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Research article

The odd coloring of some Cartesian product graphs

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Abstract: An odd c-coloring of a graph is a proper c-coloring such that each non-isolated vertex has at least one color appearing an odd number of times in its neighborhood. The minimum number of colors in any odd coloring of G, denoted $\chi_o(G)$, is called the odd chromatic number. This concept was introduced by Petruševski and Škrekovski, who conjectured that every planar graph G is odd 5-colorable and observed that $\chi_o(G\square H) \leq \chi_o(G) \cdot \chi_o(H)$ for connected nontrivial graphs G and H. In this paper, for specific Cartesian product graphs $G\square H$, such as $P_m\square P_n$, $C_m\square P_n$, and $C_m\square C_n$, we determine the exact value of $\chi_o(G\square H)$, which establishes tighter upper bounds than the multiplicative bound $\chi_o(G) \cdot \chi_o(H)$. We show that $\chi_o(P_m\square P_n) \leq 4$ with a complete characterization of all cases; $\chi_o(C_m\square P_n) \leq 5$ with a full classification for even and odd m; and $\chi_o(C_m\square C_n) \leq 5$ with necessary and sufficient conditions for 3-, 4-, and 5-colorability under parity and divisibility constraints. These results significantly improve upon the multiplicative upper bound and provide new constructive methods and theoretical insights for studying odd colorings in Cartesian product graphs. Additionally, we determine that $\chi_o(K_m\square P_n) = \chi_o(K_m\square C_n) = m$ for $m = 3$ and C_n is an even cycle or $m \geq 4$.

Keywords: odd coloring; Cartesian product; coloring matrix
Mathematics Subject Classification: 05C15, 05C10

1. Introduction

All graphs in this paper are simple, finite, and undirected. We follow Bondy's work [1] for any terminology and notation not defined here. An n-path (n-cycle) is a path (cycle) with n vertices, denoted by $P_n(C_n)$. An odd c-coloring of a graph is a proper c-coloring such that each nonisolated vertex must have a color appearing an odd number of times within its neighbors. The odd chromatic number $\chi_o(G)$ is the minimum integer c for which G admits an odd c-coloring. A graph is odd c-colorable if it has an odd c-coloring. Let $\varphi : V(G) \rightarrow \{1, 2, \ldots, c\}$ be an odd coloring of G. Then $\varphi(v)$ denotes the color of

v, and $\varphi_o(v)$ denotes an odd color of v; if v has many odd colors, then choose an arbitrary one.

Odd coloring was introduced by Petruševski and Škrekovski [11]. They showed that planar graphs are odd 9-colorable and proposed the following conjecture.

Conjecture 1.1. *[11] Every planar graph G has the odd chromatic number at most 5.*

Further studies on the odd chromatic number were conducted by Caro et al. [2], where they showed that 8 colors are enough for a significant family of planar graphs. Building upon these results, Petr and Portier [12] improved the upper bound, proving that every planar graph is odd 8-colorable.

Beyond planar graphs, odd colorings have been investigated in more general graph classes. Metrebian [9] proved the following result about toroidal graphs.

Theorem 1.2. *[9] Every toroidal graph is odd 9-colorable.*

Cranston et al. [4] demonstrated that every 1-planar graph is odd 23-colorable, a bound later improved by Niu and Zhang [10] to 16-colorability and subsequently reduced to 13-colorability by Liu, Wang, and Yu [8]. For odd colorings of k-planar graphs, we refer the reader to [5, 6]. Outerplanar graphs have also been a subject of investigation. Caro, Petruševski, and Škrekovski [2] proved that every outerplanar graph is odd 5-colorable, with the bound being tight due to the existence of the cycle C_5, which is not odd 4-colorable. Recently, Kashima and Zhu [7] strengthened this result by showing that every maximal outerplanar graph is odd 4-colorable. Furthermore, they proved that a connected outerplanar graph G is odd 4-colorable if and only if G contains a block that is not a copy of C_5. A particular focus has been given to planar graphs with high girth. For each positive integer c, determine the minimum girth g_c such that a planar graph with girth at least g_c is odd c-colorable. Cho et al. [3] showed that $6 \le g_4 \le 11$, $g_5 \le 7$, $g_7 \le g_6 \le 5$, and $g_8 \le 3$. Wang and Yang [13] proposed enhancements to the existing proof framework, proving that every planar graph without 4^--cycles adjacent to 7^--cycles is odd 6-colorable.

A critical direction of odd coloring research involves analyzing Cartesian product graphs. For a graph G, we denote its vertex and edge sets by $V(G)$ and $E(G)$, respectively. Given two graphs G and H, the **Cartesian product** $G\square H$ is defined as the graph with vertex set $V(G\square H) = V(G) \times V(H) = \{(u_i, v_j) \mid u_i \in V(G), v_j \in V(H)\}$. Two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if either $u_i = u_k$ and $v_jv_l \in E(H)$ or $v_j = v_l$ and $u_iu_k \in E(G)$. For simplicity, we denote (u_i, v_j) by (i, j), where $1 \le i \le |V(G)|$ and $1 \le j \le |V(H)|$, interpreting the vertex as lying in the i-th row and j-th column. The subgraph induced by vertices $\{(i, j)|j \in \{1, 2, \ldots, |V(H)|\}\}$ is called the H-direction, and that induced by vertices $\{(i, j)|i \in \{1, 2, \ldots, |V(G)|\}\}$ is called the G-direction.

Such Cartesian products often form gridlike structures. The subgraph induced by the vertices $\{(i, j), (i, j+1), (i+1, j), (i+1, j+1)\}$ constitutes a 4-cycle, which we refer to as **cell**(i, j).

Given a vertex $v \in V(G)$, its **neighborhood** is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its **degree** is $d_G(v) = |N_G(v)|$. A vertex is called a d-vertex if its degree is d, and a d-**neighbor** of v is a d-vertex in $N_G(v)$. When no confusion arises, we suppress the graph subscript and write $N(v)$, $d(v)$, and so on. Similarly, we abbreviate $\varphi(i, j)$ for $\varphi((i, j))$ and $N(i, j)$ for $N((i, j))$.

For Cartesian products, Caro, Petruševski, and Škrekovski obtained a foundational bound:

Lemma 1.3. *[2] If G and H are connected nontrivial graphs, then*

$$\chi_o(G\square H) \le \min\{\chi(G) \cdot \chi_o(H), \chi_o(G) \cdot \chi(H)\} \le \chi_o(G) \cdot \chi_o(H).$$

However, this bound is often loose for structured products such as grids $P_m \square P_n$ or toroidal graphs $C_m \square P_n$ and $C_m \square C_n$. Path P_n ($n \geq 1$) and cycle C_n ($n \geq 3$) exhibit simple odd chromatic behaviors:

Lemma 1.4. *[2]*

$$\chi_o(P_n) = \begin{cases} n, & n \leq 2, \\ 3, & n \geq 3; \end{cases} \quad \chi_o(C_n) = \begin{cases} 3, & 3 \mid n, \\ 4, & 3 \nmid n \text{ and } n \neq 5, \\ 5, & n = 5. \end{cases}$$

In this paper, we investigate the odd coloring of graphs $P_m \square P_n$, $C_m \square P_n$, and $C_m \square C_n$ and obtain the following results.

Theorem 1.5. *Every $P_m \square P_n$ $(1 \leq m \leq n)$ is odd 4-colorable; moreover, we have the following result:*

$$\chi_o(P_m \square P_n) = \begin{cases} n & \text{if } m = 1, 1 \leq n \leq 2, \\ 3 & \text{if } m = 1, n \geq 3 \text{ or } m = 3, n \geq 4 \text{ or } m = n = 4, \\ 4 & \text{if } m \geq 4, n \geq 5 \text{ or } m = 2, n \geq 2 \text{ or } m = n = 3. \end{cases}$$

Theorem 1.6. *Every $C_m \square P_n$ $(m \geq 3, n \geq 1)$ is odd 5-colorable; moreover, we have the following result:*

$$\chi_o(C_m \square P_n) = \begin{cases} 2, & \text{if } m = 2k,\ n = 2; \\ 3, & \text{if } m = 2k+1,\ n = 2 \quad \text{or } m = 3k,\ n \neq 2 \\ & \quad \text{or } m = 2k,\ n \geq 3; \\ 5, & \text{if } m = 5,\ n = 1; \\ 4, & \text{otherwise}. \end{cases}$$

Theorem 1.7. *Every $C_m \square C_n$ $(m, n \geq 3)$ is odd 5-colorable; moreover, we have the following result:*

$$\chi_o(C_m \square C_n) = \begin{cases} 3, & \text{if } \{m, n\} = \{3k, 2t\} (k \geq 1,\ t \geq 2); \\ 5, & \text{if } m = n = 3 \text{ or } \{m, n\} = \{3, 5\}; \\ 4, & \text{otherwise}. \end{cases}$$

These results indicate that Cartesian products of paths and cycles may admit tighter bounds on their odd chromatic number than the multiplicative bound $\chi_o(G) \cdot \chi_o(H)$. Theorems 1.5 and 1.6 provide support for Conjecture 1.1. For toroidal graphs $C_m \square P_n$, $C_m \square C_n$, the odd chromatic number is 5, which is smaller than the upper bound 9 in Theorem 1.2.

We employ coloring matrices to study the odd chromatic number of Cartesian product graphs $G \square H$. By designing compact, regular coloring templates, we build periodic global colorings that reduce global coloring to finite local patterns. This method not only simplifies the construction and verification of colorings but also guarantees scalability to graphs of arbitrary size. Such a structured periodic approach is crucial for improving the classical product bound and obtaining tight odd chromatic numbers.

The paper is organized as follows. Section 2 focuses on the study of $P_m \square P_n$, where we determine the minimum odd chromatic number for different values of m and n. In Section 3, we analyze $C_m \square P_n$ and obtain the corresponding results. Section 4 then turns to an investigation of $C_m \square C_n$, and under a range of conditions, we establish the minimum odd chromatic number for this Cartesian product. Finally, Section 5 presents new findings on the odd chromatic number of Cartesian products involving a complete graph—specifically, products of a complete graph with either a path or a cycle.

2. Odd coloring of $P_m \square P_n$

In this section, we begin by establishing an upper bound of 4 for the odd coloring number of the grid graph $P_m \square P_n$ when $m, n \geq 1$. Subsequently, we exploit the inherent properties of odd 3-colorings to derive the core rules that govern such colorings and on this basis, determine the chromatic number across distinct cases.

Lemma 2.1. *If a graph contains a vertex of even degree, then it is not odd 2-colorable.*

Proof. Suppose, for contradiction, that a graph G containing at least one even-degree vertex is odd 2-colorable. Let v be an even-degree vertex in G. Define the neighborhood of v as $N(v) = \{u_1, u_2, \ldots, u_{2k}\}$. Because the coloring uses only two colors, we first color v. To maintain a proper coloring, all neighbors $u_1, u_2, \ldots, u_{2k}$ must be assigned the other color, say $\varphi(u_i) \neq \varphi(v)$. However, this assignment implies that v has exactly one color appearing in its neighborhood, and it appears an even number of times (because $d(v)$ is even). This contradicts the definition of an odd coloring. Thus, no such 2-coloring can exist, proving that G is not odd 2-colorable. □

Lemma 2.2. *For any integers $m, n \geq 1$, we have $\chi_o(P_m \square P_n) \leq 4$.*

Proof. Set the coloring matrix $A = (a_{p,q}) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$.

For the vertices in the Cartesian product $P_m \square P_n$, we define a 4-coloring $\varphi : V(P_m \square P_n) \to \{1, 2, 3, 4\}$ as follows. The periodic coloring scheme is illustrated in Figure 1:

$$\varphi(i, j) = a_{p,q},$$

where $p = (i-1) \bmod 4 + 1$ and $q = (j-1) \bmod 2 + 1$.

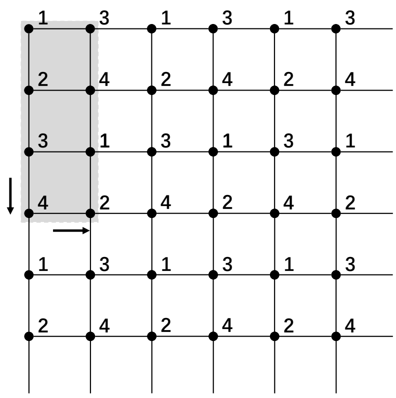


Figure 1. The odd 4-coloring of $P_m \square P_n$.

From the coloring scheme, we can observe that every cell in the graph contains vertices with four distinct colors. If we color (i, j) with color 1, then we should color its four neighbors $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, and $(i, j-1)$ with colors 2, 4, 3, and 3, respectively. This ensures that the vertex (i, j) satisfies the odd coloring condition. If (i, j) is assigned a different color, a similar pattern follows, ensuring a valid odd coloring. In a more specific case, if the vertex is located on the boundary of $P_m \square P_n$, it will have fewer neighbors, but the odd coloring condition is still satisfied. Thus, for any integers m, n, we conclude that $\chi_o(P_m \square P_n) \leq 4$. □

Lemma 2.3. *For $n \geq 4$, $\chi_o(P_3 \square P_n) = 3$.*

Proof. We begin by defining two coloring matrices,

$$A_1 = (a_{i,j}^{(1)}) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = (a_{i,j}^{(2)}) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

For Cartesian product $P_3 \square P_n$, there are four 2-vertices. To satisfy the odd coloring requirement, we define the following scheme:

$$\varphi(i, j) = \begin{cases} a_{i,j}^{(1)}, & \text{if } j < 3, \\ a_{i,j \bmod 3+1}^{(2)}, & \text{if } j \geq 3. \end{cases}$$

The periodic coloring scheme is illustrated in Figure 2.

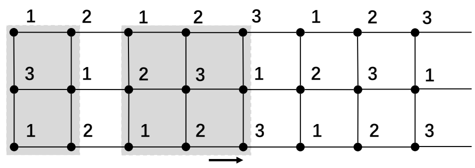


Figure 2. The odd 3-coloring of $P_3 \square P_n$.

To verify the validity of this coloring, consider a vertex $(2, j)$. If $j \leq 3$, then it is easily observed that each $(2, j)$ has a color which occurs an odd number of times in its neighbors from Figure 2. Then assume $j \geq 4$. If we assign $(2, j)$ the color 1, then its four neighbors, $(1, j), (3, j), (2, j-1)$, and $(2, j+1)$ receive colors 3, 3, 3, and 2, respectively. This guarantees that $(2, j)$ satisfies the odd coloring condition. If $(2, j)$ is assigned a different color, a similar argument holds, ensuring that every vertex maintains the required property.

For boundary vertices, all of them have an odd degree except for the four corner vertices $(1, 1), (3, 1), (1, n)$, and $(3, n)$, which are 2-vertices. To confirm that these corner vertices also satisfy the odd coloring condition, we check their adjacent vertices. Because $\varphi(1, 2) \neq \varphi(2, 1)$, vertex $(1, 1)$ maintains the odd coloring condition. The same reasoning applies to $(3, 1), (1, n)$, and $(3, n)$, confirming that all corner vertices are properly colored. This ensures that every vertex satisfies the odd coloring condition.

Thus, we conclude that for $n \geq 4$, the odd chromatic number of $P_3 \square P_n$ is 3, completing the proof. □

Observation 2.4. *Let G be a Cartesian product graph of two graphs. If G is odd 3-colorable, then it can be colored following the next rules:*

Rule 1: *If three vertices in cell(i, j) are assigned three distinct colors, then the fourth vertex must take the color of its diagonal opposite.*

Rule 2: *If three vertices in $N(i, j)$ of G have been assigned colors, then the fourth neighbor should take a color that appears either zero or twice in $N(i, j)$, and this color must be different from the color of (i, j).*

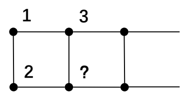
Rule 3: *At each coloring step, Rule 1 should be applied first, followed by Rule 2.*

Lemma 2.5. *A Cartesian product graph $P_m \square P_n$ $(1 \leq m \leq n)$ is not odd 3-colorable if it belongs to either of the following cases: (i) $m = 2, n \geq 2$; (ii) $m \geq 4$, $n \geq 5$.*

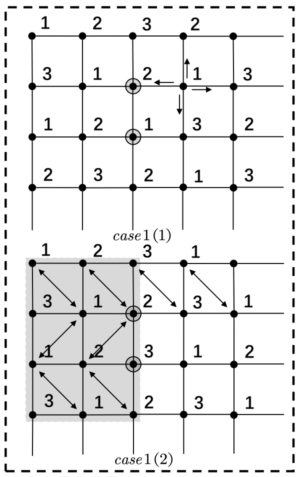
Proof. Each of (i) and (ii) is illustrated in Figure 3.

(i) Suppose $P_2 \square P_n$ is odd 3-colorable. We assign colors to a few vertices and derive a contradiction. First, we color $(1, 1)$ and $(2, 1)$ with two different colors. Then we color $(1, 2)$ with a color not in $\{\varphi(1, 1), \varphi_o(1, 1)\}$ and color $(2, 2)$ with a color not in $\varphi(N(2, 2)) \cup \varphi_o(N(2, 2))$. Note that because $|\varphi(N(2, 2)) \cup \varphi_o(N(2, 2))| = 3$, there are no more colors available for $(2, 2)$. Thus, we reach a contradiction, proving that $P_2 \square P_n$ is not odd 3-colorable.

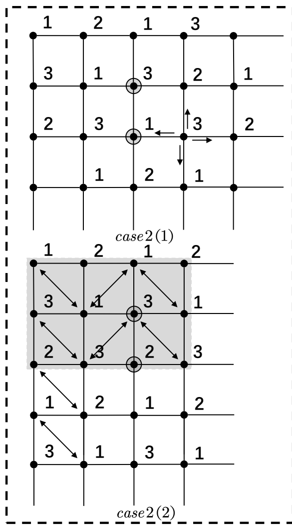
(ii) Suppose $P_m \square P_n$ is odd 3-colorable. We attempt to construct a valid odd 3-coloring and derive a contradiction. By Observation 2.4, we begin by coloring $(1, 1)$, $(1, 2)$, and $(2, 1)$ with three distinct colors. By Rule 1, we assign the color $\varphi(1, 1)$ to $(2, 2)$. While coloring the vertex $(2, 3)$, two distinct cases emerge. The specific vertex coloring order can be seen in Figure A1.



(a) The illustration for (i).



(b) The illustration for Case 1.



(c) The illustration for Case 2.

Figure 3. The illustrations for Lemma 2.5.

Case 1: Color $(2, 3)$ with $\varphi(1, 2)$. Applying Rule 2, we color $(3, 2)$ with $\varphi(1, 2)$. Now, for $(3, 3)$, we have two choices:

(1) If the vertex $(3,3)$ is assigned the color $\varphi(1,1)$, then applying the coloring rules leads to $(2,4)$ having two neighbors with color $\varphi(2,1)$ and two with color $\varphi(1,2)$, which violates the odd coloring condition;

(2) If $(3,3)$ is colored with $\varphi(2,1)$, then the coloring on cell$(1, j)$ has $\varphi(1, j) = \varphi(2, j+1)$ for $j = 1,2$ and the coloring on cell$(2, j)$ has $\varphi(3, j) = \varphi(2, j+1)$ for $j = 1,2$. By applying Rule 2 to $N(3,2)$, we have $\varphi(4,2) = \varphi(3,1)$. Then, applying Rule 1 to cell$(3,2)$, we have $\varphi(4,3) = \varphi(3,2)$. If we color $(1,3)$ with color $\varphi(2,2) = 1$, by Rule 2 on $N(2,3)$ and $N(3,3)$, we have $\varphi(2,4) = \varphi(3,4) = 1$, contrary to the proper coloring. Then $\varphi(1,3) = 3 \neq \varphi(2,2)$. Similarly, we have $\varphi(4,1) = 3 \neq \varphi(3,2)$. For the six cells, we find that (i) each cell(i, j) uses three colors for $i \in \{1,2,3\}$ and $j \in \{1,2\}$; (ii) $\varphi(i, j) = \varphi(i+1, j+1)$ for each $i \in \{1,3\}, j \in \{1,2\}$ and $\varphi(3, j) = \varphi(2, j+1)$ for each $j \in \{1,2\}$. Rule 1 is repeatedly applied to cell(i, j) and Rule 2 to $N(i, j)$ for $i \in \{1,2,3\}$; with the process carried out step by step for columns $j = 3,4,5,\ldots,n-1$, we can color vertices $(i, j+1)$ with $\varphi(i,k)$ for $i \in \{1,2,3,4\}$ and $k = j \mod 3+1$. As a result, we find that $\varphi(1, j) = \varphi(2, j+1)$ for each $j \in \{1,2,\ldots,n-1\}$. This implies that the vertex $(1,n)$ has two identically colored neighbors, violating the odd coloring condition because $d(1,n) = 2$.

Case 2: Color $(2,3)$ with $\varphi(2,1)$. By Rule 2, we color $(3,2)$ with $\varphi(2,1)$. Again, for $(3,3)$, we consider two cases:

(1) If $(3,3)$ is colored with $\varphi(1,1)$, then after the coloring rules are propagated, $(3,4)$ ends up with two neighbors colored with $\varphi(1,1)$ and the other two with $\varphi(1,2)$, violating the odd coloring constraint;

(2) If $(3,3)$ is colored with $\varphi(1,2)$, then further application of the rules leads to the fact that for the six cells, (i) each cell(i, j) uses three colors for $i \in \{1,2\}$ and $j \in \{1,2,3\}$; (ii) $\varphi(i, j) = \varphi(i+1, j+1)$ for each $i \in \{1,2\}, j \in \{1,3\}$ and $\varphi(i+1,2) = \varphi(i,3)$ for each $i \in \{1,2\}$. Similar to the discussion in Case 1 (2), Rule 1 is repeatedly applied to cell(i, j) and Rule 2 to $N(i, j)$ for $j \in \{1,2,3\}$; with the process carried out step by step for columns $i = 3,4,\ldots,m-1$, we can color vertices $(i+1, j)$ with $\varphi(k, j)$ for $j \in \{1,2,3,4\}$ and $k = i \mod 3+1$. As a result, we find that $\varphi(i,1) = \varphi(i+1,2)$ for each $i \in \{1,2,\ldots,m-1\}$, causing $(m,1)$ to have an invalid odd coloring configuration because $d(m,1) = 2$.

All cases lead to contradictions, showing that it is impossible to construct a valid odd 3-coloring for $P_m \Box P_n$. Thus, $P_m \Box P_n$ is not odd 3-colorable when $m \geq 4, n \geq 5$. □

Proof of Theorem 1.5. By Lemma 2.2, each $P_m \Box P_n$ $(m,n \geq 1)$ is odd 4-colorable. It is easy to see that $\chi_o(P_1 \Box P_n) = \chi_o(P_n)$. According to Lemma 1.4,

$$\chi_o(P_1 \Box P_n) = \begin{cases} n & \text{if } 1 \leq n \leq 2, \\ 3 & \text{if } n \geq 3. \end{cases}$$

By Lemma 2.3, $\chi_o(P_3 \Box P_n) = 3$ for $n \geq 4$. By Lemmas 2.5 and 2.2, it follows that for $n \geq 2$, $\chi_o(P_2 \Box P_n) = 4$; for $m \geq 4$ and $n \geq 5$, $\chi_o(P_m \Box P_n) = 4$.

Additionally, it is easy to construct an odd 3-coloring for $P_4 \Box P_4$, and the specific coloring result is shown in Figure 4(a). Therefore, $\chi_o(P_4 \Box P_4) = 3$.

We assume that $P_3 \Box P_3$ has an odd 3-coloring φ. From the coloring procedure, it follows that if all vertices in $N(2,2)$ are colored, then three of them must have the same color. By symmetry, we may assume that $\varphi(1,2) = \varphi(2,1) = \varphi(3,2)$, as shown in Figure 4(b). However, this results in the even-degree vertices $(1,1)$ and $(3,1)$ not having an odd coloring, contradicting our assumption. Therefore, $P_3 \Box P_3$ is not odd 3-colorable. By Lemma 2.2, we have $\chi_o(P_3 \Box P_3) = 4$. Finally, we can deduce Theorem 1.5. □

(a) The odd 3-coloring of $P_4 \square P_4$. **(b)** $P_3 \square P_3$ is not odd 3-colorable.

Figure 4. Coloring with 3 colors.

3. The odd chromatic number of $C_m \square P_n$

In this section, we investigate the odd chromatic number of the Cartesian product $C_m \square P_n$. Our analysis begins with the special case $n = 2$, followed by a general classification for arbitrary $n \geq 3$. We provide constructive colorings for various values of m and conclude with a sharp upper bound.

Lemma 3.1. *For all integers $m \geq 3$,*

$$\chi_o(C_m \square P_2) = \chi(C_m) = \begin{cases} 2, & \textit{if } m \textit{ is even}, \\ 3, & \textit{if } m \textit{ is odd}. \end{cases}$$

Proof. Observe that the degree of each vertex in $C_m \square P_2$ is 3. Because all vertices have odd degrees, any proper coloring automatically satisfies the condition for an odd coloring. Thus, $\chi_o(C_m \square P_2) = \chi(C_m \square P_2)$. Let ψ be a proper $\chi(C_m)$-coloring of C_m and $C_m = 12 \ldots m1$. Define

$$\varphi(i, j) = \begin{cases} \psi(i) & \text{if } j = 1; \\ \psi(i \ mod \ m + 1) & \text{if } j = 2. \end{cases}$$

Clearly, $\varphi(i, 1) \neq \varphi(i, 2)$ for $i(i+1) \in E(C_m)$. This construction ensures that adjacent vertices receive different colors, satisfying the odd coloring requirement. Thus, we conclude:

$$\chi_o(C_m \square P_2) = \chi(C_m) = \begin{cases} 2, & \text{if } m \text{ is even;} \\ 3, & \text{if } m \text{ is odd.} \end{cases}$$

□

Lemma 3.2. *For all integers $n \geq 3$,*
(i) $C_m \square P_n$ is odd 3-colorable, where $m \in 2N^+ \cup 3N^+$;
(ii) $C_{6k \pm 1} \square P_n$ is not odd 3-colorable.

Proof. We now consider the general case $n \geq 3$, where $C_m \square P_n$ contains vertices with degree 4. By Lemma 2.1, we attempt to apply an odd coloring using three colors. For any 4-vertex v, $N(v)$ satisfies $|\varphi(N(v))| = 2$, with one color appearing three times and the other once. This leads to two distinct types for $\varphi(N(v))$:

Type I: The two neighbors of v along the P_n-direction receive the same color (see Figure 5(a));

Type II: The two neighbors along the P_n-direction receive different colors (see Figure 5(b)).

Starting from the initial coloring shown in Figure 5, we apply the coloring rules described in Observation 2.4. The color of every vertex is uniquely determined by the coloring rules and the configuration type. Hence, the coloring process is deterministic and completely governed by local parity constraints.

(a) Two neighbors along the P_n-direction receive the same color.

(b) Two neighbors along the P_n-direction receive different colors.

Figure 5. Two distinct configurations for $\varphi(N(v))$.

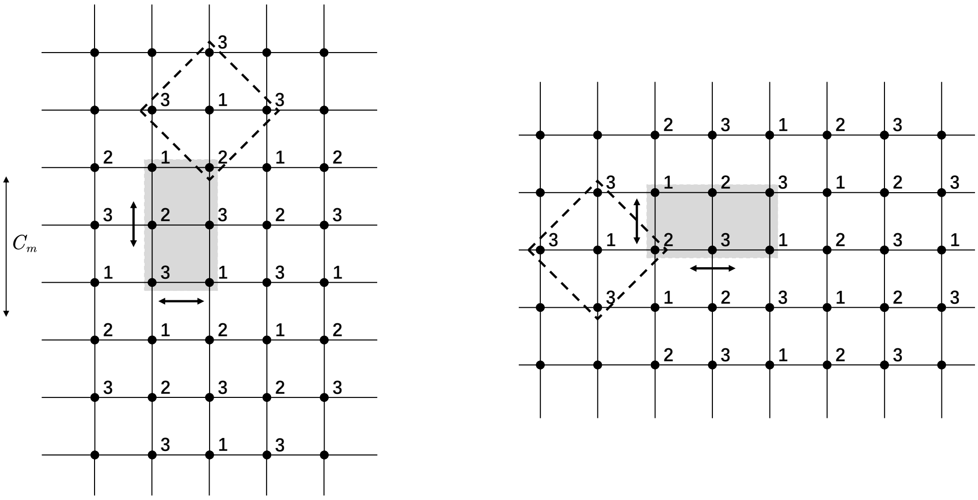
Starting from one vertex as shown in Figure 5(a) and then applying the coloring rules described in Observation 2.4, we can obtain the resulting coloring as seen in Figure 6(a). In the resulting coloring, all the vertices have two neighbors receiving the same color along the P_n-direction. A repeating coloring pattern with period 3 along the cycle-direction, period 2 along the path-direction can be found. Based on the repeating coloring pattern, we define the coloring scheme for $C_{3k}\square P_n$ as

$$\varphi(i,j) = \begin{cases} (i-1) \bmod 3 + 1, & \text{if } j \equiv 1 \pmod 2, \\ i \bmod 3 + 1, & \text{if } j \equiv 0 \pmod 2. \end{cases}$$

Based on the above discussion, we assume that all 4-vertices are Type II. A repeating coloring pattern with period 2 along the C_m-direction and period 3 along the P_n-direction can then be identified, as shown in Figure 6(b). Based on the repeating coloring pattern, we define the coloring scheme for $C_{2k}\square P_n$ as

$$\varphi(i,j) = \begin{cases} (j-1) \bmod 3 + 1, & \text{if } i \equiv 1 \pmod 2, \\ j \bmod 3 + 1, & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Therefore, we conclude that $C_m\square P_n$ is odd 3-colorable if $m = 2k$ or $m = 3k$. Any attempt to apply an odd 3-coloring to $C_{6k\pm1}\square P_n$ will inevitably force some vertex to have no color appearing an odd number of times in its neighborhood. Thus, $C_m\square P_n$ is odd 3-colorable if and only if $m \in \{0, 2, 3, 4\} \pmod 6$. This implies that $C_{6k\pm1}\square P_n$ is not odd 3-colorable. □



(a) The resulting coloring of **Case I**. **(b)** The resulting coloring of **Case II**.

Figure 6. The odd coloring of Case I and Case II.

Lemma 3.3. $\chi_o(C_{6k\pm1}\square P_n) = 4$ *for all* $n \geq 3,\ k \in \mathbb{Z}^+$.

Proof. We define a periodic coloring block composed of two parts. The first five rows follow a 6-column periodic matrix A_1, while the remaining rows are filled with a smaller 2-row, 3-column matrix A_2. Define

$$A_1 = (a_{ij}^{(1)}) = \begin{bmatrix} 1 & 2 & 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 3 & 4 & 1 \\ 2 & 1 & 2 & 1 & 2 & 3 \\ 3 & 2 & 4 & 2 & 4 & 2 \\ 4 & 1 & 3 & 4 & 3 & 1 \end{bmatrix}, \quad A_2 = (a_{ij}^{(2)}) = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 3 & 1 \end{bmatrix}.$$

The coloring is defined as

$$\varphi(i,j) = \begin{cases} a^{(1)}_{i,(j-1) \bmod 6+1}, & 1 \leq i \leq 5; \\ a^{(2)}_{(i-6) \bmod 2+1,(j-1) \bmod 3+1}, & i > 5. \end{cases}$$

Due to the periodic structure and the fact that $(6k \pm 1) - 5$ is always even, the remaining rows can be tiled by A_2 without conflict. The coloring scheme is verified on $C_{11}\square P_9$, and this construction ensures that all adjacent vertices receive distinct colors and that each vertex has at least one color appearing an odd number of times in its neighborhood. The specific odd coloring scheme of $C_{11}\square P_9$ can be seen in Figure A2(a). By periodicity, it can be easily extended to arbitrary $C_{6k\pm1}\square P_n$. By Lemma 3.2 (ii), $\chi_o(C_{6k\pm1}\square P_n) = 4$. □

Proof of Theorem 1.6. Because $C_m\square P_1 \cong C_m$, $\chi_o(C_m\square P_1) = \chi_o(C_m)$. By Lemma 1.4, $\chi_o(C_m\square P_1) = 3$ when $m = 3k$, $\chi_o(C_m\square P_1) = 5$ when $m = 5$; otherwise we have $\chi_o(C_m\square P_1) = 4$. By Lemma 3.1, we characterize the odd chromatic number of $C_m\square P_2$. Thus, we have $\chi_o(C_{2k}\square P_2) = 2$ and $\chi_o(C_{2k-1}\square P_2) = 3$ when $k \geq 2$. In the case where $n \geq 3$, we consider m in the context of modulo 6. By Lemmas 3.2

and 3.3, $\chi_o(C_m \square P_n) = 3$ when $m \in \{0, 2, 3, 4\} \pmod 6$ and $\chi_o(C_m \square P_n) = 4$ when $m \in \{1, 5\} \pmod 6$. □

We conclude with a tight upper bound for all $m, n \in \mathbb{Z}^+$ in the following.

Corollary 3.4. $\chi_o(C_m \square P_n) \le \chi_o(C_m)$ $(m \ge 3, n \ge 1)$.

This bound significantly improves upon the multiplicative upper bound $\chi_o(C_m) \cdot \chi_o(P_n) \le 15$, demonstrating how periodic structures and matrix-based constructions can effectively reduce the coloring complexity of Cartesian product graphs.

4. The odd chromatic number of $C_m \square C_n$

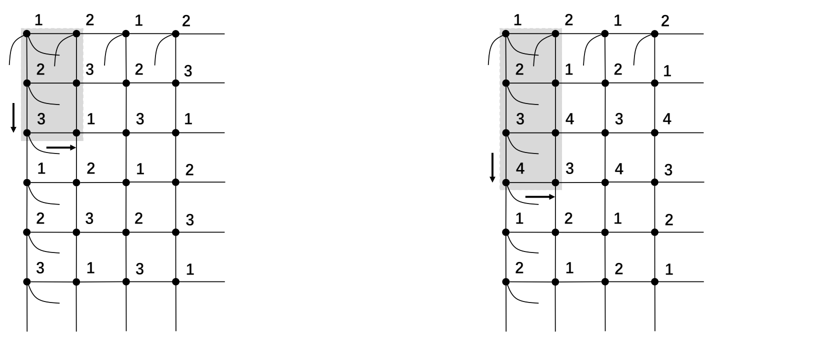
Lemma 4.1. *The odd chromatic number of Cartesian product $C_m \square C_n$ is 3 if and only if $\{m, n\} = \{3k, 2t\}$ for integers $k \ge 1$ and $t \ge 2$.*

Proof. In the Cartesian product $C_m \square C_n$, each vertex is a 4-vertex. Hence, to achieve an odd 3-coloring, we must assign colors in such a way that for every vertex, one color appears on exactly three neighbors and another appears on exactly one neighbor.

According to the classification of 4-vertex configurations in Figure 5, namely Type I and Type II, and the corresponding colorings shown in Figure 6(a), the coloring admits a repeating pattern with period 3 along the C_m-direction and period 2 along the C_n-direction. As illustrated in the figures, the coloring scheme in Figure 6(b) can be obtained by rotating the scheme in Figure 6(a) counterclockwise by 90°. This suffices to consider only one type when dealing with $C_m \square C_n$, because both directions are cycles. As seen in Figure 7(a), we define the coloring scheme for $C_{3k} \square C_{2t}$ as

$$\varphi(i, j) = \begin{cases} (i-1) \bmod 3 + 1, & \text{if } j \equiv 1 \pmod 2, \\ i \bmod 3 + 1, & \text{if } j \equiv 0 \pmod 2. \end{cases}$$

Therefore, such an odd 3-coloring is only possible when $m = 3k$ and $n = 2t$. By symmetry, $m = 2t$ and $n = 3k$ is the same case. Hence, we have $\chi_o(C_m \square C_n) = 3$ if and only if $\{m, n\} = \{3k, 2t\}$ for integers $k \ge 1$ and $t \ge 2$. □



(a) The coloring of $C_{3k} \square C_{2t}$ with 3 colors. **(b)** The coloring of $C_{4k} \square C_{2t}$ with 4 colors.

Figure 7. The odd coloring for $C_{3k} \square C_{2t}$ and $C_{4k} \square C_{2t}$.

Lemma 4.2. *If at least one of m or n is even, then $C_m \square C_n$ is odd 4-colorable.*

Proof. By symmetry, we assume $n = 2t$ ($t \geq 2$) is even. By Lemma 4.1, we have $\chi_o(C_m \square C_{2t}) = 3 < 4$ when $m \equiv 0 \pmod 3$, and we assume $m \not\equiv 0 \pmod 3$. In this case, $C_m \square C_{2t}$ is not odd 3-colorable by Lemma 4.1. Then, we consider using four colors to obtain an odd coloring. We discuss two cases based on the value of m.

Case 1: $m = 5$**.**

Let $n \equiv p$ (mod4). Because $n = 2t$, $p \in \{0, 2\}$, define two color matrices as follows:

$$A_1 = (a^{(1)}_{ij}) = \begin{bmatrix} 4 & 3 \\ 2 & 1 \\ 3 & 2 \\ 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_2 = (a^{(2)}_{ij}) = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 3 & 4 & 3 \end{bmatrix}.$$

Then, we define a coloring scheme based on A_1 and A_2. We remark that for $p = 0$, the constraint $j \leq p$ yields an empty set of valid indices. In this scenario, the corresponding branch of the piecewise function is discarded, and only the case where $j > p$ is retained for analysis.

$$\varphi(i, j) = \begin{cases} a^{(1)}_{i,j} & \text{if } j \leq p, \\ a^{(2)}_{i,\ (j-k-1) \bmod 4+1} & \text{if } j > p. \end{cases}$$

This construction combines two blocks to cover all columns of C_n, satisfying the coloring condition. For example, if $\varphi(i, j) = 1$, the colors of its neighbors could be $\{2, 2, 2, 3\}$, $\{2, 2, 2, 4\}$, $\{2, 2, 3, 4\}$, or $\{2, 3, 3, 4\}$, ensuring that some color appears an odd number of times. By symmetry, the same holds for color 2, 3, 4.

Case 2: $m \neq 5$**.**

In this case, we can define $m = 4k + q$, where $k \geq 1, q \in \{0, 3, 6, 9\}$. We add a transitional coloring block in the C_m-direction on top of the periodic construction. It should be noted that when $q = 0$, the condition $j \leq q$ corresponds to an empty index range and thus can be disregarded, meaning only the case of $j > p$ in the piecewise function needs to be considered.

$$\varphi(i, j) = \begin{cases} a^{(3)}_{(i-1) \bmod 3+1,\ (j-1) \bmod 2+1} & \text{if } i \leq q, \\ a^{(4)}_{(i-q-1) \bmod 4+1,\ (j-1) \bmod 2+1} & \text{if } i > q, \end{cases}$$

with

$$A_3 = (a^{(3)}_{ij}) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad A_4 = (a^{(4)}_{ij}) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}.$$

As shown in Figure 7(b), it is easily obtained that $C_{4k} \square C_{2t}$ is odd 4-colorable. The coloring scheme is verified on $C_{10} \square C_4$, and this construction ensures that all adjacent vertices receive distinct colors and that each vertex has at least one color appearing an odd number of times in its neighborhood. The specific odd coloring scheme of $C_{10}\square C_4$ can be seen at Figure A2(b). By periodicity, it can easily extend to arbitrary $C_{4k+q}\square C_{2t}$. Thus, $C_m\square C_{2t}$ is odd 4-colorable. □

Lemma 4.3. *If both m and n are odd, then*

$$\chi_o(C_m\square C_n) = \begin{cases} 5, & \text{if } m = n = 3 \text{ or } \{m, n\} = \{3, 5\}; \\ 4, & \text{otherwise.} \end{cases}$$

Proof. For $C_3\square C_3$ and $C_3\square C_5$, it can be verified by exhaustive checking that no odd 4-coloring exists. The odd 5-coloring is shown in Figure 8; hence $\chi_o(C_3\square C_3) = \chi_o(C_3\square C_5) = 5$.

(a) $C_3\square C_3$

(b) $C_3\square C_5$

Figure 8. Odd 5-coloring for $C_3\square C_3$ and $C_5\square C_3$.

For $C_3\square C_{2t+1}$, where $t \geq 3$, we define two coloring matrices $A_1 = (a_{ij}^{(1)}) = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 3 & 4 & 2 & 3 \\ 2 & 4 & 1 & 4 & 1 \end{bmatrix}$, $A_2 = (a_{ij}^{(2)}) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$. To satisfy the odd coloring requirement, we define the following scheme:

$$\varphi(i, j) = \begin{cases} a_{i,j}^{(1)} & \text{if } j \leq 5, \\ a_{i,\,(j-6) \bmod 2+1}^{(2)} & \text{if } j > 5. \end{cases}$$

As shown in Figure 9, the coloring scheme is verified on $C_3\square C_9$, and this construction ensures that all adjacent vertices receive distinct colors and that each vertex has at least one color appearing an odd number of times in its neighborhood. By periodicity, it can easily extend to arbitrary $C_3\square C_{2t+1}$. Thus, $C_3\square C_{2t+1}$ is odd 4-colorable.

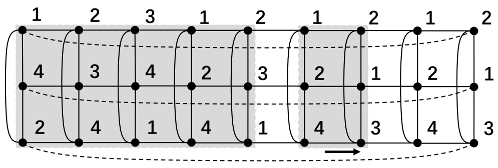


Figure 9. Odd 4-coloring of $C_3\square C_9$.

For $C_5\square C_7$ and $C_7\square C_7$, we design appropriate odd coloring schemes for them as shown in Figure A4.

Because the graphs $C_m\square C_n$ and $C_n\square C_m$ are isomorphic, for a more general case, we may assume without loss of generality that $m \le n$ for $C_m\square C_n$. Under this assumption, we define the coloring scheme for $C_{2k+1}\square C_{2t+1}$ as follows: $\varphi(i,j) = a^{(g)}_{(i-s_i^{(g)}) \bmod r_i^{(g)}+1,\ (j-s_j^{(g)}) \bmod c_j^{(g)}+1}$, where g denotes the index of the region to which the vertex (i, j) belongs, with $g \in \{1, \dots, 9\}$. This ensures the coloring is repeated modularly within the assigned block. The 9 regions are shown in Figure A3.

- $A^{(g)}$ is the coloring matrix assigned to region g;
- $(s_i^{(g)}, s_j^{(g)})$ is the starting offset of region g, which is the top left corner of the region;
- $(r_i^{(g)}, c_j^{(g)})$ is the size of the coloring matrix $A^{(g)}$ for region g.

The coloring matrices are defined as follows:

$$A^{(1)} = \begin{bmatrix} 2 & 1 & 4 & 1 & 3 \\ 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 3 & 1 & 2 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 1 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 4 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} 2 & 1 & 2 & 4 & 3 \\ 4 & 2 & 3 & 1 & 2 \end{bmatrix}, \quad A^{(5)} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad A^{(6)} = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix},$$

$$A^{(7)} = \begin{bmatrix} 2 & 1 & 2 & 4 & 3 \\ 4 & 3 & 4 & 2 & 1 \\ 2 & 4 & 2 & 4 & 3 \\ 4 & 2 & 3 & 1 & 2 \end{bmatrix}, \quad A^{(8)} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 3 \\ 4 & 2 \end{bmatrix}, \quad A^{(9)} = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

The regions and corresponding rules are summarized in Table 1:

Table 1. The coloring matrices and the coloring rules.

Region g	**Applicable Range**	**Matrix** $A^{(g)}$	**Offset** $(s_i^{(g)}, s_j^{(g)})$	**Size** $(r_i^{(g)}, c_j^{(g)})$
1	$i \le 3,\ j \le 5$	base block	$(1, 1)$	$(3, 5)$
2	$i \le 3,\ 5 < j \le 5 + \beta$	base block	$(1, 6)$	$(3, 2)$
3	$i \le 3,\ 5 + \beta < j \le n$	periodic block	$(1, 6 + \beta)$	$(3, 4)$
4	$3 < i \le 3 + \alpha,\ j \le 5$	base block	$(4, 1)$	$(2, 5)$
5	$3 < i \le 3 + \alpha,\ 5 < j \le 5 + \beta$	base block	$(4, 6)$	$(2, 2)$
6	$3 < i \le 3 + \alpha,\ 5 + \beta < j \le n$	periodic block	$(4, 6 + \beta)$	$(2, 4)$
7	$3 + \alpha < i \le m,\ j \le 5$	periodic block	$(4 + \alpha, 1)$	$(4, 5)$
8	$3 + \alpha < i \le m,\ 5 < j \le 5 + \beta$	periodic block	$(4 + \alpha, 6)$	$(4, 2)$
9	$3 + \alpha < i \le m,\ 5 + \beta < j \le n$	periodic block	$(4 + \alpha, 6 + \beta)$	$(4, 4)$

Note: $\alpha = (m - 3) \bmod 4 \in \{0, 2\}$, $\beta = (n - 5) \bmod 4 \in \{0, 2\}$.

Special case: (i) When $m = 5$: if $\beta = 0$, then set $\varphi(5, 4) = 2$ and $\varphi(5, 5) = 1$; if $\beta = 2$ and $n \ne 7$, then set $\varphi(5, 4) = \varphi(3, 7) = 2$ and $\varphi(5, 5) = 4$.
(ii) When $m \ge 7$, set $\varphi(m, 4) = 2, \varphi(m, 5) = 1$.

(iii) And when $m \geq 7$: if $\alpha = 0$ and $\beta = 2$, then set $\varphi(3+4c, 8) = 3$, $\varphi(3+4c, 9) = 4$, $\varphi(4+4c, 8) = 1$, and $\varphi(4+4c, 9) = 2$; if $\alpha = \beta = 2$, then reset $A^{(2)}$ as A_3, $A^{(5)}$ as A_4, and $A^{(9)}$ as A_5 and set $\varphi(4, 8) = \varphi(5+4c, 8) = 3$, $\varphi(5+4c, 9) = 4$, where $c \in N^+$ and $c \leq \dfrac{m-7-\alpha}{4}$.

The matrices A_3, A_4, and A_5 are given as follows:

$$A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

The coloring scheme is verified on $C_5 \square C_5$, $C_5 \square C_{13}$, $C_5 \square C_{15}$, $C_{11} \square C_{13}$, $C_{11} \square C_{15}$, $C_{13} \square C_{13}$, and $C_{13} \square C_{15}$, and this construction ensures that all adjacent vertices receive distinct colors and that each vertex has at least one color appearing an odd number of times in its neighborhood. By periodicity, it can be easily extended to arbitrary $C_{2k+1} \square C_{2t+1}$. The odd 4-coloring of $C_5 \square C_5$, $C_5 \square C_{13}$, and $C_5 \square C_{15}$ can be seen in Figure A5. The odd 4-coloring of $C_{11} \square C_{13}$, $C_{11} \square C_{15}$, $C_{13} \square C_{13}$, and $C_{13} \square C_{15}$ can be seen in Figure A6. □

By Lemmas 4.1–4.3, Theorem 1.7 can be obtained.

Proof of Theorem 1.7. By Lemma 4.1, $\chi_o(C_m \square C_n) = 3$ if $\{m, n\} = \{3k, 2t\}$; otherwise, we have $\chi_o(C_m \square C_n) \geq 4$. By Lemmas 4.2 and 4.3, $\chi_o(C_3 \square C_3) = \chi_o(C_3 \square C_5) = 5$, and $\chi_o(C_m \square C_n) = 4$ in other cases. Thus, $C_m \square C_n$ is odd 5-colorable. □

In general toroidal graphs, Theorem 1.2 provides an upper bound of 9 for the odd chromatic number, which is a universal estimate based on the most complex possible structures of toroidal graphs. For the special class of Cartesian product graphs $C_m \square C_n$, we refine this upper bound to 5 through detailed structural analysis and the construction of periodic odd colorings. It provides a solid theoretical foundation for further study of odd colorings in Cartesian product graphs.

5. Odd coloring of Cartesian product with special simple graphs

Lemma 5.1. *Let G be a simple graph with no isolated vertices and suppose that $\chi_o(G) = k$. Let φ be an odd k-coloring of G.*

(i) If every nonisolated vertex has at least two colors appearing an odd number of times in its neighborhood under φ, then $\chi_o(G \square H) \leq \chi_o(G)$, where H is a path or an even cycle.

(ii) If every nonisolated vertex has at least three colors appearing an odd number of times in its neighborhood under φ, then $\chi_o(G \square C_n) \leq \chi_o(G)$.

Proof. Because φ is an odd k-coloring of G, define $\varphi : V(G) \to \{1, 2, \ldots, k\}$. Let $V(G) = \{u_1, \ldots, u_m\}$ and $P_n = v_1 v_2 \cdots v_n$, $C_n = v_1 v_2 \cdots v_n v_1$. For the Cartesian products $G \square P_n$ and $G \square C_n$, we represent vertices by (i, j), where the first coordinate corresponds to the vertex $u_i \in V(G)$, and the second coordinate $j \in \{1, \ldots, n\}$ represents v_j of the path (or cycle). We now construct colorings for the two cases and verify that they are valid odd k-colorings.

(i) Define

$$\varphi'(i, j) = \begin{cases} \varphi(u_i), & j \equiv 1 \pmod 2, \\ \varphi(u_i) \bmod k + 1, & j \equiv 0 \pmod 2. \end{cases}$$

This coloring clearly uses only k colors. By the above definition, this coloring constitutes a proper k-coloring regardless of whether H is a path or an even cycle - provided that $1 \not\equiv n \pmod 2$ when H is specified as an even cycle. For any nonisolated vertex (i, j), its neighbors in the G-direction have exactly the same color distribution as the neighbors of u_i in G under φ. Hence, without considering the H-direction, the neighborhood of (i, j) already contains at least two colors occurring an odd number of times. Along the H-direction, each vertex has at most two neighbors. If it has two neighbors along the H-direction, they receive the same color under the construction, so they do not alter the parity of any color counts. If it has only one neighbor along the H-direction, the worst case is that this neighbor's color coincides with one of the odd colors in the G-direction. But since there are at least two odd colors under coloring φ' in the G-direction, at least one odd color remains after adding this neighbor. Therefore, φ' is a valid odd k-coloring, implying $\chi_o(G\square H) \leq \chi_o(G)$, where H is a path or an even cycle.

(ii) By (i), we only need to prove the case of $G\square C_n$, where n is odd. To handle the cyclic closure, we introduce a short offset for the first three layers and then proceed with a 2-periodic alternating pattern. Define

$$\varphi'(i, j) = \begin{cases} (\varphi(u_i) + j - 2) \bmod k + 1, & j \leq 3; \\ \varphi(u_i), & j \geq 4, (j-3) \equiv 1 \pmod 2; \\ \varphi(u_i) \bmod k + 1, & j \geq 4, (j-3) \equiv 0 \pmod 2. \end{cases}$$

Clearly, this coloring uses only k colors. For any nonisolated vertex (i, j), its neighbors in the G-direction already guarantee that at least three colors appear an odd number of times. Along the C_n-direction, each vertex v has exactly two neighbors. The two neighbors are colored by at most two odd colors of v. Because there were at least three odd colors under coloring φ' in the G-direction, at least one odd color remains. Therefore, φ' is a valid odd k-coloring, and we conclude that $\chi_o(G\square C_n) \leq k$. □

Corollary 5.2. *If $m = 3$ and C_n is an even cycle or $m \geq 4$, then we have*

$$\chi_o(K_m\square P_n) = \chi_o(K_m\square C_n) = \chi_o(K_m) = \chi(K_m) = m.$$

Proof. Because every vertex of K_m is adjacent to all the other $m - 1$ vertices, any proper m-coloring φ of K_m is automatically an odd m-coloring. Hence, $\chi_o(K_m) = \chi(K_m) = m$.

Moreover, under φ, each nonisolated vertex has $m - 1$ colors appearing an odd number of times in its neighborhood. Note that $m - 1 \geq 3$ for $m \geq 4$ and $m - 1 = 2$ for $m = 3$. By Lemma 5.1, we have $\chi_o(K_m\square P_n)$, $\chi_o(K_m\square C_n) \leq \chi_o(K_m) = m$ when $m \geq 4$ or $m = 3$ and C_n is an even cycle.

Finally, since both $K_m\square P_n$ and $K_m\square C_n$ contain K_m as an induced subgraph, we must have

$$\chi_o(K_m\square P_n),\ \chi_o(K_m\square C_n) \geq \chi(K_m) = \chi_o(K_m) = m.$$

Combining with the previous inequalities gives $\chi_o(K_m\square P_n) = \chi_o(K_m\square C_n) = \chi_o(K_m)$ when $m = 3$ and C_n is an even cycle or $m \geq 4$. □

6. Conclusions

This paper has systematically determined the exact odd chromatic numbers for fundamental Cartesian products of paths and cycles, specifically $P_m\square P_n$, $C_m\square P_n$, and $C_m\square C_n$. Theorems 1.5–1.7

present a complete characterization, demonstrating that these structured graphs admit significantly tighter bounds than the general multiplicative bound $\chi_o(G) \cdot \chi_o(H)$ established in prior work.

A central contribution of this research is the development of a constructive methodology. We introduced explicit coloring matrices and leveraged modular arithmetic to generate periodic coloring schemes. This approach is not merely an existence proof but provides a practical and extensible algorithm for odd coloring these infinite graph families. The core technical challenge involved ensuring that these periodic patterns form proper odd colorings when the indices wrap around in cycles, a condition we thoroughly analyzed and guaranteed through careful coloring matrix design and parity arguments.

Our results for the toroidal graphs $C_m \square P_n$ and $C_m \square C_n$ are particularly noteworthy, refining the known universal upper bound of 9 for general toroidal graphs down to 5 for this specific Cartesian product family. Furthermore, the facts that all grid graphs $P_m \square P_n$ admit odd 4-coloring and that all graphs $C_m \square P_n$ admit odd 5-coloring provide supporting evidence for the Petruševski-Škrekovski conjecture that all planar graphs are odd 5-colorable.

In summary, this work provides a comprehensive solution to the odd coloring problem for these foundational graph products. The matrix-based constructive framework we established offers a powerful paradigm for analyzing parity constraints in graph colorings and is poised to inspire future investigations into more complex graph products and other variants of proper coloring with local constraints.

Author contributions

Baojie Liu: methodology, formal analysis, investigation, writing original draft; Qihang Dou: conceptualization, methodology, software, validation; Fan Yang: conceptualization, methodology, validation, writing review and editing, supervision, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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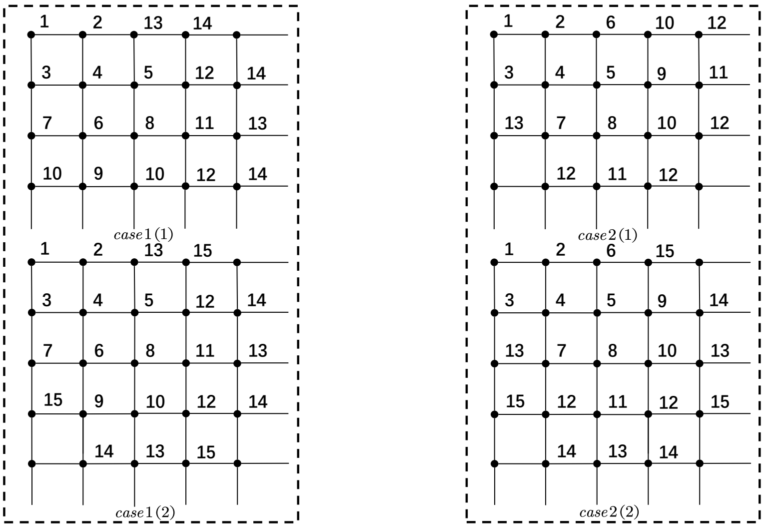
Conflict of interest

The authors declare no conflicts of interest.

Appendix

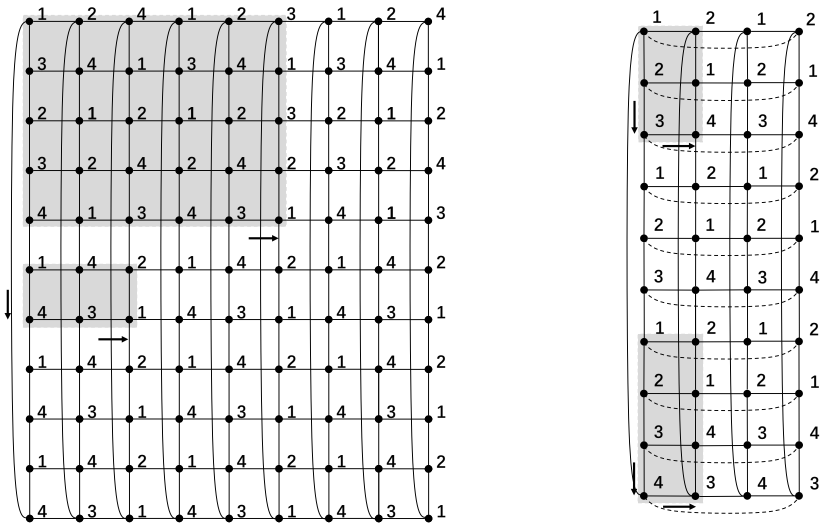
The vertex coloring order of the graph in Lemma 2.5.



(a) The coloring order for Case 1. **(b)** The coloring order for Case 2.

Figure A1. The vertex coloring order of the graph in Lemma 2.5 (ii).

The odd coloring of $C_{11}\square P_9$ and $C_{10}\square C_4$.



(a) The odd coloring of $C_{11}\square P_9$ in Lemma 3.3. **(b)** The odd coloring of $C_{10}\square C_4$ in Lemma 4.2.

Figure A2. The odd 4-coloring of two graphs.

The 9 regions of odd coloring for $C_{2k+1} \square C_{2t+1}$.

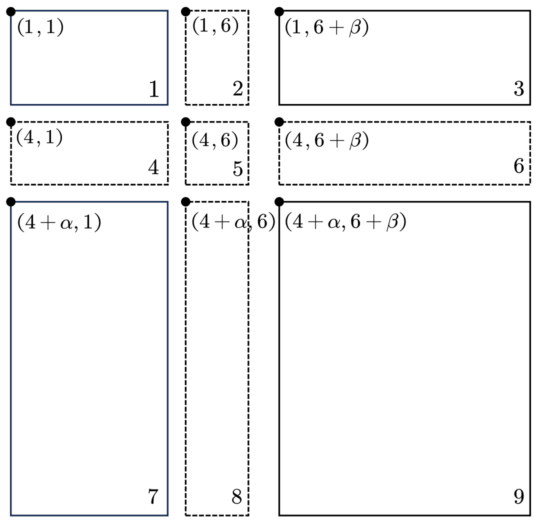
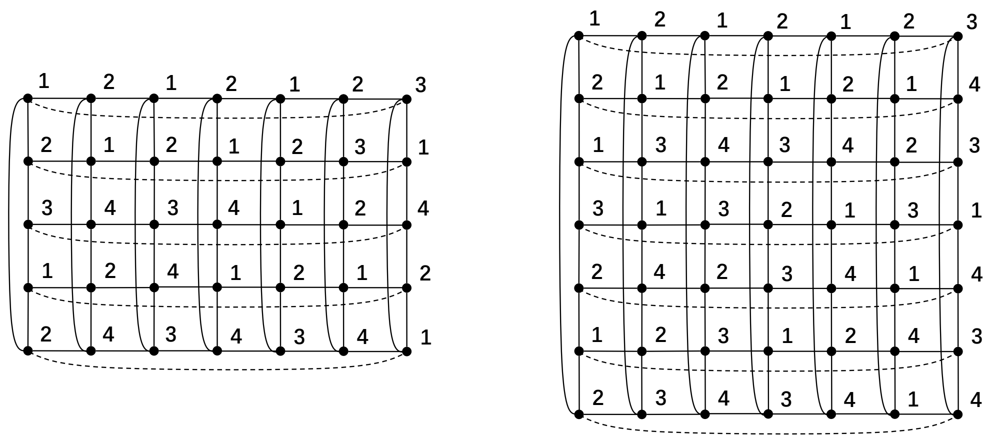


Figure A3. The 9 regions of odd coloring for $C_{2k+1} \square C_{2t+1}$.

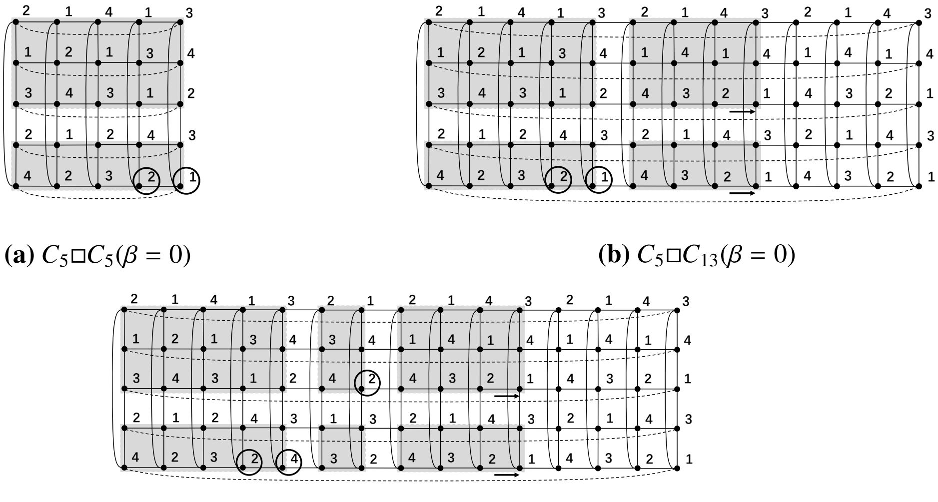
The designed odd coloring schemes for $C_5 \square C_7$ and $C_7 \square C_7$.



(a) $C_5 \square C_7$ **(b)** $C_7 \square C_7$

Figure A4. Odd 4-coloring of $C_5 \square C_7$ and $C_7 \square C_7$.

The odd coloring of three typical cases of $C_5 \square C_{2t+1}$.



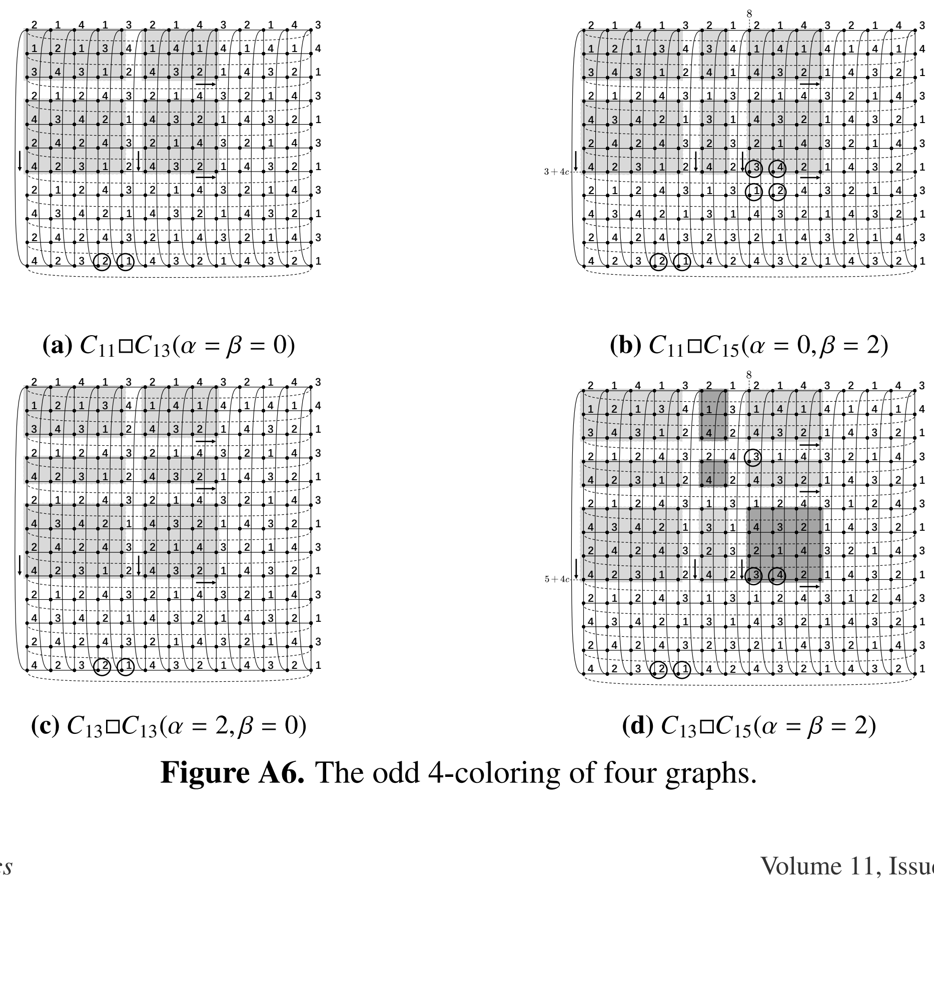
(a) $C_5 \square C_5 (\beta = 0)$

(b) $C_5 \square C_{13} (\beta = 0)$

(c) $C_5 \square C_{15} (\beta = 2)$

Figure A5. The odd 4-coloring of $C_5 \square C_{2t+1}$.

The odd coloring of four typical cases of $C_{2k+1} \square C_{2t+1}$.



(a) $C_{11} \square C_{13} (\alpha = \beta = 0)$

(b) $C_{11} \square C_{15} (\alpha = 0, \beta = 2)$

(c) $C_{13} \square C_{13} (\alpha = 2, \beta = 0)$

(d) $C_{13} \square C_{15} (\alpha = \beta = 2)$

Figure A6. The odd 4-coloring of four graphs.

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