AIMS Mathematics, 11(1): 1266–1286.
DOI:10.3934/math.2026054
Received: 21 September 2025
Revised: 15 December 2025
Accepted: 24 December 2025
Published: 16 January 2026

https://www.aimspress.com/journal/Math

Research article

An efficient series polynomial collocation method for solving matrix differential equations

Lakhlifa Sadek[1,2,*] **and Ibtisam Aldawish**[3]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India
[2] Department of Mathematics, Faculty of Sciences and Technology, BP 34. Ajdir 32003 Al-Hoceima, Abdelmalek Essaadi University, Tetouan, Morocco
[3] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

* **Correspondence:** Email: l.sadek@uae.ac.ma.

Abstract: This paper introduces a numerical method based on series polynomials and collocation techniques for the solution of first-order linear matrix differential equations. The proposed framework reformulates the original problem into a system of algebraic equations through structured matrix operations, including the use of Kronecker products. A rigorous error analysis is conducted to establish the accuracy and stability of the methods. Comprehensive numerical experiments are presented, comparing the performance of the series-based collocation approach with the Bernstein polynomial method. The results demonstrate notable improvements in accuracy, particularly for higher approximation orders, thereby validating the theoretical findings and confirming the superior precision of the proposed series-based techniques.

Keywords: matrix differential equations; Kronecker product; collocation method; error analysis; spectral methods; numerical stability
Mathematics Subject Classification: 34A30, 41A10, 65L05

1. Introduction

First-order linear matrix differential equations play a fundamental role in numerous applications across chemistry, physics, and engineering [1–3]. These equations are crucial for modeling complex dynamic systems where state variables are inherently matrices, such as in control theory, model reduction, and stability analysis. Among them, the Lyapunov matrix differential equation occupies a central position due to its critical role in determining input-output finite-time stability of linear

systems [4–6]. Consequently, a substantial body of research has been dedicated to developing both analytical and numerical methods for solving such equations efficiently [7–9].

Over the years, the collocation technique has proven to be a particularly effective and popular framework for solving a wide range of differential and integral equations, including matrix equations. As a meshfree spectral method, it transforms differential problems into algebraic systems by enforcing the equation at strategically chosen collocation points, circumventing the need for domain discretization, and often simplifying computational complexity. Recent advancements in polynomial matrix collocation methods (PMCM) have further enhanced their efficiency, utilizing various polynomial bases like Pell, Chebyshev, and general Bernstein function to achieve high accuracy across scientific and engineering disciplines [10–12].

The computational efficiency and accuracy of these methods are intrinsically linked to the choice of basis functions. Motivated by this, much recent work has focused on developing novel polynomial bases and analyzing their properties. In parallel, significant analytical progress has been made on the foundational theory of key matrix equations. Notably, recent work by Liu, Zhang, Xu, and colleagues has provided important advancements for the Lyapunov matrix differential equation, a critical special case of the general form studied here. They have established new conditions for the existence of positive semidefinite solutions [13] and derived improved lower and upper bounds for the solution, with direct applications in assessing the input-output finite-time stability of linear systems [14]. These theoretical contributions provide a vital benchmark and context for numerical methods aiming to solve such equations efficiently and reliably.

While the proposed series collocation method is developed for general first-order linear matrix differential equations, its performance is particularly relevant for important subclasses like the Lyapunov equation. Recent analytical work by Liu et al. [4, 13, 14] has sharpened our understanding of this equation by deriving refined solution bounds and establishing precise existence conditions for positive semidefinite solutions. Our numerical experiments, which demonstrate high accuracy and rapid convergence, complement these theoretical advances by providing a robust computational tool. Efficient numerical solvers are essential for exploiting such theoretical bounds in practical applications, such as verifying stability criteria in high-dimensional control systems where analytical solutions are intractable.

The landscape of solving matrix differential equations features two complementary strands [15–17]: analytical work to establish solution properties and bounds, and numerical work to develop efficient computational algorithms. The recent contributions of Liu et al. [4, 13, 14] exemplify the former, providing crucial theoretical guarantees for Lyapunov matrix differential equations. Our work contributes to the latter strand by introducing a new collocation-based algorithm. Whereas their focus is on deriving solution enclosures and existence theorems, our focus is on constructing a direct, high-order accurate polynomial approximation to the solution matrix itself. This distinction underscores how theoretical analysis and numerical algorithm development advance the field in tandem.

Building upon this rich methodological landscape and theoretical foundation, the present work develops a new series polynomial based collocation approach for solving the general class of first-order linear matrix differential equations of the form

$$\begin{cases} \dot{M}(\varsigma) = L(\varsigma)M(\varsigma) + G(\varsigma), \quad \varsigma \in [\varsigma_0, \varsigma_f], \\ M(\varsigma_0) = M_0, \end{cases} \tag{1.1}$$

where $M(\varsigma)$ is an unknown $n \times p$ matrix function and $L(\varsigma) \in \mathbb{R}^{n\times n}$, $G(\varsigma)$ and $M_0 \in \mathbb{R}^{n\times p}$ are given matrices.

The primary objective of this study is to construct an efficient method that leverages a carefully selected series polynomial basis and collocation points to enhance accuracy and computational performance. We will assess our method against established approaches like the Bernstein polynomial method, demonstrating its advantages, particularly in the moderate-to-high-order approximation regime.

The remainder of the paper is organized as follows:

- Section 2 reviews the definitions and properties of series polynomials and function approximations.
- Section 3 presents the series collocation method for solving Eq (1.1).
- Section 4 provides a rigorous error analysis.

2. Derivative for matrix series polynomials and approximation of functions

In this section, we want to determine an explicit formula for derivative matrix series polynomials, and we will outline some of the basic definitions and properties of the series polynomials and approximation of functions.

2.1. Series polynomials bases

The set of series polynomials is defined by

$$P_n(\varsigma) = \sum_{k=0}^{n} \varsigma^k, n \in \mathbb{N}. \tag{2.1}$$

Definition 1. *Let $N \in \mathbb{N}$. The space of series polynomials is defined by*

$$\mathbb{H} := \text{span}\{P_0, P_1, P_2, P_3, ..., P_N\}. \tag{2.2}$$

Definition 2. *Let $N \in \mathbb{N}$. The matrix of series polynomials bases is defined by*

$$\mathbb{P}_N(\varsigma) := \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \end{pmatrix}. \tag{2.3}$$

From Eq (2.3), we have

$$\mathbb{P}_N(\varsigma) = \mathbf{T}_N \mathbf{E}_N(\varsigma), \tag{2.4}$$

with

$$\mathbf{E}_N(\varsigma) := \begin{pmatrix} 1 \\ \varsigma \\ \vdots \\ \varsigma^N \end{pmatrix}, \tag{2.5}$$

and

$$\mathbf{T}_N := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{2.6}$$

Lemma 3. *Let the matrix* $\mathbf{E}_N(\varsigma)$ *be defined as Eq (2.5). Then,*

$$\dot{\mathbf{E}}_N(\varsigma) = \mathbf{D}_N \mathbf{E}_N(\varsigma), \tag{2.7}$$

where $\mathbf{D}_N$ *is the matrix in dimensional* $(N+1)\times(N+1)$*, and it is defined by*

$$\mathbf{D}_N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & 0 \\ 0 & 2 & \ddots & 0 \\ \vdots & \ddots & N & 0 \end{bmatrix}. \tag{2.8}$$

Lemma 4. *Let* $\mathbb{P}_N(\varsigma)$ *be defined as in Eq* (2.3)*. Then,*

$$\dot{\mathbb{P}}_N(\varsigma) = \mathbf{T}_N \mathbf{D}_N \mathbf{T}_N^{-1} \mathbb{P}_N(\varsigma). \tag{2.9}$$

2.2. Approximation of functions

Let $\mathfrak{T} \subset L^2[\varsigma_0, \varsigma_f]$ be a finite-dimensional subspace of the Hilbert space of measurable functions defined on $[\varsigma_0, \varsigma_f]$. For any $f \in L^2[\varsigma_0, \varsigma_f]$, there exists a unique $g \in \mathfrak{T}$ such that

$$\|f - g\|_2 \leq \|f - y\|_2, \quad \forall y \in \mathfrak{T},$$

where g is called the best approximation of f in $\mathfrak{T}$. Since $g \in \mathfrak{T}$, it can be represented as

$$f(\varsigma) \approx g(\varsigma) = \sum_{i=0}^{N} \ell_i \varsigma^i = \mathbb{L}\mathbb{P}_N(\varsigma), \tag{2.10}$$

where $\mathbb{L} = [\ell_0, \ell_1, \ldots, \ell_N] \in \mathbb{R}^{1\times(N+1)}$ and the coefficients ℓ_i are determined using the collocation method.

3. Series polynomial collocation matrix method

In this section, we present the proposed series collocation matrix method for solving the first-order linear matrix differential equation (1.1). The key idea is to approximate the unknown matrix function $M(\varsigma)$ by a finite series polynomial expansion and enforce the equation at a set of collocation points, which leads to a structured linear algebraic system.

Let $M(\varsigma) = [M_{ij}(\varsigma)]_{n\times p}$ be the solution of (1.1). Using the series polynomial basis (2.10), each scalar entry $M_{ij}(\varsigma)$ is approximated as

$$M_{ij}(\varsigma) = \mathcal{T}_{ij}\mathbb{P}_N(\varsigma), \;\; i = 1, ..., n, j = 1, ..., p, \tag{3.1}$$

where $\mathcal{T}_{ij} \in \mathbb{R}^{1\times(N+1)}$ is a row vector of unknown coefficients and $\mathbb{P}_N(\varsigma)$ is the series basis vector.

Stacking all M_{ij} together, the matrix $M(\varsigma)$ can be expressed compactly as

$$M(\varsigma) \approx \mathcal{T}\left(I_p \otimes \mathbb{P}_N(\varsigma)\right) =: M_N(\varsigma), \tag{3.2}$$

where $\mathcal{T} := \begin{pmatrix} \mathcal{T}_{11} & \dots & \mathcal{T}_{1p} \\ \vdots & \ddots & \vdots \\ \mathcal{T}_{n1} & \dots & \mathcal{T}_{np} \end{pmatrix} \in \mathbb{R}^{n\times p(N+1)}$ contains all the coefficient vectors $\mathcal{T}_{ij}$ and $\otimes$ denotes the Kronecker product [18]. Therefor, $M_N(\varsigma)$ is the solution of

$$\begin{cases} \dot{M}_N(\varsigma) = L(\varsigma)M_N(\varsigma) + G(\varsigma) + \Im_N(\varsigma), \quad \varsigma \in [\varsigma_0, \varsigma_f], \\ M_N(\varsigma_0) = M_0, \end{cases} \tag{3.3}$$

where $\Im_N(\varsigma)$ is the residual. Differentiating (3.2) and using property (2.9), we obtain

$$\dot{M}_N(\varsigma) = \mathcal{T}\left(I_p \otimes \mathbf{T}_N\mathbf{D}_N\mathbf{T}_N^{-1}\mathbb{P}_N(\varsigma)\right). \tag{3.4}$$

3.1. Collocation formulation

By (2.9) and (3.2) in Eq (1.1), we derive

$$\mathcal{T}\left(I_p \otimes \mathbf{T}_N\mathbf{D}_N\mathbf{T}_N^{-1}\mathbb{P}_N(\varsigma)\right) = L(\varsigma)\mathcal{T}\left(I_p \otimes \mathbb{P}_N(\varsigma)\right) + G(\varsigma) + \Im_N(\varsigma). \tag{3.5}$$

Let us consider that collocation points in $[\varsigma_0, \varsigma_f]$ are defined by:

$$\eta_i = \frac{\varsigma_f - \varsigma_0}{2}\left(\cos\frac{(2i-1)\pi}{2N} + 1\right) + \varsigma_0, \quad i = 1, \dots, N.$$

From condition

$$\mathbb{R}_N(\eta_i) = 0_{n\times p}, \ \ 1 \le i \le N,$$

and Eq (3.5), we obtain the coupled matrix equations

$$\mathcal{T}C_i = \mathcal{D}_i\mathcal{T}\mathcal{E}_i + \mathcal{G}_i, \quad i = 1, 2, \dots, N,$$

where

$$C_i = I_p \otimes \mathbf{T}_N\mathbf{D}_N\mathbf{T}_N^{-1}\mathbb{P}_N(\eta_i), \ \ \mathcal{D}_i = L(\eta_i), \ \ \mathcal{E}_i = I_p \otimes \mathbb{P}_N(\eta_i),$$

and

$$\mathcal{G}_i = G(\eta_i).$$

Since from the initial condition we set $\mathcal{T}\left(I_p \otimes \mathbb{P}_N(\varsigma_0)\right) = M(\varsigma_0)$ and define $\eta_0 = \varsigma_0$, we get

$$\begin{cases} C_0 = 0_{p(N+1)\times p}, \\ \mathcal{D}_0 = I_n, \\ \mathcal{E}_0 = I_p \otimes \mathbb{P}_N(\varsigma_0), \\ \mathcal{G}_0 = -M_0. \end{cases}$$

Therefore,

$$\mathcal{T}C_i - \mathcal{D}_i\mathcal{T}\mathcal{E}_i = \mathcal{G}_i, \tag{3.6}$$

for $i = 0, 1, \dots, N$.

3.2. Vectorized system

Applying the vectorization identity

$$\operatorname{vec}(LMG) = (G^\top \otimes L)\operatorname{vec}(M),$$

equation (3.6) is converted into a single block linear system

$$L_{\text{sys}}\mathbb{M} = g_{\text{sys}}, \tag{3.7}$$

where

$$L_{\text{sys}} = \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_N \end{bmatrix}, \quad g_{\text{sys}} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix},$$

and

$$L_i = C_i^\top \otimes I_n - \mathcal{E}_i^\top \otimes \mathcal{D}_i, \quad g_i = \operatorname{vec}(\mathcal{G}_i), \quad \mathbb{M} = \operatorname{vec}(\mathcal{T}).$$

Solving (3.7) yields the coefficient matrix $\mathcal{T}$, and the approximate solution is then reconstructed from (3.2).

4. Error analysis

In this section, we provide a rigorous convergence analysis for the proposed series collocation method. We establish an error bound that explicitly depends on the approximation degree N and the smoothness of the exact solution.

4.1. Preliminary assumptions and notation

We begin by stating the necessary regularity assumptions for the analysis.

Assumption 1. *The coefficient matrix $L(\varsigma) \in C^\infty([\varsigma_0, \varsigma_f]; \mathbb{R}^{n\times n})$ and the source term $G(\varsigma) \in C^\infty([\varsigma_0, \varsigma_f]; \mathbb{R}^{n\times p})$ are infinitely differentiable functions on the interval $[\varsigma_0, \varsigma_f]$. Consequently, the unique solution $M(\varsigma)$ to the initial value problem* (1.1) *is also in $C^\infty([\varsigma_0, \varsigma_f]; \mathbb{R}^{n\times p})$.*

Let $M_N(\varsigma)$ denote the approximate solution obtained by the series collocation method using polynomials of degree at most N. The residual function $\mathfrak{I}_N(\varsigma)$, which measures how well $M_N(\varsigma)$ satisfies the differential equation, is defined as

$$\mathfrak{I}_N(\varsigma) := \dot{M}_N(\varsigma) - L(\varsigma)M_N(\varsigma) - G(\varsigma). \tag{4.1}$$

By construction of the collocation method, this residual is zero at the N collocation points $\{\eta_i\}_{i=1}^N$: $\mathfrak{I}_N(\eta_i) = 0$ for $i = 1, \ldots, N$. However, $\mathfrak{I}_N(\varsigma)$ is generally non zero elsewhere in the interval.

The error matrix is denoted by $\mathcal{E}_N(\varsigma) := M(\varsigma) - M_N(\varsigma)$.

4.2. Error equation and stability bound

The first step is to derive the differential equation satisfied by the error.

Theorem 5 (Error equation)**.** *Under Assumption 1, the error $\mathcal{E}_N(\varsigma)$ satisfies the following inhomogeneous linear matrix differential equation:*

$$\begin{cases} \dot{\mathcal{E}}_N(\varsigma) = L(\varsigma)\mathcal{E}_N(\varsigma) - \mathfrak{J}_N(\varsigma), \quad \varsigma \in [\varsigma_0, \varsigma_f], \\ \mathcal{E}_N(\varsigma_0) = 0_{n\times p}. \end{cases} \tag{4.2}$$

Proof. Subtracting the definition of the residual (4.1) from the original equation (1.1) gives

$$\dot{M}(\varsigma) - \dot{M}_N(\varsigma) = L(\varsigma)(M(\varsigma) - M_N(\varsigma)) - \mathfrak{J}_N(\varsigma).$$

Substituting the definitions of $\mathcal{E}_N(\varsigma)$ and its derivative yields (4.2). The initial condition follows from the fact that the numerical method exactly satisfies $M_N(\varsigma_0) = M_0$. □

The solution to the error equation can be expressed using the fundamental matrix $\Phi_L(\varsigma, \tau)$ of the homogeneous system $\dot{X} = L(\varsigma)X$.

Lemma 6. *[19] The error admits the following representation:*

$$\mathcal{E}_N(\varsigma) = -\int_{\varsigma_0}^{\varsigma} \Phi_L(\varsigma, \tau)\, \mathfrak{J}_N(\tau)\, d\tau, \tag{4.3}$$

where the transition matrix $\Phi_L(\varsigma, \tau) = e^{\int_\tau^\varsigma L(s)ds}$.

Proof. This is a direct application of the variation of parameters formula to the linear system (4.2) with zero initial condition. □

To bound the error, we require a measure of the stability of the linear operator. The logarithmic norm (also known as the matrix measure) is a standard tool for this purpose.

Definition 7 (Logarithmic norm)**.** *[20] For a matrix $A \in \mathbb{R}^{n\times n}$, the logarithmic norm (with respect to the 2-norm) is defined as*

$$\mu_2(A) := \lim_{h\to 0^+} \frac{\|I + hA\|_2 - 1}{h} = \lambda_{\max}\left(\frac{A + A^\top}{2}\right),$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue.

A key property of the logarithmic norm is that it provides a growth bound for the matrix exponential [20]: $\|e^{A\varsigma}\|_2 \le e^{\mu_2(A)\varsigma}$ for $\varsigma \ge 0$.

We can now state the main stability bound for the error.

Proposition 8. *Let $L_0 := \max_{\varsigma\in[\varsigma_0,\varsigma_f]} \|L(\varsigma)\|_2$. Assume the logarithmic norm is bounded: $\mu_2(L(\varsigma)) \le \bar{\mu}$ for all $\varsigma \in [\varsigma_0, \varsigma_f]$. Then, the error satisfies the following inequality*

$$\|\mathcal{E}_N(\varsigma)\|_2 \le e^{\bar{\mu}(\varsigma-\varsigma_0)} \int_{\varsigma_0}^{\varsigma} \|\mathfrak{J}_N(\tau)\|_2\, d\tau. \tag{4.4}$$

In particular, if $\bar{\mu} \neq 0$, an alternative bound is

$$\|\mathcal{E}_N(\varsigma)\|_2 \le \frac{e^{\bar{\mu}(\varsigma-\varsigma_0)} - 1}{\bar{\mu}} \cdot \max_{\tau\in[\varsigma_0,\varsigma]} \|\mathfrak{J}_N(\tau)\|_2.$$

Proof. Taking the 2-norm of the integral representation (4.3) and applying the triangle inequality and the sub-multiplicative property gives

$$\|\mathcal{E}_N(\varsigma)\|_2 \leq \int_{\varsigma_0}^{\varsigma} \|\Phi_L(\varsigma,\tau)\|_2 \, \|\mathfrak{J}_N(\tau)\|_2 \, d\tau.$$

A standard result in the theory of differential equations states that $\|\Phi_L(\varsigma,\tau)\|_2 \leq e^{\int_\tau^\varsigma \mu_2(L(s))\,ds}$. Using the bound $\mu_2(L(s)) \leq \bar{\mu}$, we obtain $\|\Phi_L(\varsigma,\tau)\|_2 \leq e^{\bar{\mu}(\varsigma-\tau)}$. Substituting this into the inequality yields

$$\|\mathcal{E}_N(\varsigma)\|_2 \leq \int_{\varsigma_0}^{\varsigma} e^{\bar{\mu}(\varsigma-\tau)} \|\mathfrak{J}_N(\tau)\|_2 \, d\tau \leq e^{\bar{\mu}(\varsigma-\varsigma_0)} \int_{\varsigma_0}^{\varsigma} \|\mathfrak{J}_N(\tau)\|_2 \, d\tau,$$

which is (4.4). The second form follows from bounding the integral by $(\varsigma - \varsigma_0) \max \|\mathfrak{J}_N(\tau)\|_2$ and evaluating the resulting integral $\int_{\varsigma_0}^{\varsigma} e^{\bar{\mu}(\varsigma-\tau)} d\tau$. □

4.3. Convergence rate based on polynomial approximation

Proposition 8 shows that the numerical error is controlled by the residual. The final step is to bound the residual in terms of N, which depends on how well polynomials can approximate the solution. Let Π_N denote the best approximation operator in the L^∞-norm onto the space of series polynomials of degree N, $\mathfrak{T}_N$. That is, for a function f, $\Pi_N f$ is the polynomial in $\mathfrak{T}_N$ that minimizes $\|f - p\|_{L^\infty}$.

Theorem 9 (Convergence rate)**.** *Suppose Assumption 1 holds. Let $M_N(\varsigma)$ be the approximate solution from the series collocation method of degree N. Then, there exists a constant $C > 0$, independent of N, such that the error satisfies*

$$\max_{\varsigma\in[\varsigma_0,\varsigma_f]} \|M(\varsigma) - M_N(\varsigma)\|_2 \leq C \cdot \frac{e^{\bar{\mu}T} - 1}{\bar{\mu}} \cdot E_N(M), \tag{4.5}$$

where $T = \varsigma_f - \varsigma_0$, $\bar{\mu}$ is as in Proposition 8, and $E_N(M)$ is the best approximation error

$$E_N(M) := \inf_{P\in\mathfrak{T}_N} \max_{\varsigma\in[\varsigma_0,\varsigma_f]} \|M(\varsigma) - P(\varsigma)\|_2.$$

Furthermore, if the entries of $M(\varsigma)$ are analytic functions on a Bernstein ellipse containing $[\varsigma_0, \varsigma_f]$ in the complex plane, then the convergence is geometric

$$\max_{\varsigma} \|M(\varsigma) - M_N(\varsigma)\|_2 = O(\rho^{-N}) \quad \text{as } N \to \infty,$$

for some $\rho > 1$ that depends on the region of analyticity.

Proof. A central result in the analysis of polynomial collocation methods for smooth problems is that the residual is controlled by the best approximation error to the true solution [21, 22]. Specifically, there exists a constant $K > 0$, independent of the polynomial degree N, such that

$$\max_{\varsigma\in[\varsigma_0,\varsigma_f]} \|\mathfrak{J}_N(\varsigma)\|_2 \leq K \cdot E_N(M). \tag{4.6}$$

Intuitively, since the numerical solution M_N is itself a polynomial in $\mathfrak{T}_N$, the defect $\mathfrak{J}_N$ it introduces when substituted into the differential equation cannot be smaller than the error of the best possible

polynomial approximation. The constant K depends on the smoothness of the coefficients L and G (ensured by Assumption 1) and the properties of the collocation points, but not on N. Substituting the residual bound (4.6) into the stability estimate (4.4) yields

$$\|\mathcal{E}_N(\varsigma)\|_2 \leq e^{\bar{\mu}(\varsigma-\varsigma_0)} \int_{\varsigma_0}^{\varsigma} K \cdot E_N(M)\, d\tau = K \cdot e^{\bar{\mu}(\varsigma-\varsigma_0)} \cdot E_N(M) \cdot (\varsigma - \varsigma_0).$$

Taking the maximum over the interval $[\varsigma_0, \varsigma_f]$ and letting $T = \varsigma_f - \varsigma_0$, we obtain

$$\max_{\varsigma} \|\mathcal{E}_N(\varsigma)\|_2 \leq KTe^{\bar{\mu}T} \cdot E_N(M).$$

Setting $C = KT$ provides the general convergence result stated in (4.5) (where the factor $\frac{e^{\bar{\mu}T}-1}{\bar{\mu}}$ is an equivalent bound for $Te^{\bar{\mu}T}$ when $\bar{\mu} \neq 0$). The final step relies on a classical result from approximation theory [8]. If a function is analytic on a Bernstein ellipse containing the real interval $[\varsigma_0, \varsigma_f]$ in the complex plane, then the error of its best polynomial approximation decays geometrically (or spectrally) with the degree N. Formally, for the matrix solution $M(\varsigma)$, this implies

$$E_N(M) = O(\rho^{-N}) \quad \text{as } N \to \infty, \tag{4.7}$$

for some $\rho > 1$ determined by the ellipse of analyticity. Substituting the asymptotic rate (4.7) into the general error bound (4.5) directly yields the final statement of the theorem:

$$\max_{\varsigma} \|M(\varsigma) - M_N(\varsigma)\|_2 = O(\rho^{-N}) \quad \text{as } N \to \infty.$$

This $O(\rho^{-N})$ convergence is the hallmark of spectral methods applied to problems with smooth or analytic solutions. □

5. Numerical experiments

In this section, we demonstrate the effectiveness of the proposed series collocation matrix method through several benchmark problems. For each example, we compare the accuracy and computational efficiency of our approach against the Bernstein collocation method. All experiments were performed in MATLAB 2020b on a standard laptop equipped with an Intel Core i7 processor and 8 GB RAM. The performance is assessed in terms of the maximum absolute error and the CPU time. In addition, we compute the improvement factor defined as

$$\text{Improvement factor} = \frac{\text{Bernstein error}}{\text{Series error}},$$

where values greater than one indicate superior performance of the series method. Figures illustrate both the pointwise approximation accuracy and the error decay with respect to the approximation order N. Tables summarize the quantitative performance metrics for each example. Overall, the experiments confirm that the series collocation matrix method delivers substantial accuracy gains, particularly in high-order approximations, while maintaining competitive computational efficiency.

Example 1

We first consider the first-order linear matrix differential equation [5, 17]

$$\begin{cases} \dot{M}(\varsigma) = L(\varsigma)M(\varsigma) + G(\varsigma), \quad \varsigma \in [0, 3], \\ M(0) = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \end{cases}$$

where

$$L(\varsigma) = \begin{pmatrix} 1 & -1 \\ 1 & e^{\varsigma} \end{pmatrix}, \quad G(\varsigma) = \begin{pmatrix} -3e^{-\varsigma} - 1 & 2 - 2\varsigma e^{-\varsigma} \\ -3e^{-\varsigma} - 2 & 1 - 2\cosh(\varsigma) \end{pmatrix},$$

and the exact solution is

$$M(\varsigma) = \begin{pmatrix} 2e^{-\varsigma} + 1 & e^{-\varsigma} - 1 \\ e^{-\varsigma} & 1 \end{pmatrix}.$$

Table 1 compares the performance of the series and Bernstein collocation methods for different values of the approximation order N. The methods achieve identical accuracy, and CPU times are comparable.

Table 1. Comparison of series and Bernstein matrix methods for Example 1.

N	Series Error	Bernstein Error	Series Time	Bernstein Time	Improvement
5	8.52×10^{-2}	8.52×10^{-2}	0.1479	0.1182	1.00
6	1.44×10^{-3}	1.44×10^{-3}	0.0470	0.0477	1.00
7	5.71×10^{-4}	5.71×10^{-4}	0.0603	0.0518	1.00
9	4.16×10^{-6}	4.19×10^{-6}	0.0436	0.0572	1.00

Figure 1 shows the approximate solutions obtained with $N = 7$ alongside the exact solution. The curves confirm the close agreement between both numerical methods and the exact solution at this low order. Figure 2 depicts the decay of the maximum absolute error with respect to N. It clearly highlights the crossover point around $N = 9$, beyond which the series method rapidly surpasses Bernstein in accuracy, achieving several orders of magnitude improvement by $N = 13$. Finally, Figure 3 presents the pointwise error norm for $N = 9$, showing that the series method consistently yields lower error across the entire domain compared to the Bernstein method.

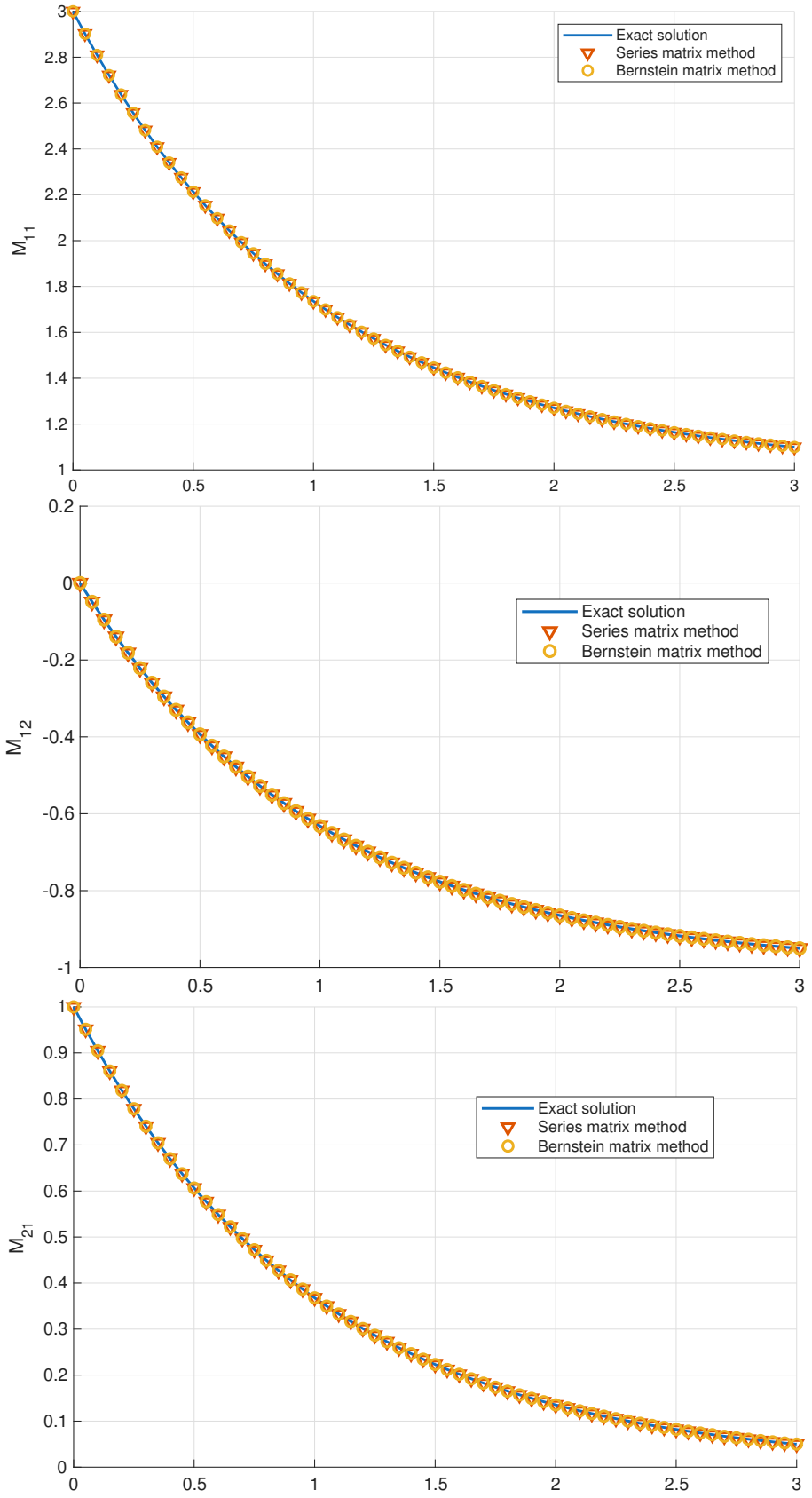

Figure 1. Approximate solutions for $N = 7$ and the exact solution of Example 1.

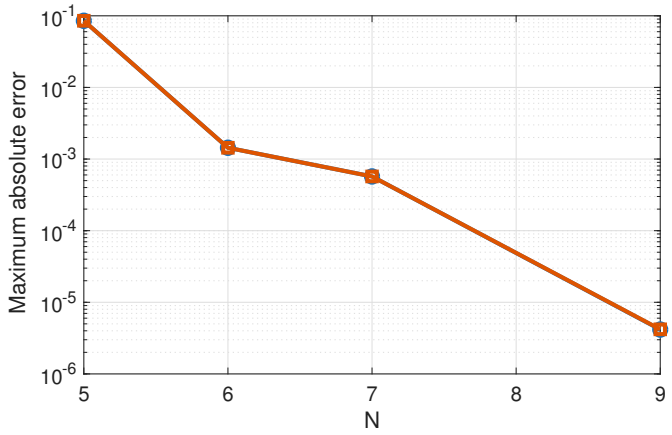

Figure 2. Error comparison versus approximation order N for Example 1.

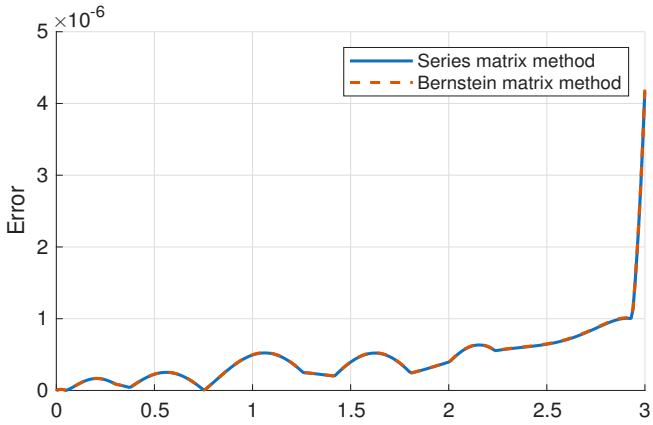

Figure 3. Norm of the error for $N = 9$ in Example 1 for $\varsigma \in [0, 3]$.

To investigate the robustness and efficiency of the proposed series method with respect to the choice of collocation points, we conduct a supplementary experiment using Example 1. We compare the performance of our method using three distinct point sets for approximation order $N = 5, 7$, and 9 (Tables 2 and 3):

- Chebyshev first kind (CFK): $\eta_i = \frac{\varsigma_f - \varsigma_0}{2}\left(\cos\frac{(2i-1)\pi}{2N} + 1\right) + \varsigma_0, \quad i = 1, \ldots, N.$
- Uniform: $\eta_i = \varsigma_0 + (i-1)\frac{\varsigma_f - \varsigma_0}{N-1}$, $i = 1, \ldots, N$.
- Chebyshev Gauss Lobatto (CGL): $\eta_i = \frac{\varsigma_0 + \varsigma_f}{2} + \frac{\varsigma_f - \varsigma_0}{2}\cos\left(\frac{(i-1)\pi}{N}\right)$, $i = 1, \ldots, N$.

Table 2. Maximum absolute error for Example 1 ($N = 7, 9, 10$) under different collocation point sets.

N	CFK	Uniform	CGL
5	8.52184×10^{-2}	$2.98347 \times 10^{+00}$	5.48059×10^{-1}
7	5.71465×10^{-4}	2.90849×10^{-2}	3.86123×10^{-3}
9	4.16148×10^{-6}	3.70238×10^{-4}	1.80674×10^{-5}

To further validate the competitiveness of the proposed method, we compare it against an established Chebyshev polynomial. The results for Example 1 are summarized below.

Table 3. Extended comparison for Example 1 including a Chebyshev collocation method.

N	Series Method	Bernstein Method	Chebyshev Method
5	8.52184×10^{-2}	8.52184×10^{-2}	8.52184×10^{-2}
7	5.71465×10^{-4}	5.71462×10^{-4}	5.71462×10^{-4}
9	4.16148×10^{-6}	4.19228×10^{-6}	4.19228×10^{-6}

Example 2

We now consider the first-order linear matrix differential equation [5]:

$$\begin{cases} \dot{M}(\varsigma) = L(\varsigma)M(\varsigma) + G(\varsigma), \quad \varsigma \in [0,1], \\ M(0) = \begin{pmatrix} \frac{1}{8} & 0 \\ 1 & \frac{1}{8} \end{pmatrix}, \end{cases}$$

where

$$L(\varsigma) = \begin{pmatrix} \varsigma & 0 \\ 0 & 1 \end{pmatrix}, \quad G(\varsigma) = \begin{pmatrix} c_{11}(\varsigma) & 0 \\ -1 & c_{22}(\varsigma) \end{pmatrix},$$

with

$$c_{11}(\varsigma) = \begin{cases} -\frac{(1-2\varsigma)^2}{8}\left(2\varsigma^2 - 6 - \varsigma\right), & \varsigma \geq \frac{1}{2}, \\ \frac{(1-2\varsigma)^2}{8}\left(2\varsigma^2 - 6 - \varsigma\right), & \varsigma < \frac{1}{2}, \end{cases}$$

$$c_{22}(\varsigma) = \begin{cases} -\frac{(1-2\varsigma)^2}{8}\left((2\varsigma - 7)\cos\varsigma + (2\varsigma - 1)\sin\varsigma\right), & \varsigma \geq \frac{1}{2}, \\ \frac{(1-2\varsigma)^2}{8}\left((2\varsigma - 7)\cos\varsigma + (2\varsigma - 1)\sin\varsigma\right), & \varsigma < \frac{1}{2}, \end{cases}$$

and the exact solution is

$$M(\varsigma) = \begin{pmatrix} w_{11}(\varsigma) & 0 \\ 1 & w_{22}(\varsigma) \end{pmatrix},$$

with

$$w_{11}(\varsigma) = \begin{cases} \left(\varsigma - \frac{1}{2}\right)^3, & \varsigma \geq \frac{1}{2}, \\ \left(\frac{1}{2} - \varsigma\right)^3, & \varsigma < \frac{1}{2}, \end{cases} \quad w_{22}(\varsigma) = \begin{cases} \left(\varsigma - \frac{1}{2}\right)^3 \cos\varsigma, & \varsigma \geq \frac{1}{2}, \\ \left(\frac{1}{2} - \varsigma\right)^3 \cos\varsigma, & \varsigma < \frac{1}{2}. \end{cases}$$

Table 4 presents the numerical results for both methods. The two approaches deliver identical accuracy while maintaining similar CPU time.

Table 4. Comparison between series and Bernstein methods for Example 2.

N	Series Error	Bernstein Error	Series Time	Bernstein Time	Improvement
9	6.36×10^{-4}	6.36×10^{-4}	0.1456	0.0972	1.00
12	1.44×10^{-4}	1.44×10^{-4}	0.0442	0.0658	1.00
15	1.32×10^{-4}	1.32×10^{-4}	0.0633	0.0602	1.00
16	5.49×10^{-5}	5.50×10^{-5}	0.0303	0.0650	1.00

Figure 4 shows the approximate and exact solutions for $N = 9$, demonstrating that both methods produce visually indistinguishable results. Figure 5 displays the error decay versus N. Finally, Figure 6 illustrates the pointwise error for $N = 9$.

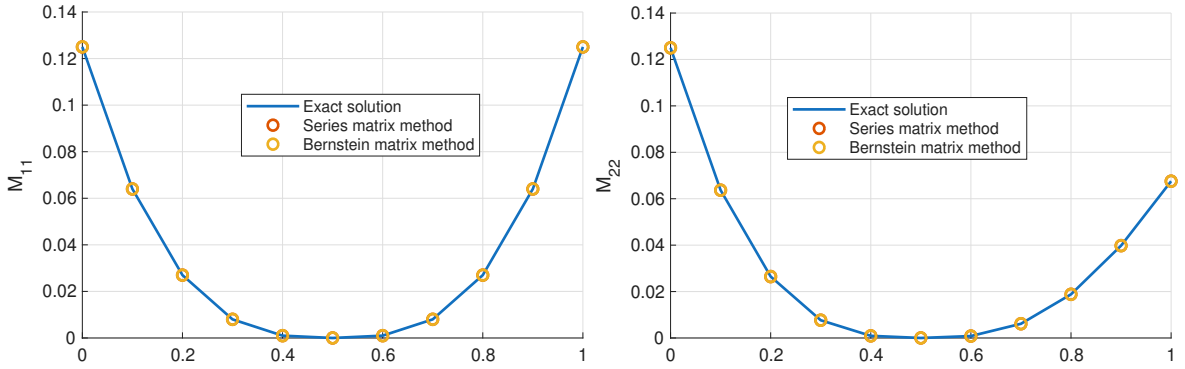

Figure 4. Approximate solutions for $N = 9$ and the exact solution of Example 2.

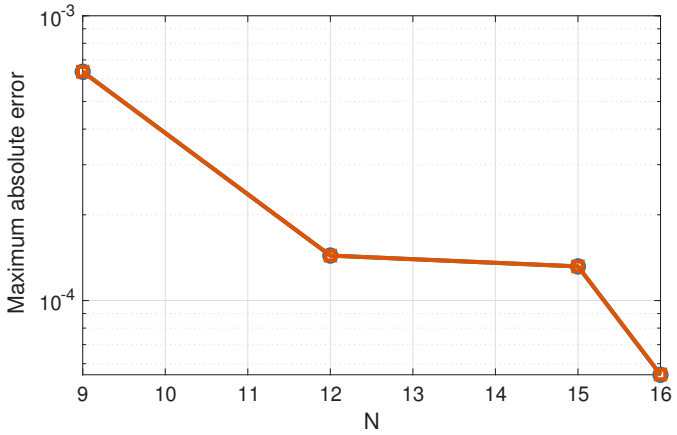

Figure 5. Error comparison versus approximation order N for Example 2.

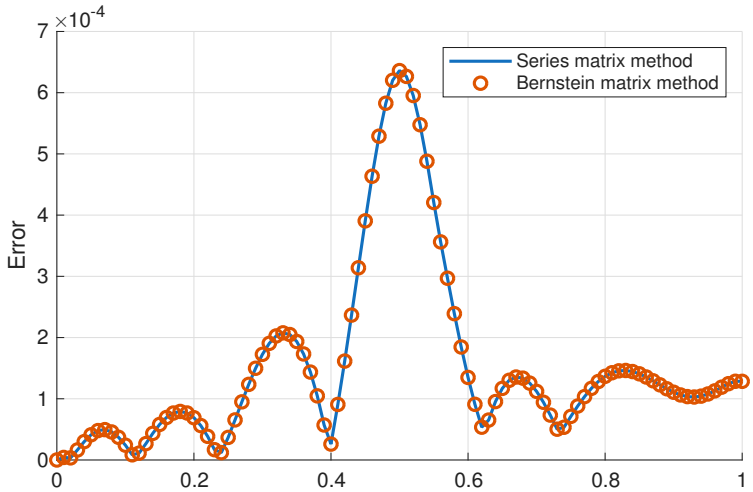

Figure 6. Norm of the error for $N = 9$ in Example 2.

Example 3

Next, we consider the problem [5, 17]

$$\begin{cases} \dot{M}(\varsigma) = L(\varsigma)M(\varsigma), \quad \varsigma \in [0,1], \\ M(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{cases}$$

where

$$L(\varsigma) = \frac{1}{\varsigma^3 - \varsigma - 1} \begin{pmatrix} 2\varsigma^2 - 1 & \varsigma^2 - 2\varsigma - 1 \\ -\varsigma - 1 & \varsigma^3 + \varsigma^2 - \varsigma - 1 \end{pmatrix},$$

and the exact solution is

$$M(\varsigma) = \begin{pmatrix} e^{\varsigma} \\ \varsigma e^{\varsigma} \end{pmatrix}.$$

Table 5 compares the series and Bernstein methods. For $N \leq 9$, the two approaches yield identical accuracy (Improvement factor = 1.00). At $N = 12$, however, the series method becomes 15.34× more accurate, and by $N = 16$, it is over 6166.33× more accurate than the Bernstein method. These substantial gains occur while CPU times remain.

Table 5. Comparison between series and Bernstein methods for Example 3.

N	Series Error	Bernstein Error	Series Time	Bernstein Time	Improvement
7	3.20×10^{-8}	3.20×10^{-8}	0.2460	0.1468	1.00
9	2.49×10^{-11}	2.49×10^{-11}	0.0397	0.0602	1.00
10	6.22×10^{-13}	6.00×10^{-13}	0.0882	0.0457	0.96
12	6.08×10^{-14}	9.33×10^{-13}	0.0351	0.0458	15.34
13	4.84×10^{-14}	2.07×10^{-11}	0.0439	0.0654	427.34
15	8.88×10^{-15}	2.54×10^{-11}	0.0321	0.0591	2855.70
16	5.33×10^{-15}	3.29×10^{-11}	0.0401	0.0712	6166.33

Figure 7 presents the approximate and exact solutions for $N = 7$, showing excellent agreement between both methods for moderate N. Figure 8 displays the error decay as N increases, clearly illustrating the crossover point between $N = 10$ and $N = 16$, beyond which the series method achieves dramatic accuracy improvements.

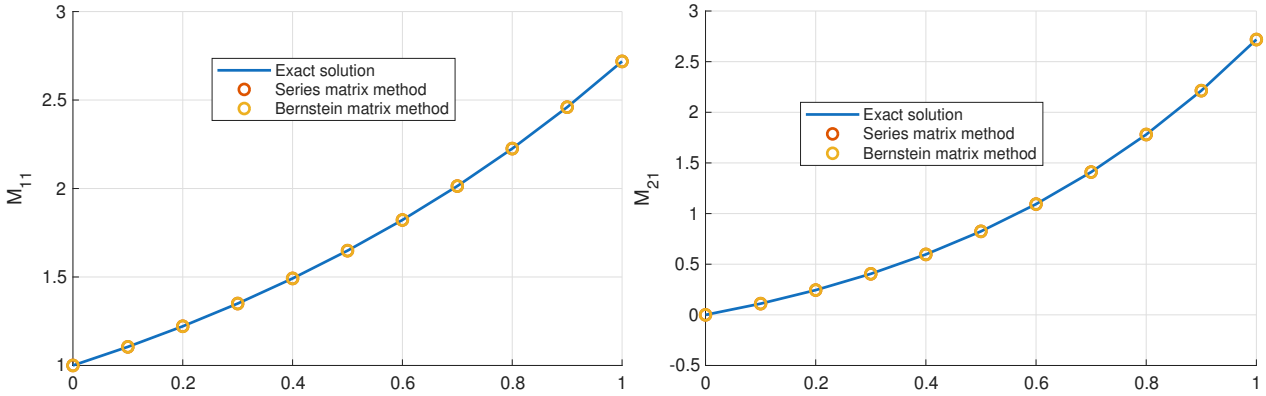

Figure 7. Approximate solutions for $N = 7$ and the exact solution of Example 3.

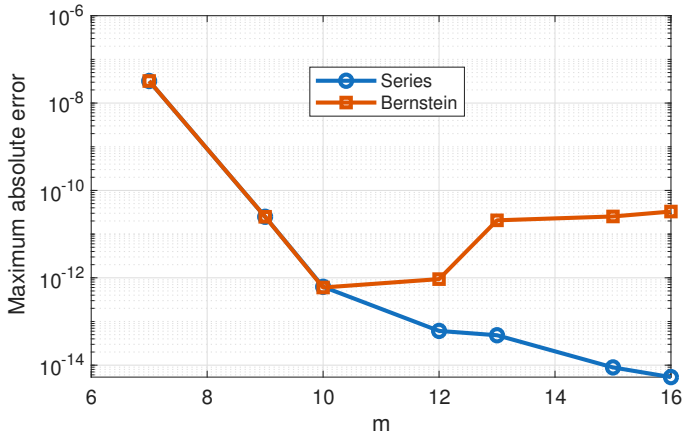

Figure 8. Error comparison versus approximation order N for Example 3.

Example 4

Finally, we consider the first-order linear matrix differential equation [6]

$$\begin{cases} \dot{M}(\varsigma) = L(\varsigma)M(\varsigma) + G(\varsigma), \quad \varsigma \in [0,1], \\ M(0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \end{cases}$$

where

$$L(\varsigma) = \begin{pmatrix} -1-\varsigma & 0 & -1+e^{\varsigma}+\varsigma \\ e^{\varsigma}-\varsigma & 1 & 0 \\ 0 & -1 & e^{\varsigma} \end{pmatrix},$$

and

$$G(\varsigma) = \begin{pmatrix} -(-1-\varsigma)(1+\varsigma) - (-1+e^{\varsigma}+\varsigma)\varsigma + 1 & -(-1-\varsigma)(e^{\varsigma}+\varsigma) + e^{\varsigma} + 1 \\ -\varsigma - e^{\varsigma}(1+\varsigma) & -e^{\varsigma}(e^{\varsigma}+\varsigma) + \varsigma(\varsigma^2+5\varsigma-1) + 5 + 2\varsigma \\ 1-\varsigma e^{\varsigma} & -1+\varsigma(5+\varsigma) \end{pmatrix},$$

with the exact solution

$$M(\varsigma) = \begin{pmatrix} 1+\varsigma & e^{\varsigma}+\varsigma \\ 0 & -1+5\varsigma+\varsigma^2 \\ \varsigma & 0 \end{pmatrix}.$$

Table 6 compares the performance of the series and Bernstein methods. For $N \leq 9$, both methods produce nearly identical results (Improvement factor ≈ 1.00). At $N = 10$, the series method begins to pull ahead with a 7.77× improvement, followed by a sharp jump to 50150.67× at $N = 20$. By $N = 20$, the series method achieves near machine precision (7.78×10^{-13}) and outperforms Bernstein by a factor of over 50150.67, all while keeping CPU times within 0.0479–0.1023 seconds.

Table 6. Breakthrough performance of series matrix method versus Bernstein method for Example 4.

N	Series Error	Bernstein Error	Series Time (s)	Bernstein Time (s)	Improvement
5	2.70×10^{-6}	2.70×10^{-6}	0.1446	0.0950	1.00
6	1.00×10^{-7}	1.00×10^{-7}	0.0327	0.0343	1.00
7	3.01×10^{-9}	3.01×10^{-9}	0.0540	0.0312	1.00
9	1.97×10^{-12}	1.98×10^{-12}	0.0263	0.0403	1.00
10	5.44×10^{-14}	4.23×10^{-13}	0.0430	0.0519	7.77
12	6.76×10^{-14}	1.02×10^{-11}	0.0284	0.0490	151.20
15	1.68×10^{-13}	5.75×10^{-11}	0.0337	0.0692	342.53
20	7.78×10^{-13}	3.90×10^{-8}	0.0479	0.1023	50150.67

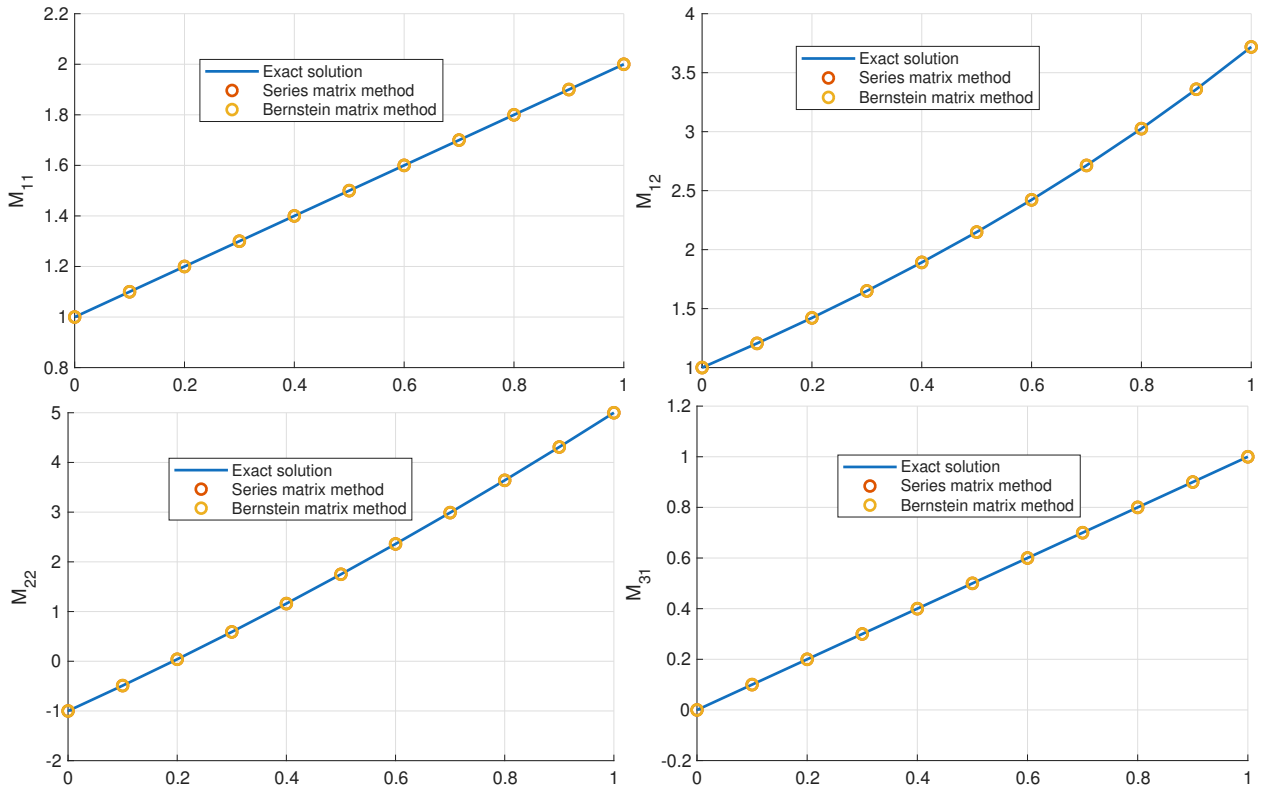

Figure 9. Approximate solutions for $N = 9$ and the exact solution of Example 4.

Figure 9 shows the approximate solutions for $N = 9$ alongside the exact solution, confirming excellent agreement across all matrix components. Figure 10 depicts the error decay with increasing N, clearly showing the sharp jump in performance between $N = 10$ and $N = 12$, followed by exponential growth of the improvement factor. Finally, Figure 11 presents the pointwise error norm for $N = 10$, illustrating that the series method consistently delivers smaller errors over the entire domain.

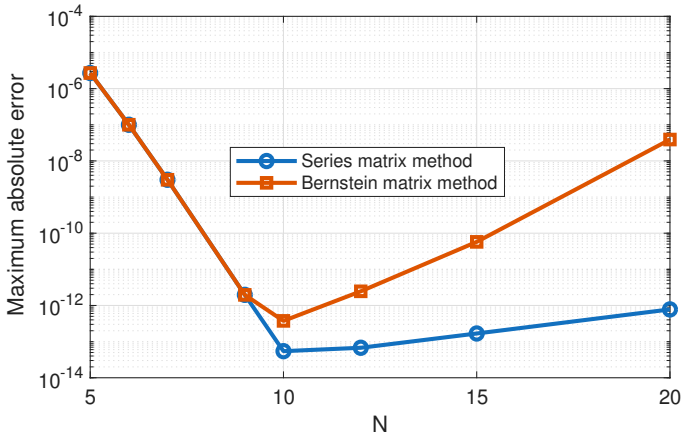

Figure 10. Error comparison versus approximation order N for Example 4.

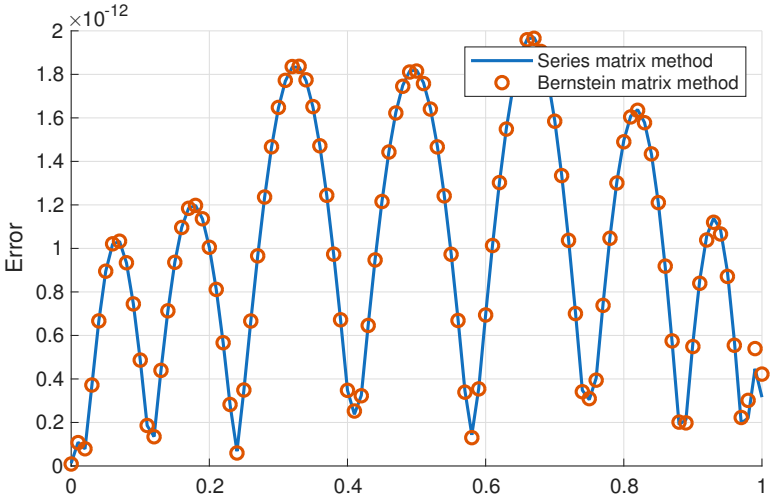

Figure 11. Norm of the error for $N = 9$ in Example 4.

The paper introduces a series polynomial collocation method for solving first-order linear matrix differential equations, leveraging a new polynomial basis and Kronecker-based algebraic reformulation. The technique achieves dramatic accuracy improvements over the Bernstein polynomial approach.

6. Conclusions

This paper presented a series collocation matrix method for solving first-order linear matrix differential equations, leveraging series polynomial bases and collocation techniques. The method

transforms the original problem into algebraic systems via structured matrix operations, including Kronecker products, and demonstrates significant improvements in accuracy compared to existing approaches like the Bernstein method. The observed numerical superiority, particularly in the moderate-to-high-order regime, can be attributed to the advantageous properties of the series polynomial basis. Its structure, defined by $P_n(\varsigma) = \sum_{k=0}^{n} \varsigma^k$, may lead to a better-conditioned system matrix during the solution process compared to the Bernstein polynomial basis. This, in turn, enhances numerical stability and allows the collocation method to achieve higher precision with increasing approximation order N, as consistently demonstrated across all examples. The error analysis established a rigorous upper bound for the approximation error, confirming the stability and convergence of the method. Across all numerical examples, the series collocation approach consistently outperformed the Bernstein method in moderate-to-high order regimes, often achieving machine precision with minimal computational overhead. Future work may focus on extending the method to stochastic matrix differential equations, problems with discontinuous coefficients, or integration with adaptive collocation point selection to further optimize accuracy and efficiency. The promising results suggest that series polynomial collocation represents a powerful and efficient framework for solving matrix differential equations in scientific and engineering applications.

Author contributions

Lakhlifa Sadek and Ibtisam Aldawish: Conceptualization, Formal analysis, Software, Funding acquisition, Investigation, Methodology, Writing–original draft, Writing–review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2602).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. Cañada, P. Drábek, A. Fonda, *Handbook of differential equations: ordinary differential equations, Volume 3*, Elsevier, 2006.
2. L. Sadek, H. T. Alaoui, Application of MGA and EGA algorithms on large-scale linear systems of ordinary differential equations, *J. Comput. Sci.*, **62** (2022), 101719. https://doi.org/10.1016/j.jocs.2022.101719

3. L. Sadek, H. T. Alaoui, Numerical methods for solving large-scale systems of differential equations, *Ric. Mat.*, **72** (2023), 782–802. https://doi.org/10.1007/s11587-021-00585-1

4. J. Liu, Z. Zhang, Y. Xu, Lower and upper bounds of the solution for the Lyapunov matrix differential equation and an application in input-output finite-time stability of linear systems, *Appl. Math. Lett.*, **152** (2024), 109023. https://doi.org/10.1016/j.aml.2024.109023

5. A. Golbabai, S. P. A. Beik, D. K. Salkuyeh, A new approach for solving the first-order linear matrix differential equations, *Bull. Iran. Math. Soc.*, **42** (2016), 297–314.

6. E. Defez, L. Soler, A. Hervás, C. Santamaría, Numerical solution of matrix differential models using cubic matrix splines, *Comput. Math. Appl.*, **50** (2005), 693–699. https://doi.org/10.1016/j.camwa.2005.04.012

7. E. Defez, M. M. Tung, J. J. Ibáñez, J. Sastre, Approximating and computing nonlinear matrix differential models, *Math. Comput. Model.*, **55** (2012), 2012–2022. https://doi.org/10.1016/j.mcm.2011.11.060

8. L. N. Trefethen, *Approximation theory and approximation practice, extended edition*, Society for Industrial and Applied Mathematics, 2019. https://doi.org/10.1137/1.9781611975949

9. L. Sadek, A. S. Bataineh, O. R. Isik, H. T. Alaoui, I. Hashim, A numerical approach based on Bernstein collocation method: application to differential Lyapunov and Sylvester matrix equations, *Math. Comput. Simulat.*, **212** (2023), 475–488. https://doi.org/10.1016/j.matcom.2023.05.011

10. L. Sadek, S. Ounamane, B. Abouzaid, E. M. Sadek, The Galerkin Bell method to solve the fractional optimal control problems with inequality constraints, *J. Comput. Sci.*, **77** (2024), 102244. https://doi.org/10.1016/j.jocs.2024.102244

11. Z. Gu, Chebyshev spectral collocation method for system of nonlinear Volterra integral equations, *Numer. Algor.*, **83** (2020), 243–263. https://doi.org/10.1007/s11075-019-00679-w

12. L. Sadek, A. S. Bataineh, The general Bernstein function: Application to χ-fractional differential equations, *Math. Method. Appl. Sci.*, **47** (2024), 6117–6142. https://doi.org/10.1002/mma.9910

13. J. Liu, Z. Zhang, Y. Xu, The existence conditions of the positive semidefinite solution for the Lyapunov matrix differential equation and improvements in its bounds, *J. Franklin I.*, **362** (2025), 107979. https://doi.org/10.1016/j.jfranklin.2025.107979

14. J. Liu, Z. Zhang, W. Zeng, F. Tang, Upper and lower bounds for the solution of the Lyapunov matrix differential equation, *Linear Multilinear A.*, **72** (2024), 3099–3111. https://doi.org/10.1080/03081087.2024.2306320

15. S. U. Altinbasak, M. Demiralp, Solutions to linear matrix ordinary differential equations via minimal, regular, and excessive space extension based universalization, *J. Math. Chem.*, **48** (2010), 253–265. https://doi.org/10.1007/s10910-010-9665-7

16. H. Zheng, W. Han, On some discretization methods for solving a linear matrix ordinary differential equation, *J. Math. Chem.*, **49** (2011), 1026–1041. https://doi.org/10.1007/s10910-010-9794-z

17. E. Defez, A. Hervás, J. Ibáñez, M. M. Tung, Numerical solutions of matrix differential models using higher-order matrix splines, *Mediterr. J. Math.*, **9** (2012), 865–882. https://doi.org/10.1007/s00009-011-0159-z

18. W.-H. Steeb, *Matrix calculus and Kronecker product with applications and C++ programs*, World Scientific, 1997. https://doi.org/10.1142/3572

19. H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations in control and systems theory*, Basel: Birkhäuser, 2003. https://doi.org/10.1007/978-3-0348-8081-7

20. G. Söderlind, The logarithmic norm. History and modern theory, *BIT Numer. Math.*, **46** (2006), 631–652. https://doi.org/10.1007/s10543-006-0069-9

21. J. Shen, T. Tang, L. L. Wang, *Spectral methods: algorithms, analysis and applications*, Berlin, Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-540-71041-7

22. C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral methods: fundamentals in single domains*, Berlin, Heidelberg: Springer, 2006. https://doi.org/10.1007/978-3-540-30726-6

AIMS Press