(\lambda^{-\frac{1}{\alpha}}),\ \overline{\omega}_\alpha(\lambda) = \sum_{j=1}^{\infty}(-1)^{j-1}\lambda^{-j\alpha-1}\frac{\Gamma(j\alpha+1)}{\pi j!}\sin(j\pi\alpha),\ \lambda\in(0,\infty),$$

we recall that Ψ_α is a probability density function that satisfies

$$\Psi_\alpha(\lambda) \ge 0,\ \lambda\in(0,\infty) \textit{ and } \int_0^\infty \Psi_\alpha(\lambda)d\lambda = 1.$$

Remark 2.5. *We provide a concise overview of the procedure necessary to acquire this mild solution, as referenced in [36]. We begin by applying the Laplace transform to system* (1.3)*, which changes it into an algebraic system. We then use resolvent operator theory to get the state variable in the modified equation. This shows the basic solution structure through the operators $S_{\alpha,\beta}(t)$ and $\mathcal{Y}_{\alpha,\beta}(t)$ with $t \geq 0$. Finally, we use the inverse Laplace transform to go back to the time-domain outcome.*

Lemma 2.6. *[38] The families of operators $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ and $\{\mathcal{Y}_{\alpha,\beta}(t)\}_{t\geq 0}$ have the following properties.*

(1) For all $t \geq 0$, $S_{\alpha,\beta}(t)$ and $\mathcal{Y}_{\alpha,\beta}(t)$ are bounded linear operators, that is

$$\|S_{\alpha,\beta}(t)f\| \leq \varpi\|f\|, \; f \in X \text{ and } \|\mathcal{Y}_{\alpha,\beta}(t)f\| \leq \frac{\alpha\varpi}{\Gamma(\alpha+1)}\|f\|, \; \text{for all } f \in X.$$

(2) The families of operators $\{S_{\alpha,\beta}(t), t \geq 0\}$ and $\{\mathcal{Y}_{\alpha,\beta}(t), t \geq 0\}$ are strongly continuous semigroups of operators, see [37].

(3) If $S(t)$ is compact for every $t > 0$, it follows that $S_{\alpha,\beta}(t)$ and $\mathcal{Y}_{\alpha,\beta}(t)$ are compact operators as well for all $t > 0$.

Definition 2.7. *The collection of all attainable states at the terminal time L is called the reachable set of system* (1.3)*. Formally, it is defined as*

$$B_L(\eta) = \{f(L) : f(L) \text{ is the mild solution of } (1.3) \text{ at time } L \\ \text{corresponding to some admissible control } u\}.$$

In the particular case $\eta = 0$, system (1.3) *simplifies to the associated linear control system:*

$$\begin{cases} D^{\alpha,\beta}f(t) = \Im f(t) + Ku(t) + \varphi(t)\dfrac{dw^H(t)}{dt}, & t \in I = [0, L], \\ f(0) = f_0. \end{cases} \tag{2.6}$$

The corresponding set of reachable states is denoted by $B_L(0)$.

Definition 2.8. *System* (1.3) *is said to be approximately controllable on the interval I if the closure of its reachable set coincides with the entire state space; that is, if*

$$\overline{B_L(\eta)} = V^2(\Omega, X).$$

This means that for any desired final state $\Psi \in V^2(\Omega, X)$ and any tolerance $\varepsilon > 0$, one can find a control $u \in V_F^2(I, Y)$ such that

$$\mathbb{E}\|f(L) - \Psi\|^2 < \varepsilon,$$

where $f(L)$ is the terminal state generated by u. Analogously, system (2.6) *is approximately controllable if $\overline{B_L(0)} = V^2(\Omega, X)$.*

3. Existence and uniqueness

In this section, we present Ex-Un results.

We make the following assumptions.

$(H1)$: The function $\eta : I \times X \to X$ is measurable and there is $\Phi_1 > 0$ such that for each $f \in X$ and any $t \in I$,

$$\|\eta(t, f)\|^2 \leq \Phi_1(1 + \|f\|^2).$$

$(H2)$: There exists a constant $\Phi_2 > 0$ such that for all $f_1, f_2 \in X$, $t \in I$,

$$\|\eta(t, f_1) - \eta(t, f_2)\|^2 \leq \Phi_2\|f_1 - f_2\|^2.$$

$(H3)$: The mapping $\varphi : I \to V_J^0(Z, X)$ is assumed to be Lebesgue measurable, with a constant $\Phi_3 > 0$ such that

(1) $\sup_{0\leq\gamma\leq L} \|\varphi(\gamma)\|^2_{V_J^0(Z,X)} \leq \Phi_3$,

(2) $\sum_{J=1}^{\infty} \|\varphi J^{\frac{1}{2}} e_J\|_{V^{\frac{1}{H}}(I,X)} < \infty$,

(3) $\sum_{J=1}^{\infty} \|\varphi(t) J^{\frac{1}{2}} e_J\|_X$ is uniformly convergent for $t \in I$.

$(H4)$: For all $t > 0$, the operator $S(t)$ is compact.

Remark 3.1. *Assumption (H4) is introduced mainly for technical reasons to facilitate the use of compactness arguments in the analysis. While standard in the study of fractional evolution equations, relaxing this assumption would require alternative mathematical techniques and is left for future investigation.*

We introduce an operator U on A, $U : A \to A$,

$$\begin{aligned}(Uf)(t) =& S_{\alpha,\beta}(t)f_0 + \int_0^t \left(\frac{t^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t^\beta - \gamma^\beta}{\beta}\right)[\eta(\gamma, f(\gamma)) + Ku(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}} \\ &+ \int_0^t \left(\frac{t^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t^\beta - \gamma^\beta}{\beta}\right)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}. \end{aligned} \tag{3.1}$$

Lemma 3.2. *Assume that $(H1)$, $(H3)$, and $(H4)$ hold. Then, for each $f \in A$, the mapping $t \mapsto (Uf)(t)$ is continuous on I with respect to the $V^2(\Omega, X)$-norm.*

Proof. For every $f \in A$ and all $0 \leq t_1 < t_2 \leq L$, it holds that

$$\begin{aligned}&E\|(Uf)(t_2) - (Uf)(t_1)\|^2 \leq 4E\|S_{\alpha,\beta}(t_2)f_0 - S_{\alpha,\beta}(t_1)f_0\|^2 \\ &+ 4E\bigg\|\int_0^{t_2} \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)\eta(\gamma, f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}} - \int_0^{t_1} \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)\eta(\gamma, f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}}\bigg\|^2 \\ &+ 4E\bigg\|\int_0^{t_2} \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}} - \int_0^{t_1} \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}\bigg\|^2\end{aligned}$$

$$+ 4E\left\| \int_0^{t_2} \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}} - \int_0^{t_1} \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}} \right\|^2$$
$$= \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4. \tag{3.2}$$

Due to the strong continuity of $S_{\alpha,\beta}(t)$, it follows that

$$\lim_{t_2 \to t_1} \|S_{\alpha,\beta}(t_2) f_0 - S_{\alpha,\beta}(t_1) f_0\| = 0. \tag{3.3}$$

By Lemma 2.6,

$$\|S_{\alpha,\beta}(t_2) f_0 - S_{\alpha,\beta}(t_1) f_0\| \leq 2\varpi \|f_0\| \ \text{ for } f_0 \in V^2(\Omega, \mathbb{R}^+). \tag{3.4}$$

Through the Lebesgue dominated convergence theorem,

$$\lim_{t_2 \to t_1} \Upsilon_1 = 0.$$

Moreover,

$$\Upsilon_2 \leq 12E\left\| \int_0^{t_1} \left[\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} - \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1}\right] \mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)\eta(\gamma, f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}} \right\|^2$$
$$+ 12E\left\| \int_0^{t_1} \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \left[\mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right) - \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)\right]\eta(\gamma, f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}} \right\|^2$$
$$+ 12E\left\| \int_{t_1}^{t_2} \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)\eta(\gamma, f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}} \right\|^2$$
$$= \Upsilon_{21} + \Upsilon_{22} + \Upsilon_{23}. \tag{3.5}$$

By $(H1)$ and Hölder inequality (Höl-In),

$$\Upsilon_{21} \leq \frac{12\varpi^2}{\Gamma^2(\alpha)}\left(\int_0^{t_1} \left[\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} - \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1}\right]^2 \frac{d\gamma}{\gamma^{2-2\beta}}\right)\int_0^{t_1} E\|\eta(\gamma, f(\gamma))\|^2 d\gamma$$
$$\leq \frac{12\varpi^2}{\Gamma^2(\alpha)}\left(\sup_{0\leq\gamma\leq L}(\gamma^{\beta-1})\right)\int_0^{t_1} \left[\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{2\alpha-2} - \left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{2\alpha-2}\right]\frac{d\gamma}{\gamma^{2-2\beta}} \times \int_0^{t_1} \Phi_1(1 + E\|f(\gamma)\|^2) d\gamma$$
$$\leq Q\frac{12\varpi^2\Phi_1 t_1(1 + \sup_{\gamma\in I} E\|f(\gamma)\|^2)}{\Gamma^2(\alpha)(2\alpha - 1)} \times \left[\left(\frac{t_1^\beta}{\beta}\right)^{2\alpha-1} + \left(\frac{t_2^\beta - t_1^\beta}{\beta}\right)^{2\alpha-1} - \left(\frac{t_2^\beta}{\beta}\right)^{2\alpha-1}\right], \tag{3.6}$$

where $Q = \sup_{0\leq\gamma\leq L}(\gamma^{\beta-1})$ with $\beta \geq 1$. Then, we have

$$\Upsilon_{22} \leq 12E\left(\int_0^{t_1} \left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \left\|\mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right) - \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)\right\| \|\eta(\gamma, f(\gamma))\| \frac{d\gamma}{\gamma^{1-\beta}}\right)^2$$
$$\leq Q\frac{12(\frac{t_1^\beta}{\beta})^{2\alpha-1}\Phi_1(1 + \sup_{\gamma\in I} E\|f(\gamma)\|^2)}{(2\alpha - 1)}\left(\sup_{\gamma\in[0,t_1]}\left\|\mathcal{Y}_{\alpha,\beta}\left(\frac{t_2^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} - \mathcal{Y}_{\alpha,\beta}\left(\frac{t_1^\beta - \gamma^\beta}{\beta}\right)\right\|\right)^2. \tag{3.7}$$

Again invoking Lemma 2.6 together with hypothesis $(H4)$, we conclude that the operator $\mathcal{Y}_{\alpha,\beta}(t)$ is continuous in the uniform operator topology for $t > 0$. As an immediate consequence, we have $\lim_{t_2\to t_1} \Upsilon_{21} = \lim_{t_2\to t_1} \Upsilon_{22} = 0$.

Further,

$$\Upsilon_{23} \leq \frac{12\varpi^2}{\Gamma^2(\alpha)}\Big(\int_{t_1}^{t_2}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_{t_1}^{t_2}E\|\eta(\gamma,f(\gamma))\|^2d\gamma\Big)$$
$$\leq Q\frac{12\varpi^2\Phi_1(\frac{t_2^\beta-t_1^\beta}{\beta})^{2\alpha-1}(t_2-t_1)(1+\sup_{\gamma\in I}E\|f(\gamma)\|^2)}{\Gamma^2(\alpha)(2\alpha-1)}$$
$$\to 0 \ as \ t_2\to t_1. \tag{3.8}$$

A similar computation yields that

$$\Upsilon_3 \leq 12E\Big\|\int_0^{t_1}\Big[\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}-\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big]\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2$$
$$+12E\Big\|\int_0^{t_1}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big[\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big]Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2$$
$$+12E\Big\|\int_{t_1}^{t_2}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2$$
$$=\Upsilon_{31}+\Upsilon_{32}+\Upsilon_{33}. \tag{3.9}$$

Similarly,

$$\Upsilon_{31} \leq Q\frac{12\varpi^2\|Ku\|_{V^2(I,X)}[(\frac{t_1^\beta}{\beta})^{2\alpha-1}+(\frac{t_2^\beta-t_1^\beta}{\beta})^{2\alpha-1}-(\frac{t_2^\beta}{\beta})^{2\alpha-1}]}{\Gamma^2(\alpha)(2\alpha-1)} \to 0 \ as \ t_2\to t_1,$$
$$\Upsilon_{32} \leq Q\frac{12(\frac{t_1^\beta}{\beta})^{2\alpha-1}\|Ku\|_{V^2(I,X)}}{2\alpha-1}\Big(\sup_{\gamma\in[0,t_1]}\Big\|\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big\|\Big)^2 \to 0 \ as \ t_2\to t_1,$$
$$\Upsilon_{33} \leq Q\frac{12\varpi^2(\frac{t_2^\beta-t_1^\beta}{\beta})^{2\alpha-1}\|Ku\|_{V^2(I,X)}}{\Gamma^2(\alpha)(2\alpha-1)} \to 0 \ as \ t_2\to t_1. \tag{3.10}$$

Along the same lines, one obtains

$$\Upsilon_4 \leq 12E\Big\|\int_0^{t_1}\Big[\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}-\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big]\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\Big\|^2$$
$$+12E\Big\|\int_0^{t_1}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big[\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big]\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\Big\|^2$$
$$+12E\Big\|\int_{t_1}^{t_2}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\Big\|^2$$
$$=\Upsilon_{41}+\Upsilon_{42}+\Upsilon_{43}. \tag{3.11}$$

Combining Lemma 2.3 and $(H3)$, we have

$$\Upsilon_{41} \leq 12\Phi_H\Big(\frac{t_1^\beta}{\beta}\Big)^{2H-1}\int_0^{t_1}\Big\|\Big[\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}-\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big]\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)\frac{\varphi(\gamma)}{\gamma^{1-\beta}}\Big\|^2_{V_J^0(Z,X)}d\gamma$$

$$
\begin{aligned}
&\leq \frac{12\Phi_H(\frac{t_1^\beta}{\beta})^{2H-1}\Phi_3\varpi^2}{(\Gamma(\alpha))^2}\int_0^{t_1}\Big[\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}-\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\Big]\frac{d\gamma}{\gamma^{2-2\beta}}\\
&\leq Q\frac{12\Phi_H(\frac{t_1^\beta}{\beta})^{2H-1}\Phi_3\varpi^2}{(2\alpha-1)(\Gamma(\alpha))^2}\Big[\Big(\frac{t_1^\beta}{\beta}\Big)^{2\alpha-1}+\Big(\frac{t_2^\beta-t_1^\beta}{\beta}\Big)^{2\alpha-1}-\Big(\frac{t_2^\beta}{\beta}\Big)^{2\alpha-1}\Big]\\
&\to 0\ as\ t_2\to t_1,
\end{aligned}\tag{3.12}
$$

$$
\begin{aligned}
\Upsilon_{42}&\leq 12\Phi_H\Big(\frac{t_1^\beta}{\beta}\Big)^{2H-1}\int_0^{t_1}\Big\|\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\Big[\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big]\frac{\varphi(\gamma)}{\gamma^{1-\beta}}\Big\|^2_{V_J^0(Z,X)}d\gamma\\
&\leq 12\Phi_H\Big(\frac{t_1^\beta}{\beta}\Big)^{2H-1}\Phi_3\sup_{\gamma\in[0,t_1]}\Big\|\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big\|^2\int_0^{t_1}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\\
&\leq Q\frac{12\Phi_H(\frac{t_1^\beta}{\beta})^{2H+2\alpha-2}\Phi_3}{(2\alpha-1)}\sup_{\gamma\in[0,t_1]}\Big\|\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)-\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_1^\beta-\gamma^\beta}{\beta}\Big)\Big\|^2\\
&\to 0\ as\ t_2\to t_1,
\end{aligned}\tag{3.13}
$$

$$
\begin{aligned}
\Upsilon_{43}&\leq 12\Phi_H\Big(\frac{t_2^\beta-t_1^\beta}{\beta}\Big)^{2H-1}\int_{t_1}^{t_2}\Big\|\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t_2^\beta-\gamma^\beta}{\beta}\Big)\frac{\varphi(\gamma)}{\gamma^{1-\beta}}\Big\|^2_{V_J^0(Z,X)}d\gamma\\
&\leq Q\frac{12\Phi_H\varpi^2\Phi_3(\frac{t_2^\beta-t_1^\beta}{\beta})^{2H+2\alpha-2}}{(2\alpha-1)(\Gamma(\alpha))^2}\\
&\to 0\ as\ t_2\to t_1,
\end{aligned}\tag{3.14}
$$

so,

$$\lim_{t_2\to t_1}\mathbb{E}\,\|(Uf)(t_2)-(Uf)(t_1)\|^2=0,$$

which shows that the mapping $t\to(Uf)(t)$ is continuous on I in the $V^2(\Omega,X)$-sense. □

Lemma 3.3. *Assuming* $(H1)$, $(H3)$, *and* $(H4)$ *hold, the operator U maps A into itself.*

Proof. When $f\in A$,

$$
\begin{aligned}
&E\|(Uf)(t)\|^2\\
&\leq 4E\|S_{\alpha,\beta}(t)f_0\|^2+4E\Big\|\int_0^t\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)\eta(\gamma,f(\gamma))\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2\\
&+4E\Big\|\int_0^t\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2\\
&+4E\Big\|\int_0^t\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\Big\|^2\\
&=r_1+r_2+r_3+r_4.
\end{aligned}\tag{3.15}
$$

By Lemma 2.6,

$$r_1 \leq 4\varpi^2 E\|f_0\|^2. \tag{3.16}$$

Using $(H1)$ and Höl-In,

$$\begin{aligned} r_2 &\leq \frac{4\varpi^2}{\Gamma^2(\alpha)} E\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\|\eta(\gamma,f(\gamma))\|\frac{d\gamma}{\gamma^{1-\beta}}\Big)^2 \\ &\leq \frac{4\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t E\|\eta(\gamma,f(\gamma))\|^2 d\gamma\Big) \\ &\leq Q\frac{4\varpi^2(\frac{l^\beta}{\beta})^{2\alpha-1}L\Phi_1(1+\sup_{\gamma\in I}E\|f(\gamma)\|^2)}{(2\alpha-1)\Gamma^2(\alpha)}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} r_3 &\leq \frac{4\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t \|Ku(\gamma)\|^2 d\gamma\Big) \\ &\leq Q\frac{4\varpi^2(\frac{l^\beta}{\beta})^{2\alpha-1}\|Ku\|_{V^2(I,X)}}{(2\alpha-1)\Gamma^2(\alpha)}. \end{aligned} \tag{3.18}$$

Combining Lemma 2.3, $(H3)$ and Höl-In,

$$\begin{aligned} r_4 &\leq 4\Phi_H\Big(\frac{t^\beta}{\beta}\Big)^{2H-1}\int_0^t \Big\|\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)\frac{\varphi(\gamma)}{\gamma^{1-\beta}}\Big\|^2_{V_J^0(Z,X)}d\gamma \\ &\leq \frac{4\Phi_H\varpi^2(\frac{l^\beta}{\beta})^{2H-1}}{\Gamma^2(\alpha)}\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\|\varphi(\gamma)\|^2_{V_J^0(Z,X)}\frac{d\gamma}{\gamma^{2-2\beta}} \\ &\leq Q\frac{4\Phi_H\varpi^2(\frac{l^\beta}{\beta})^{2H+2\alpha-2}\Phi_3}{(2\alpha-1)\Gamma^2(\alpha)}. \end{aligned} \tag{3.19}$$

Therefore, $\|Uf\|_a^2 = \sup_{t\in I}E\|(Uf)(t)\|^2 < \infty$. The $V^2(\Omega,X)$-continuity of $(Uf)(t)$ on I (Lemma 3.2) ensures that $U(A)\subseteq A$. □

Theorem 3.4. *If the assumptions* $(H1)$–$(H4)$ *hold, then system (1.3) admits a unique mild solution belonging to the space A.*

Proof. We employ the PIM to establish the Ex-Un result. For $j \geq 0$, we assume

$$\begin{cases} f_{j+1}(t) = (Uf_j)(t), \; j = 0,1,2,\cdots, \\ f_0(t) = f_0. \end{cases} \tag{3.20}$$

By Lemma 3.3, we have $f_j \in A$, $j = 0,1,2,\cdots$. By Lemma 2.6 and $(H2)$,

$$\begin{aligned} &E\|f_{j+1}(t)-f_j(t)\|^2 \\ &\quad = E\Big\|\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)[\eta(\gamma,f_j(\gamma))-\eta(\gamma,f_{j-1}(\gamma))]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t E\|\eta(\gamma, f_j(\gamma)) - \eta(\gamma, f_{j-1}(\gamma))\|^2 d\gamma\Big)\\
&\le Q\frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t E\|f_j(\gamma) - f_{j-1}(\gamma)\|^2 d\gamma\\
&\le \Big(\frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{(2\alpha-1)\Gamma^2(\alpha)}\Big)^2\int_0^t\int_0^\gamma E\|f_{j-1}(\gamma_1) - f_{j-2}(\gamma_1)\|^2 d\gamma_1 d\gamma\\
&\le \dots\\
&\le \Big(\frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{(2\alpha-1)\Gamma^2(\alpha)}\Big)^j\int_0^t\int_0^\gamma\cdots\int_0^{\gamma_{j-2}} E\|f_1(\gamma_{j-1}) - f_0(\gamma_{j-1})\|^2 d\gamma_{j-1}\dots d\gamma_1 d\gamma\\
&\le \Big(\frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{(2\alpha-1)\Gamma^2(\alpha)}\Big)^j\frac{\sup_{\gamma\in I} E\|f_1(\gamma) - f_0(\gamma)\|^2}{j!},
\end{aligned}
\tag{3.21}
$$

which implies that

$$
\sup_{t\in I} E\|f_{j+1}(t) - f_j(t)\|^2 \le \Big(\frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{(2\alpha-1)\Gamma^2(\alpha)}\Big)^j\frac{\sup_{\gamma\in I} E\|f_1(\gamma) - f_0(\gamma)\|^2}{j!}. \tag{3.22}
$$

Thus, the sequence $\{f_j(t)\}_{j\ge 0} \subseteq V^2(\Omega, X)$ forms a Cauchy sequence. Consequently, there exists a limit function $f \in V^2(\Omega, X)$ such that

$$
\sup_{t\in I} E\|f_j(t) - f(t)\|^2 \to 0 \quad \text{as } j \to \infty. \tag{3.23}
$$

Passing to the limit as $j \to \infty$ in relation (3.20), we conclude that f is a mild solution of the system, establishing its existence.

To verify uniqueness, suppose that system (1.3) admits two mild solutions.

$$
\begin{aligned}
&E\|f(t) - f'(t)\|^2\\
&\quad = E\Big\|\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1}\mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)[\eta(\gamma, f(\gamma)) - \eta(\gamma, f'(\gamma))]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2\\
&\quad \le \frac{\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t E\|\eta(\gamma, f(\gamma)) - \eta(\gamma, f'(\gamma))\|^2 d\gamma\Big)\\
&\quad \le \frac{\varpi^2\Phi_2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t E\|f(\gamma) - f'(\gamma)\|^2 d\gamma.
\end{aligned}
\tag{3.24}
$$

Using Gronwall's lemma,

$$
\sup_{t\in I} E\|f(t) - f'(t)\|^2 = 0. \tag{3.25}
$$

□

4. Controllability results

In this section, we establish approximate controllability.

Consider $\varrho : A \to V^2(I, X)$ as

$$(\varrho f)(t) = \eta(t, f(t)),\ t \in I.$$

The linear operator $\hbar : V^2(I, X) \to X$ is

$$\hbar(\aleph) = \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\aleph(\gamma)\frac{d\gamma}{\gamma^{1-\beta}},\ \aleph \in V^2(I, X).$$

The kernel of the operator $\hbar$ is denoted by $M_0(\hbar)$. One can verify that $M_0(\hbar)$ constitutes a closed subspace of $V^2(I, X)$. Its orthogonal complement is written as $M_0^\perp(\hbar)$, leading to the unique orthogonal decomposition

$$V^2(I, X) = M_0(\hbar) \bigoplus M_0^\perp(\hbar).$$

Let $Ran(K)$ represent the range of the operator K. We impose the following additional hypothesis:

$(H5)$: For every $\aleph \in V^2(I, X)$, there exists a function $\zeta \in \overline{Ran(K)}$ such that $\hbar_\aleph = \hbar_x$.

We also assume that $Ran(\hbar) = X$.

Now, take an arbitrary $\aleph \in V^2(I, X)$. By hypothesis $(H5)$, we can select $\zeta \in \overline{Ran(K)}$ satisfying

$$\hbar_\aleph = \hbar_\zeta.$$

This equality implies

$$\hbar(\aleph - \zeta) = 0 \quad \text{and consequently} \quad \aleph - \zeta \in M_0(\hbar).$$

Thus, $\aleph$ admits the decomposition

$$\aleph = (\aleph - \zeta) + \zeta,$$

with $\aleph - \zeta \in M_0(\hbar)$ and $\zeta \in \overline{Ran(K)}$.

Assuming further that

$$M_0(\hbar) \cap \overline{Ran(K)} = \{0\},$$

the sum becomes direct. Combined with $(H5)$, this yields the direct sum decomposition

$$V^2(I, X) = M_0(\hbar) \bigoplus \overline{Ran(K)}.$$

We now define a linear, continuous operator

$$\rho : M_0^\perp(\hbar) \to \overline{Ran(K)} \tag{4.1}$$

by setting $\rho s^* = \zeta^*$. Here, ζ^* is the unique element in the intersection $\{s^* + M_0(\hbar)\} \cap \overline{Ran(K)}$ possessing the minimal norm; that is,

$$\|\rho s^*\| = \|\zeta^*\| = \min\Big\{\|\theta\| :\ \theta \in \{s^* + M_0(\hbar)\} \cap \overline{Ran(K)}\Big\}. \tag{4.2}$$

By virtue of $(H5)$, the set $\{s^* + M_0(\hbar)\} \cap \overline{Ran(K)}$ is non-empty for each $s^* \in M_0^\perp(\hbar)$. Moreover, every element $s \in V^2(I, X)$ can be expressed uniquely as $s = h + h^*$ with $h \in M_0(\hbar)$ and $h^* \in M_0^\perp(\hbar)$. Consequently, the operator ρ is well-defined, and its norm is bounded by some constant τ, i.e., $\|\rho\| \le \tau$ [39].

Remark 4.1. *Assumption (H5) is restrictive in nature, as it imposes structural compatibility between the nonlinear dynamics and the control operator. While it is essential for the fixed-point and controllability arguments, relaxing this condition is an important direction for future research.*

Lemma 4.2. *[40] For every $s \in V^2(I, X)$ and its associated component $h \in M_0(\hbar)$, one can find a constant $K > 0$ satisfying the bound*

$$\|h\|_{V^2(I,X)} \leq (1 + K)\|s\|_{V^2(I,X)}.$$

Next, we introduce the operator $C : V^2(I, X) \to V^2(I, X)$ defined by

$$(Cp)(t) = \int_0^t \left(\frac{t^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t^\beta - \gamma^\beta}{\beta}\right) p(\gamma) \frac{d\gamma}{\gamma^{1-\beta}}.$$

Using this operator, we define the subspace

$$\Theta_0 = \{b \in V^2(I, X) : b(t) = (Ch)(t), \text{ for some } h \in M_0(\hbar) \text{ and all } t \in I\}.$$

Note that for any $b \in \Theta_0$, we have $b(L) = (Ch)(L) = 0$.

Now, let $f(\cdot)$ be a mild solution of system (2.6). We define the operator $\mathbb{G}_f : \Theta_0 \to \Theta_0$ by

$$(\mathbb{G}_f b)(t) = (Ch)(t), \qquad t \in I,$$

where the element h is uniquely determined via the decomposition

$$\varrho(f + b) = h + \zeta, \quad h \in M_0(\hbar), \ \zeta \in \overline{Ran(K)}.$$

Theorem 4.3. *Assume that conditions $(H1)$–$(H5)$ hold. Then the linear fractional stochastic system (2.6) is approximately controllable on I; that is,*

$$\overline{B_L(0)} = V^2(\Omega, X).$$

Proof. For every $\Psi \in V^2(\Omega, X)$, the expression

$$\Psi - S_{\alpha,\beta}(L)f_0 - \int_0^L \left(\frac{L^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{L^\beta - \gamma^\beta}{\beta}\right) \varphi(\gamma) \frac{dw^H(\gamma)}{\gamma^{1-\beta}}$$

belongs to $V^2(\Omega, X)$. In particular, for almost every $u \in \Omega$, the expression

$$\Psi - S_{\alpha,\beta}(L)f_0 - \int_0^L \left(\frac{L^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{L^\beta - \gamma^\beta}{\beta}\right) \varphi(\gamma) \frac{dw^H(\gamma)}{\gamma^{1-\beta}}$$

defines an element of X. By hypothesis $(H5)$, we can select $\aleph \in V^2(I, X)$ such that

$$\begin{aligned} \Psi - S_{\alpha,\beta}(L)f_0 &- \int_0^L \left(\frac{L^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{L^\beta - \gamma^\beta}{\beta}\right) \varphi(\gamma) \frac{dw^H(\gamma)}{\gamma^{1-\beta}} \\ &= \int_0^L \left(\frac{L^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{L^\beta - \gamma^\beta}{\beta}\right) \aleph(\gamma) \frac{d\gamma}{\gamma^{1-\beta}}. \end{aligned} \tag{4.3}$$

A second application of $(H5)$ guarantees the existence of $\zeta \in \overline{Ran(K)}$ satisfying

$$\int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\aleph(\gamma)\frac{d\gamma}{\gamma^{1-\beta}} = \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\zeta(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}. \tag{4.4}$$

Consequently, the target state Ψ can be represented as

$$\begin{aligned}\Psi = S_{\alpha,\beta}(L)f_0 &+ \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\\ &+ \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\zeta(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}.\end{aligned} \tag{4.5}$$

Since $\zeta \in \overline{Ran(K)}$, for any $\varepsilon > 0$ there exists a control u_ε such that

$$\sup_{t\in I} E\|Ku_\varepsilon(t) - x(t)\|^2 < \frac{\Gamma^2(\alpha)(2\alpha-1)\varepsilon}{\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}. \tag{4.6}$$

Now define the approximating state

$$\begin{aligned}\Psi_\varepsilon =& S_{\alpha,\beta}(L)f_0 + \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}\\ &+ \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)Ku_\epsilon(\gamma)\frac{d\gamma}{\gamma^{1-\beta}}.\end{aligned} \tag{4.7}$$

By construction, $\Psi_\varepsilon \in B_L(0)$. Finally, we estimate the approximation error:

$$\begin{aligned}&E\|\Psi - \Psi_\varepsilon\|^2\\ &= E\Big\| \int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)[Ku_\varepsilon(\gamma) - \zeta(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2\\ &\le \frac{\varpi^2}{\Gamma^2(\alpha)}E\Big(\int_0^L \Big(\frac{L^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1}\|Ku_\varepsilon - \zeta\|\frac{d\gamma}{\gamma^{1-\beta}}\Big)^2\\ &\le \frac{\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ}{\Gamma^2(\alpha)(2\alpha-1)}\sup_{\gamma\in I} E\|Ku_\varepsilon(\gamma) - \zeta(\gamma)\|^2\\ &< \varepsilon.\end{aligned} \tag{4.8}$$

This inequality confirms that system (2.6) is approximately controllable. □

Lemma 4.4. *Assume that conditions* $(H1)$–$(H4)$ *are satisfied. Then the operator* $\mathbb{G}_f$ *possesses a fixed point* $b_0 \in \Theta_0$*, provided that*

$$\frac{4\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(1+K)^2\Phi_2}{\Gamma^2(\alpha)(2\alpha-1)} < 1. \tag{4.9}$$

Proof. For a given $\nu > 0$, define the closed ball $\varsigma_\nu = \{\mathrm{s} \in \Theta_0 : \|\mathrm{s}\|_{V^2(I,X)} \le \nu\}$. Our next goal is to establish that $\mathbb{G}_f(\varsigma_\nu) \subseteq \varsigma_\nu$. We argue by contradiction. If this containment fails for some ν, then there must exist an element $b \in \varsigma_\nu$ for which $\|\mathbb{G}_f(b)\|^2_{V^2(I,X)} > \nu$. It follows that

$$\nu^2 < \|\mathbb{G}_f(b)\|^2_{V^2(I,X)} = \|C\mathrm{h}\|^2_{V^2(I,X)}. \tag{4.10}$$

In fact, by Lemma 4.2, we have

$$
\begin{aligned}
\|(C\mathrm{h})(t)\|^2 &= \Big\| \int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)\mathrm{h}(\gamma)\frac{d\gamma}{\gamma^{1-\beta}} \Big\|^2 \\
&\le \frac{\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{\alpha-1} \|\mathrm{h}(\gamma)\| \frac{d\gamma}{\gamma^{1-\beta}}\Big)^2 \\
&\le \frac{\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta-\gamma^\beta}{\beta}\Big)^{2\alpha-2} \frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t \|\mathrm{h}(\gamma)\|^2 d\gamma\Big) \\
&\le \frac{\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{\Gamma^2(\alpha)(2\alpha-1)}\|\mathrm{h}\|^2_{V^2(I,X)} \\
&\le \frac{\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\|\varrho(f+b)\|^2_{V^2(I,X)} \\
&\le \frac{\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\Big(\int_0^L \|\eta(t,(f+b)(t))-\eta(t,0)+\eta(t,0)\|^2 dt\Big) \\
&\le \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\Big(\int_0^L \|\eta(t,(f+b)(t))-\eta(t,0)\|^2+\|\eta(t,0)\|^2 dt\Big) \\
&\le \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\Big[\int_0^L (\Phi_2\|(f+b)(t)\|^2+\tau_\eta^2)dt\Big] \\
&\le \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\Big[2\Phi_2\Big(\int_0^L \|f(t)\|^2 dt+\nu^2\Big)+\tau_\eta^2 L\Big],
\end{aligned}
\tag{4.11}
$$

where $\tau_\eta = \max_{t\in I}\|\eta(t,0)\|$. Hence,

$$
\begin{aligned}
\nu^2 &< \|\mathbb{G}_f(b)\|^2_{V^2(I,X)} \\
&= \|Cn\|^2_{V^2(I,X)} \\
&\le \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)}\Big[2\Phi_2\int_0^L \|f(t)\|^2 dt+2\Phi_2\nu^2+\tau_\eta^2 L\Big].
\end{aligned}
\tag{4.12}
$$

Dividing by ν^2 on both sides of (4.12) and taking limitation as $\nu \to \infty$, it follows that

$$
\frac{4\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q(K+1)^2}{\Gamma^2(\alpha)(2\alpha-1)} \ge 1, \tag{4.13}
$$

which contradicts (4.9). Therefore, the operator $\mathbb{G}_f$ maps ς_ν into itself.

Hypothesis $(H4)$ together with Lemma 2.6 implies that the operator $\mathcal{Y}_{\alpha,\beta}(t)$ is compact. This property, in turn, ensures the compactness of the operator $\mathbb{G}_f$.

Now, we prove that the operator

$$
\mathbb{G}_f : \Theta_0 \to \Theta_0, \qquad \mathbb{G}_f(b) = Ch,
$$

is continuous in $V^2(I,X)$.

Let $\{b_\iota\} \subset \Theta_0$ be a sequence such that

$$b_\iota \to b \quad \text{in } V^2(I, X).$$

By assumption (H2), the nonlinear mapping is Lipschitz continuous from $V^2(I, X)$ into itself. Consequently,

$$\varrho(f + b_\iota) \to \varrho(f + b) \quad \text{in } V^2(I, X).$$

By assumption (H5), the space $V^2(I, X)$ admits the direct sum decomposition

$$V^2(I, X) = M_0(\hbar) \oplus Ran(K).$$

Hence, for each ι, there exist unique elements $h_\iota \in M_0(\hbar)$ and $\zeta_\iota \in Ran(K)$ such that

$$\varrho(f + b_\iota) = h_\iota + \zeta_\iota,$$

and similarly,

$$\varrho(f + b) = h + \zeta,$$

with $h \in M_0(\hbar)$ and $\zeta \in Ran(K)$. Moreover, the mapping that assigns to each element of $V^2(I, X)$ its component in $M_0(\hbar)$ is linear and bounded. Therefore,

$$h_\iota \to h \quad \text{in } V^2(I, X).$$

Since the operator

$$(Cp)(t) = \int_0^t \left(\frac{t^\beta - \gamma^\beta}{\beta}\right)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\left(\frac{t^\beta - \gamma^\beta}{\beta}\right) p(\gamma) \frac{d\gamma}{\gamma^{1-\beta}},$$

is linear and bounded on $V^2(I, X)$, it follows that

$$\mathbb{G}_f(b_\iota) = Ch_\iota \to Ch = \mathbb{G}_f(b) \quad \text{in } V^2(I, X).$$

Thus, $\mathbb{G}_f$ is continuous on Θ_0. □

Therefore, all the hypotheses of Schauder's fixed point theorem, namely boundedness, compactness, and continuity of $\mathbb{G}_f$, are satisfied. The proof of Lemma 4.4 is thus complete.

□

Theorem 4.5. *Assume that* (*H*1) *through* (*H*5) *hold and that condition* (4.9) *is satisfied. Then, the fractional stochastic system* (1.3) *is approximately controllable on I.*

Proof. The proof is based on transferring approximate controllability from the associated linear fractional stochastic system to the nonlinear system. More precisely, the argument proceeds as follows. First, by Theorem 4.3, the linear system is approximately controllable on I, so that any prescribed terminal state can be approximated arbitrarily well by a suitable control. Next, for a fixed mild solution of the linear system, Lemma 4.4 is used to construct a correction function $b_0 \in \Theta_0$ as a fixed point of an auxiliary operator. This correction incorporates the nonlinear term while preserving the initial and terminal values. Finally, it is shown that the corrected trajectory $f'(\cdot) = x(\cdot) + b_0(\cdot)$ is a mild solution of the nonlinear system. Consequently, the approximate controllability of the linear system implies the approximate controllability of the nonlinear system.

We have

$$\begin{aligned} f(t) =& S_{\alpha,\beta}(t)f_0 + \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big) Ku(\gamma)\frac{d\gamma}{\gamma^{1-\beta}} \\ &+ \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}. \end{aligned} \tag{4.14}$$

Since we have $\mathbb{G}_f(b_0) = C\mathrm{h}_0 = b_0$, it follows that

$$\varrho(f + b_0)(t) = \mathrm{h}_0(t) + \zeta_0(t). \tag{4.15}$$

Applying C yields

$$C\varrho(f + b_0)(t) = C\mathrm{h}_0(t) + C\zeta_0(t) = b_0(t) + C\zeta_0(t). \tag{4.16}$$

Hence,

$$f(t) + C\varrho(f + b_0)(t) = f(t) + b_0(t) + C\zeta_0(t). \tag{4.17}$$

Denoting $f'(t) = f(t) + b_0(t)$, then

$$f(t) + C\varrho f'(t) = f'(t) + C\zeta_0(t). \tag{4.18}$$

Hence,

$$\begin{aligned} &f'(t) = f(t) + C\varrho f'(t) - C\zeta_0(t) \\ &= S_{\alpha,\beta}(t)f_0 + \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[Ku(\gamma) - \zeta_0(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}} \\ &+ \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)\eta(\gamma, f'(\gamma))\frac{d\gamma}{\gamma^{1-\beta}} + \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}. \end{aligned} \tag{4.19}$$

Consequently, for $f' = f + b_0$, we have

$$\begin{cases} D^{\alpha,\beta} f''(t) = Af'(t) + (Ku - \zeta_0)(t) + \eta(t, f''(t)) + \varphi(t)\frac{dw^H(t)}{dt}, \ t \in [0, L], \\ f'(0) = f_0. \end{cases} \tag{4.20}$$

Recalling that $\mathbb{G}_f(b_0) = C\mathrm{h}_0 = b_0$, and using the definitions of C and the fact that $\mathrm{h}_0 \in M_0(\hbar)$, we obtain the boundary conditions $b_0(0) = b_0(L) = 0$. Consequently,

$$\begin{aligned} &f'(0) = f(0) + b_0(0) = f_0, \\ &f'(L) = f(L) + b_0(L) = f(L) \in C_L(0). \end{aligned} \tag{4.21}$$

Our next objective is to prove the set inclusion $C_L(0) \subseteq C_L(\eta)$. Given $\zeta_0 \in \overline{Ran(K)}$, we can choose a control $p \in V_F^2(I, Y)$ such that

$$\sup_{t\in I} E\|Kp - \zeta_0\|^2 < \varepsilon. \tag{4.22}$$

Now, define the adjusted control $\tilde{u} = u - p$. Let $\phi = f_{\tilde{u}}$ denote the mild solution corresponding to $\tilde{u}$, which satisfies the system

$$\begin{cases} D^{\alpha,\beta}\phi(t) = \Im\phi(t) + K\tilde{u}(t) + \eta(t,\phi(t)) + \varphi(t)\dfrac{dw^H(t)}{dt}, \quad t \in [0, L], \\ \phi(0) = f_0. \end{cases} \tag{4.23}$$

Thus,

$$\begin{aligned} f_{\tilde{u}}(t) =& S_{\alpha,\beta}(t) f_0 + \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[\eta(\gamma, f_{\tilde{u}}(\gamma)) + K\tilde{u}(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}} \\ &+ \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)\varphi(\gamma)\frac{dw^H(\gamma)}{\gamma^{1-\beta}}, \ t \in [0, L], \end{aligned} \tag{4.24}$$

and $f_{\tilde{u}}(L) \in C_L(\eta)$.
On the other hand,

$$\begin{aligned} & E\|f'(t) - f_{\tilde{u}}(t)\|^2 \\ &= E\Big\| \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[(Kp)(\gamma) - \zeta(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}} \\ &+ \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[\eta(\gamma, f'(\gamma)) - \eta(\gamma, f_{\tilde{u}}(\gamma))]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2 \\ &\leq 2E\Big\| \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[(Kp)(\gamma) - \zeta(\gamma)]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2 \\ &+ 2E\Big\| \int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{\alpha-1} \mathcal{Y}_{\alpha,\beta}\Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)[\eta(\gamma, f'(\gamma)) - \eta(\gamma, f_{\tilde{u}}(\gamma))]\frac{d\gamma}{\gamma^{1-\beta}}\Big\|^2 \\ &\leq \frac{2\varpi^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t E\|Kp - x\|^2 d\gamma\Big) \\ &+ \frac{2\varpi^2\tau_1^2}{\Gamma^2(\alpha)}\Big(\int_0^t \Big(\frac{t^\beta - \gamma^\beta}{\beta}\Big)^{2\alpha-2}\frac{d\gamma}{\gamma^{2-2\beta}}\Big)\Big(\int_0^t E\|f'(\gamma) - f_{\tilde{u}}(\gamma)\|^2 d\gamma\Big) \\ &\leq \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ\varepsilon}{\Gamma^2(\alpha)(2\alpha-1)} + \frac{2\varpi^2\tau_1^2(\frac{L^\beta}{\beta})^{2\alpha-1}Q}{\Gamma^2(\alpha)(2\alpha-1)}\int_0^t E\|f'(\gamma) - f_{\tilde{u}}(\gamma)\|^2 d\gamma. \end{aligned} \tag{4.25}$$

Define $\mathbb{A}(t) = E\|f'(t) - f_{\tilde{u}}(t)\|^2$. Using Gronwall's inequality,

$$E\|f'(t) - f_{\tilde{u}}(t)\|^2 \leq \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ\varepsilon}{\Gamma^2(\alpha)(2\alpha-1)}\exp\Big\{\frac{2\varpi^2\tau_1^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ}{\Gamma^2(\alpha)(2\alpha-1)}\Big\}. \tag{4.26}$$

Moreover,

$$\begin{aligned} E\|f'(L) - f_{\tilde{u}}(L)\|^2 &\leq \sup_{t\in I} E\|f'(t) - f_{\tilde{u}}(t)\|^2 \\ &\leq \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ\varepsilon}{\Gamma^2(\alpha)(2\alpha-1)}\exp\Big\{\frac{2\varpi^2\tau_1^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ}{\Gamma^2(\alpha)(2\alpha-1)}\Big\}. \end{aligned} \tag{4.27}$$

Therefore,

$$\begin{aligned} E\|f(L)-f_{\tilde{u}}(L)\|^2 &= E\|f'(L)-f_{\tilde{u}}(L)\|^2 \\ &\le \frac{2\varpi^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ\varepsilon}{\Gamma^2(\alpha)(2\alpha-1)}\exp\left\{\frac{2\varpi^2\tau_1^2(\frac{L^\beta}{\beta})^{2\alpha-1}LQ}{\Gamma^2(\alpha)(2\alpha-1)}\right\}, \end{aligned} \tag{4.28}$$

which implies that $C_L(0)\subseteq C_L(\eta)$. By Theorem 4.3, $\overline{C_L(0)}=V^2(\Omega,X)$. Therefore, $\overline{C_L(\eta)}=V^2(\Omega,X)$.

To complete the proof, it remains to show that the reachable set of the nonlinear system is dense in $V^2(\Omega,X)$.

From Theorem 4.3, the corresponding linear system is approximately controllable on I. Hence,

$$\overline{B_L(0)}=V^2(\Omega,X).$$

Let $x(\cdot)$ be the mild solution of the linear system and let b_0 be the fixed point of the operator $\mathbb{G}_x$ obtained in Lemma 4.4. Define

$$f'(t)=x(t)+b_0(t),\quad t\in I.$$

Then $f'(\cdot)$ is a mild solution of the nonlinear system (1.3) corresponding to a suitably modified control.

Moreover, by the construction of b_0, we have

$$f'(L)=x(L),$$

which implies that every terminal state reachable by the linear system is also reachable, or can be approximated arbitrarily closely, by the nonlinear system. Consequently,

$$B_L(0)\subset\overline{B_L(\eta)}.$$

Since $\overline{B_L(0)}=V^2(\Omega,X)$, it follows that

$$\overline{B_L(\eta)}=V^2(\Omega,X).$$

Therefore, for any $\xi\in L^2(\Omega,Y)$ and any $\varepsilon>0$, there exists a control $u\in V_F^2(I,Y)$ such that the corresponding mild solution $f'(t)$ of system (1.3) satisfies

$$\mathbb{E}\|f'(L)-\xi\|^2<\varepsilon.$$

Hence, system (1.3) is approximately controllable on I.

□

5. Examples

We now present an illustrative example to validate the theoretical results established in the preceding sections.

Consider the fractional stochastic control system governed by the equation

$$\begin{cases} D^{\alpha,\beta}s(t,y)=\frac{\partial^2}{\partial y^2}s(t,y)+\eta(t,s(t,y))+Ku(t,y)+\varphi(t)\dfrac{dw^H(t)}{dt}, & t\in[0,1],\ y\in(0,\pi),\\ s(t,0)=s(t,\pi)=0, & t\in[0,1],\\ s(0,y)=s_0(y), & y\in(0,\pi), \end{cases} \tag{5.1}$$

where $D^{\alpha,\beta}$ denotes the CK fractional derivative.

Take the state space $X = Z = V^2(0,\pi)$ and the time interval $I = [0,1]$. Introduce the operator $\Im : N(\Im) \subset X \to X$ defined by $\Im s = \partial^2 s/\partial y^2$, with domain

$$N(\Im) = \left\{ s \in X : s, \frac{\partial s}{\partial y} \text{ are absolutely continuous, } \frac{\partial^2 s}{\partial y^2} \in X,\ s(0) = s(\pi) = 0 \right\}.$$

For $j = 1, 2, \ldots$, define the functions $g_j(y) = \sqrt{2/\pi}\sin(jy)$. The family $\{g_j\}_{j\geq 1}$ constitutes a complete orthonormal eigenbasis for $\Im$. It is known that $\Im$ generates a strongly continuous semigroup $\{S_t\}_{t\geq 0}$ that is compact, analytic, and self-adjoint [41]. Hence, hypothesis $(H4)$ is satisfied.

Next, construct the control space Y as

$$Y = \left\{ u : \mathbb{R} \to \mathbb{R} : u = \sum_{j=2}^{\infty} u_j g_j \text{ with } \sum_{j=2}^{\infty} u_j^2 < \infty \right\}.$$

The norm on Y is given by $\|u\|_Y = (\sum_{j=1}^{\infty} u_j^2)^{1/2}$. We define the bounded linear control operator $K : Y \to X$ by

$$Ku = 2u_2 g_1 + \sum_{j=2}^{\infty} u_j g_j, \qquad \text{for } u = \sum_{j=2}^{\infty} u_j g_j \in Y.$$

Choose a sequence of positive real numbers $\{\xi_j\}_{j=1}^{\infty}$ and define the covariance operator $J : Z \to Z$ through its action on the basis: $Jg_j = \xi_j g_j$. We assume that

$$\mathrm{tr}(J) = \sum_{j=1}^{\infty} \sqrt{\xi_j} < \infty.$$

The cylindrical fBm process w^H is then defined as

$$w^H(t) = \sum_{j=1}^{\infty} \sqrt{\xi_j} W_j^H(t) g_j, \qquad t \geq 0,\ \frac{1}{2} < H < 1,$$

where $\{W_j^H\}_{j\in\mathbb{N}}$ is a family of mutually independent one-dimensional fBm.

Finally, we identify the abstract functions corresponding to the system variables:

$$f(t)(y) = s(t,y), \quad \eta(t, f(t))(y) = \eta(t, s(t,y)), \quad u(t)(y) = u(t,y).$$

Then, (5.1) can be reformulated as

$$\begin{cases} D^{\alpha,\beta} f(t) = \Im f(t) + Ku(t) + \eta(t, f(t)) + \varphi(t)\frac{dw^H(t)}{dt},\ t \in [0,1], \\ f(0) = f_0. \end{cases} \tag{5.2}$$

Define $\eta(t, \mathrm{s}(t,y)) = \frac{\exp^{-t}|\mathrm{s}(t,y)|}{(1+\exp^t)(1+|\mathrm{s}(t,y)|)}$. Clearly, we have

$$\|\eta(t, \mathrm{s}(t,y))\| \leq |\mathrm{s}(t,y)|, \tag{5.3}$$

and

$$
\begin{aligned}
&\|\eta(t, s_1(t))(y) - \eta(t, s_2(t))(y)\| \\
&= \frac{\exp^{-t} \| |s_1(t, y)| - |s_2(t, y)| \|}{(1 + \exp^{t})(1 + |s_1(t, y)|)(1 + |s_2(t, y)|)} \\
&\leq \frac{\exp^{-t}}{1 + \exp^{t}} |s_1(t, y) - s_2(t, y)| \\
&\leq \frac{1}{2} |s_1(t, y) - s_2(t, y)|. \qquad (5.4)
\end{aligned}
$$

Therefore, the conditions $(H1)$ and $(H2)$ hold. If $(H3)$, $(H5)$, and Eq (4.9) are also fulfilled, then, by applying Theorem 4.5, system (5.1) is approximately controllable on the interval $[0, 1]$.

Figure 1 shows the state trajectory of the controlled system 5.1, obtained by solving it using the Euler–Maruyama method over $t \in [0, 1]$. Under the designed control input, the state is driven toward the target value 0.5 and enters a small neighborhood around it at the final time, illustrating the approximate controllability of the system. The reported optimal control value used to achieve this behavior is $u = 5.547$.

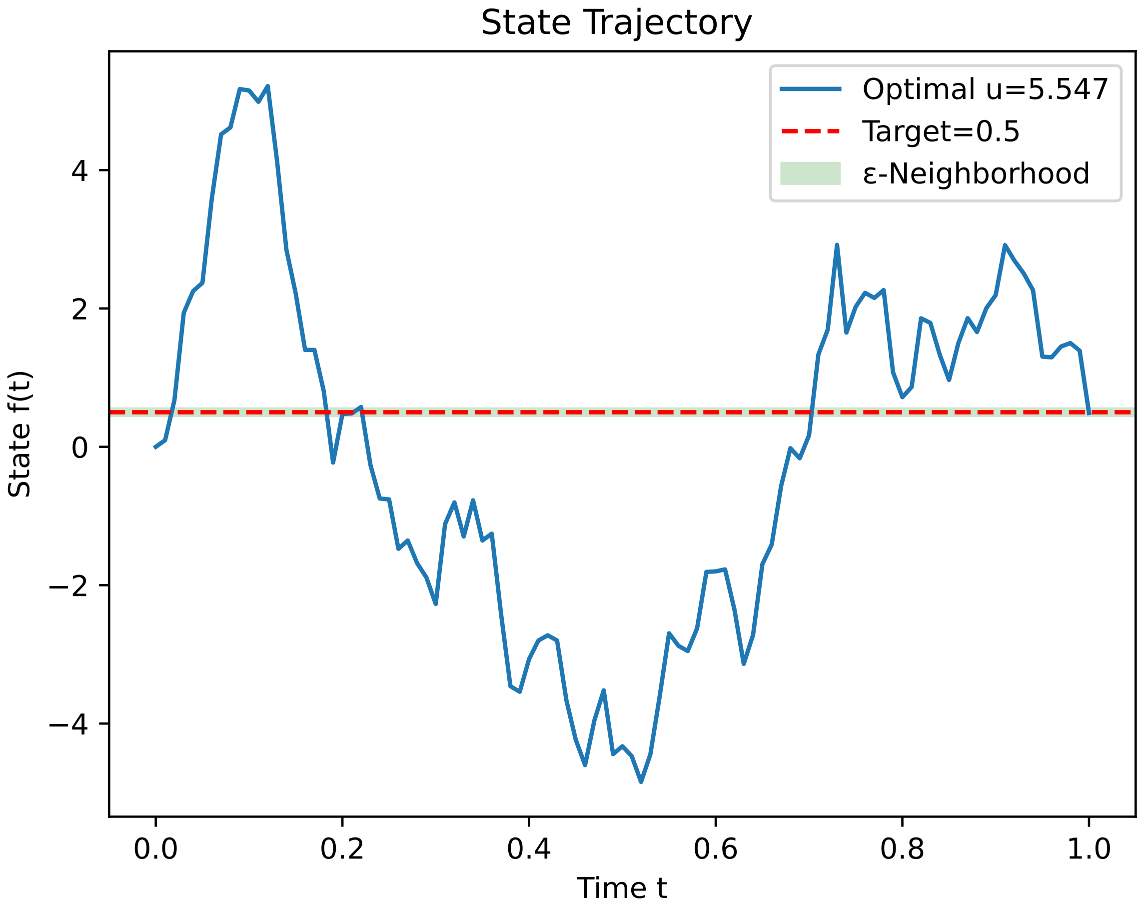

Figure 1. State trajectory under the optimal control $u = 5.547$, showing convergence toward the target state 0.5 and illustrating approximate controllability on $[0, 1]$.

6. Conclusions

This research examines the controllability of FSDEs defined by the CKFD and driven by fBm. The PIM is used for investigating the Ex-Un solutions in the context of CKFD. We use the Schauder fixed-point method to find the appropriate criteria for controllability. To illustrate the efficacy of the theoretical findings, an example is provided. Consequently, FSDEs incorporating CKFD and fBm offer a resilient mathematical framework for analyzing the dynamics of intricate systems.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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