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Research article

Dynamic properties and α-path of an uncertain SIRS epidemic model

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Abstract: The dynamic analysis of epidemic models plays a crucial role in understanding disease transmission mechanisms and prevention strategies. Building upon the research of Tan et al. [1], this paper investigates novel dynamic properties of solutions and α-paths of an uncertain SIRS model. First, we prove the existence, uniqueness, and stability of a solution for the SIRS model. Using the Yao-Chen formula, we derive α-paths of the model and prove that both the disease-free and endemic equilibria are globally asymptotically stable, thereby refining existing research. When the threshold $\mathscr{R}_0^u \leq 1$, the disease-free equilibrium is globally asymptotically stable, while the endemic equilibrium is globally asymptotically stable if $\mathscr{R}_0^u > 1$. Finally, numerical simulations demonstrate that increasing the recovery rate and decreasing the disease-induced mortality rate can effectively reduce the disease spread. The results show that the disease transmission rate and the intensity of the Liu process are essential to prevent the disease spread.

Keywords: uncertain SIRS model; α-path; Lyapunov function; globally asymptotically stability
Mathematics Subject Classification: 60G51, 60G57, 92B05

1. Introduction

Infectious diseases pose a significant challenge to global public health governance, thereby continually threatening human health and hindering social development. Epidemic models have become essential analytical tools to study the disease transmission patterns and to devise effective prevention and control strategies. Kermack and McKendrick [2] were the first to propose a deterministic SIRS epidemic model. Let N_t denote the total population at time t, divided into three compartments: susceptible, infectious, and recovered individuals. Define S_t, I_t, and R_t as the number of susceptible, infectious, and recovered individuals at time t, respectively. The classical SIRS model with a bilinear incidence rate is formulated as follows [3]:

$$\begin{cases} dS_t = (\Lambda - \beta S_t I_t - v S_t + \delta R_t)dt, \\ dI_t = (\beta S_t I_t - (r + v + \mu)I_t)dt, \\ dR_t = (rI_t - (\delta + v)R_t)dt, \end{cases} \tag{1.1}$$

where Λ is a recruitment rate, β is the transmission efficiency of infection, r corresponds to the recovery rate of infected individuals, v is the natural mortality rate unrelated to disease, μ measures the fatality rate induced by the infection, and δ represents the coefficient of immune loss. Adding the three equations of model (1.1), we obtain that N_t meets the differential equation $dN_t = [\Lambda - vN_t - \mu I_t]dt$. Thus, $\lim_{t\to+\infty}\{S_t + I_t + R_t\} = \max\{N_0, \Lambda/v\}$. It means that the solutions of system (1.1) will either enter or remain within the region Γ over time:

$$\Gamma = \left\{(S_t, I_t, R_t) \,\middle|\, S_t, I_t, R_t \geq 0, S_t + I_t + R_t \leq \Lambda/v, \forall t > 0\right\}.$$

For model (1.1), define a basic reproduction number $\mathscr{R}_0 = \beta\Lambda/(v(r + v + \mu))$. If $\mathscr{R}_0 < 1$, then there exists only a disease-free equilibrium $E_0 = (\Lambda/v, 0, 0)$, which is globally asymptotically stable. If $\mathscr{R}_0 > 1$, then there exists an endemic equilibrium $E_1^* = (S_1^*, I_1^*, R_1^*)$, which is globally asymptotically stability, where

$$S_1^* = \frac{r + v + \mu}{\beta}, I_1^* = \Lambda\left(1 - \frac{1}{R_0}\right)\left(\mu + v\left(1 + \frac{r}{\delta + v}\right)\right)^{-1}, R_1^* = rI_1^*/(\delta + v).$$

Especially, if $\mu = 0$, then $dN_t = (\Lambda - vN_t)dt$, that is, $N_t = \Lambda(1 - e^{-vt})/v + N_0 e^{-vt}$. As $t \to \infty$, $N_t \to \Lambda/v$. Model (1.1) is simplified by the following:

$$\begin{cases} dS_t = (\Lambda - \beta S_t I_t - v S_t + \delta(\Lambda/v - S_t - I_t))dt, \\ dI_t = (\beta S_t I_t - (r + v)I_t)dt. \end{cases} \tag{1.2}$$

Since model (1.2) is a special case of model (1.1), the basic reproduction number, disease-free, and endemic equilibria of model (1.2) are the special forms which correspond to $\mu = 0$ in model (1.1).

Compared with deterministic models, stochastic dynamical models, which incorporate Brownian motion or Markov chains, account for the effects of random disturbances on the disease transmission process and are more realistic. In recent years, numerous researchers have explored a range of stochastic SIRS epidemic models. For example, Lakhal et al. [4] constructed a threshold value for a stochastic SIRS model with a general incidence rate, and derived the conditions for disease extinction and persistence. Zhi et al. [5] proposed a reaction-diffusion SIRS model and discussed dynamic properties in a spatially heterogeneous environment. Lan et al. [6] presented a stochastic SIRS epidemic model with a non-monotone incidence rate and regime-switching. Zhi et al. [7] investigated a periodic waterborne disease model with environmental pollution. In this paper, we obtain a stochastic SIRS model by introducing a stochastic perturbation into the transmission rate β of model (1.1), where the stochastic SIRS model (1.3) is described by the following equations:

$$\begin{cases} dS_t = (\Lambda - vS_t - \beta S_t I_t + \delta R_t)\,dt - \sigma S_t I_t\,dB_t, \\ dI_t = (\beta S_t I_t - (r + v + \mu)I_t)\,dt + \sigma S_t I_t\,dB_t, \\ dR_t = (\lambda I_t - (\delta + v)R_t)\,dt, \end{cases} \tag{1.3}$$

where σ is a constant that represents the environmental stochastic perturbation of the transmission rate β, and B_t denotes a standard Brownian motion. Under the condition $\sigma < \sqrt{\beta v/\Lambda}$, the dynamic properties of model (1.3) have been studied by the threshold $\mathscr{R}_0^s = \frac{\beta\Lambda}{v(r+v+\mu)} - \frac{\sigma^2\Lambda^2}{2v^2(r+v+\mu)}$. Subsequent studies have extensively investigated and refined the types of SIRS models and their variants [8–10].

Although stochastic methods have numerous applications in daily life, the characteristics of some phenomena to be studied are often closer to uncertainty rather than randomness, and thus are more suitable for analysis within the theoretical framework of uncertainty [11]. The uncertainty theory is a branch of mathematics for modeling human uncertainty based on four axioms: normality, self-duality, sub-additivity, and product measure. To better describe and analyze uncertain dynamic systems, Liu [12] extended several fundamental concepts of fuzzy processes to uncertain processes and introduced a class of uncertain differential equations driven by canonical processes. Uncertain differential equations have since attracted increasing attention from researchers [13–15]. For example, Liu [16] derived analytical solutions for a type of uncertain differential equations. However, it is often challenging to find analytical solutions. Yao and Chen [17] proposed a numerical method to solve uncertain differential equations based on the α-path, a deterministic function of time t, which does not require the extensive repeated simulations of the Monte Carlo method (Ng and Willcox [18], Korostil et al. [19], and Du and Su [20]).

In recent years, uncertain differential equations have been widely applied in epidemic modeling. Li et al. [21] studied a class of uncertain SIS epidemic models with nonlinear incidence rates and revealed the relationship between deterministic models and uncertain models. Chen et al. [22] proposed an α-path-based method to handle a high-dimensional uncertain SIR model. Tan et al. [1] established an uncertain SIRS model. However, many problems remain unsolved. Based on the research of Tan et al. [1], this paper studies a class of uncertain SIRS epidemic models with population input and temporary immunity. Including the population input term more realistically captures the impact of population movement on disease transmission in the real world. Additionally, the observation that recovered individuals only have temporary immunity is more consistent with the actual transmission characteristics of some infectious diseases. Compared with deterministic and stochastic SIS or SIRS models [23–25], this paper establishes a class of SIRS epidemic models described by uncertain differential equations. During outbreaks of sudden or unknown diseases, there are often numerous uncertain factors within the system, which makes it challenging to accurately characterize their behavior using traditional methods such as the probability theory. Therefore, in such situations, it is more reasonable and applicable to construct epidemic models using the uncertainty theory.

By the Liu process, the disease transmission coefficient β in the model (1.1) is transformed into uncertain disturbances, $\beta dt \to \beta dt + \sigma dC_t$, where C_t is a Liu process. Thus,

$$\begin{cases} dS_t = (\Lambda - \beta S_t I_t - vS_t + \delta R_t)dt - \sigma S_t I_t dC_t, \\ dI_t = (\beta S_t I_t - (r+v+\mu)I_t)dt + \sigma S_t I_t dC_t, \\ dR_t = (rI_t - (\delta + v)R_t)dt, \end{cases} \tag{1.4}$$

where σ represents the intensity of the Liu process. For model (1.4), Tan et al. [1] proved some properties of the α-path. More dynamic properties, such as solution dynamics and the global stability of the endemic equilibrium, remain unsettled. Building on the work of Tan et al. [1], this paper further improves the properties of the model (1.4) and yields interesting results. The follow-up research of this paper is as follows: we first review basic concepts of uncertain differential equations in Section 2; in Section 3, we prove the existence and uniqueness of the solution, as well as the stability of the uncertain SIRS model; in Section 4, we use the Yao-Chen formula to derive the ordinary differential equation (ODE) associated with the α-path, present a threshold to characterize the disease's extinction and persistence, and prove the global stability of the equilibria; in Section 5, a series of examples is given for numerical simulations; and finally, the theoretical results are concluded in Section 6.

2. Preliminaries

In this section, we review the definitions of uncertain measure, uncertain variable, uncertain distribution, and uncertain differential equations, as well as useful lemmas (Liu [11], Yao [26]).

Let Γ be a nonempty set, and $\mathscr{L}$ be a σ-algebra over Γ. A set function $\mathscr{M} : \mathscr{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathscr{M}(\Gamma) = 1$ for the universal set Γ.

Axiom 2. (Duality Axiom) For any event $A \in \mathscr{L}$, $\mathscr{M}(A) + \mathscr{M}(A^c) = 1$.

Axiom 3. (Subadditivity Axiom) For a sequence of events $A_1, A_2, \ldots$, we have $\mathscr{M}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathscr{M}(A_i)$.

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathscr{L}_k, \mathscr{M}_k)$ be uncertainty spaces $(k = 1, 2, \ldots)$. The product uncertain measure $\mathscr{M}$ satisfies

$$\mathscr{M}(\prod_{k=1}^{\infty} A_k) = \bigwedge_{k=1}^{\infty} \mathscr{M}_k(A_k),$$

where $A_k \in \mathscr{L}_k$ for each k.

An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers (i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event). The uncertainty distribution $\Phi(x)$ of an uncertain variable ξ is defined by $\Phi(x) = \mathscr{M}(\xi \leq x), x \in \mathbb{R}$. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x, thus satisfying $\lim_{x\to-\infty} \Phi(x) = 0, \lim_{x\to+\infty} \Phi(x) = 1$. If ξ is an uncertain variable with a regular uncertainty distribution $\Phi(x)$, then the inverse function $\Phi^{-1}(\alpha) = \inf\{x \mid \Phi(x) \geq \alpha\}, \alpha \in (0,1)$ is called the inverse uncertainty distribution of ξ.

An uncertain variable ξ is said to be normal if it has a normal uncertainty distribution

$$\Psi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R},$$

denoted by $\mathscr{N}(e, \sigma)$. The inverse uncertainty distribution of ξ is represented as follows:

$$\Psi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right), \quad \alpha \in (0,1).$$

For $e = 0$ and $\sigma = 1$, the inverse standard normal uncertainty distribution is expressed as follows:

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1). \tag{2.1}$$

Definition 2.1 (Liu [27]). *An uncertain process C_t is called a Liu process provided it satisfies the following properties:*

(i) $C_0 = 0$ and almost all its sample paths are Lipschitz continuous;

(ii) C_t possesses stationary and independent increments; and

(iii) $C_{s+t} - C_s$ follows a normal uncertainty distribution with mean 0 and variance t^2.

Lemma 2.1 (Yao et al. [14]). *Let C_t be a canonical Liu process. Then, there exists an uncertain variable K such that for each γ, $K(\gamma)$ is a Lipschitz constant of the sample path $C_t(\gamma)$ and*

$$\lim_{x\to+\infty} \mathscr{M}\{K(\gamma) \le x\} = 1.$$

Suppose f and g are two measurable functions. Then,

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dC_t \tag{2.2}$$

is called an uncertain differential equation (UDE).

UDEs are an important tool to model dynamic systems in uncertain environments. However, finding analytic solutions is often challenging. To solve UDEs, Yao and Chen [17] proposed the concept of the α-path. Unlike the sample paths of stochastic differential equations, the uncertainty distribution of a UDE is obtained by a spectrum of α-paths.

Definition 2.2 (Yao, Chen [17]). *For a UDE (2.2) with a given initial condition X_0, its α-path ($0 < \alpha < 1$) is defined as the solution to the corresponding ODE*

$$dX_t^\alpha = f(t, X_t^\alpha)\,dt + |g(t, X_t^\alpha)|\,\Phi^{-1}(\alpha)\,dt,$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution (2.1).

Lemma 2.2 (Yao-Chen Formula [17]). *Consider a UDE (2.2), and let X_t be its solution and X_t^α its α-path. Then, the measure of the event that every trajectory of X_t remains below the α-path satisfies the following:*

$$\mathscr{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha, \mathscr{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

3. Solution of the uncertain SIRS model

In this section, we discuss the existence, uniqueness, and stability of the solution for the uncertain SIRS (USIRS) model (1.4) based on the existence and uniqueness theorem and the stability theorem for solutions of UDEs. The USIRS model (1.4) is equivalent to the following uncertain integral equation:

$$\begin{cases} S_t = S_0 + \int_0^t (\Lambda + vS_s - \beta S_s I_s + \delta R_s)\,ds - \int_0^t \sigma S_s I_s\,dC_s, \\ I_t = I_0 + \int_0^t (\beta S_s I_s - (r + v + \mu) I_s)\,ds + \int_0^t \sigma S_s I_s\,dC_s, \\ R_t = R_0 + \int_0^t (r I_s - (\delta + v) R_s)\,ds. \end{cases} \tag{3.1}$$

Lemma 3.1. *In the USIRS model (1.4), for a sample path $C_t(\gamma)$ with a Lipschitz constant $K(\gamma)$, we have the following:*

$$\left| \int_0^t \sigma S_s(\gamma) I_s(\gamma)\,dC_s(\gamma) \right| \le K(\gamma) \int_a^b |\sigma S_s(\gamma) I_s(\gamma)|\,ds. \tag{3.2}$$

Proof. For a sample path C_t with a Lipschitz constant $K(\gamma)$, we have the following:

$$\left|\int_0^t \sigma S_s(\gamma) I_s(\gamma)\, dC_s(\gamma)\right| \le \lim_{\Delta t\to 0}\sum_{i=1}^{n}\left|\sigma S_{t_{i-1}}(\gamma) I_{t_{i-1}}(\gamma)\right|\cdot\left|C_{t_i}(\gamma)-C_{t_{i-1}}(\gamma)\right|.$$

According to the definition of a canonical process, for any sample γ, $C_t(\gamma)$ is Lipschitz continuous in t. As a result, there exists a finite constant $K(\gamma)$ such that for all t_1 and t_2, we have $|C_{t_1}(\gamma)-C_{t_2}(\gamma)| \le K(\gamma)|t_1 - t_2|$. Thus, $|C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)| \le K(\gamma)|t_i - t_{i-1}|, i = 1, 2, \dots, n$. Consequently, we derive the following:

$$\begin{aligned}\lim_{\Delta t\to 0}\sum_{i=1}^{n}|\sigma S_{t_{i-1}}(\gamma) I_{t_{i-1}}(\gamma)|\cdot|C_{t_i}(\gamma)-C_{t_{i-1}}(\gamma)| &\le K(\gamma)\lim_{\Delta t\to 0}\sum_{i=1}^{n}|\sigma S_{t_{i-1}}(\gamma) I_{t_{i-1}}(\gamma)|\cdot|t_i-t_{i-1}|\\ &\le K(\gamma)\int_0^t|\sigma S_s(\gamma) I_s(\gamma)|\,\mathrm{d}s.\end{aligned}$$

Theorem 3.1. *The USIRS model (1.4) has a unique solution if the coefficients of the system satisfy the Lipschitz condition*

$$\begin{aligned}&\max\Big\{\left|-v(S_t-S'_t)-\beta(S_tI_t-S'_tI'_t)+\delta(R_t-R'_t)\right|, \left|\beta(S_tI_t-S'_tI'_t)-(r+v+\mu)(I_t-I'_t)\right|,\\ &\qquad \left|r(I_t-I'_t)-(\delta+v)(z-z')\right|\Big\}+\left|\sigma(S_tI_t-S'_tI'_t)\right|\\ &\quad\le L\cdot\max\left\{\left|S_t-S'_t\right|,\left|I_t-I'_t\right|,\left|R_t-R'_t\right|\right\},\end{aligned}$$

and the linear growth condition

$$\begin{aligned}&\max\{|\Lambda+vS_t-\beta S_tI_t+\delta R_t|, |\beta S_tI_t-(r+v+\mu)I_t|, |rI_t-(\delta+v)R_t|\}+|\sigma S_tI_t|\\ &\quad\le L_0(1+\max\{|S_t|,|I_t|,|R_t|\})\end{aligned}$$

for some constants L and L_0, where $(S_t, I_t, R_t), (S'_t, I'_t, R'_t) \in \mathbb{R}^3$ are two solutions to the USIRS model (1.4) for any $t > 0$.

Proof. Let $\Delta S_t = S_t - S'_t$, $\Delta I_t = I_t - I'_t$, and $\Delta R_t = R_t - R'_t$. The bilinear term $S_tI_t - S'_tI'_t$ can be decomposed as follows:

$$S_tI_t - S'_tI'_t = S_t(I_t - I'_t) + I'_t(S_t - S'_t) = S_t\Delta I_t + I'_t\Delta S_t.$$

By making the above variable substitutions for the system of the USIRS model (1.4), we can obtain the following inequality relationships:

$$\begin{aligned}&|-v\Delta S_t-\beta(S_t\Delta I_t+I'_t\Delta S_t)+\delta\Delta R_t| \le (v+\beta|I'_t|)|\Delta S_t|+\beta|S_t||\Delta I_t|+\delta|\Delta R_t|,\\ &|\beta(S_t\Delta I_t+I'_t\Delta S_t)-(r+v+\mu)\Delta I_t| \le \beta|I'_t||\Delta S_t|+(\beta|S_t|+r+v+\mu)|\Delta I_t|,\\ &|r\Delta I_t-(\delta+v)\Delta R_t| \le r|\Delta I_t|+(\delta+v)|\Delta R_t|, \quad |\sigma(S_tI_t-S'_tI'_t)| \le \sigma|S_t||\Delta I_t|+\sigma|I'_t||\Delta S_t|.\end{aligned}$$

We assume there exists a constant M that satisfies $\Lambda/v > M > 0$ such that $S_t, I_t, R_t, S'_t, I'_t, R'_t < M$. Taking the maximum values of the coefficients and combining them, we can get the following:

$$\begin{aligned}
&\max\Big\{\big|-v(S_t - S'_t) - \beta(S_tI_t - S'_tI'_t) + \delta(R_t - R'_t)\big|, \big|\beta(S_tI_t - S'_tI'_t) - (r+v+\mu)(I_t - I'_t)\big|, \\
&\quad \big|r(I_t - I'_t) - (\delta+v)(R_t - R'_t)\big|\Big\} + \big|\sigma(S_tI_t - S'_tI'_t)\big| \\
&\quad \leq \max\Big\{\big|S_t - S'_t\big|, \big|I_t - I'_t\big|, \big|R_t - R'_t\big|\Big\}\Big(\max\Big\{v+\beta M+\delta, \beta M + r + v + \mu, r + \delta + v\Big\} + 2\sigma M\Big).
\end{aligned}$$

Denote a Lipschitz constant $L = \max\{v + \beta M + \delta, \beta M + r + v + \mu, r + \delta + v\} + 2\sigma M$. Under the boundedness of the solution and the constraints on the parameters, the coefficients of the USIRS model (1.4) satisfy the Lipschitz condition.

Then, we prove by the method of successive approximations that for any given real number T, the USIRS model (1.4) has a solution on the interval $[0, T]$. For each $\gamma \in \Gamma$, define $S_t^{(0)}(\gamma) = S_0$, $I_t^{(0)}(\gamma) = I_0$, and $R_t^{(0)}(\gamma) = R_0$. Hence,

$$\begin{cases}
S_t^{(n+1)}(\gamma) = S_0 + \int_0^t \Big(\Lambda + vS_s^{(n)}(\gamma) - \beta S_s^{(n)}(\gamma)I_s^{(n)}(\gamma) + \delta R_s^{(n)}(\gamma)\Big)\, ds - \int_0^t \sigma S_s^{(n)}(\gamma)I_s^{(n)}(\gamma)\, dC_s(\gamma), \\
I_t^{(n+1)}(\gamma) = I_0 + \int_0^t \Big(\beta S_s^{(n)}(\gamma)I_s^{(n)}(\gamma) - (r+v+\mu)I_s^{(n)}(\gamma)\Big)\, ds + \int_0^t \sigma S_s^{(n)}(\gamma)I_s^{(n)}(\gamma)\, dC_s(\gamma), \\
R_t^{(n+1)}(\gamma) = R_0 + \int_0^t \Big(rI_s^{(n)}(\gamma) - (\delta+v)R_s^{(n)}(\gamma)\Big)\, ds,
\end{cases}$$

and

$$Q_t^{(n)}(\gamma) = \sup_{0\leq s\leq t}\Big\{\big|S_s^{(n+1)}(\gamma) - S_s^{(n)}(\gamma)\big|, \big|I_s^{(n+1)}(\gamma) - I_s^{(n)}(\gamma)\big|, \big|R_s^{(n+1)}(\gamma) - R_s^{(n)}(\gamma)\big|\Big\} \tag{3.3}$$

for $n = 1, 2, 3, \cdots$. By mathematical induction, we prove that

$$Q_t^{(n)}(\gamma) \leq (1 + \max\{|S_0|, |I_0|, |R_0|\})\frac{L^{n+1}(1+K(\gamma))^{(n+1)}}{(n+1)!}t^{n+1} \tag{3.4}$$

for almost every $\gamma \in \Gamma$ and for every nonnegative integer n, where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$. Since the right term of (3.4) satisfies

$$\sum_{n=0}^{\infty}(1 + \max\{|S_0|, |I_0|, |R_0|\})\frac{L^{n+1}(1+K(\gamma))^{(n+1)}}{(n+1)!}t^{n+1} < +\infty, \forall t \in [0, T],$$

it follows from the Weierstrass criterion that $X_t^{(n)}(\gamma)$ uniformly converges on $[0, T]$, whose limit is denoted by $X_t(\gamma)$. Then, we have the following:

$$\begin{cases}
S_t(\gamma) = S_0 + \int_0^t (\Lambda + vS_s(\gamma) - \beta S_s(\gamma)I_s(\gamma) + \delta R_s(\gamma))\, ds - \int_0^t \sigma S_s(\gamma)I_s(\gamma)\, dC_s(\gamma), \\
I_t(\gamma) = I_0 + \int_0^t (\beta S_s(\gamma)I_s(\gamma) - (r+v+\mu)I_s(\gamma))\, ds + \int_0^t \sigma S_s(\gamma)I_s(\gamma)\, dC_s(\gamma), \\
R_t(\gamma) = R_0 + \int_0^t (rI_s(\gamma) - (\delta+v)R_s(\gamma))\, ds.
\end{cases}$$

Therefore, the inequality (3.4) is proven as follows. For Eq (3.3), when $n = 0$, we have the following:

$$Q_t^{(0)}(\gamma) = \sup_{0\leq s\leq t}\left\{\max\left\{\left|S_u^{(1)}(\gamma) - S_0\right|, \left|I_u^{(1)}(\gamma) - I_0\right|, \left|R_u^{(1)}(\gamma) - R_0\right|\right\}\right\}$$

$$
\begin{aligned}
&= \sup_{0\le s\le t}\left\{\max\left\{\left|\int_0^s(\Lambda+vS_0-\beta S_0I_0+\delta R_0)\,du-\int_0^s\sigma S_0I_0\,dC_u(\gamma)\right|,\right.\right.\\
&\left.\left.\left|\int_0^s(\beta S_0I_0-(r+v+\mu)I_0)\,du+\int_0^s\sigma S_0I_0\,dC_u(\gamma)\right|,\left|\int_0^s(rI_0-(\delta+v)R_0)\,du\right|\right\}\right\}\\
&\le \sup_{0\le s\le t}\left\{\max\left\{\left|\int_0^s(\Lambda+vS_0-\beta S_0I_0+\delta R_0)\,du\right|,\left|\int_0^s(\beta S_0I_0-(r+v+\mu)I_0)\,du\right|,\right.\right.\\
&\left.\left.\left|\int_0^s(rI_0-(\delta+v)R_0)\,du\right|\right\}\right\}+\sup_{0\le s\le t}\left|\int_0^s\sigma S_0I_0\,dC_u(\gamma)\right|\\
&\le \sup_{0\le s\le t}\left\{\max\left\{\left|\int_0^s(\Lambda+vS_0-\beta S_0I_0+\delta R_0)\,du\right|,\left|\int_0^s(\beta S_0I_0-(r+v+\mu)I_0)\,du\right|,\right.\right.\\
&\left.\left.\left|\int_0^s(rI_0-(\delta+v)R_0)\,du\right|\right\}\right\}+K(\gamma)\sup_{0\le s\le t}\int_0^s|\sigma S_0I_0|\,du\\
&\le \int_0^t\max\left\{|\Lambda+vS_0-\beta S_0I_0+\delta R_0|,|\beta S_0I_0-(r+v+\mu)I_0|,|rI_0-(\delta+v)R_0|\right\}du\\
&\quad+K(\gamma)\int_0^t|\sigma S_0I_0|\,du\\
&\le\left(\max\left\{|\Lambda+vS_0-\beta S_0I_0+\delta R_0|,|\beta S_0I_0-(r+v+\mu)I_0|,|rI_0-(\delta+v)R_0|\right\}+|\sigma S_0I_0|\right)\\
&\quad\cdot(1+K(\gamma))t.
\end{aligned}
$$

At this time, if we let

$$
L_0\ge\frac{g(S_0,I_0,R_0)}{1+\max\{|S_0|,|I_0|,|R_0|\}},
$$

where

$$
g(S_0,I_0,R_0)=\max\left\{|\Lambda+vS_0-\beta S_0I_0+\delta R_0|,|rI_0-(\delta+v)R_0|,|\beta S_0I_0-(r+v+\mu)I_0|\right\}+|\sigma S_0I_0|,
$$

then the linear growth condition will be satisfied. Hence, by Lemma 3.1, it follows that

$$
Q_t^{(0)}(\gamma)\le L\,(1+\max\{|S_0|,|I_0|,|R_0|\})\,(1+K(\gamma))\,t.
$$

Assume the inequality (3.4) holds for the integer n, i.e.,

$$
\begin{aligned}
Q_t^{(n)}(\gamma)&=\sup_{0\le s\le t}\left\{\left|S_s^{(n+1)}(\gamma)-S_s^{(n)}(\gamma)\right|,\left|I_s^{(n+1)}(\gamma)-I_s^{(n)}(\gamma)\right|,\left|R_s^{(n+1)}(\gamma)-R_s^{(n)}(\gamma)\right|\right\}\\
&\le(1+\max\{|S_0|,|I_0|,|R_0|\})\cdot\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}\,t^{n+1}.
\end{aligned}
$$

Then, we have the following:

$$
\begin{aligned}
Q_t^{(n+1)}(\gamma)&=\sup_{0\le s\le t}\left\{\max\left\{\left|S_u^{(n+2)}(\gamma)-S_u^{(n+1)}(\gamma)\right|,\left|I_u^{(n+2)}(\gamma)-I_u^{(n+1)}(\gamma)\right|,\left|R_u^{(n+2)}(\gamma)-R_u^{(n+1)}(\gamma)\right|\right\}\right\}\\
&\le\sup_{0\le s\le t}\left\{\max\left\{\left|\int_0^s\left(-v\Delta S_u^{(n+1)}(\gamma)-\beta\Delta(S_uI_u)^{(n+1)}(\gamma)+\delta\Delta R_u^{(n+1)}(\gamma)\right)du\right|,\right.\right.
\end{aligned}
$$

$$\left|\int_0^s \Big(\beta\Delta(S_uI_u)^{(n+1)}(\gamma) - (r+v+\mu)\Delta I_u^{(n+1)}(\gamma)\Big)\,du\right|, \left|\int_0^s \Big(r\Delta I_s^{(n+1)}(\gamma) - (\delta+v)\Delta R_s^{(n+1)}(\gamma)\Big)\,du\right|\Bigg\}$$
$$+ \sup_{0\le s\le t}\left|\int_0^s \sigma\Delta(S_uI_u)^{(n+1)}(\gamma)\,dC_u(\gamma)\right|$$
$$\le \int_0^t \max\Big\{\Big|v\Delta S_u^{(n+1)}(\gamma) + \beta\Delta(S_uI_u)^{(n+1)}(\gamma) - \delta\Delta R_u^{(n+1)}(\gamma)\Big|,$$
$$\Big|\beta\Delta(S_uI_u)^{(n+1)}(\gamma) - (r+v+\mu)\Delta I_u^{(n+1)}(\gamma)\Big|,$$
$$\Big|r\Delta I_s^{(n+1)}(\gamma) - (\delta+v)\Delta R_s^{(n+1)}(\gamma)\Big|\Big\}\,du + K(\gamma)\int_0^t \Big|\sigma\Delta(S_uI_u)^{(n+1)}(\gamma)\Big|\,du$$
$$\le L\int_0^t Q_u^{(n+1)}(\gamma)\,du + K(\gamma)L\int_0^t Q_u^{(n+1)}(\gamma)\,du$$
$$\le L(1+K(\gamma))\int_0^t \Big((1+\max\{|S_0|,|I_0|,|R_0|\})\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}s^{n+1}\Big)\,ds$$
$$= (1+\max\{|S_0|,|I_0|,|R_0|\})\cdot\frac{L^{n+1}(1+K(\gamma))^{n+2}}{(n+2)!}t^{n+2}.$$

This implies that the inequality (3.4) also holds for the integer $n+1$. Therefore, inequality (3.4) holds for all nonnegative integers. Next, we prove the uniqueness of the solution under the given conditions. Assume that (S_t, I_t, R_t) and (S_t^*, I_t^*, R_t^*) are two solutions of the USIRS model (1.4) with the same initial values (S_0, I_0, R_0). Then, for almost every $\gamma\in\Gamma$, we have the following:

$$\sup_{0\le s\le t}\Big\{\Big|S_t(\gamma) - S_t^*(\gamma)\Big|, \Big|I_t(\gamma) - I_t^*(\gamma)\Big|, \Big|R_t(\gamma) - R_t^*(\gamma)\Big|\Big\}$$
$$= \sup_{0\le s\le t}\Bigg\{\Bigg|\int_0^t \Big(-v(S_s(\gamma) - S_s^*(\gamma)) - \beta(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)) + \delta(R_s(\gamma) - R_s^*(\gamma))\Big)\,ds$$
$$+ \int_0^t -\sigma\Big(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)\Big)\,dC_s(\gamma)\Bigg|,$$
$$\Bigg|\int_0^t \Big(\beta(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)) - (r+v+\mu)(I_s(\gamma) - I_s^*(\gamma))\Big)\,ds$$
$$+ \int_0^t \sigma\Big(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)\Big)\,dC_s(\gamma)\Bigg|,$$
$$\Bigg|\int_0^t \Big(r(I_s(\gamma) - I_s^*(\gamma)) - (\delta+v)(R_s(\gamma) - R_s^*(\gamma))\Big)\,ds\Bigg|\Bigg\}$$
$$\le \sup_{0\le s\le t}\Bigg\{\Bigg|\int_0^t \Big(v(S_s(\gamma) - S_s^*(\gamma)) + \beta(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)) - \delta(R_s(\gamma) - R_s^*(\gamma))\Big)\,ds\Bigg|,$$
$$\Bigg|\int_0^t \Big(\beta(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)) - (r+v+\mu)(I_s(\gamma) - I_s^*(\gamma))\Big)\,ds\Bigg|,$$
$$\Bigg|\int_0^t \Big(r(I_s(\gamma) - I_s^*(\gamma)) - (\delta+v)(R_s(\gamma) - R_s^*(\gamma))\Big)\,ds\Bigg|\Bigg\}$$
$$+ \Bigg|\int_0^t \sigma\Big(S_s(\gamma)I_s(\gamma) - S_s^*(\gamma)I_s^*(\gamma)\Big)\,dC_s(\gamma)\Bigg|$$

$$\leq \int_0^t \sup_{0\leq s\leq t}\Big\{\Big|\Big(v(S_s(\gamma)-S_s^*(\gamma))+\beta\Big(S_s(\gamma)I_s(\gamma)-S_s^*(\gamma)I_s^*(\gamma)\Big)-\delta\Big(R_s(\gamma)-R_s^*(\gamma)\Big)\Big)\Big|,$$

$$\Big|\Big(\beta\Big(S_s(\gamma)I_s(\gamma)-S_s^*(\gamma)I_s^*(\gamma)\Big)-(r+v+\mu)\Big(I_s(\gamma)-I_s^*(\gamma)\Big)\Big)\Big|,$$

$$\Big|\Big(r\Big(I_s(\gamma)-I_s^*(\gamma)\Big)-(\delta+v)\Big(R_s(\gamma)-R_s^*(\gamma)\Big)\Big)\Big|\Big\}\,ds+K(\gamma)\int_0^t\Big|\sigma\Big(S_s(\gamma)I_s(\gamma)-S_s^*(\gamma)I_s^*(\gamma)\Big)\Big|\,ds$$

$$\leq \int_0^t L_s\sup_{0\leq s\leq t}\Big\{\Big|S_t(\gamma)-S_t^*(\gamma)\Big|,\Big|I_t(\gamma)-I_t^*(\gamma)\Big|,\Big|R_t(\gamma)-R_t^*(\gamma)\Big|\Big\}\,ds$$

$$+\int_0^t K(\gamma)L_s\sup_{0\leq s\leq t}\Big\{\Big|S_t(\gamma)-S_t^*(\gamma)\Big|,\Big|I_t(\gamma)-I_t^*(\gamma)\Big|,\Big|R_t(\gamma)-R_t^*(\gamma)\Big|\Big\}\,ds$$

$$=(1+K(\gamma))\int_0^t L_s\sup_{0\leq s\leq t}\Big\{\Big|S_t(\gamma)-S_t^*(\gamma)\Big|,\Big|I_t(\gamma)-I_t^*(\gamma)\Big|,\Big|R_t(\gamma)-R_t^*(\gamma)\Big|\Big\}\,ds.$$

By Grönwall's inequality, we can obtain the following:

$$\sup_{0\leq s\leq t}\Big\{\Big|S_t(\gamma)-S_t^*(\gamma)\Big|,\Big|I_t(\gamma)-I_t^*(\gamma)\Big|,\Big|R_t(\gamma)-R_t^*(\gamma)\Big|\Big\}\leq 0\cdot\exp\Big(\Big(1+K(\gamma)\Big)\int_0^t L_s\,ds\Big)=0.$$

Theorem 3.2. *Assume the USIRS model (1.4) has a unique solution for each given initial value. Then, it is stable in measure if the coefficients satisfy the strong Lipschitz condition*

$$\begin{aligned}&\max\Big\{\Big|-v(S_t-S_t')-\beta(S_tI_t-S_t'I_t')+\delta(R_t-R_t')\Big|,\Big|\beta(S_tI_t-S_t'I_t')-(r+v+\mu)(I_t-I_t')\Big|,\\&\quad\Big|r(I_t-I_t')-(\delta+v)(R_t-R_t')\Big|\Big\}+\Big|\sigma(S_tI_t-S_t'I_t')\Big|\\&\quad\leq \mathrm{L_t}\max\Big\{\Big|S_t-S_t'\Big|,\Big|I_t-I_t'\Big|,\Big|R_t-R_t'\Big|\Big\}\end{aligned}\tag{3.5}$$

for $(S_t,I_t,R_t),(S_t',I_t',R_t')\in\mathbb{R}^3, t\geq 0,$ *where* L_t *is some positive function that satisfies* $\int_0^{+\infty}L_t\,dt<+\infty$.

Proof. Let (S_t,I_t,R_t) and (S_t',I_t',R_t') be the solutions of the USIRS model (1.4) with different initial values (S_0,I_0,R_0) and (S_0',I_0',R_0'), respectively. Then, for a Lipschitz continuous sample path $C_t(\gamma)$, we have the following:

$$\begin{cases}S_t(\gamma)=S_0+\int_0^t(\Lambda+vS_s(\gamma)-\beta S_s(\gamma)I_s(\gamma)+\delta R_s(\gamma))\,ds-\int_0^t\sigma S_s(\gamma)I_s(\gamma)\,dC_s(\gamma),\\ I_t(\gamma)=I_0+\int_0^t(\beta S_s(\gamma)I_s(\gamma)-(r+v+\mu)I_s(\gamma))\,ds+\int_0^t\sigma S_s(\gamma)I_s(\gamma)\,dC_s(\gamma),\\ R_t(\gamma)=R_0+\int_0^t(rI_s(\gamma)-(\delta+v)R_s(\gamma))\,ds,\end{cases}\tag{3.6}$$

and the equations for $(S_t'(\gamma),I_t'(\gamma),R_t'(\gamma))$ with initial values (S_0',I_0',R_0') satisfy the same system of integral equations (3.6). By the strong Lipschitz condition, we have the following:

$$\max\Big\{\Big|S_t(\gamma)-S_t'(\gamma)\Big|,\Big|I_t(\gamma)-I_t'(\gamma)\Big|,\Big|R_t(\gamma)-R_t'(\gamma)\Big|\Big\}$$

$$\leq\max\Big\{\Big|S_0-S_0'\Big|,\Big|I_0-I_0'\Big|,\Big|R_0-R_0'\Big|\Big\}$$

$$+\int_0^t\max\Big\{\Big|v(S_s(\gamma)-S_s'(\gamma))-\beta(S_s(\gamma)I_s(\gamma)-S_s'(\gamma)I_s'(\gamma))+\delta(R_s(\gamma)-R_s'(\gamma))\Big|,$$

$$\left|\beta(S_s(\gamma)I_s(\gamma) - S'_s(\gamma)I'_s(\gamma)) - (r + v + \mu)(I_s(\gamma) - I'_s(\gamma))\right|,$$
$$\left.\left|r(I_s(\gamma) - I'_s(\gamma)) - (\delta + v)(R_s(\gamma) - R'_s(\gamma))\right|\right\}ds$$
$$+ \int_0^t \max\left\{\left| - \sigma(S_s(\gamma)I_s(\gamma) - S'_s(\gamma)I'_s(\gamma))\right|, \left|\sigma(S_s(\gamma)I_s(\gamma) - S'_s(\gamma)I'_s(\gamma))\right|\right\}dC_s(\gamma)$$
$$\leq \max\left\{\left|S_0 - S'_0\right|, \left|I_0 - I'_0\right|, \left|R_0 - R'_0\right|\right\}$$
$$+ \int_0^t L_s \max\left\{\left|S_t(\gamma) - S'_t(\gamma)\right|, \left|I_t(\gamma) - I'_t(\gamma)\right|, \left|R_t(\gamma) - R'_t(\gamma)\right|\right\}ds$$
$$+ \int_0^t K(\gamma)L_s \max\left\{\left|S_t(\gamma) - S'_t(\gamma)\right|, \left|I_t(\gamma) - I'_t(\gamma)\right|, \left|R_t(\gamma) - R'_t(\gamma)\right|\right\}ds$$
$$= \max\left\{\left|S_0 - S'_0\right|, \left|I_0 - I'_0\right|, \left|R_0 - R'_0\right|\right\}$$
$$+ (1 + K(\gamma)) \int_0^t L_s \max\left\{\left|S_t(\gamma) - S'_t(\gamma)\right|, \left|I_t(\gamma) - I'_t(\gamma)\right|, \left|R_t(\gamma) - R'_t(\gamma)\right|\right\}ds,$$

where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$. It follows from Grönwall's inequality that

$$\max\left\{\left|S_t(\gamma) - S'_t(\gamma)\right|, \left|I_t(\gamma) - I'_t(\gamma)\right|, \left|R_t(\gamma) - R'_t(\gamma)\right|\right\}$$
$$\leq \max\left\{\left|S_0 - S'_0\right|, \left|I_0 - I'_0\right|, \left|R_0 - R'_0\right|\right\} \exp\left((1 + K(\gamma)) \int_0^t L_s ds\right)$$
$$\leq \max\left\{\left|S_0 - S'_0\right|, \left|I_0 - I'_0\right|, \left|R_0 - R'_0\right|\right\} \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right)$$

for any $t \geq 0$, and we obtain the following:

$$\sup_{t\geq 0}\left\{ \max\left\{\left|S_t - S'_t\right|, \left|I_t - I'_t\right|, \left|R_t - R'_t\right|\right\}\right\}$$
$$\leq \max\left\{\left|S_0 - S'_0\right|, \left|I_0 - I'_0\right|, \left|R_0 - R'_0\right|\right\} \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right),$$

almost surely, where K is a nonnegative uncertain variable. It follows from Lemma 2.1 that

$$\lim_{x\to\infty} \mathscr{M}\{\gamma \in \tau | K(\gamma) \leq x\} = 1.$$

For any given $\epsilon > 0$, there exists $H > 0$ such that $\mathscr{M}\{\gamma | K(\gamma) \leq H\} \geq 1 - \epsilon$. Take

$$\delta = \exp\left(- (1 + H) \int_0^{+\infty} L_s ds\right)\varepsilon.$$

Then, $\max\{|S_t(\gamma) - S'_t(\gamma)|, |I_t(\gamma) - I'_t(\gamma)|, |R_t(\gamma) - R'_t(\gamma)|\} \leq \varepsilon$ for any t provide that $\max\{|S_0 - S'_0|, |I_0 - I'_0|, |R_0 - R'_0|\} \leq \delta$, and $K(\gamma) \leq H$. Thus,

$$\mathscr{M}\left\{ \sup_{t\geq 0} \max\{|S_t - S'_t|, |I_t - I'_t|, |R_t - R'_t|\} \leq \varepsilon\right\} > 1 - \epsilon.$$

Therefore,

$$\lim_{\sup\{|\Delta S_0|,|\Delta I_0|,|\Delta R_0|\}\to 0} \mathscr{M}\left\{\sup_{t\geq 0}\max\{|\Delta S_t|,|\Delta I_t|,|\Delta R_t|\}\leq\varepsilon\right\}=1,$$

where $\Delta S_t = S_t - S'_t$, $\Delta I_t = I_t - I'_t$, $\Delta R_t = R_t - R'_t$, and the USIRS model (1.4) is stable in measure.

4. Dynamic properties of α-path

Based on Definition 2.2, we deduce that the solutions S_t, I_t, and R_t of the UDEs (1.4) are contour processes with α-path S_t^α, I_t^α, and R_t^α, respectively. Consequently, we can obtain the following:

$$\begin{cases} dS_t^\alpha = (\Lambda - vS_t^\alpha - \beta S_t^\alpha I_t^\alpha + \delta R_t^\alpha)dt + |\sigma S_t^\alpha I_t^\alpha|\Phi^{-1}(\alpha)dt, \\ dI_t^\alpha = (\beta S_t^\alpha I_t^\alpha - (r+v+\mu)I_t^\alpha)dt + |\sigma S_t^\alpha I_t^\alpha|\Phi^{-1}(\alpha)dt, \\ dR_t^\alpha = (rI_t^\alpha - (\delta+v)R_t^\alpha)dt. \end{cases} \tag{4.1}$$

Remark 4.1. *When $\alpha = 0.5$, $\Phi^{-1}(\alpha) = 0$, and system (4.1) reduces to the classical deterministic epidemic dynamics framework, namely model (1.1).*

Define a threshold of α-path as follows:

$$\mathscr{R}_0^u = \mathscr{R}_0 + \frac{\sigma\Phi^{-1}(\alpha)(\Lambda/v)}{(r+v+\mu)}.$$

Theorem 4.1. *If $\mathscr{R}_0^u \leq 1$, $I_t^\alpha = 0$, then $R_t^\alpha = 0, S_t^\alpha = \Lambda/v$. Moreover, there exists a unique disease-free equilibrium state $E_0(\Lambda/v, 0, 0)$, which is globally asymptotically stable.*

Proof. Define a non-negative Lyapunov function $V = I_t^\alpha$, and differentiate it with respect to time t. Then, we obtain the following:

$$\begin{aligned} \dot{V} = \frac{dI_t^\alpha}{dt} &= (\beta S_t^\alpha I_t^\alpha - (r+v+\mu)I_t^\alpha) + \sigma S_t^\alpha I_t^\alpha \Phi^{-1}(\alpha) \\ &\leq ((\beta + \sigma\Phi^{-1}(\alpha))(\Lambda/v) - (r+v+\mu))I_t^\alpha \\ &= (r+v+\mu)(R_0^u - 1)I_t^\alpha \leq 0. \end{aligned}$$

When $\mathscr{R}_0^u \leq 1$, we have $\dot{V} < 0$, with equality $\dot{V} = 0$ if and only if $I_t^\alpha = 0$. According to Lyapunov's stability theorem, the disease-free equilibrium $E_0(\Lambda/v, 0, 0)$ is globally asymptotically stable.

Theorem 4.2. *If $\mathscr{R}_0^u > 1$, then the results holds:*
(i) Model (4.1) has a unique endemic equilibrium $E_2^* = (S_2^*, I_2^*, R_2^*)$ for $\alpha \in (0,1)$ as follows:

$$S_2^* = \frac{r+v+\mu}{\beta_2}, I_2^* = \Lambda\left(1 - \frac{1}{\mathscr{R}_0^u}\right)\left((r+v+\mu)\frac{\beta_1}{\beta_2} - \frac{\delta}{\delta+v}r\right)^{-1}, R_2^* = \frac{r}{\delta+v}I_2^*,$$

where $\beta_1 = \beta - \sigma\Phi^{-1}(\alpha)$, $\beta_2 = \beta + \sigma\Phi^{-1}(\alpha)$, which is locally asymptotically stable.
(ii) If $\beta_1(r+v+\mu) \geq \beta_2 r$, then the endemic equilibrium E_2^* is globally asymptotically stable.

Proof. (i) To analyze the dynamical behavior near the equilibrium point E_2^* in model (4.1), we apply linearization techniques to the nonlinear system by constructing and examining the Jacobian matrix at $E_2^* = (S_2^*, I_2^*, R_2^*)$. We can obtain the following:

$$J(E_2^*) = \begin{pmatrix} -v - \beta_1 I_2^* & -\beta_1 S_2^* & \delta \\ \beta_2 I_2^* & 0 & 0 \\ 0 & r & -(\delta + v) \end{pmatrix}.$$

The characteristic equation is given by $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where

$$a_1 = -\mathrm{tr}(J(E_2^*)) = v + \beta_1 I_2^*, \quad a_2 = (v + \beta_1 I_2^*)(\delta + v) + \beta_1\beta_2 S_2^* I_2^*,$$
$$a_3 = -\det(J(E_2^*)) = \beta_2 I_2^*((\delta + v)\beta_1 S_2^* - \delta r).$$

When the following condition is met, the above equations a_1, a_2, and a_3 are all positive.

$$\frac{\beta_1}{\beta_2} > \frac{r}{r + v + \mu} > \frac{\delta r}{(\delta + v)(r + v + \mu)}.$$

We can get the range of α

$$0 < \alpha < \frac{1}{1 + \exp\left(-\frac{\pi}{\sqrt{3}} \cdot \frac{\beta K_1}{\sigma K_2}\right)},$$

where $K_1 = 1 - \frac{r}{r+v+\mu}$, $K_2 = 1 + \frac{r}{r+v+\mu}$. Furthermore, we have the following:

$$a_1a_2 - a_3 = (v + \beta_1 I_2^*)\left((v + \beta_1 I_2^*)(\delta + v) + \beta_1(r + v + \mu)I_2^*\right) + (\delta + v)\left(v + \beta_1 I_2^*(\delta + v) + v + \beta_2 I_2^*\delta r\right) > 0.$$

Based on the Routh-Hurwitz criterion, if the coefficients of the characteristic polynomial satisfy certain positivity conditions, or if all eigenvalues have negative real parts, then the system is locally asymptotically stable. Therefore, when $\mathscr{R}_0^u > 1$, there exists an endemic equilibrium E_2^* that is locally asymptotically stable.

(ii) Obviously, we can obtain the stability condition of the local equilibrium point E_2^*, which satisfies the following:

$$\begin{cases} 0 = \Lambda - vS_2^* - \beta S_2^* I_2^* + \delta R_2^* + \left|\sigma S_2^* I_2^*\right| \Phi^{-1}(\alpha), \\ 0 = \beta S_2^* I_2^* - (r + v + \mu)I_2^* + \left|\sigma S_2^* I_2^*\right| \Phi^{-1}(\alpha), \\ 0 = rI_2^* - (\delta + v)R_2^*. \end{cases} \tag{4.2}$$

By substituting the corresponding stability conditions of the equilibrium point into Eq (4.1), we can obtain a new set of equations as follows:

$$\begin{cases} \dfrac{dS_t^\alpha}{dt} = -(v + \beta_1 I_t^\alpha)(S_t^\alpha - S_2^*) - \dfrac{\beta_1}{\beta_2}(r + v + \mu)(I_t^\alpha - I_2^*) + \delta(R_t^\alpha - R_2^*), \\ \dfrac{dI_t^\alpha}{dt} = \beta_2 I_t^\alpha(S_t^\alpha - S_2^*), \\ \dfrac{dR_t^\alpha}{dt} = r(I_t^\alpha - I_2^*) - (\delta + v)(R_t^\alpha - R_2^*). \end{cases} \tag{4.3}$$

By summing up each part of the above equations, we can obtain the following:

$$\frac{dS_t^\alpha}{dt}+\frac{dI_t^\alpha}{dt}+\frac{dR_t^\alpha}{dt}=-\left(v+(\beta_1-\beta_2)I_t^\alpha\right)(S_t^\alpha-S_2^*)+\left(r-\frac{\beta_1}{\beta_2}(r+v+\mu)\right)(I_t^\alpha-I_2^*)-v(R_t^\alpha-R_2^*).$$

To discuss the global asymptotic stability of the endemic equilibrium E_2^*, we take the Lyapunov function as follows:

$$\begin{aligned}V=&\frac{1}{2}(S_t^\alpha-S_2^*+I_t^\alpha-I_2^*+R_t^\alpha-R_2^*)^2+\frac{1}{2}\left(\left(\delta+\frac{\beta_1}{\beta_2}(r+v+\mu)\right)^{-1}+1\right)(S_t^\alpha-S_2^*)^2\\&+\frac{1}{\beta_2}\left(2\frac{\beta_1}{\beta_2}(r+v+\mu)-r-v+1\right)\left(I_t^\alpha-I_2^*-I_2^*\ln\frac{I_t^\alpha}{I_2^*}\right)\\&+\frac{1}{2r}\left(\frac{\beta_1}{\beta_2}(r+v+\mu)+v-r\right)(R_t^\alpha-R_2^*)^2.\end{aligned}\tag{4.4}$$

By calculation, we can obtain the following:

$$\begin{aligned}\dot V=&(S_t^\alpha-S_2^*+I_t^\alpha-I_2^*+R_t^\alpha-R_2^*)\left(\frac{dS_t^\alpha}{dt}+\frac{dI_t^\alpha}{dt}+\frac{dR_t^\alpha}{dt}\right)+\left(\left(\delta+\frac{\beta_1}{\beta_2}(r+v+\mu)\right)^{-1}+1\right)(S_t^\alpha-S_2^*)\frac{dS_t^\alpha}{dt}\\&+\frac{1}{\beta_2}\left(2\frac{\beta_1}{\beta_2}(r+v+\mu)-r-v+1\right)\left(1-\frac{I_2^*}{I_t^\alpha}\right)\frac{dI_t^\alpha}{dt}+\frac{1}{r}\left(\frac{\beta_1}{\beta_2}(r+v+\mu)+v-r\right)(R_t^\alpha-R_2^*)\frac{dR_t^\alpha}{dt}\\=&(S_t^\alpha-S_2^*+I_t^\alpha-I_2^*+R_t^\alpha-R_2^*)\\&\cdot\left(-(v+(\beta_1-\beta_2)I_t^\alpha)(S_t^\alpha-S_2^*)+\left(r-\frac{\beta_1}{\beta_2}(r+v+\mu)\right)(I_t^\alpha-I_2^*)-v(R_t^\alpha-R_2^*)\right)\\&+\left(\left(\delta+\frac{\beta_1}{\beta_2}(r+v+\mu)\right)^{-1}+1\right)(S_t^\alpha-S_2^*)\left(-(v+\beta_1)(S_t^\alpha-S_2^*)-\frac{\beta_1}{\beta_2}(r+v+\mu)(I_t^\alpha-I_2^*)\right.\\&\left.+\delta(R_t^\alpha-R_2^*)\right)+\left(2\frac{\beta_1}{\beta_2}(r+v+\mu)-r-v+1\right)(S_t^\alpha-S_2^*)(I_t^\alpha-I_2^*)\\&+\frac{1}{r}\left(\frac{\beta_1}{\beta_2}(r+v+\mu)+v-r\right)(R_t^\alpha-R_2^*)\left(r(I_t^\alpha-I_2^*)-(\delta+v)(R_t^\alpha-R_2^*)\right)\\=&-\left(\left(\left(\delta+\frac{\beta_1}{\beta_2}(r+v+\mu)\right)^{-1}+1\right)(v+\delta+\beta_1I_t^\alpha)-v\right)(S_t^\alpha-S_2^*)^2-\left(\frac{\beta_1}{\beta_2}(r+v+\mu)-r\right)(I_t^\alpha-I_2^*)^2\\&-\left(v+\frac{\delta+v}{r}\left(\frac{\beta_1}{\beta_2}(r+v+\mu)+v-r\right)\right)(R_t^\alpha-R_2^*)^2\\\le&-(\delta+\beta_1I_t^\alpha)(S_t^\alpha-S_2^*)^2-\left(\frac{\beta_1}{\beta_2}(r+v+\mu)-r\right)(I_t^\alpha-I_2^*)^2\\&-\left(v+\frac{\delta+v}{r}\left(\frac{\beta_1}{\beta_2}(r+v+\mu)+v-r\right)\right)(R_t^\alpha-R_2^*)^2.\end{aligned}$$

When $\beta_1(r+v+\mu)>\beta_2 r$, we have $\dot V\le 0$. The equality only holds when $S_t^\alpha=S_2^*$, $I_t^\alpha=I_2^*$, and $R_t^\alpha=R_2^*$. Therefore, the endemic equilibrium E_2^* is globally asymptotically stable.

Remark 4.2. *In their study, Tan et al. [1] demonstrated the local stability of the epidemic equilibrium using the α-path. In this paper, Theorem 4.2 extends and strengthens the results of Tan et al. [1].*

For model (1.2), by considering the environmental fluctuations in disease transmission, we can obtain the uncertain SIS (USIS) model as follows:

$$\begin{cases} dS_t = (\Lambda - \beta S_t I_t - v S_t + \delta(\Lambda/v - S_t - I_t))dt - \sigma S_t I_t dC_t, \\ dI_t = \beta S_t I_t - (r+v)I_t + \sigma S_t I_t dC_t. \end{cases} \tag{4.5}$$

Similarly, we employ the α-path method to solve the simplified USIS (4.5) as follows:

$$\begin{cases} dS_t^\alpha = (\Lambda - vS_t^\alpha - \beta S_t^\alpha I_t^\alpha + \delta(\Lambda/v - S_t^\alpha - I_t^\alpha))dt + |-\sigma S_t^\alpha I_t^\alpha|\Phi^{-1}(\alpha)dt, \\ dI_t^\alpha = (\beta S_t^\alpha I_t^\alpha - (r+v)I_t^\alpha)dt + |\sigma S_t^\alpha I_t^\alpha|\Phi^{-1}(\alpha)dt. \end{cases} \tag{4.6}$$

For the ODE (4.6), we use the next generation matrix [28] to obtain the basic reproduction number as follows:

$$\mathscr{R}_0^* = \mathscr{R}_0 + \frac{\sigma\Phi^{-1}(\alpha)\Lambda}{v(r+v)}.$$

Corollary 4.1. *For the ODE (4.6), the following results hold:*

(i) If $\mathscr{R}_0^* \leq 1$, then the disease-free equilibrium $E_0 = (S_0, I_0) = (\Lambda/v, 0)$ is globally asymptotically stable.

(ii) If $\mathscr{R}_0^* > 1$, then there exists an endemic equilibrium $E_3^* = (S_3^*, I_3^*)$, which has local progressive stability, where

$$S_3^* = \frac{r+v}{\beta + \sigma\Phi^{-1}(\alpha)}, \quad I_3^* = \frac{\Lambda}{v}\left(1 - \frac{1}{\mathscr{R}_0^*}\right)(r+v)\left(\delta + (r+v)\beta_1/\beta_2\right)^{-1}.$$

According to Theorem 4.2, for the global stability of the endemic equilibrium of the ODE (4.6), we construct a Lyapunov function of the following form:

$$V = \frac{\beta_2}{2(\beta_2\delta + \beta_1(r+v))}(S_t^\alpha - S_3^*)^2 + \frac{1}{\beta_2}(I_t^\alpha - I_3^* - I_3^* \ln \frac{I_t^\alpha}{I_3^*}).$$

By the Lyapunov function, we directly obtain the following result.

Corollary 4.2. *If $\mathscr{R}_0^* > 1$ in the system (4.6), then the equilibrium E_3^* is globally asymptotically stable.*

5. Numerical simulation

In this section, we compare the trajectory variations of the susceptible compartment S, infected compartment I, and recovered compartment R across the deterministic SIRS model (1.1), the stochastic SIRS model (1.3), and the uncertain SIRS model (1.4). The key to numerically solving UDEs lies in obtaining their α-path spectrum. To this end, Yao and Chen [17] developed an Euler method based on the α-path for the numerical solution of UDEs, which we employ in our numerical simulations.

Example 5.1. Assume the following parameters: $\Lambda = 0.4$, $\mu = 0.15$, $\beta = 0.39$, $r = 0.2$, $\sigma = 0.1$, $v = 0.05$, and $\delta = 0.005$. Figure 1 shows the solution trajectories of model (1.4), compared with models (1.1), (1.3) under the same parameter settings. Under uncertainty disturbances, the solution curves of the uncertain SIRS model (1.4) are highly correlated with those of the deterministic SIRS model (1.1), compared to the trajectories of the stochastic SIRS model (1.3).

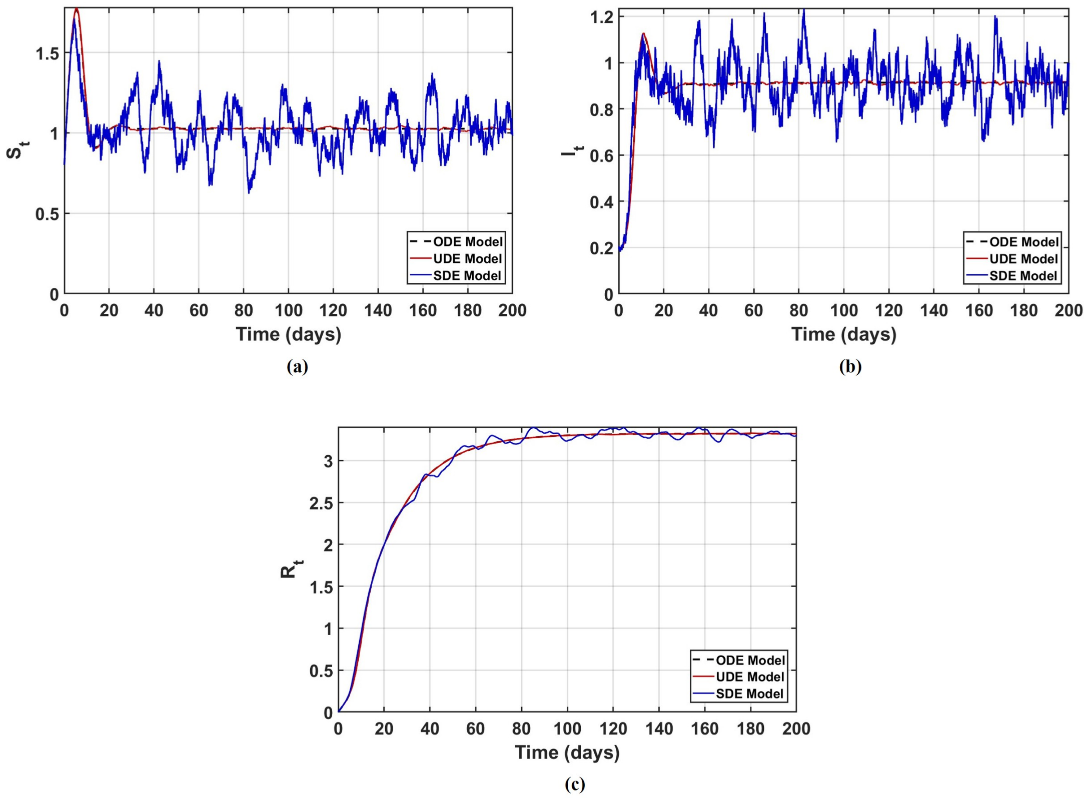
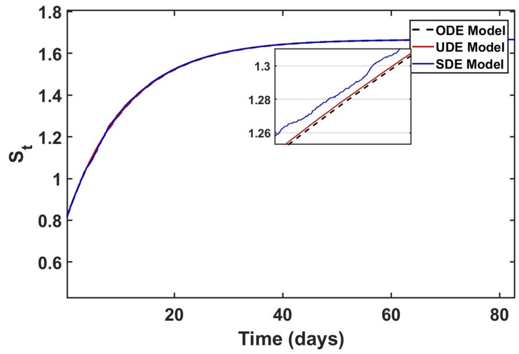
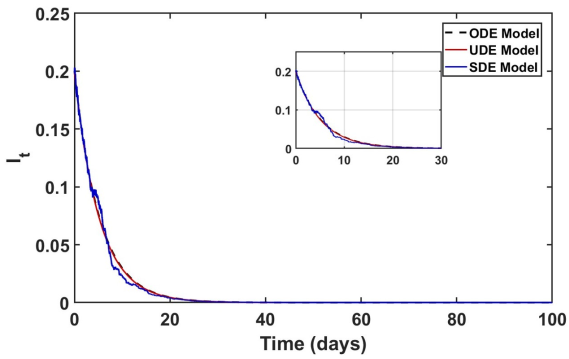


Figure 1. The trajectories of S_t, I_t, and R_t in models (1.1), (1.3), and (1.4).

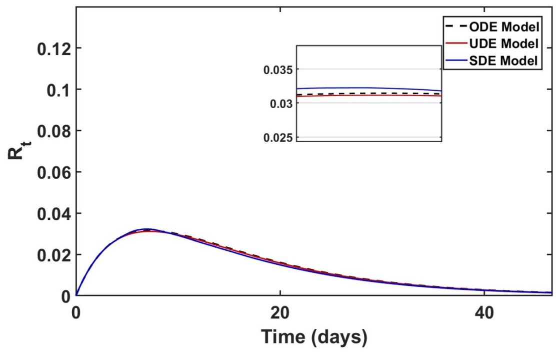
Example 5.2. Assume the following parameters: $\Lambda = 0.15$, $\mu = 0.05$, $\beta = 0.009$, $r = 0.06$, $\sigma = 0.09$, $v = 0.09$, and $\delta = 0.005$. From Figure 2, it can be observed that the trajectories of S_t, I_t, and R_t fluctuate around the disease-free equilibrium $E_0 = (1.67, 0, 0)$.



(a)



(b)



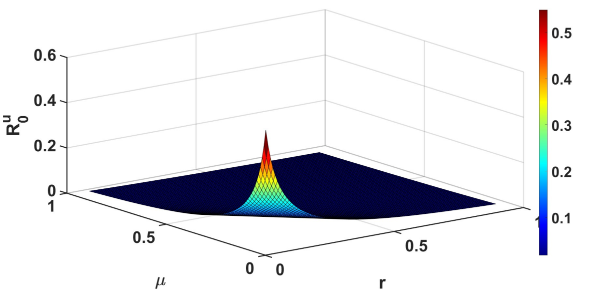
(c)

Figure 2. The trajectories of S_t, I_t, and R_t in models (1.1), (1.3), and (1.4) around E_0.

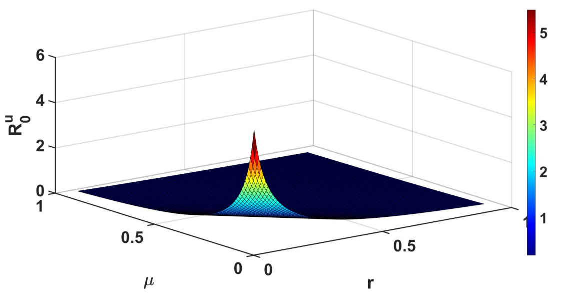
Example 5.3. Consider the impact of the recovery rate r and the disease-induced mortality rate μ on the threshold $\mathscr{R}_0^u$. The sensitivities of r and μ are represented as follows:

$$\frac{\partial \mathscr{R}_0^u}{\partial r}\frac{r}{\mathscr{R}_0^u} = -\frac{r}{r+v+\mu}, \quad \frac{\partial \mathscr{R}_0^u}{\partial r}\frac{\mu}{\mathscr{R}_0^u} = -\frac{\mu}{r+v+\mu}.$$

Assume the following parameters: $\sigma = 0.06$, $\Lambda = 0.2$, $\mu = 0.005$, $\beta = 0.02$, $r = 0.05$, $\delta = 0.005$, and $v = 0.04$. In Figure 3, the recovery rate r and the disease-induced mortality rate μ have negative sensitivity indices.



(a) $\alpha = 0.4$



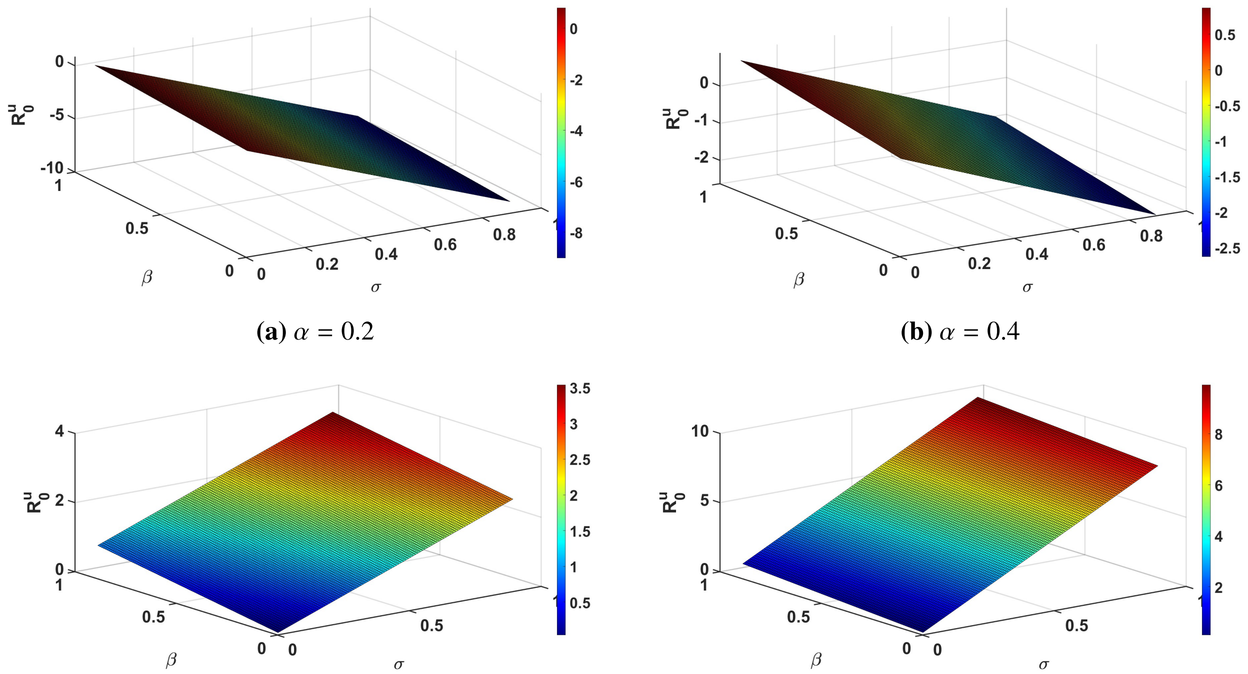
(b) $\alpha = 0.8$

Figure 3. The sensitivity of the recovery rate r and the mortality rate μ under $\alpha = 0.2, 0.8$.

Example 5.4. Assume the parameters are defined by Example 5.3. The sensitivities of σ and β are represented as follows:

$$\frac{\partial \mathscr{R}_0^u}{\partial \sigma}\frac{\sigma}{\mathscr{R}_0^u} = \frac{\sigma\Phi^{-1}(\alpha)}{\beta + \sigma\Phi^{-1}(\alpha)}, \quad \frac{\partial \mathscr{R}_0^u}{\partial \beta}\frac{\beta}{\mathscr{R}_0^u} = \frac{\beta}{\beta + \sigma\Phi^{-1}(\alpha)}.$$

Figure 4 shows the sensitivity analysis of the threshold $\mathscr{R}_0^u$ to the parameters σ and β under $\alpha = 0.2, 0.4, 0.6, 0.8$. We observe that β has the positive sensitivity index, and the sensitivity of σ is related to the value of α.

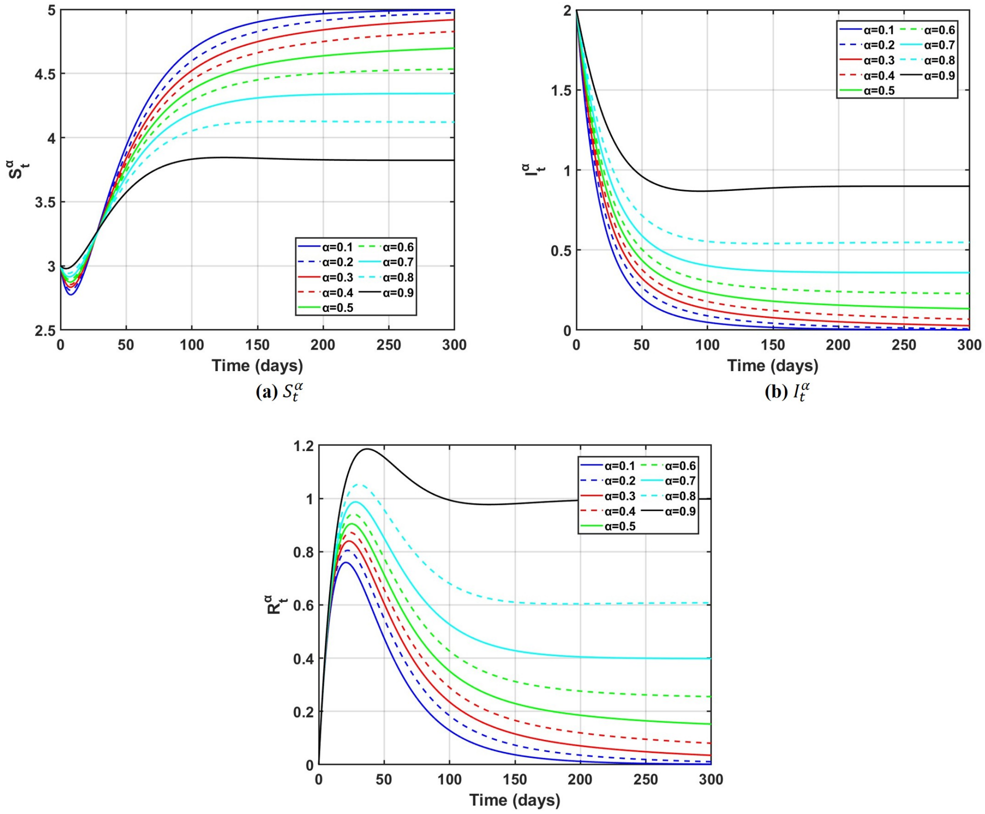


(a) $\alpha = 0.2$ **(b)** $\alpha = 0.4$

(c) $\alpha = 0.6$ **(d)** $\alpha = 0.8$

Figure 4. (a) represents the sensitivity of σ and β to $\mathscr{R}_0^u$ when $\alpha = 0.2$. (b), (c), and (d) correspond to the sensitivities when $\alpha = 0.4, 0.6, 0.8$, respectively.

Example 5.5. In the uncertain SIRS model (1.4), take $S_0 = 3.0$, $I_0 = 2.0$, $R_0 = 0$, with parameters $\Lambda = 0.2$, $\mu = 0.005$, $\beta = 0.02$, $r = 0.05$, $\delta = 0.005$, $\sigma = 0.004$, and $v = 0.04$. For each given α, we can obtain the corresponding α-paths S_t^α, I_t^α, and R_t^α. Figure 5 displays the trajectory curves of S_t^α, I_t^α, and R_t^α when $\alpha = 0.1, 0.2, \ldots, 0.9$.



(a) S_t^α (b) I_t^α

(c) R_t^α

Figure 5. The trajectories of S_t^α, I_t^α, and R_t^α under different α-paths.

6. Conclusions

Based on the work of Tan et al. [1], this paper discussed the properties of the uncertain SIRS model and the dynamic characteristics of the α-path. First, the existence, uniqueness, and stability of the solutions to the uncertain SIRS epidemic model were established. Next, we applied the Yao-Chen formula to derive the corresponding ODEs and their equilibrium points. By constructing Lyapunov functions, we demonstrated the global asymptotic stability of the equilibrium points for both models. Additionally, we defined the threshold $\mathscr{R}_0^u$ to characterize disease extinction and persistence. When $\mathscr{R}_0^u \leq 1$, the disease-free equilibrium is globally asymptotically stable, and the disease will eventually die out. Conversely, when $\mathscr{R}_0^u > 1$, the disease-free equilibrium becomes unstable, and the endemic equilibrium is globally asymptotically stable. Finally, numerical simulations were presented to validate the theoretical results, with illustrative examples provided.

Our proposed method can be extended to other uncertain epidemic models with general incidence rates and multiple uncertain parameters. Exploring these problems will be a focus of our future research.

Author contributions

Jianye Zhang: Conceptualization, Formaly analysis, Methodology, Software, Visualization, Writing–original draft; Zhiming Li: Funding acquisition, Methodology, Supervision, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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