AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1287–1310.
DOI: 10.3934/math.2026055
Received: 20 November 2025
Revised: 05 January 2026
Accepted: 08 January 2026
Published: 16 January 2026

Research article

The wrapped Lindley-exponential distribution: properties and application in circular data appearing in geological context

Hazem Al-Mofleh*

Department of Mathematics, Tafila Technical University, Tafila 66110, Jordan

* **Correspondence:** Email: almof1hm@ttu.edu.jo; Tel: +962795593416.

Abstract: In this article, the circular (wrapped) version of Lindley-exponential (WRLE) distribution was introduced, and its different distributional, statistical, and mathematical properties were studied. The maximum likelihood method was used to estimate the model parameters. Simulation results were used to investigate the performance of the estimates. The importance of the proposed distribution was illustrated over: wrapped Lindley, wrapped exponential, new wrapped exponential, and the wrapped xgamma distributions by means of real directional data set.

Keywords: circular distribution; life distribution; trigonometric moments; Monte-Carlo simulation; axial data
Mathematics Subject Classification: 62E10, 62F12

1. Introduction

Angles measured with respect to a zero direction, either clockwise or anticlockwise in a rotational sense, are typically used to describe two-dimensional directions. *Circular data* [1,2] refers to such two-dimensional directed measurements with a circular representation, namely points on the circumference of a unit circle centered at the origin.

Consequently, a probability distribution of a random angle with its whole probability concentrated on a unit circle's circumference is called a *circular distribution*. By using the "wrapping" technique, any univariate probability distribution can be converted to a circular distribution; such distributions are known as wrapped probability distributions.

In many natural and physical sciences, including biology, medicine, ecology, geology, and others, directional data in two dimensions can be modeled using circular distributions. For instance, studies of bird migrations display the flight paths of recently released birds as they vanish over the horizon; the angle of knee flexion is measured to evaluate orthopedic patients' recuperation, paleocurrents are studied in geology to determine the historical direction of river flow; and wind direction at a specific location during a time period are examined. Additionally, any periodic phenomena with a known period, such as, a day, a month, or a year, can be depicted on a circle whose radius matches this period for a certain individual.

The author in [3] introduced wrapped distributions by applying this method to the Cauchy distribution, thereby proposing the wrapped Cauchy distribution. Since then, a substantial amount of research has been conducted in this area. To name a few, authors in [4] introduced new families of wrapped distributions for modeling skewed circular data, they proposed wrapped exponential and wrapped Laplace distributions; authors in [5, 6] introduced the wrapped weighted exponential and the generalized Gompertz distributions; the work [7] proposed the wrapped variance gamma distribution and applied it to wind direction data; authors in [8] introduced the wrapped Lindley distribution; and authors in [9] presented the wrapped version of xgamma distribution and applied it on directional data.

Wrapping univariate probability distribution is a very useful and effective approach for obtaining a circular representation of an underlying density.

The work [10] proposed a new generalization of the exponential (Ex) distribution, named the Lindley-exponential (LE) distribution. They studied its properties as a special case of the alpha power transformed Ex distribution and examined its suitability for modeling lifetime data sets.

In consideration of the LE distribution's growing significance and popularity, the objective of this article is to introduce the wrapped version of this distribution, investigate its various characteristics, discuss the distribution parameters estimation process, and demonstrate its applicability in modeling circular data. The wrapped version of the LE distribution has one more shape parameter. Compared with the wrapped versions of the Lindley and exponential distributions, the wrapped LE distribution includes an additional shape parameter. This extra parameter allows the wrapped version of the LE distribution more flexibility to model a wide range of directional data sets.

This is how the remainder of the article is structured. Section 2 presents the wrapped version of LE distribution. Section 3 examines several distributional features, including the distribution function, limit behavior, first and second derivatives, modality behavior, characteristic function, and trigonometric moments with related measures. Parameter estimates and a simulation study are covered in Section 4. In Section 5, an application is demonstrated using geological data. Finally, Section 6 concludes significant discoveries and with some directions for future research.

2. The wrapped LE distribution

This section outlines a synthesis procedure for the circular LE distribution based on the wrapping of univariate probability density methodology. The probability density function (PDF) of the LE distribution is provided as follows: If X has the parameters $\alpha(> 0)$ and $\beta(> 0)$ and follows the LE distribution [10],

$$f(x;\boldsymbol{\varphi}) = \frac{\alpha^2}{(1+\alpha)}\left(\beta+\beta^2 x\right)e^{-\alpha\beta x} \quad \text{for} \quad x > 0. \tag{2.1}$$

We denoted it by $X \sim LE(\boldsymbol{\varphi})$, where

$$\boldsymbol{\varphi} = (\alpha,\beta)^{\intercal}.$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x;\boldsymbol{\varphi}) = 1 - \left(1 + \frac{\alpha\beta}{1+\alpha}x\right)e^{-\alpha\beta x} \quad \text{for} \quad x > 0. \tag{2.2}$$

With the definition of the circular (wrapped) random variable as $\theta \equiv X \pmod{2\pi}$, the wrapped LE PDF can be obtained as follows:

$$\begin{aligned} g(\theta;\boldsymbol{\varphi}) &= \sum_{k=0}^{\infty} f(\theta + 2k\pi) \\ &= \frac{\alpha^2\beta e^{-\alpha\beta\theta}}{(1+\alpha)\left(1-e^{-2\pi\alpha\beta}\right)}\left[1 + \beta\theta + \frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}\right], \end{aligned} \tag{2.3}$$

for $\theta \in [0, 2\pi)$, where

$$\boldsymbol{\varphi} = (\alpha, \beta)^\intercal.$$

The wrapped LE distribution is defined as follows.

Definition 2.1. A circular random variable θ; that is defined on $[0, 2\pi)$, is said to have a wrapped LE (*WRLE*) distribution with parameters $(\alpha > 0, \beta > 0)$ if its PDF can be written as

$$g(\theta;\boldsymbol{\varphi}) = \frac{\alpha^2\beta e^{-\alpha\beta\theta}}{(1+\alpha)\left(1-e^{-2\pi\alpha\beta}\right)}\left[1 + \beta\theta + \frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}\right]. \tag{2.4}$$

It is denoted by $\theta \sim WRLE(\boldsymbol{\varphi})$, where

$$\boldsymbol{\varphi} = (\alpha, \beta)^\intercal.$$

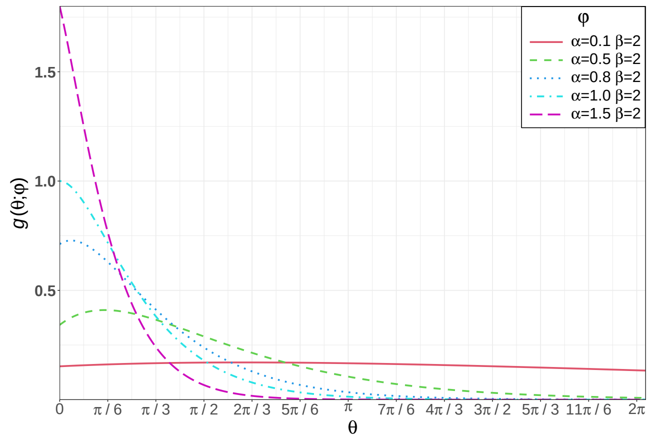
Definition 2.2. Following [1, 2], the CDF of $WRLE(\theta;\boldsymbol{\varphi})$ can be obtained as follows.

$$\begin{aligned} G(\theta;\boldsymbol{\varphi}) &= \sum_{k=0}^{\infty} \left[F(\theta + 2\pi k) - F(2\pi k)\right] \\ &= \frac{1}{1-e^{-2\pi\alpha\beta}}\left[1 - \left(1 + \frac{\alpha\beta\theta}{1+\alpha}\right) e^{-\alpha\beta\theta}\right] + \frac{2\pi\alpha\beta\, e^{-2\pi\alpha\beta}}{(1+\alpha)\left(1-e^{-2\pi\alpha\beta}\right)^2}\left[1 - e^{-\alpha\beta\theta}\right], \end{aligned} \tag{2.5}$$

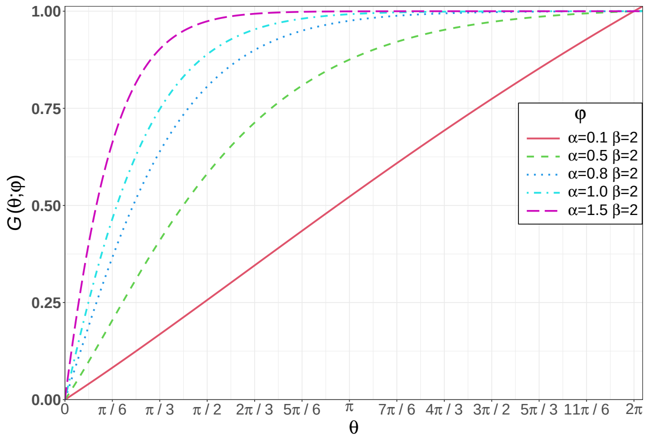
where $\theta \in [0, 2\pi)$ and

$$\boldsymbol{\varphi} = (\alpha, \beta)^\intercal.$$

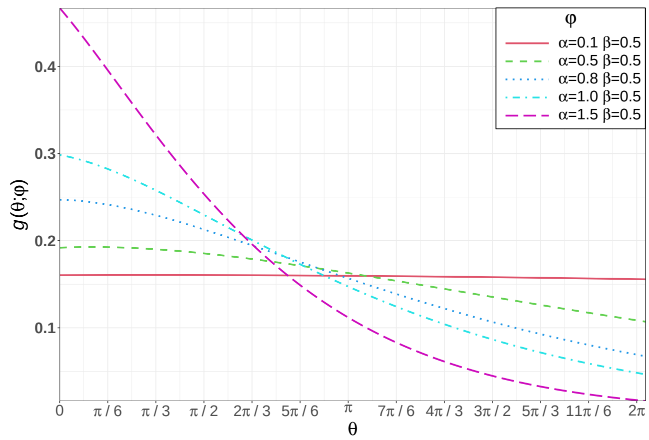
Plots of the unwrapped PDF and CDF of the *WRLE* distribution are displayed in Figure 1, and its wrapped (circular) PDF and CDF are displayed in Figures 2 to 3, for various values of the vector of parameters $\boldsymbol{\varphi}$, respectively. These plots show the *WRLE* distribution is uni-modal and right skewed, also it can be decreasing or increasing-decreasing shapes.



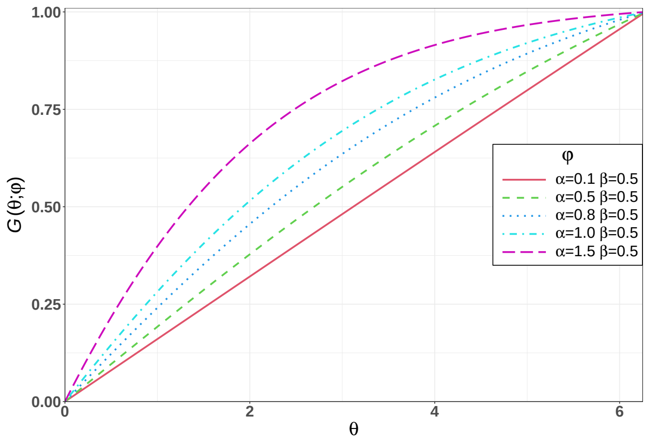
(a) Probability density function



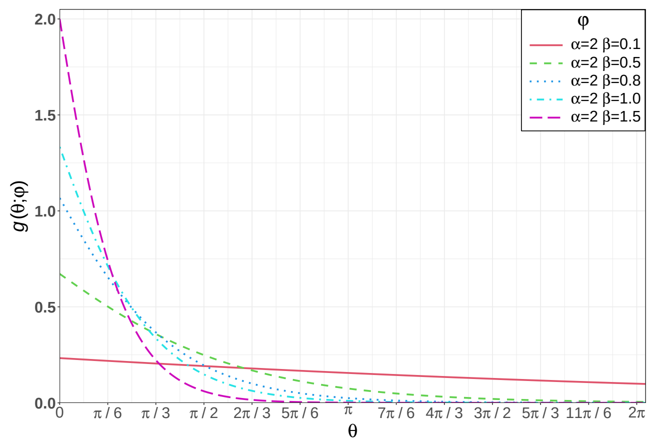
(b) Cumulative distribution function



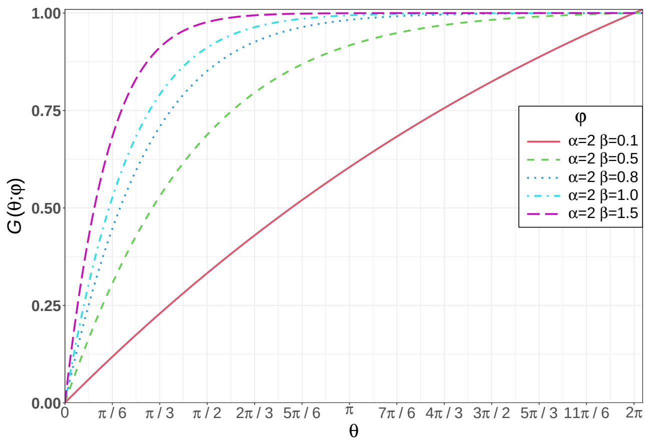
(c) Probability density function



(d) Cumulative distribution function



(e) Probability density function



(f) Cumulative distribution function

Figure 1. Plots of unwrapped PDF and CDF of the $WRLE(\theta;\boldsymbol{\varphi})$ for selected values of the vector of parameters $\boldsymbol{\varphi}$.

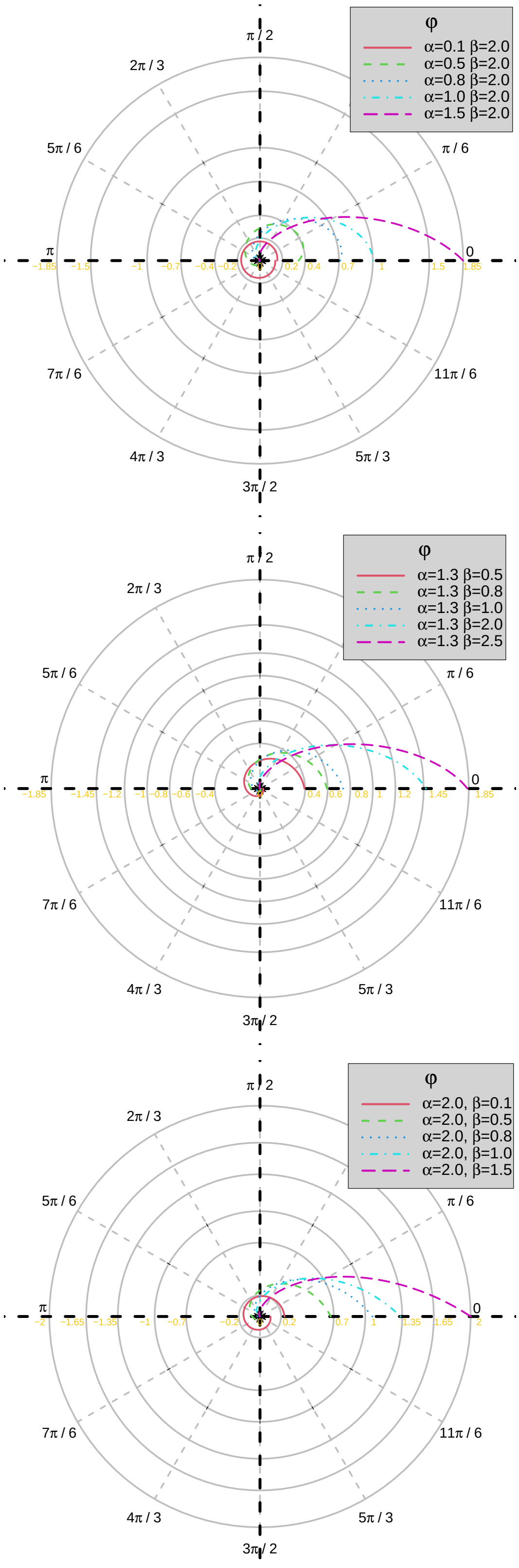


Figure 2. Plots of wrapped (circular) PDF $WRLE(\theta;\boldsymbol{\varphi})$ for different values of the vector of parameters $\boldsymbol{\varphi}$.

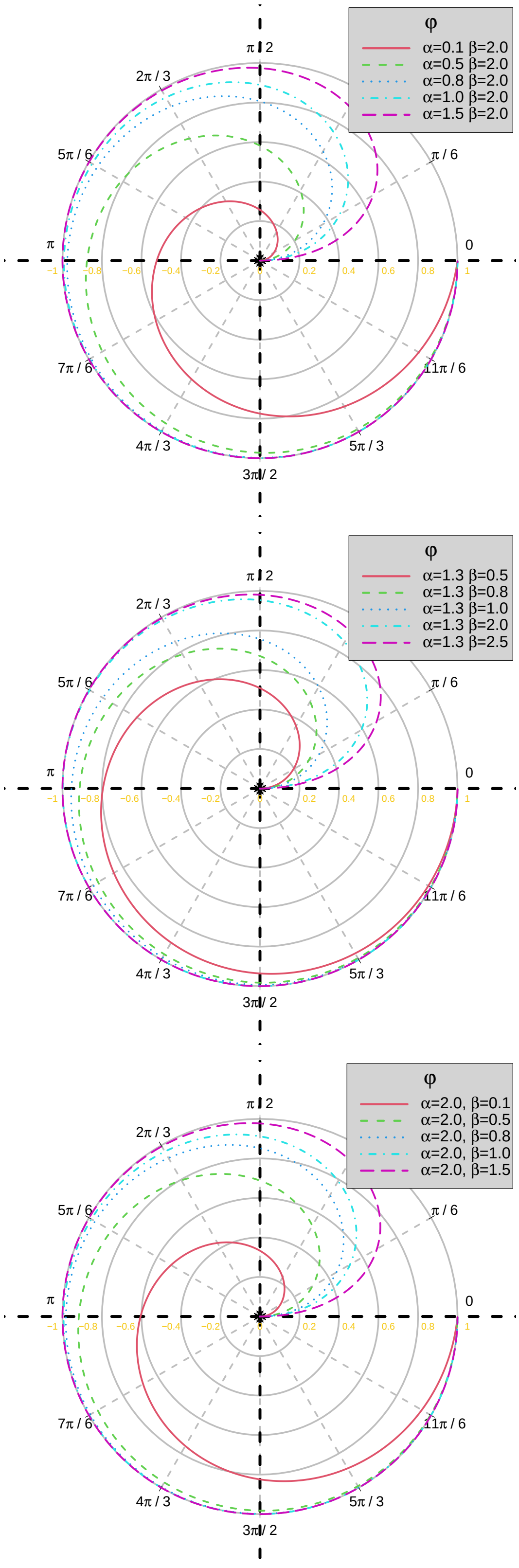


Figure 3. Plots of wrapped (circular) CDF of the $WRLE(\theta;\boldsymbol{\varphi})$ for different values of the vector of parameters $\boldsymbol{\varphi}$.

The *WRLE* density function's survival function (sf), $S(\cdot)$, and hazard rate function (hrf), $h(\cdot)$, for a lifetime random variable t are, respectively, provided by

$$\begin{aligned}S(t;\boldsymbol{\varphi}) &= 1 - G(t;\boldsymbol{\varphi})\\ &= 1 - \frac{1}{1-e^{-2\pi\alpha\beta}}\left[1-\left(1+\frac{\alpha\beta t}{1+\alpha}\right)e^{-\alpha\beta t}\right]\\ &\quad - \frac{2\pi\alpha\beta\, e^{-2\pi\alpha\beta}}{(1+\alpha)(1-e^{-2\pi\alpha\beta})^2}\left[1-e^{-\alpha\beta t}\right],\end{aligned}$$

and

$$\begin{aligned}h(t;\boldsymbol{\varphi}) &= \frac{g(t;\boldsymbol{\varphi})}{S(t;\boldsymbol{\varphi})}\\ &= \frac{\alpha^2\beta e^{-\alpha\beta t}\left[1+\beta t+\frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}\right]}{\left[1+\alpha(1+\beta t)\right]\, e^{-\alpha\beta t} - 2\pi\alpha\beta\frac{e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}(1-e^{-\alpha\beta t}) - (1+\alpha)e^{-2\pi\alpha\beta}},\end{aligned} \tag{2.6}$$

where

$$t \in [0, 2\pi).$$

The cumulative hazard rate function (chrf), $H(\cdot)$, is given by

$$\begin{aligned}H(t;\boldsymbol{\varphi}) &= -\log(S(t;\boldsymbol{\varphi}))\\ &= -\log\left(1-\frac{1}{1-e^{-2\pi\alpha\beta}}\left[1-\left(1+\frac{\alpha\beta t}{1+\alpha}\right)e^{-\alpha\beta t}\right] - \frac{2\pi\alpha\beta\, e^{-2\pi\alpha\beta}}{(1+\alpha)(1-e^{-2\pi\alpha\beta})^2}\left[1-e^{-\alpha\beta t}\right]\right),\end{aligned}$$

where

$$t \in [0, 2\pi).$$

Figure 4 shows hrf plots of the *WRLE* distribution for different values of the vector of parameters $\boldsymbol{\varphi}$. Since the hazard function of a wrapped distribution shows the instantaneous rate of occurrence at a specific angle provided that no event occurred before that angle.

The plots in Figure 4 show how the parameters α and β affect the patterns of hazard rates, with increased value of one of the parameters with fixed value of the other, thus denoting higher instantaneous rates at particular angular locations.

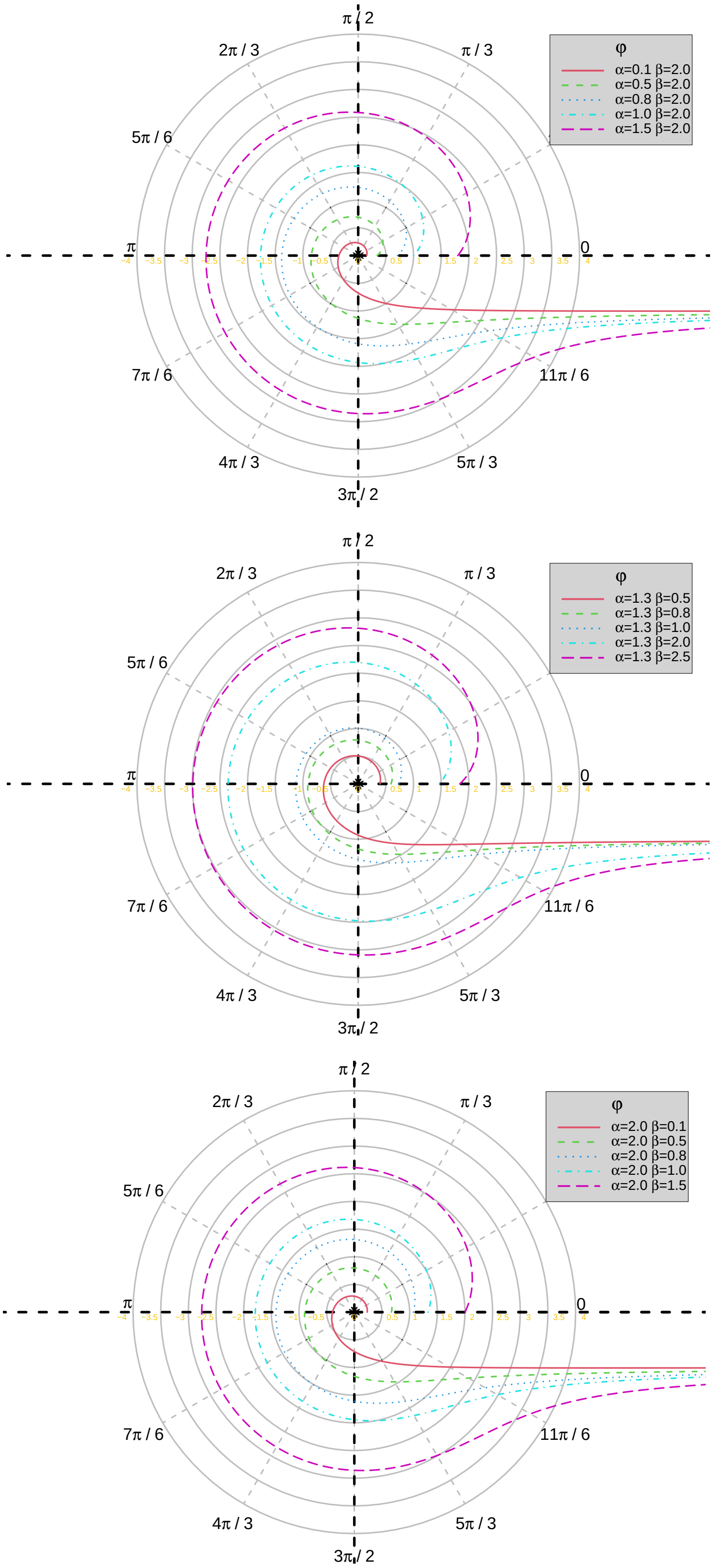


Figure 4. The hrf of the $WRLE(\theta; \boldsymbol{\varphi})$ for different values of the vector of parameters $\boldsymbol{\varphi}$.

3. Distributional and mathematical properties

This section examines several aspects of *WRLE* distribution, including the limit behavior, the first and the second derivatives, the modality, characteristic function, and trigonometric moments.

3.1. The limit behavior and the first and the second derivatives

The limits of PDF (2.3) of the *WRLE* distribution as θ approaches 0^+ and $2\pi^-$ are, respectively,

$$\frac{\alpha^2\beta}{(\alpha+1)\left(1-e^{-2\pi\alpha\beta}\right)^2}\left(1-(1-2\pi\beta)e^{-2\pi\alpha\beta}\right)$$

and

$$\frac{\alpha^2\beta}{(\alpha+1)\left(1-e^{-2\pi\alpha\beta}\right)^2}e^{-2\pi\alpha\beta}\left(1+2\pi\beta-e^{-2\pi\alpha\beta}\right).$$

The limit of hrf (2.6) of the *WRLE* distribution as t approaches $2\pi^-$ is $+\infty$, while its limit as t approaches 0^+ is

$$\frac{\alpha^2\beta}{(\alpha+1)\left(1-e^{-2\pi\alpha\beta}\right)^2}\left(1-(1-2\pi\beta)e^{-2\pi\alpha\beta}\right).$$

The first and the second derivatives of the PDF of the *WRLE* distribution with respect to θ are, respectively, given as

$$g'(\theta)=\frac{d}{d\theta}(g(\theta;\boldsymbol{\varphi}))=\frac{\alpha^2\beta^2e^{-\alpha\beta\theta}}{(\alpha+1)\left(1-e^{-2\pi\alpha\beta}\right)}\left[1-\alpha\left(1+\beta\theta+\frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}\right)\right] \tag{3.1}$$

and

$$g''(\theta)=\frac{d^2}{d\theta^2}(g(\theta;\boldsymbol{\varphi}))=-\frac{\alpha^3\beta^3e^{-\alpha\beta\theta}}{(\alpha+1)\left(1-e^{-2\pi\alpha\beta}\right)}\left[2-\alpha\left(\frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}+\beta\theta+1\right)\right]. \tag{3.2}$$

3.2. Modality behavior

The *WRLE* distribution is a uni-modal distribution. By putting the first derivative (3.1) of the *WRLE* distribution to 0 and solve it to θ, the critical value (unique value) of *WRLE* distribution is obtained and given as

$$\theta_0=\frac{1-\alpha}{\alpha\beta}-\frac{2\pi e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}. \tag{3.3}$$

Now, by substituting θ_0 into the second derivative (3.2), the result is given as

$$g''(\theta_0)=-\frac{\alpha^3\beta^3}{(\alpha+1)\left(e^{2\pi\alpha\beta}-1\right)}e^{\frac{e^{2\pi\alpha\beta}(2\pi\alpha\beta+\alpha-1)-\alpha+1}{e^{2\pi\alpha\beta}-1}},$$

which is negative for any value of α and β. Thus, the value θ_0 in (3.3) is considered the mode of the *WRLE* distribution.

3.3. Characteristic function

According to [2], the trigonometric moments of order p for a wrapped circular distribution, say $\phi^*_\theta(p)$, are equal to the value of the characteristic function of the unwrapped random variable X, say $\phi_X(t)$, at the integer value p, i.e.,

$$\phi^*_\theta(p) = \phi_X(p).$$

The LE distribution's characteristic function is given by

$$\phi_X(t) = E\left(e^{itX}\right) = \frac{\alpha^2\beta}{\alpha+1}(\alpha\beta+\beta-it)(\alpha\beta-it)^{-2}, \quad i = \sqrt{-1},$$

where $t \in \mathbb{R}$ is the argument of the characteristic function.

The wrapped LE distribution's characteristic function is given by

$$\phi^*_\theta(p) = E\left(e^{ip\theta}\right) = \frac{\alpha^2\beta}{\alpha+1}(\alpha\beta+\beta-ip)(\alpha\beta-ip)^{-2},$$

where

$$p = \pm1, \pm2, \cdots.$$

By DeMoivre's formula, any complex number to the power r, say $(a-ib)^r$ for $a, b, r \in R$, can be written as $\left(\sqrt{a^2+b^2}\right)^r e^{ir\tan^{-1}(b/a)}$, so $(\alpha\beta-ip)^{-2}$ and $((\alpha\beta+\beta)-ip)$ can be written as

$$(\alpha\beta-ip)^{-2} = \left(\sqrt{(\alpha\beta)^2+p^2}\right)^{-2} e^{i2\tan^{-1}(p/(\alpha\beta))},$$
$$((\alpha\beta+\beta)-ip) = \sqrt{(\alpha\beta+\beta)^2+p^2}\, e^{-i\tan^{-1}(p/(\alpha\beta+\beta))}.$$

Therefore, $\varphi(p)$ becomes

$$\phi^*_\theta(p) = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)(\alpha^2\beta^2+p^2)}\, e^{i(2\tan^{-1}(p/(\alpha\beta))-\tan^{-1}(p/(\alpha\beta+\beta)))}. \tag{3.4}$$

3.4. Trigonometric moments and related measures

According to [2], we have

$$\phi^*_\theta(p) = \rho_p\, e^{i\mu_p}.$$

We can now draw the following conclusion by comparing the right side of Eq (3.4) with $\rho_p\, e^{i\mu_p}$

$$\rho_p = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)(\alpha^2\beta^2+p^2)} \tag{3.5}$$

and

$$\mu_p = 2\tan^{-1}\left(\frac{p}{\alpha\beta}\right) - \tan^{-1}\left(\frac{p}{\alpha\beta+\beta}\right). \tag{3.6}$$

The mean direction, the resultant length, the circular variance, and the circular standard deviation, respectively, are given by

$$\mu = \mu_1 = 2\tan^{-1}\left(\frac{1}{\alpha\beta}\right) - \tan^{-1}\left(\frac{1}{\alpha\beta+\beta}\right), \tag{3.7}$$

$$\rho = \rho_1 = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)(\alpha^2\beta^2+1)}, \tag{3.8}$$

$$V_0 = 1-\rho_1 = 1 - \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)(\alpha^2\beta^2+1)} \tag{3.9}$$

and

$$\sigma_0 = \sqrt{-2\log(1-V_0)} = \sqrt{-2\log\left(\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)(\alpha^2\beta^2+1)}\right)}. \tag{3.10}$$

Since each complex number can be expressed as $\alpha_p + i\beta_p$ according to the definition of trigonometric moments, so the $\phi^*_\theta(p)$ in (3.4) can be rewritten as

$$\phi^*_\theta(p) = \alpha_p + i\beta_p,$$

where

$$\begin{aligned}\alpha_p &= \rho_p\cos(\mu_p)\\ &= \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)(\alpha^2\beta^2+p^2)}\cos\left\{2\tan^{-1}\left(\frac{p}{\alpha\beta}\right)-\tan^{-1}\left(\frac{p}{\alpha\beta+\beta}\right)\right\}\end{aligned} \tag{3.11}$$

and

$$\beta_p = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)(\alpha^2\beta^2+p^2)}\sin\left\{2\tan^{-1}\left(\frac{p}{\alpha\beta}\right)-\tan^{-1}\left(\frac{p}{\alpha\beta+\beta}\right)\right\}, \tag{3.12}$$

where

$$p = 0, \pm1, \pm2, \cdots.$$

The distribution's non-central trigonometric moments are denoted by α_p and β_p. The formula to define the central trigonometric moments is

$$\bar{\alpha_p} = \rho_p\cos(\mu_p - p\mu_1),$$

as well as

$$\bar{\beta_p} = \rho_p\sin(\mu_p - p\mu_1).$$

Therefore, the corresponding distribution's central trigonometric moments are are given by

$$\bar{\alpha_p} = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)(\alpha^2\beta^2+p^2)}\cos\left(\kappa_{\alpha,\beta,p}\right) \tag{3.13}$$

and

$$\bar{\beta}_p = \frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+p^2}}{(\alpha+1)\left(\alpha^2\beta^2+p^2\right)}\sin\left(\kappa_{\alpha,\beta,p}\right), \tag{3.14}$$

where

$$\kappa_{\alpha,\beta,p} = 2\left\{\tan^{-1}\left(\frac{p}{\alpha\beta}\right) - p\tan^{-1}\left(\frac{1}{\alpha\beta}\right)\right\} - \tan^{-1}\left(\frac{p}{\alpha\beta+\beta}\right) + p\tan^{-1}\left(\frac{1}{\alpha\beta+\beta}\right).$$

The coefficient of skewness,

$$\zeta_1^0 = \frac{\bar{\beta}_2}{V_0^{3/2}},$$

and the coefficient of kurtosis,

$$\zeta_2^0 = \frac{\bar{\alpha}_2 - (1-V_0)^4}{V_0^2},$$

respectively, are given by

$$\zeta_1^0 = \frac{\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+4}}{(\alpha+1)\left(\alpha^2\beta^2+4\right)}\sin\left(\kappa_{\alpha,\beta,2}\right)}{\left(1-\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)\left(\alpha^2\beta^2+1\right)}\right)^{3/2}} \tag{3.15}$$

and

$$\zeta_2^0 = \frac{\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+4}}{(\alpha+1)\left(\alpha^2\beta^2+4\right)}\cos\left(\kappa_{\alpha,\beta,2}\right) - \left(\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)\left(\alpha^2\beta^2+1\right)}\right)^4}{\left(1-\frac{\alpha^2\beta\sqrt{(\alpha\beta+\beta)^2+1}}{(\alpha+1)\left(\alpha^2\beta^2+1\right)}\right)^2}, \tag{3.16}$$

where

$$\kappa_{\alpha,\beta,2} = 2\left\{\tan^{-1}\left(\frac{2}{\alpha\beta}\right) - 2\tan^{-1}\left(\frac{1}{\alpha\beta}\right)\right\} - \tan^{-1}\left(\frac{2}{\alpha\beta+\beta}\right) + 2\tan^{-1}\left(\frac{1}{\alpha\beta+\beta}\right).$$

Table 1 displays the values of these *WRLE* distribution characteristics for different α and β values. It can be concluded from Table 1 that for a fixed α value, ρ increases when β increases, and V_0 decreases when β increases. For a fixed β value, μ, V_0 and σ_0 decrease when α increases, while the ρ increases when α increases.

Table 1. Values of different characteristics of $WRLE$ distribution for varying values of α and β.

Characteristics of $WREL$ distribution			β					
		$\alpha\downarrow$	0.1	0.7	1	2.5	4	8
Mean direction	μ	0.1	1.66036	2.08720	2.20444	2.30286	2.15710	1.67896
		0.5	1.61977	1.70723	1.62629	1.08888	0.76215	0.40682
		1	1.56885	1.29989	1.10715	0.56362	0.36560	0.18629
		3	1.36839	0.54581	0.39852	0.16543	0.10386	0.05205
Resultant length	ρ	0.1	0.00092	0.00799	0.01338	0.06259	0.14145	0.39276
		0.5	0.01681	0.15071	0.24037	0.63106	0.81104	0.94444
		1	0.05049	0.40414	0.55902	0.87914	0.94850	0.98654
		3	0.22232	0.86558	0.92770	0.98743	0.99504	0.99875
Circular variance	V_0	0.1	0.99909	0.99201	0.98662	0.93741	0.85855	0.60725
		0.5	0.98319	0.84930	0.75963	0.36894	0.18897	0.05556
		1	0.94952	0.59586	0.44098	0.12086	0.05150	0.01346
		3	0.77768	0.13442	0.07230	0.01257	0.00496	0.00125
Circular standard deviation	σ_0	0.1	3.74090	3.10782	2.93732	2.35420	1.97779	1.36716
		0.5	1.68854	1.94547	1.68854	0.95953	0.64722	0.33813
		1	1.07850	1.34611	1.07850	0.50756	0.32518	0.16465
		3	0.38742	0.53731	0.38742	0.15904	0.09971	0.04993
Non-central trigonometric moments	α_1	0.1	-0.00008	-0.00395	-0.00792	-0.04184	-0.07826	-0.04240
		0.5	-0.00082	-0.02050	-0.01333	0.29249	0.58667	0.86736
		1	0.00010	0.10815	0.25000	0.74316	0.88581	0.96947
		3	0.04469	0.73982	0.85500	0.97395	0.98968	0.99740
	α_2	0.1	-0.00002	-0.00100	-0.00203	-0.01216	-0.02878	-0.07826
		0.5	-0.00021	-0.00873	-0.01499	0.01157	0.16667	0.58667
		1	0.00001	0.01191	0.04000	0.37180	0.64000	0.88581
		3	0.01125	0.39967	0.58580	0.90262	0.95982	0.98968
	β_1	0.1	0.00091	0.00695	0.01078	0.04656	0.11783	0.39046
		0.5	0.24000	0.14931	0.24000	0.55919	0.56000	0.37370
		1	0.05049	0.38940	0.50000	0.46968	0.33910	0.18272
		3	0.21779	0.44934	0.36000	0.16261	0.10316	0.05196
	β_2	0.1	0.00046	0.00326	0.00476	0.01463	0.03093	0.11783
		0.5	0.00835	0.06333	0.09689	0.31814	0.50000	0.56000
		1	0.02506	0.18993	0.28000	0.54134	0.52000	0.33910
		3	0.11164	0.50549	0.50592	0.30294	0.20051	0.10316
Central trigonometric moments	$\bar{\alpha}_1$	0.1	0.00091	0.00799	0.01338	0.06259	0.14145	0.39276
		0.5	0.01681	0.15071	0.24037	0.63106	0.81104	0.94444
		1	0.05049	0.40414	0.55902	0.87914	0.94850	0.98654
		3	0.22232	0.86558	0.92770	0.98743	0.99504	0.99875
	$\bar{\alpha}_2$	0.1	-0.00006	-0.00228	0.00394	-0.01325	-0.01735	0.05115
		0.5	-0.00061	-0.00866	0.00417	0.25472	0.50721	0.80996
		1	0.00009	0.08774	0.20000	0.64852	0.82364	0.94848
		3	0.03363	0.63283	0.77125	0.95208	0.98054	0.99504
	$\bar{\beta}_1$	0.1	0	0	0	0	0	0
		0.5	0	0	0	0	0	0
		1	0	0	0	0	0	0
		3	0	0	0	0	0	0
	$\bar{\beta}_2$	0.1	-0.00045	-0.00253	-0.00336	0.01365	-0.03852	-0.13188
		0.5	-0.00833	-0.06334	-0.09795	-0.19097	-0.14324	-0.04174
		1	-0.02506	-0.16887	-0.20000	-0.10350	-0.04030	-0.00662
		3	-0.10704	-0.12161	-0.06547	-0.00672	-0.00175	-0.00023
Coefficient of skewness	ζ_1^0	0.1	-0.00045	-0.00256	-0.00343	-0.01504	-0.04842	-0.27869
		0.5	-0.00854	-0.08092	-0.14795	-0.85218	-1.74382	-3.18696
		1	-0.02709	-0.36714	-0.68296	-2.46333	-3.44810	-4.23800
		3	-0.15609	-2.46769	-3.36770	-4.77113	-5.01220	-5.13680
Coefficient of kurtosis	ζ_2^0	0.1	-0.00006	-0.00232	-0.00404	-0.01510	-0.02408	0.07418
		0.5	-0.00063	-0.01272	0.00144	0.70617	2.08738	4.65150
		1	0.00010	0.17199	0.52628	3.50263	5.37620	6.91590
		3	0.05157	3.95590	5.84880	8.88363	9.41040	9.68300

4. Parameter estimation

This section examines the estimating of the unknown parameter of the *WRLE* distribution using the maximum likelihood approach. Let $\theta_1, \theta_2, \cdots, \theta_n$ be a random sample of size n from the $WRLE(\boldsymbol{\varphi})$ distribution. The log-likelihood function for $\boldsymbol{\varphi}$ can be written as

$$\ell(\boldsymbol{\varphi}) = n\log\left(\frac{\alpha^2\beta}{\alpha+1}\right) - \alpha\beta\sum_{i=1}^{n}\theta_i - n\log\left(1-e^{-2\pi\alpha\beta}\right) + \sum_{i=1}^{n}\log\left(1+\beta\theta_i+\frac{2\pi\beta e^{-2\pi\alpha\beta}}{1-e^{-2\pi\alpha\beta}}\right), \tag{4.1}$$

where $\boldsymbol{\varphi} = (\alpha, \beta)^{\intercal}$.

The following equations are obtained by partially differentiating Eq (4.1) with respect to α and β, respectively

$$\frac{\partial\ell(\boldsymbol{\varphi})}{\partial\alpha} = \frac{n(\alpha+2)}{\alpha(\alpha+1)} - \frac{2n\pi\beta}{e^{2\pi\alpha\beta}-1} - n\beta\bar{\theta} - 4\pi^2\beta^2\sum_{i=1}^{n}\left\{\frac{1+\frac{1}{e^{2\pi\alpha\beta}-1}}{2\pi\beta+(\beta\theta_i+1)(e^{2\pi\alpha\beta}-1)}\right\}$$

and

$$\frac{\partial\ell(\boldsymbol{\varphi})}{\partial\beta} = \frac{n}{\beta} - \frac{2\pi\alpha n}{e^{2\pi\alpha\beta}-1} - n\alpha\bar{\theta} - 2\pi e^{-2\pi\alpha\beta}\sum_{i=1}^{n}\frac{2\pi\alpha\beta-\left(e^{2\pi\alpha\beta}-1\right)\theta_i - 1 + \frac{2\pi\alpha\beta}{e^{2\pi\alpha\beta}-1}}{2\pi\beta e^{-2\pi\alpha\beta}+\left(1-e^{-2\pi\alpha\beta}\right)(\beta\theta_i+1)}.$$

By setting the nonlinear system of Eq (4.1) as

$$\partial\ell(\boldsymbol{\varphi})/\partial\alpha = 0 \quad \text{and} \quad \partial\ell(\boldsymbol{\varphi})/\partial\beta = 0.$$

The maximum likelihood estimators (MLEs) of the unknown parameters α and β of the WRLE distribution can be found by solving them simultaneously; this can be accomplished by using various programs, such as `R`, `SAS`, and `Python`. This system of nonlinear equations cannot be solved analytically, but it can be solved numerically using iterative methods like the Newton-Raphson algorithm by mathematical or statistical software.

4.1. A simulation study

In order to examine the behavior of the maximum likelihood estimators and the performance of the WRLE distribution, 10,000 samples from the WRLE distribution with sample sizes of $n = \{30, 50, 80, 100, 120, 200\}$ were created for each parameter's combination from $\alpha = \{0.5, 0.7, 0.8, 1.0, 2.5, 8.0\}$ and $\beta = \{0.7, 1.0, 2.5, 4.0\}$. For each parameter's combination and each sample, the following measures have been calculated: The average of absolute value of biases ($|Bias(\widehat{\boldsymbol{\varphi}})|$):

$$|Bias(\widehat{\boldsymbol{\varphi}})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}|,$$

the mean square error of the estimates ($MSE(\widehat{\boldsymbol{\varphi}})$),

$$MSE(\widehat{\boldsymbol{\varphi}}) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi})^2,$$

and the mean relative estimates ($MRE(\widehat{\varphi})$)

$$MRE(\widehat{\varphi}) = \frac{1}{N}\sum_{i=1}^{N} |\widehat{\varphi} - \varphi|/\varphi.$$

R software (version 4.5.1) [11] is used for the simulation study's computations. The MLEs' $|Bias(\widehat{\varphi})|$, $MSE(\widehat{\varphi})$, and $MRE(\widehat{\varphi})$ are shown in Tables 2–4.

Table 2. Result of simulation study showing $|Bias(\widehat{\varphi})|$, MSE, and MRE for varying φ and n.

α	β	n	$\lvert Bias(\widehat{\alpha})\rvert$	$\lvert Bias(\widehat{\beta})\rvert$	$MSE(\widehat{\alpha})$	$MSE(\widehat{\beta})$	$MRE(\widehat{\alpha})$	$MRE(\widehat{\beta})$
0.5	0.7	30	0.06678	0.04728	0.01358	0.00677	0.13355	0.06755
		50	0.05229	0.03711	0.00833	0.00418	0.10459	0.05302
		80	0.04376	0.03109	0.00573	0.00288	0.08752	0.04441
		100	0.03987	0.02834	0.00472	0.00238	0.07973	0.04048
		120	0.03692	0.02625	0.00405	0.00204	0.07384	0.03750
		200	0.02814	0.02003	0.00250	0.00126	0.05627	0.02861
	1.0	30	0.05805	0.02876	0.01081	0.00264	0.11610	0.02876
		50	0.04668	0.02318	0.00671	0.00165	0.09337	0.02318
		80	0.03676	0.01828	0.00427	0.00105	0.07352	0.01828
		100	0.03291	0.01638	0.00342	0.00085	0.06583	0.01638
		120	0.02910	0.01449	0.00281	0.00069	0.05821	0.01449
		200	0.02353	0.01173	0.00172	0.00043	0.04706	0.01173
	2.5	30	0.04650	0.00925	0.00749	0.00030	0.09299	0.00370
		50	0.03486	0.00695	0.00418	0.00017	0.06972	0.00278
		80	0.02633	0.00525	0.00231	0.00009	0.05266	0.00210
		100	0.02338	0.00466	0.00181	0.00007	0.04675	0.00187
		120	0.02123	0.00424	0.00151	0.00006	0.04246	0.00169
		200	0.01578	0.00315	0.00083	0.00003	0.03156	0.00126
	4.0	30	0.04835	0.00602	0.00826	0.00013	0.09670	0.00151
		50	0.03471	0.00433	0.00417	0.00006	0.06941	0.00108
		80	0.02646	0.00330	0.00237	0.00004	0.05292	0.00083
		100	0.02353	0.00294	0.00190	0.00003	0.04707	0.00073
		120	0.02113	0.00264	0.00151	0.00002	0.04225	0.00066
		200	0.01557	0.00194	0.00082	0.00001	0.03115	0.00049
0.7	0.7	30	0.05287	0.05232	0.00879	0.00856	0.07553	0.07474
		50	0.04205	0.04172	0.00547	0.00536	0.06008	0.05960
		80	0.03262	0.03242	0.00334	0.00329	0.04660	0.04631
		100	0.03013	0.02996	0.00282	0.00278	0.04304	0.04280
		120	0.02739	0.02725	0.00239	0.00236	0.03912	0.03892
		200	0.02142	0.02133	0.00147	0.00145	0.03059	0.03047
	1.0	30	0.05316	0.03690	0.00940	0.00450	0.07595	0.03690
		50	0.04222	0.02936	0.00562	0.00271	0.06032	0.02936
		80	0.03221	0.02243	0.00331	0.00160	0.04602	0.02243
		100	0.02959	0.02061	0.00285	0.00138	0.04227	0.02061
		120	0.02664	0.01857	0.00224	0.00109	0.03806	0.01857
		200	0.02050	0.01430	0.00135	0.00066	0.02929	0.01430
	2.5	30	0.06345	0.01769	0.01381	0.00107	0.09064	0.00708
		50	0.04642	0.01296	0.00721	0.00056	0.06631	0.00518
		80	0.03464	0.00968	0.00410	0.00032	0.04948	0.00387
		100	0.03140	0.00877	0.00329	0.00026	0.04485	0.00351
		120	0.02807	0.00784	0.00263	0.00021	0.04010	0.00314
		200	0.02107	0.00589	0.00149	0.00012	0.03010	0.00236
	4.0	30	0.06591	0.01151	0.01550	0.00047	0.09416	0.00288
		50	0.04972	0.00868	0.00834	0.00025	0.07102	0.00217
		80	0.03646	0.00637	0.00452	0.00014	0.05208	0.00159
		100	0.03205	0.00560	0.00347	0.00011	0.04579	0.00140
		120	0.02955	0.00516	0.00292	0.00009	0.04221	0.00129
		200	0.02202	0.00385	0.00163	0.00005	0.03146	0.00096

Table 3. Result of simulation study showing $|Bias(\widehat{\boldsymbol{\varphi}})|$, MSE, and MRE for varying $\boldsymbol{\varphi}$ and n.

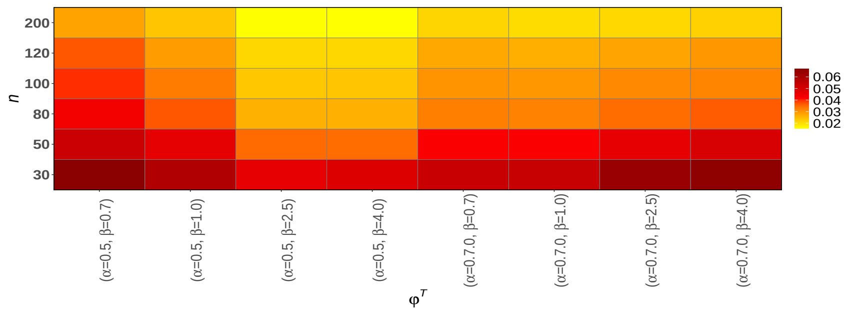
α	β	n	$\lvert Bias(\widehat{\alpha})\rvert$	$\lvert Bias(\widehat{\beta})\rvert$	$MSE(\widehat{\alpha})$	$MSE(\widehat{\beta})$	$MRE(\widehat{\alpha})$	$MRE(\widehat{\beta})$
0.8	0.7	30	0.04795	0.05426	0.00729	0.00928	0.05994	0.07751
		50	0.03755	0.04258	0.00443	0.00567	0.04693	0.06083
		80	0.02902	0.03297	0.00269	0.00346	0.03628	0.04710
		100	0.02629	0.02988	0.00224	0.00288	0.03286	0.04269
		120	0.02420	0.02752	0.00186	0.00240	0.03024	0.03931
		200	0.01901	0.02165	0.00115	0.00149	0.02377	0.03093
	1.0	30	0.05304	0.04209	0.00926	0.00580	0.06629	0.04209
		50	0.04126	0.03280	0.00548	0.00345	0.05157	0.03280
		80	0.03117	0.02482	0.00317	0.00200	0.03896	0.02482
		100	0.02834	0.02257	0.00263	0.00166	0.03543	0.02257
		120	0.02524	0.02011	0.00211	0.00134	0.03155	0.02011
		200	0.01938	0.01546	0.00122	0.00078	0.02422	0.01546
	2.5	30	0.07297	0.02327	0.01815	0.00184	0.09121	0.00931
		50	0.05169	0.01650	0.00914	0.00093	0.06461	0.00660
		80	0.03973	0.01269	0.00533	0.00054	0.04966	0.00507
		100	0.03557	0.01136	0.00430	0.00044	0.04446	0.00454
		120	0.03174	0.01014	0.00342	0.00035	0.03967	0.00406
		200	0.02339	0.00747	0.00187	0.00019	0.02924	0.00299
	4.0	30	0.07581	0.01513	0.02013	0.00080	0.09476	0.00378
		50	0.05616	0.01121	0.01086	0.00043	0.07020	0.00280
		80	0.04251	0.00849	0.00619	0.00025	0.05314	0.00212
		100	0.03687	0.00736	0.00466	0.00019	0.04608	0.00184
		120	0.03334	0.00666	0.00388	0.00015	0.04168	0.00167
		200	0.02494	0.00498	0.00211	0.00008	0.03118	0.00125
1.0	0.7	30	0.04118	0.05832	0.00548	0.01093	0.04118	0.08331
		50	0.03163	0.04487	0.00323	0.00648	0.03163	0.06410
		80	0.02467	0.03505	0.00196	0.00395	0.02467	0.05007
		100	0.02217	0.03152	0.00161	0.00324	0.02217	0.04503
		120	0.01993	0.02835	0.00128	0.00259	0.01993	0.04049
		200	0.01566	0.02229	0.00079	0.00159	0.01566	0.03185
	1.0	30	0.05153	0.05118	0.00890	0.00874	0.05153	0.05118
		50	0.03889	0.03869	0.00502	0.00495	0.03889	0.03869
		80	0.03037	0.03024	0.00306	0.00303	0.03037	0.03024
		100	0.02813	0.02803	0.00257	0.00254	0.02813	0.02803
		120	0.02508	0.02499	0.00206	0.00204	0.02508	0.02499
		200	0.01857	0.01852	0.00114	0.00113	0.01857	0.01852
	2.5	30	0.08353	0.03332	0.02500	0.00397	0.08353	0.01333
		50	0.06513	0.02599	0.01432	0.00228	0.06513	0.01040
		80	0.04846	0.01935	0.00786	0.00125	0.04846	0.00774
		100	0.03987	0.01593	0.00544	0.00087	0.03987	0.00637
		120	0.03775	0.01508	0.00492	0.00078	0.03775	0.00603
		200	0.02958	0.01182	0.00289	0.00046	0.02958	0.00473
	4.0	30	0.09608	0.02398	0.03258	0.00203	0.09608	0.00599
		50	0.06745	0.01684	0.01622	0.00101	0.06745	0.00421
		80	0.05019	0.01254	0.00869	0.00054	0.05019	0.00313
		100	0.04527	0.01131	0.00715	0.00045	0.04527	0.00283
		120	0.04242	0.01059	0.00603	0.00038	0.04242	0.00265
		200	0.03138	0.00784	0.00338	0.00021	0.03138	0.00196

Table 4. Result of simulation study showing $|Bias(\widehat{\boldsymbol{\varphi}})|$, MSE, and MRE for varying $\boldsymbol{\varphi}$ and n.

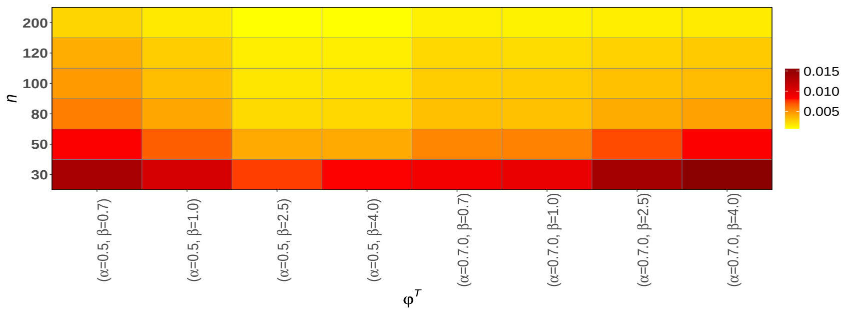
α	β	n	$\lvert Bias(\widehat{\alpha})\rvert$	$\lvert Bias(\widehat{\beta})\rvert$	$MSE(\widehat{\alpha})$	$MSE(\widehat{\beta})$	$MRE(\widehat{\alpha})$	$MRE(\widehat{\beta})$
2.5	0.7	30	0.02341	0.08329	0.00196	0.02470	0.00936	0.11899
		50	0.01690	0.06020	0.00100	0.01261	0.00676	0.08599
		80	0.01288	0.04592	0.00056	0.00712	0.00515	0.06560
		100	0.01137	0.04053	0.00044	0.00561	0.00455	0.05790
		120	0.00987	0.03520	0.00034	0.00426	0.00395	0.05028
		200	0.00758	0.02704	0.00019	0.00247	0.00303	0.03863
	1.0	30	0.04189	0.10446	0.00608	0.03774	0.01675	0.10446
		50	0.03087	0.07704	0.00320	0.01991	0.01235	0.07704
		80	0.02366	0.05906	0.00188	0.01171	0.00946	0.05906
		100	0.02043	0.05100	0.00142	0.00882	0.00817	0.05100
		120	0.01862	0.04649	0.00116	0.00720	0.00745	0.04649
		200	0.01419	0.03543	0.00068	0.00423	0.00567	0.03543
	2.5	30	0.13738	0.13725	0.05939	0.05925	0.05495	0.05490
		50	0.09762	0.09755	0.03206	0.03200	0.03905	0.03902
		80	0.07508	0.07503	0.01925	0.01922	0.03003	0.03001
		100	0.06622	0.06618	0.01511	0.01509	0.02649	0.02647
		120	0.06075	0.06072	0.01230	0.01229	0.02430	0.02429
		200	0.04537	0.04535	0.00719	0.00718	0.01815	0.01814
	4.0	30	0.18607	0.11641	0.10965	0.04304	0.07443	0.02910
		50	0.13828	0.08639	0.06446	0.02516	0.05531	0.02160
		80	0.10382	0.06487	0.03704	0.01445	0.04153	0.01622
		100	0.09263	0.05788	0.03022	0.01179	0.03705	0.01447
		120	0.08409	0.05254	0.02419	0.00944	0.03363	0.01314
		200	0.06318	0.03948	0.01383	0.00540	0.02527	0.00987
8.0	0.7	30	0.00851	0.09718	0.00026	0.03409	0.00106	0.13883
		50	0.00611	0.06975	0.00013	0.01712	0.00076	0.09965
		80	0.00456	0.05210	0.00007	0.00948	0.00057	0.07443
		100	0.00395	0.04508	0.00005	0.00714	0.00049	0.06440
		120	0.00351	0.04005	0.00004	0.00552	0.00044	0.05721
		200	0.00266	0.03038	0.00002	0.00317	0.00033	0.04341
	1.0	30	0.01699	0.13575	0.00103	0.06569	0.00212	0.13575
		50	0.01207	0.09654	0.00052	0.03337	0.00151	0.09654
		80	0.00910	0.07275	0.00029	0.01876	0.00114	0.07275
		100	0.00759	0.06073	0.00021	0.01326	0.00095	0.06073
		120	0.00691	0.05527	0.00017	0.01108	0.00086	0.05527
		200	0.00509	0.04075	0.00009	0.00600	0.00064	0.04075
	2.5	30	0.07986	0.25271	0.01988	0.19512	0.00998	0.10108
		50	0.06219	0.19832	0.01263	0.12765	0.00777	0.07933
		80	0.04632	0.14810	0.00748	0.07642	0.00579	0.05924
		100	0.04028	0.12878	0.00579	0.05918	0.00503	0.05151
		120	0.03710	0.11861	0.00490	0.05006	0.00464	0.04745
		200	0.02574	0.08227	0.00241	0.02461	0.00322	0.03291
	4.0	30	0.15492	0.30401	0.07330	0.27585	0.01937	0.07600
		50	0.12102	0.24014	0.04592	0.17879	0.01513	0.06004
		80	0.09422	0.18788	0.02903	0.11508	0.01178	0.04697
		100	0.08224	0.16411	0.02335	0.09284	0.01028	0.04103
		120	0.07258	0.14493	0.01827	0.07286	0.00907	0.03623
		200	0.05536	0.11054	0.01088	0.04343	0.00692	0.02764

It is evident that as sample size increases, the values of $|Bias(\widehat{\boldsymbol{\varphi}})|$, $MSE(\widehat{\boldsymbol{\varphi}})$, and $MRE(\widehat{\boldsymbol{\varphi}})$ decrease.

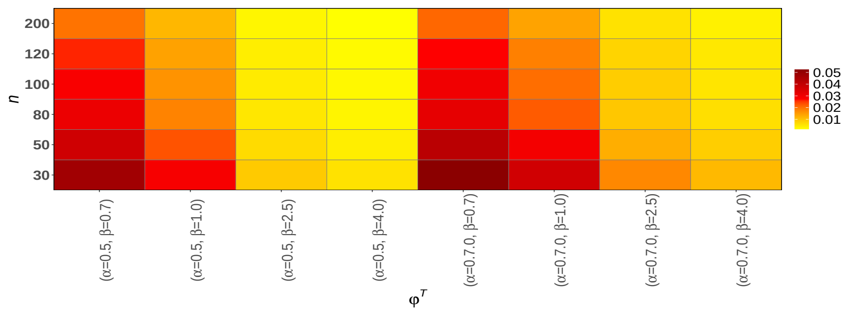
We can conclude from the results in Tables 2–4 that the consistency property is maintained for all parameter combinations by the ML estimation method. In addition, Figures 5–7 illustrate the heat maps associated with each parameter.



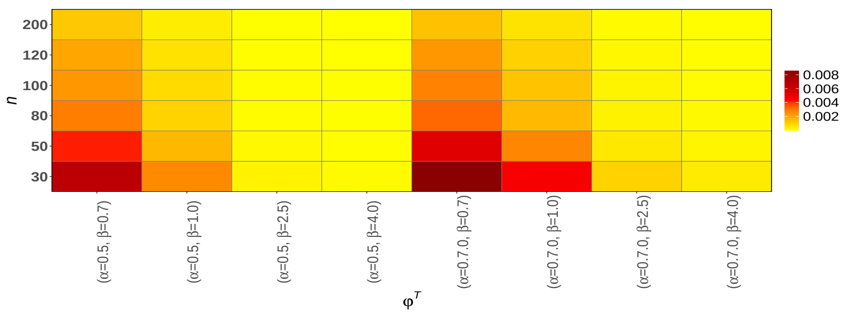
(a) Absolute value of biases of $\hat{\alpha}$.



(b) MSEs of $\hat{\alpha}$.

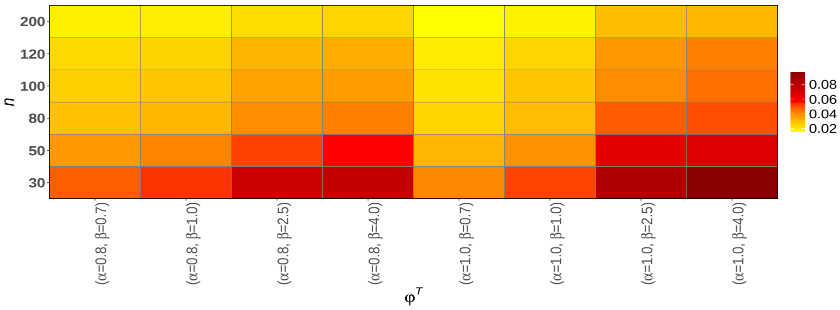


(c) Absolute value of biases of $\hat{\beta}$.

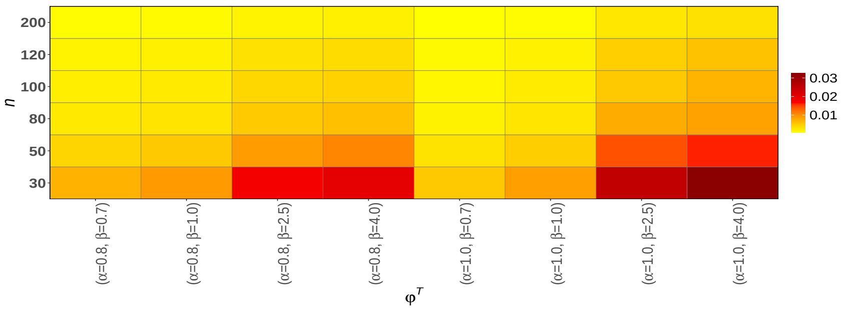


(d) MSEs of $\hat{\beta}$.

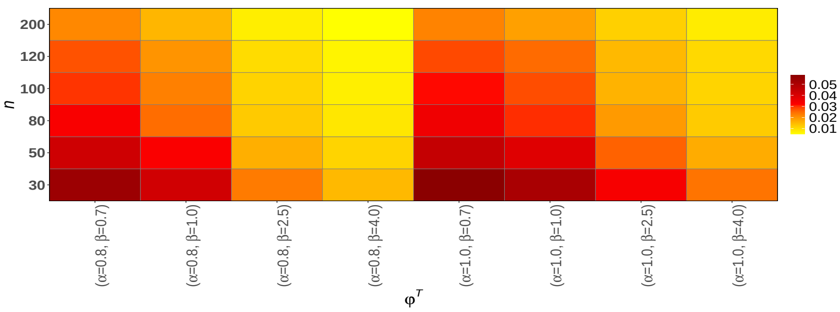
Figure 5. The heatmaps of the simulated biases and MSE of the MLE simulation method for various true values α and β.



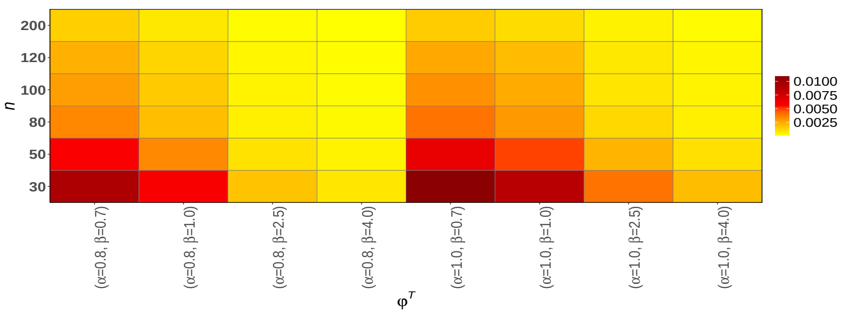
(a) Absolute value of biases of $\hat{\alpha}$.



(b) MSEs of $\hat{\alpha}$.

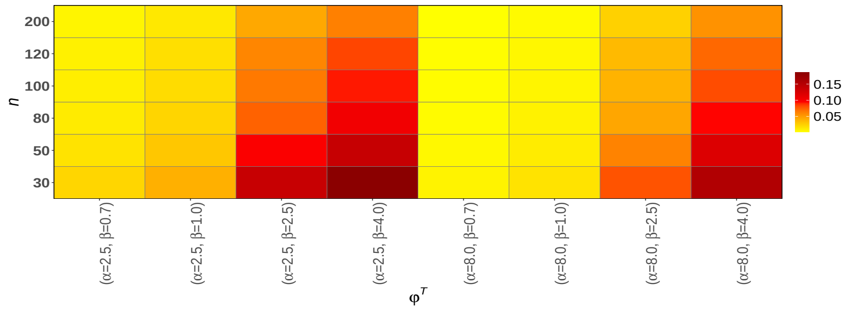


(c) Absolute value of biases of $\hat{\beta}$.

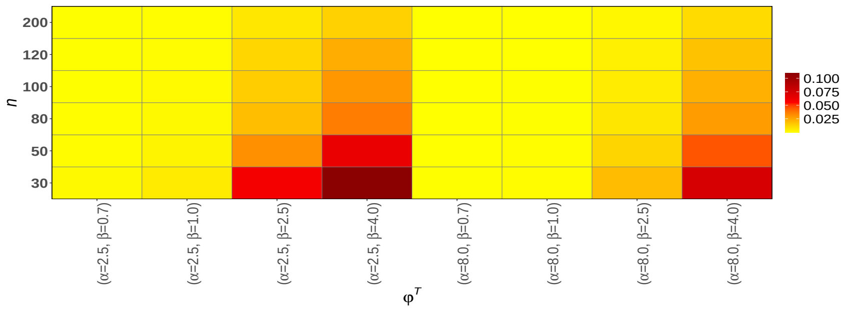


(d) MSEs of $\hat{\beta}$.

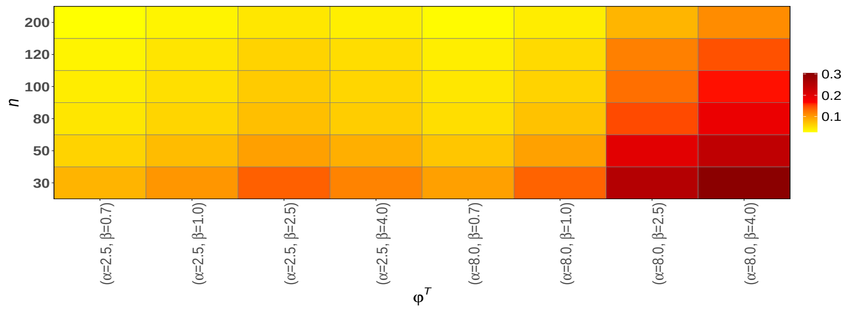
Figure 6. The heatmaps of the simulated biases and MSE of the MLE simulation method for various true values α and β.



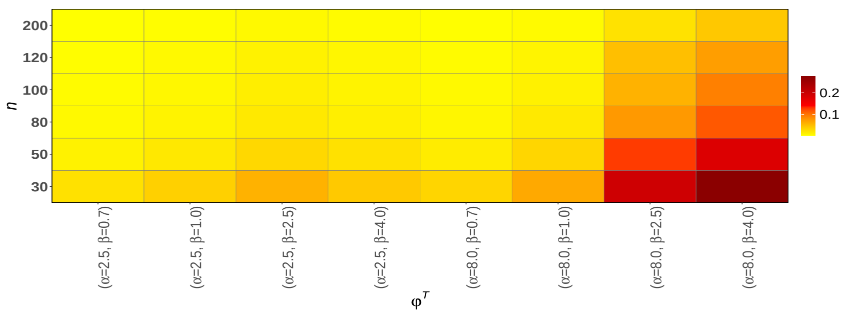
(a) Absolute value of biases of $\hat{\alpha}$.



(b) MSEs of $\hat{\alpha}$.



(c) Absolute value of biases of $\hat{\beta}$.



(d) MSEs of $\hat{\beta}$.

Figure 7. The heatmaps of the simulated biases and MSE of the MLE simulation method for various true values α and β.

5. Application

This section covers the use of the *WRLE* model to demonstrate how the model behaves on a real dataset. Wind direction data [12] has been analyzed to demonstrate how the *WRLE* distribution can be utilized with real datasets. This dataset is available within the website: `https://arcticdata.io/catalog/view/doi:10.5065/D6R49NXH` with name "seabasin200301". This data represents the daily-mean unit vector of NNR daily wind direction at 10 meters (recorded in degrees) from the period *January* 2003 at Bering Strait between the Pacific and Arctic oceans.

For $WRLE(\theta; \boldsymbol{\varphi})$ and four distribution, namely: wrapped Lindley distribution $WL(\theta; \alpha)$ [8], wrapped xgamma distribution $WRXG(\theta; \alpha)$ [9], new wrapped Ex distribution $NWE(\theta; \boldsymbol{\varphi})$ [13] and wrapped Ex distribution $WE(\theta; \alpha)$ [4], the ML estimation and corresponding standard error of the parameters are obtained for the seabasin200301 data, these results are given in Table 5. Table 6 shows the seabasin200301 data resultant length and mean direction, and the resultant length when the dataset is modeled with the aforementioned distributions. From Table 6 it can be concluded that the resultant length and mean direction with *WRLE* model are close to the sample resultant length and sample mean direction of the seabasin200301 data.

Table 5. MLEs and their standard errors (in parentheses) for seabasin200301 dataset.

The model	$\hat{\alpha}$	$\hat{\beta}$
WRLE	0.01137	118.05191
	(0.01102)	(115.23805)
NWRE	0.99326	-0.99999
	(0.37037)	(1.60973)
WRL	0.98876	—
	(0.14686)	—
WRE	0.62567	—
	(0.13582)	—
WRXG	1.29020	—
	0.18016	—

Table 6. Mean direction and Resultant length for seabasin200301 data and the given models.

	Resultant length	Mean direction
seabasin200301 data	0.77121	1.18505 (~ 67.89860°)
WRLE Model	**0.64310**	**1.27222** (~ **72.89256**°)
NWRE Model	0.62946	2.34390 (~ 134.29550°)
WRL Model	0.55333	1.11619 (~ 63.95307°)
WRE Model	0.53041	1.01172 (~ 57.96721°)
WRXG Model	0.54096	1.05560 (~ 60.48138°)

Figures 8 and 9 provide the plots of the circular data, rose diagram, fitted PDF, estimated densities and estimated cumulative of the fitted *WRLE*, *WRL*, *WRE*, *WRXG* and *NWRE* models for the given dataset.

The dashed arrow in Figure 8 indicates the mean direction vector of the fitted *WRLE* distribution, whereas the solid arrow indicates the sample mean resultant vector.

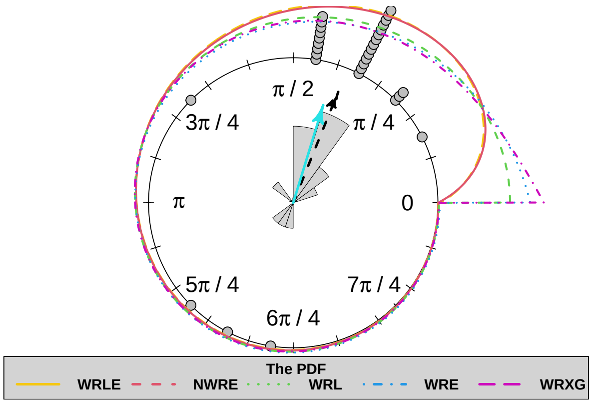
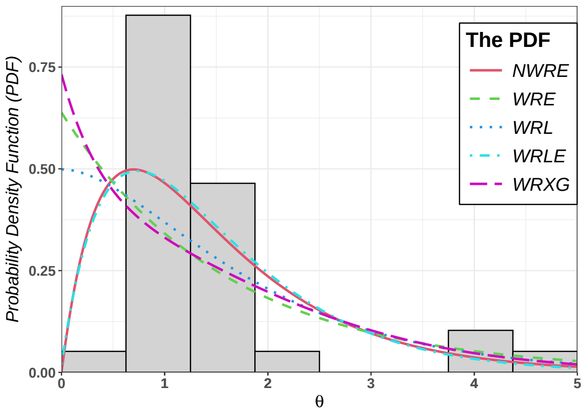
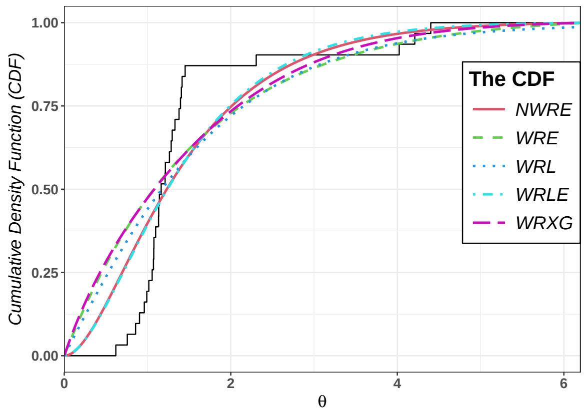


Figure 8. Plots for seabasin200301 data. Circular data plot, fitted circular PDF and rose diagram of $WRLE(\theta;\boldsymbol{\varphi})$, $NWRE(\theta;\boldsymbol{\varphi})$, $WRL(\theta;\alpha)$, $WRE(\theta;\alpha)$ and $WRXG(\theta;\alpha)$.



(a) The linear histogram and fitted pdf.



(b) The empirical CDF.

Figure 9. The linear (unwrapped) histogram, fitted PDF and empirical CDF of $NWRE(\theta;\boldsymbol{\varphi})$, $WRE(\theta;\alpha)$, $WRL(\theta;\alpha)$, $WRLE(\theta;\boldsymbol{\varphi})$ and $WRXG(\theta;\alpha)$, for seabasin200301 data.

The summary results of several measures, including -2 log-likelihood (-2ℓ), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), Watson's U^2 test (WT), Kuiper's test (KT), Rayleigh's test (RT) and Rao's spacing test (RST), for the models are provided in Table 7 for comparison.

The *WRLE* distribution provides the best fit to the data set when compared to the *WRL*, *WRE*, *WRXG* and *NWRE* distributions. This is evident from the smallest values of $-2\widehat{\ell}$, AIC, CAIC, BIC, HQIC, WT statistic, KT statistic, RT statistic and RST statistic which all clearly show superiority for the *WRLE* distribution. The fitted *WRLE* distribution seems to offer a sufficient fit to the seabasin 200301 dataset.

Table 7. ML estimates for seabasin200301 data.

The Model	$-2\widehat{\ell}$	AIC	CAIC	BIC	HQIC	KT(Stat)	WT(Stat)	RT(Stat)	RST(Stat)
WRLE	69.95691	73.95691	74.38548	76.82488	74.89180	5.05407	2.20086	0.98299	305.69280
NWRE	70.15385	74.15385	74.58242	77.02183	75.08874	5.06232	2.20793	0.98356	305.76630
WRL	80.77774	82.77774	82.91567	84.21172	83.24518	5.15487	2.27424	0.98765	310.23860
WRE	84.96779	86.96779	87.10572	88.40177	87.43523	5.19024	2.29939	0.98930	312.44660
WRXG	86.25484	88.25484	88.39277	89.68883	88.72228	5.18198	2.29387	0.98867	311.93110

6. Concluding remarks

In this paper, we investigated and studied various mathematical, distributional and circular properties of the *LE distribution* [10] using the method of "wrapping", including the distribution function, limit behavior, first and second derivatives, modality behavior, characteristic function, and trigonometric moments with related measures. We discussed maximum likelihood estimation of the model parameters and provided simulation results to assess the performance of the proposed distribution. An application to real wind directions data set illustrates that the new distribution provides consistently better fits than other nested and non-nested models.

In their next work, the authors plan to conduct a change-point investigation about the wrapped LE distribution (the change-point model in Lindley distribution in linear setup may be found in [14]), as well as Bayesian analysis for this distribution.

Use of Generative-AI tools declaration

The author declares he has not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. K. V. Mardia, P. E. Jupp, *Directional statistics*, 2 Eds., John Wiley & Sons, Inc., 1999. https://doi.org/10.1002/9780470316979
2. J. S. Rao, G. A. Sen, *Topics in circular statistics*, World Scientific, 2001. https://doi.org/10.1142/4031
3. P. Lévy, L'addition des variables aléatoires définies sur une circonférence, *Bull. Soc. Math. France*, **67** (1939), 1–41. https://doi.org/10.24033/bsmf.1288
4. S. R. Jammalamadaka, T. J. Kozubowski, New families of wrapped distributions for modeling skew circular data, *Commun. Stat.*, **33** (2004), 2059–2074. https://doi.org/10.1081/STA-200026570
5. S. Roy, M. A. S. Adnan, Wrapped generalized Gompertz distribution: an application to ornithology, *J. Biom. Biostat.*, **3** (2012), 153–156. https://doi.org/10.4172/2155-6180.1000153

6. S. Roy, M. A. S. Adnan, Wrapped weighted exponential distributions, *Stat. Probab. Lett.*, **82** (2012), 77–83. https://doi.org/10.1016/j.spl.2011.08.023
7. M. A. S. Adnan, S. Roy, Wrapped variance gamma distribution with an application to wind direction, *J. Environ. Stat.*, **6** (2014), 1–10.
8. S. Joshi, K. K. Jose, Wrapped Lindley distribution, *Commun. Stat.*, **47** (2018), 1013–1021. https://doi.org/10.1080/03610926.2017.1280168
9. H. Al-Mofleh, S. Sen, M. Zannon, The wrapped version of Xgamma distribution: properties and application with directional data, *J. Appl. Probab. Stat.*, **19** (2024), 83–98.
10. G. M. Cordeiro, A. Z. Afify, H. M. Yousof, S. Cakmakyapan, G. Ozel, The Lindley Weibull distribution: properties and applications, *Proc. Natl. Acad. Sci. Brazil*, **90** (2018), 2579–2598. https://doi.org/10.1590/0001-3765201820170635
11. R Core Team, *The R project for statistical computing*, R foundation for statistical computing, 2025. Available from: `https://www.R-project.org/`.
12. C. J. Vörösmarty, R. B. Lammers, D. H. Bromwich, Arctic RIMS: NCEP reanalysis wind direction, *Arctic Data Center*, 2016. Available from: `https://doi.org/10.5065/D6R49NXH`.
13. A. Yilmaz, C. Biçer, A new wrapped exponential distribution, *Math. Sci.*, **12** (2018), 285–293. https://doi.org/10.1007/s40096-018-0268-y
14. S. Joshi, K. K. Jose, D. Bhati, Estimation of a change-point in the hazard rate of the Lindley model under right censoring, *Commun. Stat. Simul. Comput.*, **46** (2017), 3563–3574. https://doi.org/10.1080/03610918.2015.1096381

AIMS Press