Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1239–1265.
DOI: 10.3934/math.2026053
Received: 13 November 2025
Revised: 19 December 2025
Accepted: 24 December 2025
Published: 16 January 2026

Research article

History-dependent generalized fractional differential hemivariational inequalities with application to contact mechanics

Furi Guo*

School of Mathematics and statistics Shanxi Datong University, Datong 037009, Shanxi, China

* **Correspondence:** Email: dtdxgfr@163.com.

Abstract: In this paper, we investigate a class of ψ-Caputo fractional differential hemivariational inequalities with history-dependent operators. By employing the Rothe method in conjunction with surjectivity results for multivalued pseudomonotone operators, the solvability of weak solutions to ψ-Caputo fractional differential hemivariational inequalities is obtained. As an application, a class of history-dependent viscoelastic friction contact problems that account for adhesion phenomena is investigated. In this contact model, the friction and contact conditions are described by Clarke generalized gradient of two nonconvex and nonsmooth function involving adhesion. Finally, the solvability of the solution for this friction contact model is established.

Keywords: hemivariational inequality; history-dependent operator; adhesion; ψ-Caputo fractional derivatives; viscoelastic contact model
Mathematics Subject Classification: 47J22, 49J40, 74M10, 74M15

1. Introduction

Fractional calculus, as an extension of integer-order calculus, has held a significant position in the field of mathematics since its inception. In materials science, particularly in the study of viscoelastic materials, fractional calculus has demonstrated substantial potential and advantages. By introducing fractional-order models, researchers can more accurately describe the dynamic behavior and memory characteristics of materials. For example, in 1947, G. W. Scott Blair introduced a viscoelastic element known as the Scott-Blair dashpot, whose dynamic behavior can be described by a time-fractional differential equation [1].

In the early 1980s, Greek mathematician Panagiotopoulos [2] first introduced the hemivariational inequality based on Clarke generalized gradient. As a generalization of variational inequality theory, hemivariational inequality has also attracted great attention from scholars since its inception. Considerable progress has been achieved regarding the solvability, numerical approximation, simulation of hemivariational inequalities, as well as their applications in nonsmooth contact problems, see, e.g., [3–5]. At the same time, some progress has been made in the research on the optimal control theory of hemivariational inequalities in contact mechanics problems see, e.g., [6–8]. The research results of fractional differential hemivariational inequality can be seen in [9–11]. Recently, research on ψ-fractional differential systems can be found in [12–14].

In [15, 16], the authors investigated two types of viscoelastic contact models within the framework of Caputo derivatives, obtaining two fractional hemivariational inequalities. They also demonstrated the solvability of these fractional hemivariational inequalities by applying the Rothe method in combination with monotone operator theory. In [17], the author studied a class of viscoelastic contact models within the framework of ψ-fractional calculus and established the existence of solutions. However, the viscoelastic constitutive law in [17] does not incorporate a history-dependent operator. Inspired by the aforementioned studies, this paper investigates viscoelastic contact problems with history-dependent operators within the framework of ψ-fractional calculus. A new ψ-fractional hemivariational inequality is derived, and the solvability of the contact model is established. Our research model exhibits greater generality, serving as an extension of the models presented in [15–17].

The phenomenon of adhesion is the result of the interaction between the surface forces that are at play when two separate bodies are in some form of contact. Adhesion can be demonstrated by friction between two solid material bodies during relative sliding, aggregation and the sintering of solid powder, etc. Therefore, adhesion processes often occur and have important applications in industrial settings. In [18], the author introduced the concept of the bonding field . In this paper, we use θ to denote a bonding field; this represents the intensity of adhesion, and $\theta \in [0, 1]$. When $\theta = 1$, this means that all of the bonds are active and that the adhesion is complete. When $\theta = 0$, there is no adhesion; that is to say, all of the bonds are inactive and severed. When $0 < \theta < 1$, there is partial adhesion, and only a fraction θ of the bonds are active. Various frictional contact problems with adhesion have been introduced and considered for different conditions some of them can be found in [19–21]. This paper uses a ψ-fractional differential equation to characterize the bonding field.

The current paper contains three main innovations. Firstly, we consider a viscoelastic frictional contact model featuring a long-term memory constitutive law formulated within the framework of ψ-Caputo fractional derivatives. This model is entirely new. Second, we account for the adhesion phenomenon occurring during the contact process, where the adhesion field is characterized by a differential equation based on ψ-Caputo fractional derivatives. The outcome of our study is a coupled system comprising a ψ-Caputo fractional differential hemivariational inequality and a ψ-Caputo fractional differential equation. Our results demonstrate enhanced generality and constitute an extension of the findings in [17], which also represents the novel contribution of this paper.

The organization of the rest of this article is as follows: in Section 2, some preliminaries are introduced. In Section 3, the unique solvability of ψ-Caputo fractional differential equations is presented, and the existence of solutions for ψ-Caputo fractional differential hemivariational inequalities with history-dependent operators is obtained. In Section 4, the theoretical results from Sectional 3 are applied to a class of frictional contact model in witch adhesion is taken into account.

2. Preliminaries

Let $(E, \|\cdot\|_E)$ be a Banach space and let E^* represent the dual of E. Let V be a reflexive Banach space and V^* represent the dual of V. Let Y be a separable and reflexive Banach space. The dual space of Y is denoted by Y^*.

Definition 2.1. *([3]) Assume that $j : E \to \mathbb{R}$. Then, the Clarke directional derivative of j at the point $u \in E$ in the direction $v \in E$ is defined by*

$$j^0(u; v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda},$$

where j is a locally Lipschitz function and $\lambda \in \mathbb{R}^+$. The generalized gradient of j at $u \in E$ is defined by the set

$$\partial j(u) = \{\xi \in E^* \mid j^0(u; v) \geq \langle \xi, v\rangle \text{ for all } v \in E\}.$$

Definition 2.2. *([22]) A nonempty, bounded, closed and convex multivalued operator $T : V \to 2^{V^*}$ is said to be pseudomonotone, if $v_n \rightharpoonup v$ weakly in V and $v_n^* \rightharpoonup v^*$ weakly in V^* with $v_n^* \in T v_n$ and $\limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle_{V^*\times V} \leq 0$, then $v^* \in T v$ and $\langle v_n^*, v_n\rangle_{V^*\times V} \to \langle v^*, v\rangle_{V^*\times V}$.*

Theorem 2.3. *([22]) Let the multivalued operator $T : V \to 2^{V^*}$ be pseudomonotone and coercive. Then, T is surjective, i.e., for every $\pi \in V^*$, there is $u \in V$ such that $T u \ni \pi$.*

Lemma 2.4. *([23]) Nonnegative sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ satisfy*

$$a_n \leq b_n + \sum_{k=1}^{n-1} c_k a_k \ \text{ for } n \geq 1.$$

Then, we have

$$a_n \leq b_n + \sum_{k=1}^{n-1} c_k b_k \exp\Big(\sum_{j=k+1}^{n-1} c_j\Big) \text{ for } n \geq 1.$$

Moreover, if $\{a_n\}$ and $\{c_n\}$ are such that

$$a_n \leq m + \sum_{k=1}^{n-1} c_k a_k \text{ for } n \geq 1,$$

where constant $m > 0$. Then, for all $n \geq 1$, it holds

$$a_n \leq m \exp\Big(\sum_{k=1}^{n-1} c_k\Big).$$

Definition 2.5. *([12]) Let X be a reflexive and separable Banach space. Let $\alpha > 0$, $I = (0, T)(T > 0)$, $x \in L^1(I, X)$, and let $\psi \in C^1(I)$ be an increasing function such that $\psi'(t) \neq 0$ on I. The $\alpha > 0$ order ψ-fractional integral of x is defined by*

$$I_{0,t}^{\alpha;\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) \mathrm{d}s, \quad t \in I,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.6. *([12]) Let $\alpha > 0$, $n \in \mathbb{N}$, $x \in C^n(I)$ and let $\psi \in C^n(I)$ be an increasing function such that $\psi'(t) \neq 0$, for all $t \in I$, the ψ-Caputo fractional derivative of order α for x is defined by*

$$ {}^{C}D_{0,t}^{\alpha;\psi} x(t) = I_{0,t}^{n-\alpha;\psi}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n x(t), \quad t \in I, $$

where $n = [\alpha] + 1$ for $\alpha \neq \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$. In particular, if $0 < \alpha \leq 1$, we have

$$ {}^{C}D_{0,t}^{\alpha;\psi} x(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (\psi(t) - \psi(s))^{-\alpha} x'(s) \mathrm{d}s, & \text{if } \alpha \in (0,1), \\ \frac{x'(t)}{\psi'(t)}, & \text{if } \alpha = 1. \end{cases} $$

3. Main results

Problem 1. *For a given $\phi \in L^2(I, Y)$, find $x : I \to X$ such that*

$$ \begin{cases} {}^{C}D_{0,t}^{\alpha;\psi} x(t) = G(t, x(t), \phi(s)) \text{ for a.e. } t \in I, \\ x(0) = x_0. \end{cases} \tag{3.1} $$

Taking the fractional integral of α-order to the Eq (3.1), we have

$$ \begin{aligned} x(t) &= I_{0,t}^{\alpha;\psi} G(t, x(t), \phi(t)) + c_0, \quad c_0 \in \mathbb{R} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), \phi(s)) \mathrm{d}s + c_0, \quad t \in I. \end{aligned} \tag{3.2} $$

Let $t = 0$ in (3.2) and using $x(0) = x_0$, it yields that $c_0 = x_0$. So, we obtain the integral equivalent form of the boundary value problem (3.1) as follows

$$ x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), \phi(s)) \mathrm{d}s + x_0, \quad t \in I. \tag{3.3} $$

According to [13, Lemma 2.1], we obtain that Problem 1 is equivalent to the following Problem 2.

Problem 2. *Find $x \in C(I; X)$ such that*

$$ x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), \phi(s)) \mathrm{d}s + x_0, \quad t \in I. \tag{3.4} $$

In what follows, we establish the solvability of Problem 2. To this end, we provide some necessary assumptions. We denote $K_0 = \|\widetilde{\theta}\|_{L^{\frac{1}{\hat{p}}}([0,T])}$, where $\widetilde{\theta}(s) = \|G(s, x(s), \phi(s))\|_Y$ and $0 < \alpha < \hat{p} < 1$, $M_0 = \max_{t\in I}\{\psi'(t)\}$ and $\hat{q} = \frac{\alpha-1}{1-\hat{p}}$.

(H_G): $G : I \times X \times Y \to Y$ is such that

$$ \begin{cases} \text{(i) the mapping } t \mapsto G(t, x, \eta) \text{ is continuous for all } x \in X, \eta \in Y; \\ \text{(ii) there exists } L_G > 0, \text{ such that} \\ \qquad \|G(t, x_1, \eta_1) - G(t, x_2, \eta_2)\|_Y \leq L_G(\|x_1 - x_2\|_X + (\|\eta_1 - \eta_2\|_Y) \\ \quad \text{for all } x_1, x_2 \in X, \eta_1, \eta_2 \in Y, t \in I; \\ \text{(iii) } t \mapsto G(t, 0, 0) \text{ belongs to } L^2(I; Y). \end{cases} $$

(H_0) : There exists a constant $M_1 > 0$, such that

$$\frac{M_0 L_G}{\Gamma(\alpha)}\Big(\frac{\psi(T)^{(\hat{q}+1)}}{1+\hat{q}}\Big)^{1-\hat{p}}\Big(\frac{\hat{p}}{M_1}\Big)^{\hat{p}} < 1.$$

Lemma 3.1. *Assume that* (H_0) *and* (H_G) *hold. Then, for a given* $\phi \in L^2(I, Y)$, $0 < \alpha \le 1$, $\psi(0) = 0$, *the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}G(s, x(s), \phi(s))\mathrm{d}s + x_0, \quad t \in I. \tag{3.5}$$

has a unique solution $x \in C(I, X)$.

Proof. We prove that Eq (3.5) admits a unique solution x. To this end, we consider an equivalent norm of $C(I; X)$, which defined by $\|x\|_* = \sup_{t\in I} e^{-M_1 t}\|x(t)\|$ for all $x \in C(I; X)$. We define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}G(s, x(s), \phi(s))\mathrm{d}s + x_0, \quad t \in I. \tag{3.6}$$

Next, we show that operator $\mathcal{T}$ admits a unique fixed point in $C(I; X)$. The proof process is divided into two steps.

Step1. For any $x \in C(I; X)$, $\mathcal{T}x \in C(I, X)$. In fact, for any $\delta > 0$ and $x \in C(I; X)$. By using the condition (H_G) and Holder inequality, we have

$$\begin{aligned}
&\|(\mathcal{T}x)(t+\delta) - (\mathcal{T}x)(t)\|_X \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)[(\psi(t)-\psi(s))^{\alpha-1} - (\psi(t+\delta)-\psi(s))^{\alpha-1}]\|G(s, x(s), \phi(s))\|_Y\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} \psi'(s)(\psi(t+\delta)-\psi(s))^{\alpha-1}\|G(s, x(s), \phi(s))\|_Y\mathrm{d}s \\
&\le \frac{M_0}{\Gamma(\alpha)}\int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1} - (\psi(t+\delta)-\psi(s))^{\alpha-1}\Big)\widetilde{\theta}(s)\mathrm{d}s \\
&\quad + \frac{M_0}{\Gamma(\alpha)}\int_t^{t+\delta} (\psi(t+\delta)-\psi(s))^{\alpha-1}\widetilde{\theta}(s)\mathrm{d}s \\
&\le \frac{M_0}{\Gamma(\alpha)}\Big(\int_0^t \Big((\psi(t)-\psi(s))^{\alpha-1} - (\psi(t+\delta)-\psi(s))^{\alpha-1}\Big)^{\frac{1}{1-\hat{p}}}\mathrm{d}s\Big)^{1-\hat{p}}\Big(\int_0^t (\widetilde{\theta}(s))^{\frac{1}{\hat{p}}}\mathrm{d}s\Big)^{\hat{p}} \\
&\quad + \frac{M_0}{\Gamma(\alpha)}\Big(\int_t^{t+\delta} (\psi(t+\delta)-\psi(s))^{\frac{\alpha-1}{1-\hat{p}}}\mathrm{d}s\Big)^{1-\hat{p}}\Big(\int_t^{t+\delta} (\widetilde{\theta}(s))^{\frac{1}{\hat{p}}}\mathrm{d}s\Big)^{\hat{p}} \\
&\le \frac{M_0 K_0}{\Gamma(\alpha)}\Big(\int_0^t \Big((\psi(t)-\psi(s))^{\hat{q}} - (\psi(t+\delta)-\psi(s))^{\hat{q}}\Big)\mathrm{d}s\Big)^{1-\hat{p}} \\
&\quad + \frac{M_0 K_0}{\Gamma(\alpha)}\Big(\int_t^{t+\delta} (\psi(t+\delta)-\psi(s))^{\hat{q}}\mathrm{d}s\Big)^{1-\hat{p}} \\
&\le \frac{M_0 K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\Big(\psi(t)^{1+\hat{q}} + (\psi(t+\delta)-\psi(t))^{1+\hat{q}} - \psi(t+\delta)^{1+\hat{q}}\Big)^{1-\hat{p}} \\
&\quad + \frac{M_0 K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\Big(\psi(t+\delta)-\psi(t)\Big)^{(1+\hat{q})(1-\hat{p})}
\end{aligned}$$

$$\leq \frac{M_0K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\Big((\psi'(\zeta)\delta)^{1+\hat{q}}\Big)^{1-\hat{p}} + \frac{M_0K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\Big(\psi'(\zeta)\delta\Big)^{(1+\hat{q})(1-\hat{p})}$$
$$\leq \frac{2M_0^{(1+\hat{q})(1-\hat{p})+1}K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\delta^{(1+\hat{q})(1-\hat{p})}.$$

Thus, we have $\frac{2M_0^{(1+\hat{q})(1-p)+1}K_0}{\Gamma(\alpha)(1+\hat{q})^{1-\hat{p}}}\delta^{(1+\hat{q})(1-\hat{p})} \to 0$, as $\delta \to 0$. Similarly, when $\delta < 0$, we have

$$\|(\mathcal{T}x)(t+\delta) - (\mathcal{T}x)(t)\|_X \to 0 \text{ as } \delta \to 0.$$

So, we know that $\mathcal{T}x \in C(I;X)$.

Step2. We will check $\mathcal{T}$ is a contractive map. For any $x_1, x_2 \in C(I;X)$, using (H_G) and Holder inequation, we have

$$\begin{aligned}
&\|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)\|_X \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|G(s,x_1(s),\phi(s)) - G(s,x_2(s),\phi(s))\|_Y \mathrm{d}s \\
&\leq \frac{M_0L_G}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|x_1(s)-x_2(s)\|_X \mathrm{d}s \\
&= \frac{M_0L_G}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}(\|x_1(s)-x_2(s)\|_X e^{-M_1 s})e^{M_1 s}\mathrm{d}s \\
&\leq \frac{M_0L_G}{\Gamma(\alpha)}\|x_1-x_2\|_* \int_0^t (\psi(t)-\psi(s))^{\alpha-1}e^{M_1 s}\mathrm{d}s \\
&\leq \frac{M_0L_G}{\Gamma(\alpha)}\|x_1-x_2\|_* \Big(\int_0^t (\psi(t)-\psi(s))^{\frac{\alpha-1}{1-\hat{p}}}\mathrm{d}s\Big)^{1-\hat{p}}\Big(\int_0^t (e^{M_1 s})^{\frac{1}{\hat{p}}}\mathrm{d}s\Big)^{\hat{p}} \\
&\leq \frac{M_0L_G}{\Gamma(\alpha)}\|x_1-x_2\|_* \Big(\frac{1}{1+\hat{q}}\Big)^{1-\hat{p}}(\psi(T))^{(\hat{q}+1)(1-\hat{p})}\Big(\frac{\hat{p}}{M_1}\Big)^{\hat{p}}e^{M_1 t}.
\end{aligned}$$

According to the above inequality, we have

$$\|(\mathcal{T}x_1)(t) - (\mathcal{T}x_2)(t)\|_X e^{-M_1 t} \leq \frac{M_0L_G}{\Gamma(\alpha)}\Big(\frac{\psi(T)^{(\hat{q}+1)}}{1+\hat{q}}\Big)^{1-\hat{p}}\Big(\frac{\hat{p}}{M_1}\Big)^{\hat{p}}\|x_1-x_2\|_* \text{ for all } t \in I,$$

which implies that

$$\|(\mathcal{T}x_1) - (\mathcal{T}x_2)\|_* \leq \frac{M_0L_G}{\Gamma(\alpha)}\Big(\frac{\psi(T)^{(\hat{q}+1)}}{1+\hat{q}}\Big)^{1-\hat{p}}\Big(\frac{\hat{p}}{M_1}\Big)^{\hat{p}}\|x_1-x_2\|_*.$$

Based on the assumptions (H_0) and Banach fixed point theorem, we get $\mathcal{T}$ has a unique solution $x \in C(I;X)$. □

Let W be separable and reflexive Banach spaces. W^* and $\mathcal{W}^*$ represent the dual spaces of W and $\mathcal{W}$, respectively. Define the following function spaces

$$\mathcal{W} = L^p(I;W), \quad \mathcal{W}^* = L^q(I;W^*) \quad \text{and} \quad \mathcal{V} = \{y \in \mathcal{W} \mid^C D_{0,t}^{\beta;\psi} y \in \mathcal{W}\},$$

where $0 < \beta \leq 1$, $\frac{1}{\beta} < p < +\infty$ and $q = \frac{p}{p-1}$.

Problem 3. *Find* $(x, y) \in C(I, X) \times \mathcal{V}$ *such that*

$$\begin{cases} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), M(y(t))\mathrm{d}s + x_0 \quad \text{for a.e. } t \in I, \\ \langle A({}^C D_{0,t}^{\beta;\psi} y(t)) + By(t) - f(t), v(t)\rangle_{Y\times Y^*} + \langle(\mathcal{R}({}^C D_{0,t}^{\beta;\psi} y))(t), v(t)\rangle \\ \quad + j^0(x(t), My(t); Mv(t)) \geq 0 \quad \text{for } v \in \mathcal{W}, \text{ a.e. } t \in I, \\ x(0) = x_0, \ \ y(0) = y_0. \end{cases} \tag{3.7}$$

Let $u(t) = {}^C D_{0,t}^{\beta;\psi} y(t)$ in Problem 3. One has

$$y(t) = I_{0,t}^{\beta;\psi} u(t) + y_0 \quad \text{for a.e.} \quad t \in I.$$

Therefore, Problem 3 can be rewritten in the following form.

Problem 4. *Find* $(x, u) \in C(I, X) \times \mathcal{W}$ *such that*

$$\begin{cases} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), M(I_{0,s}^{\beta;\psi} u(s) + y_0))\mathrm{d}s + x_0 \quad \text{for a.e. } t \in I, \\ \langle A(u(t)) + B(I_{0,t}^{\beta;\psi} u(t) + y_0) - f(t), v(t)\rangle_{Y\times Y^*} + \langle(\mathcal{R}u)(t), v(t)\rangle \\ \quad + j^0(x(t), M(I_{0,t}^{\beta;\psi} u(t) + y_0); Mv(t)) \geq 0 \quad \text{for } v \in \mathcal{W}, \text{ a.e. } t \in I. \end{cases} \tag{3.8}$$

Further, we know that the Problem 4 can be translated into the following inclusion Problem 5.

Problem 5. *Find* $(x, u) \in C(I, X) \times \mathcal{W}$ *such that*

$$\begin{cases} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x(s), M(I_{0,s}^{\beta;\psi} u(s) + y_0))\mathrm{d}s + x_0 \ \text{for a.e. } t \in I, \\ A(u(t)) + B(I_{0,t}^{\beta;\psi} u(t) + y_0) + (\mathcal{R}u)(t) + M^*\partial j(x(t), M(I_{0,t}^{\beta;\psi} u(t) + y_0)) \ni f(t) \\ \text{for a.e. } t \in I, \end{cases} \tag{3.9}$$

where M^* *denotes the adjoint operator of* M.

The following hypothesis are proposed regarding the data in Problem 5.
(H_A) : $A \in \mathcal{L}(W, W^*)$ and there exists a constant $m_A > 0$ such that $\langle Au, u\rangle \geq m_A\|u\|_W^2$ for all $u \in W$.
(H_B) : $B \in \mathcal{L}(W, W^*)$ and $\langle Bu, u\rangle \geq 0$ for all $u \in W$.
(H_M) : The compact operator $M \in \mathcal{L}(W, Y)$.
(H_J) : The function $J : X \times Y \to \mathbb{R}$ is such that

$$\begin{cases} \text{(i) } y \mapsto J(x, y) \text{ is locally Lipschitz for all } x \in X; \\ \text{(ii) } \|\partial J(x, y)\|_{Y^*} \leq L_J(1 + \|y\|_Y) \ \text{ for all } x \in X \text{ and } y \in Y, \text{ where contact } L_J > 0; \\ \text{(iii) } (x, y) \mapsto J^0(x, y; v) \text{ is upper semicontinuous from } X \times Y \text{ into } \mathbb{R} \text{ for all } v \in Y. \end{cases}$$

(H_h): The operator $h \in C(I \times I, \mathcal{L}(W, W^*)$ satisfies

$$\|h(t_1, s) - h(t_2, s)\| \leq L_h|t_1 - t_2| \text{ for all } s, t_1, t_2 \in I,$$

with $L_h > 0$ and $m_h = \max_{t,s \in I\times I} \|h(t,s)\|$.
$(H_{\mathcal{R}})$: The operator $\mathcal{R} \in \mathcal{L}(W, W^*)$ is defined by

$$(\mathcal{R}u)(t) = \mathcal{S}\left(\int_0^t h(t,s)u(s)\mathrm{d}s + \epsilon_{\mathcal{R}}\right),$$

and there exists constant $L_{\mathcal{R}} > 0$ such that

$$\|(\mathcal{R}u_1)(t) - (\mathcal{R}u_2)(t)\|_{W^*} \leq L_{\mathcal{R}} \int_0^t \|u_1(s) - u_2(s)\|_W \mathrm{d}s$$

for all $u_1(t), u_2(t) \in C(I; W)$, where $\mathcal{S} \in \mathcal{L}(W, W^*)$, $\epsilon_{\mathcal{R}} \in W$, and $L_{\mathcal{R}} = m_h\|\mathcal{S}\|$.
(H_1) : $f \in \mathcal{W}^*$ and $y_0 \in W$.

Theorem 3.2. *Assume that the conditions $(H_A), (H_B), (H_M), (H_J), (H_h)$, $(H_{\mathcal{R}})$, (H_G) and (H_1) hold. Then Problem 5 admits at last one solution $(x, u) \in C(I; X) \times \mathcal{W}$*

We use the Rothe method(temporally semi-discrete scheme) to prove Theorem 3.2.
Let $N \in \mathbb{N}^+$, $\tau = T/N$, and the equidistant nodes of the interval $[0, T]$ are denoted by $\{t_k\}_{k=0}^N = \{k\tau\}_{k=0}^N$, and $f_\tau(t) = f_\tau^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(s)\mathrm{d}s$ for $t \in (t_{k-1}, t_k], k = 1, 2, ..., N$. Next, we consider the discretized format of Problem 5.

Problem 6. *Find $\{u_\tau^k\}_{k=0}^N \in W$, $x_\tau \in C(I; X)$ and $\{\eta_\tau^k\}_{k=0}^N \in Y^*$ such that $y_\tau^0 = y_0$, and*

$$x_\tau(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} G(s, x_\tau(s), M\widehat{y_\tau}(s))\mathrm{d}s + x_0 \text{ for a.e. } t \in (0, t_k), \tag{3.10}$$

$$Au_\tau^k + By_\tau^k + w_\tau^k + M^*\eta_\tau^k \ni f_\tau^k \tag{3.11}$$

with $\eta_\tau^k \in \partial J(x_\tau(t_k), My_\tau^k)$ for $k = 1, 2, ..., N$, where y_τ^k and $\widehat{y_\tau}(t)$ for $t \in (0, t_k)$ are defined by

$$\begin{aligned} y_\tau^k &= y_0 + \frac{1}{\Gamma(\beta)}\sum_{j=1}^k u_\tau^j \int_{t_{j-1}}^{t_j} \psi'(s)(\psi(t_k) - \psi(s))^{\beta-1}\mathrm{d}s \\ &= y_0 + \frac{1}{\Gamma(\beta+1)}\sum_{j=1}^k \Big(((\psi(t_k) - \psi(t_{j-1}))^\beta - (\psi(t_k) - \psi(t_j))^\beta\Big)u_\tau^j \\ &= y_0 + \frac{\tau^\beta}{\Gamma(\beta+1)}\sum_{j=1}^k \Big((\psi'(\xi_{j-1}))^\beta (k-j+1)^\beta - (\psi'(\xi_j))^\beta (k-j)^\beta\Big)u_\tau^j \end{aligned} \tag{3.12}$$

where $\xi_{j-1} \in (t_k, t_{j-1})$, $\xi_j \in (t_k, t_j)$, and

$$\widehat{y_\tau}(t) = \begin{cases} \sum_{j=1}^N \mathcal{X}_{(t_{j-1}, t_j]}(t) y_\tau^{j-1}, & 0 < t \leq T, \\ y_0, & t = 0, \end{cases} \tag{3.13}$$

respectively. Here, $\mathcal{X}_{(t_{j-1}, t_j]}$ be defined by

$$\mathcal{X}_{(t_{j-1}, t_j]}(t) = \begin{cases} 1, & t \in (t_{j-1}, t_j], \\ 0, & \textit{otherwise}. \end{cases}$$

Moreover, the element w_τ^k is given as follows

$$w_\tau^k = \mathcal{S}\Big(\epsilon_{\mathcal{R}} + \sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} h(t_k,s)u_\tau^j \mathrm{d}s + \int_{t_{k-1}}^{t_k} h(t_k,s)u_\tau^k \mathrm{d}s\Big).$$

Theorem 3.3. *Assume that the conditions $(H_A), (H_B), (H_M), (H_J), (H_h), (H_{\mathcal{R}}), (H_G)$ and (H_1) hold. Then Problem 6 has at least one solution $(u_\tau^k, x_\tau) \in W \times C(I;X)$.*

Proof. For the given $y_\tau^0, y_\tau^1, ..., y_\tau^{n-1}$, we will prove the existence of y_τ^n, η_τ^n and x_τ that satisfy (3.10) and (3.11). Using the definition of y_τ^k, we can obtain $y_\tau^0, y_\tau^1, ..., y_\tau^{n-1}$. Therefor, $\widehat{y_\tau}$ is welldefined in $(0, t_n)$. It's easy to know that $\widehat{y_\tau} \in L^2(0, t_n; W)$. Furthermore, we can verify that $\widehat{y_\tau}$ satisfies all the conditions of Lemma 3.1. Now we consider inclusion problem (3.11) and write it in the following equivalent form

$$Au_\tau^k + By_\tau^k + \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k,s)u_\tau^k \mathrm{d}s) + M^*\partial J(x_\tau(t_k), My_\tau^k) \ni F_\tau^k \tag{3.14}$$

where

$$F_\tau^k = f_\tau^k - \mathcal{S}(\epsilon_{\mathcal{R}} + \sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} h(t_k,s)u_\tau^j \mathrm{d}s).$$

Let

$$\widehat{\vartheta} = y_0 + \frac{\tau^\beta}{\Gamma(\beta+1)}\sum_{j=1}^{n-1}\Big((\psi'(\xi_{j-1}))^\beta(k-j+1)^\beta - (\psi'(\xi_j))^\beta(k-j)^\beta\Big)u_\tau^j,$$

$$\widehat{c} = \frac{\tau^\beta}{\Gamma(\beta+1)}(\psi'(\xi_{n-1}))^\beta, \quad \xi_{n-1} \in (t_n, t_{n-1}).$$

It easy to see that inclusion problem (3.14) equivalent to the following inclusion problem. Find $\vartheta \in W$ such that

$$A\vartheta + B(\widehat{\vartheta} + \widehat{c}\vartheta) + \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k,s)\vartheta \mathrm{d}s) + M^*\partial J(x_\tau(t_k), M(\widehat{\vartheta} + \widehat{c}\vartheta)) \ni F_\tau^k. \tag{3.15}$$

We will show that the multivalued operator $\mathcal{G} : W \to 2^{W^*}$ defined by

$$\mathcal{G}\vartheta = A\vartheta + B(\widehat{\vartheta} + \widehat{c}\vartheta) + \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k,s)\vartheta \mathrm{d}s) + M^*\partial J(x_\tau(t_k), M(\widehat{\vartheta} + \widehat{c}\vartheta)) \tag{3.16}$$

is surjective. To this end, we first prove that operator $\mathcal{G}$ is coercive.

Since

$$\begin{aligned}&\Big|\langle B(\widehat{\vartheta} + \widehat{c}\vartheta) + \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k,s)\vartheta \mathrm{d}s) + M^*\partial J(x_\tau(t_k), M(\widehat{\vartheta} + \widehat{c}\vartheta)), \vartheta\rangle_{W^*\times W}\Big|\\ &\le \|B\|\|\widehat{\vartheta}\|\|\vartheta\| + \widehat{c}\|B\|\|\vartheta\|^2 + \|\mathcal{S}\|\int_{t_{k-1}}^{t_k}\|h(t_k,s)\|\|\vartheta\|^2 \mathrm{d}s\\ &\quad + L_J\|M\|(1 + \|\widehat{\vartheta}\|\|M\|)\|\vartheta\| + L_J\widehat{c}\|M\|^2\|\vartheta\|^2\end{aligned}$$

$$\leq \Big(\|B\|\|\widehat{\vartheta}\| + L_J\|M\|(1+\|\widehat{\vartheta}\|\|M\|)\Big)\|\vartheta\| + \tau m_h\|\mathcal{S}\|\|\vartheta\|^2 + \frac{\tau^\beta M_0{}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2)\|\vartheta\|^2. \tag{3.17}$$

Next, we choose

$$\tau_0 = \Big(\frac{(m_A - Tm_h\|\mathcal{S}\|)\Gamma(\beta+1)}{M_0{}^\beta(\|B\| + L_J\|M\|^2)}\Big)^{\frac{1}{\beta}}.$$

From the $H(A)$ and (3.17), one has

$$\begin{aligned}\langle \xi, \vartheta\rangle_{W^*\times W} \geq &\Big(m_A - \tau m_h\|\mathcal{S}\| - \frac{\tau^\beta M_0{}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2)\Big)\|\vartheta\|^2\\ &- \Big(\|B\|\|\widehat{\vartheta}\| + L_J\|M\|(1+\|\widehat{\vartheta}\|\|M\|)\Big)\|\vartheta\|\end{aligned}$$

for all $\xi \in \mathcal{G}\vartheta$ and $0 < \tau < \tau_0$. Furthermore, we get

$$\begin{aligned}\frac{\langle \xi, \vartheta\rangle_{W^*\times W}}{\|\vartheta\|} \geq &\Big(m_A - \tau m_q\|\mathcal{S}\| - \frac{\tau^\beta M_0{}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2)\Big)\|\vartheta\|\\ &- \Big(\|B\|\|\widehat{\vartheta}\| + L_J\|M\|(1+\|\widehat{\vartheta}\|\|M\|)\Big) \text{ for all } \xi \in \mathcal{G}\vartheta,\end{aligned}$$

which implies that operator $\mathcal{G}$ is coercive.

On the other hand, we show that operator $\mathcal{G}$ is pseudomonotone. Under the condition (H_A), (H_B) and $(H_{\mathcal{R}})$, we obtain that for every $\vartheta \in W$, the set $\mathcal{G}\vartheta$ is nonempty, bounded, closed and convex, this means that $\mathcal{G}$ is nonempty, bounded, closed and convex operator. Let $\vartheta_n \rightharpoonup \vartheta$ in W as $n \to \infty$, $\vartheta_n^* \in \mathcal{G}\vartheta_n$, $\vartheta_n^* \rightharpoonup \vartheta^*$ in W^* as $n \to \infty$, and $\limsup_{n\to\infty}\langle \vartheta_n^*, \vartheta_n - \vartheta\rangle_{W^*\times W} \leq 0$. We will check $\vartheta^* \in \mathcal{G}\vartheta$ and

$$\lim_{n\to\infty}\langle \vartheta_n^*, \vartheta_n\rangle_{W^*\times W} = \langle \vartheta^*, \vartheta\rangle_{W^*\times W}. \tag{3.18}$$

First, since $A, B, \mathcal{S} \in \mathcal{L}(W, W^*)$ and $\vartheta_n \rightharpoonup \vartheta$ in W as $n \to \infty$, we know that

$$A\vartheta_n \rightharpoonup A\vartheta \text{ in } W^*, \tag{3.19}$$

$$B(\widehat{\vartheta} + \widehat{c}\vartheta_n) \rightharpoonup B(\widehat{\vartheta} + \widehat{c}\vartheta) \text{ in } W^*, \tag{3.20}$$

$$\mathcal{S}(\int_{t_{k-1}}^{t_k} q(t_k, s)\vartheta_n \mathrm{d}s) \rightharpoonup \mathcal{S}(\int_{t_{k-1}}^{t_k} q(t_k, s)\vartheta \mathrm{d}s) \text{ in } W^*. \tag{3.21}$$

According to the compactness of M and $\vartheta_n \rightharpoonup \vartheta$ in W as $n \to \infty$, one has

$$M(\widehat{\vartheta} + \widehat{c}\vartheta_n) \to M(\widehat{\vartheta} + \widehat{c}\vartheta) \text{ in } W. \tag{3.22}$$

Let $\xi_n \in \partial J(M(\widehat{\vartheta} + \widehat{c}\vartheta_n))$. Based on (H_J), we obtain that $\{\xi_n\} \subset W^*$ is bounded. So, there is convergent subsequence, which is still represented by $\{\xi_n\}$. Further, one has

$$\xi_n \rightharpoonup \xi \text{ in } W^* \text{ as } n \to \infty. \tag{3.23}$$

According to convergence results (3.22) and (3.23), we get $\xi \in \partial J(M(\widehat{\vartheta} + \widehat{c\vartheta}))$. By virtue of (3.19)–(3.23), we can obtain

$$\lim_{n\to\infty} \vartheta_n^* = \lim_{n\to\infty} (A\vartheta_n + B(\widehat{\vartheta} + \widehat{c\vartheta_n}) + \mathcal{S}(\int_{t_{k-1}}^{t_k} q(t_k, s)\vartheta_n \mathrm{d}s) + M^*\xi_n)$$
$$= A\vartheta + B(\widehat{\vartheta} + \widehat{c\vartheta}) + \mathcal{S}(\int_{t_{k-1}}^{t_k} q(t_k, s)\vartheta \mathrm{d}s) + M^*\xi.$$

Using $\vartheta_n^* \rightharpoonup \vartheta^*$ as $n \to \infty$, we get

$$\vartheta^* = A\vartheta + B(\widehat{\vartheta} + \widehat{c\vartheta}) + \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta \mathrm{d}s) + M^*\xi,$$

which implies that $\vartheta^* \in \mathcal{G}\vartheta$. Subsequently, we prove that $\lim_{n\to\infty}\langle\vartheta_n^*, \vartheta_n\rangle_{W^*\times W} = \langle\vartheta^*, \vartheta\rangle_{W^*\times W}$. Using (H_B),(H_G) and $\vartheta_n \rightharpoonup \vartheta$ in W, we have

$$\limsup_{n\to\infty}\langle B(\widehat{\vartheta} + \widehat{c\vartheta_n}), \vartheta - \vartheta_n\rangle_{W^*\times W}$$
$$\le \limsup_{n\to\infty}\langle B(\widehat{\vartheta} + \widehat{c\vartheta}), \vartheta - \vartheta_n\rangle_{W^*\times W} = 0 \tag{3.24}$$

and

$$\limsup_{n\to\infty}\langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta_n \mathrm{d}s), \vartheta - \vartheta_n\rangle_{W^*\times W}$$
$$\le \limsup_{n\to\infty}\langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta \mathrm{d}s), \vartheta - \vartheta_n\rangle_{W^*\times W} = 0. \tag{3.25}$$

Applying $\vartheta_n^* \in \mathcal{G}\vartheta_n$, we have

$$\limsup_{n\to\infty}\langle\vartheta_n^*, \vartheta_n\rangle = \limsup_{n\to\infty}(\langle A\vartheta_n + B(\widehat{\vartheta} + \widehat{c\vartheta_n}), \vartheta_n\rangle_{X^*\times X} + \langle M^*\xi_n, M\vartheta_n\rangle_{W^*\times W})$$
$$+ \limsup_{n\to\infty}\langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta_n \mathrm{d}s), \vartheta_n\rangle_{W^*\times W}.$$

From, (3.22), (3.24)–(3.26) and $\limsup_{n\to\infty}\langle\vartheta_n^*, \vartheta_n - \vartheta\rangle_{W^*\times W} \le 0$, we obtain

$$\limsup_{n\to\infty}\langle A\vartheta_n, \vartheta_n - \vartheta\rangle_{W^*\times W}$$
$$\le \limsup_{n\to\infty}\langle\vartheta_n^*, \vartheta_n - \vartheta\rangle_{W^*\times W} + \limsup_{n\to\infty}\langle B(\widehat{\vartheta} + \widehat{c\vartheta_n}, \vartheta - \vartheta_n)\rangle_{W^*\times W}$$
$$+ \limsup_{n\to\infty}\langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta_n \mathrm{d}s), \vartheta - \vartheta_n\rangle_{W^*\times W} + \limsup_{n\to\infty}\langle M^*\xi_n, M(\vartheta - \vartheta_n)\rangle \le 0.$$

On the other hand, by using the monotonicity of A, it follows that

$$\limsup_{n\to\infty}\langle A\vartheta_n, \vartheta_n - \vartheta\rangle_{W^*\times W} \ge \limsup_{n\to\infty}\langle A\vartheta, \vartheta_n - \vartheta\rangle_{W^*\times W} = 0.$$

Based on the discussion above, we get

$$\lim_{n\to\infty}\langle A\vartheta_n, \vartheta_n\rangle = \langle A\vartheta, \vartheta\rangle. \tag{3.26}$$

By a similar scheme, we can get the following conclusion

$$\lim_{n\to\infty} \langle B(\widehat{\vartheta} + \widehat{c}\vartheta_n), \vartheta_n\rangle_{W^*\times W} = \langle B(\widehat{\vartheta} + \widehat{c}\vartheta), \vartheta\rangle_{W^*\times W} \tag{3.27}$$

and

$$\lim_{n\to\infty} \langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta_n \mathrm{d}s), \vartheta_n\rangle_{W^*\times W} = \langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta \mathrm{d}s), \vartheta\rangle_{W^*\times W}. \tag{3.28}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\langle \vartheta_n^*, \vartheta_n\rangle_{W^*\times W} &= \lim_{n\to\infty}\langle A\vartheta_n, \vartheta_n\rangle_{W^*\times W} + \lim_{n\to\infty}\langle B(\widehat{\vartheta} + \widehat{c}\vartheta_n), \vartheta_n\rangle_{W^*\times W}\\ &+ \langle \mathcal{S}(\int_{t_{k-1}}^{t_k} h(t_k, s)\vartheta_n \mathrm{d}s), \vartheta_n\rangle_{W^*\times W} + \lim_{n\to\infty}\langle \xi_n, M\vartheta_n\rangle_{W^*\times W} = \langle \vartheta^*, \vartheta\rangle_{W^*\times W}.\end{aligned}$$

So, operator $\mathcal{G}$ is pseudomonotone. Using the Theorem 2.3, we obtain that $\mathcal{G}$ is surjective for all $0 < \tau < \tau_0$. The proof of the Theorem 3.3 is completed. □

Next, we will provide results of priori estimates for the sequence of solution of Problem 6.

Lemma 3.4. *Assume that the hypotheses* $(H_A), (H_B), (H_M), (H_J), (H_h)$*,* $(H_{\mathcal{R}})$*,* (H_G) *and* (H_1) *hold. Then, there exist* $\tau_0 > 0$ *and* $C_i > 0 (i = 1, 2, 3, 4)$ *independent of* τ *, such that* $\tau \in (0, \tau_0)$*, the solutions of Problem 6 satisfy*

$$\max_{k=1,2,\ldots,N} \|u_\tau^k\| \le C_1, \tag{3.29}$$

$$\max_{k=1,2,\ldots,N} \|y_\tau^k\| \le C_2, \tag{3.30}$$

$$\max_{k=1,2,\ldots,N} \|w_\tau^k\| \le C_3, \tag{3.31}$$

$$\max_{k=1,2,\ldots,N} \|\eta_\tau^k\| \le C_4, \tag{3.32}$$

where $\eta_\tau^k \in \partial J(x_\tau(t_k), My_\tau^k)$ *with*

$$Au_\tau^k + By_\tau^k + w_\tau^k + M^*\eta_\tau^k = f_\tau^k \ \ (k = 1, 2, \cdots, N).$$

Proof. Based on hypothesis (H_B) and (3.12), we get

$$\begin{aligned}\langle By_\tau^n, u_\tau^n\rangle &= \Big\langle B\Big(y_0 + \frac{\tau^\beta}{\Gamma(\beta+1)}\sum_{j=1}^{n}((\psi'(\xi_{j-1}))^\beta(n-j+1)^\beta - (\psi'(\xi_j))^\beta(n-j)^\beta)u_\tau^j\Big), u_\tau^n\Big\rangle \\ &\ge -\|By_0\|_{V^*}\|u_\tau^n\| - \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}\|B\|\sum_{j=1}^{n-1}((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\|\|u_\tau^n\| - \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}\|B\|\|u_\tau^n\|^2.\end{aligned} \tag{3.33}$$

From hypothesis $(H_{\mathcal{S}})$, (H_h) and definition of w_τ^n, we obtain

$$\begin{aligned}\langle w_\tau^n, u_\tau^n\rangle &= \Big\langle \mathcal{S}\Big(\epsilon_{\mathcal{R}} + \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} h(t_n, s)u_\tau^j \mathrm{d}s + \int_{t_{n-1}}^{t_n} h(t_n, s)u_\tau^n \mathrm{d}s\Big), u_\tau^n\Big\rangle \\ &\geq -\|\mathcal{S}\|(\|\epsilon_{\mathcal{R}}\| + \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \|h(t_n, s)\|\|u_\tau^j\|\mathrm{d}s)\|u_\tau^n\| - \tau\|\mathcal{S}\|\|h(t_n, s)\|\|u_\tau^n\|^2 \\ &\geq -\|\mathcal{S}\|(\|\epsilon_{\mathcal{R}}\| + L_h\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \|u_\tau^j\|\mathrm{d}s)\|u_\tau^n\| - TL_h\|\mathcal{S}\|\|u_\tau^n\|^2. \end{aligned} \tag{3.34}$$

Using the (H_J), we obtain

$$\begin{aligned}\langle \xi_\tau^n, Mu_\tau^n\rangle &\geq -L_J(1 + \|My_\tau^n\|)\|Mu_\tau^n\| \\ &\geq -L_J\|Mu_\tau^n\|\Big(1 + \|My_0\| + \frac{\tau^\beta\|M\|}{\Gamma(\beta+1)}\sum_{j=1}^{n}((\psi'(\xi_{j-1}))^\beta(n-j+1)^\beta \\ &\quad - (\psi'(\xi_j))^\beta(n-j)^\beta)\|u_\tau^j\|\Big) \\ &\geq -L_J\Big(\|M\| + \|M\|^2\|y_0\|\Big)\|u_\tau^n\| - \frac{L_J\tau^\beta {M_0}^\beta\|M\|^2}{\Gamma(\beta+1)}\|u_\tau^n\|^2 \\ &\quad - \frac{L_J\tau^\beta {M_0}^\beta\|M\|^2}{\Gamma(\beta+1)}\sum_{j=1}^{n-1}((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\|\|u_\tau^n\|. \end{aligned} \tag{3.35}$$

According to the coercivity of A and inequalities (3.33)–(3.35), we have

$$\begin{aligned}\langle f_\tau^n, u_\tau^n\rangle &= \langle Au_\tau^n + By_\tau^n + w_\tau^n, u_\tau^n\rangle + \langle \xi_\tau^n, Mu_\tau^n\rangle \\ &\geq m_A\|u_\tau^n\|^2 - \Big(\frac{\tau^\beta M_0^\beta\|B\|}{\Gamma(\beta+1)} + \frac{L_J\tau^\beta M_0^\beta\|M\|^2}{\Gamma(\beta+1)} + TL_h\|\mathcal{S}\|\Big)\|u_\tau^n\|^2 \\ &\quad - \Big(\|By_0\|_{V^*} + L_J\|M\| + L_J\|M\|^2\|y_0\| + \|\mathcal{S}\|\|\epsilon_{\mathcal{R}}\|\Big)\|u_\tau^n\| \\ &\quad - L_h\|\mathcal{S}\|\sum_{j=1}^{n-1}\|u_\tau^j\|\int_{t_{j-1}}^{t_j}\mathrm{d}s\|u_\tau^n\| \\ &\quad - \frac{\tau^\beta M_0^\beta\|B\|}{\Gamma(\beta+1)}\sum_{j=1}^{n-1}((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\|\|u_\tau^n\| \\ &\quad - \frac{L_J\tau^\beta {M_0}^\beta\|M\|^2}{\Gamma(\beta+1)}\sum_{j=1}^{n-1}((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\|\|u_\tau^n\|, \end{aligned}$$

and

$$\begin{aligned}&\Big(m_A - \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2) - TL_h\|\mathcal{S}\|\Big)\|u_\tau^n\| \\ &\leq \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2)\sum_{j=1}^{n-1}((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\| + L_h\|\mathcal{S}\|\sum_{j=1}^{n-1}\|u_\tau^j\|\int_{t_{j-1}}^{t_j}\mathrm{d}s\end{aligned}$$

$$+ \|By_0\|_{W^*} + L_J\|M\| + L_J\|M\|^2\|y_0\| + \|\mathcal{S}\|\|\epsilon_{\mathcal{R}}\| + \|f_\tau^n\|_{W^*}.$$

Let $\tau_0 = \left(\frac{(m_A - TL_h\|\mathcal{S}\|)\Gamma(\beta+1)}{2M_0{}^\beta(\|B\|+L_J\|M\|^2)}\right)^{\frac{1}{\beta}}$, we know that $m_A - \frac{\tau^\beta M_0{}^\beta}{\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2) - TL_h\|\mathcal{S}\| \geq \frac{m_A}{2}$ for all $\tau \in (0, \tau_0)$. Furthermore, we get

$$\|u_\tau^n\| \leq \frac{2(\|By_0\|_{W^*} + L_J\|M\| + L_J\|M\|^2\|y_0\| + \|\mathcal{S}\|\|\epsilon_{\mathcal{R}}\|)}{m_A} + \frac{2L_h\|\mathcal{S}\| \sum_{j=1}^{n-1} \|u_\tau^j\| \int_{t_{j-1}}^{t_j} \mathrm{d}s}{m_A}$$
$$+ \frac{2\tau^\beta M_0{}^\beta}{m_A\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2) \sum_{j=1}^{n-1} ((n-j+1)^\beta - (n-j)^\beta)\|u_\tau^j\| + \frac{2\|f_\tau^n\|_{W^*}}{m_A}.$$

Using (H_1), one has $\|f_\tau^n\| \leq c_f$ for all $\tau > 0, n \in N$, where $c_f > 0$. Let

$$C_0 = \frac{2(\|By_0\|_{Y^*} + L_J\|M\| + L_J\|M\|^2\|y_0\| + \|\mathcal{S}\|\|\epsilon_{\mathcal{R}}\|)}{m_A} + \frac{2c_f}{m_A}.$$

By using Gronwall inequality, we know that

$$\|u_\tau^n\| \leq C_0 \exp\left(\frac{2\tau^\beta M_0^\beta}{m_A\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2) \sum_{j=1}^{n-1} ((n-j+1)^\beta - (n-j)^\beta) + 2L_h\|\mathcal{S}\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \mathrm{d}s\right)$$
$$\leq C_0 \exp\left(\frac{2M_0^\beta}{m_A\Gamma(\beta+1)}(\|B\| + L_J\|M\|^2)t_n^\beta + 2TL_h\|\mathcal{S}\|\right)$$
$$\leq C_0 \exp\left(\frac{2M_0^\beta(\|B\| + L_J\|M\|^2)T^\beta}{m_A\Gamma(\beta+1)} + 2TL_h\|\mathcal{S}\|\right) := C_1.$$

From (3.12), we know that

$$\|y_\tau^n\| = \left\|y_0 + \frac{\tau^\beta}{\Gamma(\beta+1)} \sum_{j=1}^{n} \Big((\psi'(\xi_{j-1}))^\beta(n-j+1)^\beta - (\psi'(\xi_j))^\beta(n-j)^\beta\Big)u_\tau^j\right\|$$
$$\leq \|y_0\| + \frac{C_1M_0{}^\beta}{\Gamma(\beta+1)} \sum_{j=1}^{n}(t_{n-j+1}^\beta - t_{n-j}^\beta)$$
$$\leq \|y_0\| + \frac{C_1M_0{}^\beta}{\Gamma(\beta+1)}t_n^\beta$$
$$\leq \|y_0\| + \frac{C_1M_0{}^\beta T^\beta}{\Gamma(\beta+1)} := C_2.$$

Under the hypotheses $(H_{\mathcal{S}})$ and (H_h), we have

$$\|w_\tau^n\| = \left\|\mathcal{S}\Big(\epsilon_{\mathcal{R}} + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} h(t_n, s)u_\tau^n ds\Big)\right\| \leq \|\mathcal{S}\|(\|\epsilon_{\mathcal{R}}\| + TL_hC_1) := C_3.$$

The condition $(H_J)(ii)$ ensures that

$$\|\eta_\tau^n\| \leq L_J(1 + \|My_\tau^n\|) \leq L_J(1 + C_2\|M\|) := C_4.$$

So far, the proof of the Lemma 3.4 has been completed. □

Next, in order to provide the existence result of the solution to Problem 3, we define the piecewise constant interpolant functions $\widetilde{u}_\tau, \widetilde{y}_\tau, \widetilde{w}_\tau : [0,T] \to W, \eta_\tau : [0,T] \to Y^*$ and $f_\tau : [0,T] \to W^*$ as follows

$$
\begin{aligned}
\widetilde{u}_\tau(t) &= u_\tau^n, \quad t \in (t_{n-1}, t_n], \\
\widetilde{y}_\tau(t) &= y_\tau^n, \quad t \in (t_{n-1}, t_n], \\
\widetilde{w}_\tau(t) &= w_\tau^n, \quad t \in (t_{n-1}, t_n], \\
\eta_\tau(t) &= \eta_\tau^n, \quad t \in (t_{n-1}, t_n], \\
f_\tau(t) &= f_\tau^n, \quad t \in (t_{n-1}, t_n]
\end{aligned}
$$

for $n = 1, 2, ..., N$.

Theorem 3.5. *Assume that* $(H_A), (H_B), (H_M), (H_J), (H_h)$, $(H_\mathcal{R})$, (H_G) *and* (H_1) *hold. Let* $\frac{1}{\beta} < p < +\infty$ *and* $\{\tau_n\}$ *be a sequence satisfies* $\tau_n \to 0 (n \to +\infty)$. *Here, for convenience, we still use* τ *to represent the subsequence of* $\{\tau_n\}$. *Then, the following conclusion holds:*

$$
\begin{aligned}
\widetilde{u}_\tau &\rightharpoonup u \, (\tau \to 0), \quad in \quad L^p(0,T;W), \\
\eta_\tau &\rightharpoonup \eta \, (\tau \to 0), \quad in \quad L^q(0,T;Y^*), \\
x_\tau &\to x \, (\tau \to 0), \quad in \quad C(0,T;X),
\end{aligned}
$$

where $(u, x) \in L^p(0,T;W) \times C(0,T;X)$ *is a solution of Problem 5.*

Proof. Due to $\|u_\tau^n\| \le C_1$, we have

$$
\|\widetilde{u}_\tau\|_{L^p(0,T;W)}^p = \int_0^T \|\widetilde{u}_\tau(s)\|^p ds = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|u_\tau^i\|^p ds = \tau \sum_{i=1}^N \|u_\tau^i\|^p \le C_5.
$$

So, we deduce that $\{\widetilde{u}_\tau\}$ is bounded in $L^p(0,T;W)$ which implies that there exists $u \in L^p(0,T;W)$ such that

$$
\widetilde{u}_\tau \rightharpoonup u \, (\tau \to 0), \quad \text{in} \quad L^p(0,T;W). \tag{3.36}
$$

For any $t \in [0,T]$ and $v^* \in W^*$, Let $\gamma(s) = (\psi(t) - \psi(s))^{\beta-1}\psi'(s)v^*\chi_{[0,t]}(s)$ for $s \in (0,t)$. Clearly, $\gamma \in L^q(0,T;W^*)$ because $\frac{1}{\beta} < p < +\infty$. So, we have

$$
\begin{aligned}
&\left| \langle v^*, \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \widetilde{u}_\tau(s) \mathrm{d}s - \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} u(s) \mathrm{d}s \rangle \right| \\
&\le \frac{1}{\Gamma(\beta)} \int_0^t |\langle (\psi(t) - \psi(s))^{\beta-1} \psi'(s) v^*, \widetilde{u}_\tau(s) - u(s) \rangle| \mathrm{d}s \\
&\le \frac{1}{\Gamma(\beta)} |\langle \gamma, \widetilde{u}_\tau - u \rangle_{L^q(0,T;W^*) \times L^p(0,T;W)}| \to 0, \quad \text{as } \tau \to 0.
\end{aligned}
$$

So, we can get

$$
I_{0,t}^{\beta;\psi} \widetilde{u}_\tau(t) \rightharpoonup I_{0,t}^{\beta;\psi} u(t) \quad \text{in } W, \quad \text{as } \tau \to 0 \tag{3.37}
$$

for all $t \in I$. Moreover, one has

$$\begin{aligned}
\|\widetilde{y}_\tau(t) - y_0 - I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t)\| &= \Big\|\frac{\tau^\beta}{\Gamma(\beta+1)}\sum_{j=1}^{n}\Big((\psi'(\xi_{j-1}))^\beta(n-j+1)^\beta - (\psi'(\xi_j))^\beta(n-j)^\beta\Big)u_\tau^j \\
&\quad - \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\widetilde{u}_\tau(s)\mathrm{d}s\Big\| \\
&= \frac{1}{\Gamma(\beta)}\Big\|\int_0^{t_n}(\psi(t_n)-\psi(s))^{\beta-1}\widetilde{u}_\tau(s)\psi'(s)\mathrm{d}s \\
&\quad - \int_0^t (\psi(t_n)-\psi(s))^{\beta-1}\widetilde{u}_\tau(s)\psi'(s)\mathrm{d}s\Big\| \\
&\le \frac{1}{\Gamma(\beta)}\Big\|\int_t^{t_n}(\psi(t_n)-\psi(s))^{\beta-1}\widetilde{u}_\tau(s)\psi'(s)\mathrm{d}s\Big\| \\
&\quad + \frac{1}{\Gamma(\beta)}\Big\|\int_0^t \Big[(\psi(t_n)-\psi(s))^{\beta-1} - (\psi(t)-\psi(s))^{\beta-1}\Big]\widetilde{u}_\tau(s)\psi'(s)\mathrm{d}s\Big\| \\
&\le \frac{C_5}{\Gamma(\beta+1)}\|(\psi(t_n)-\psi(t))^\beta\| \\
&\quad + \frac{C_5}{\Gamma(\beta+1)}\|\psi^\beta(t_n) - (\psi(t_n)-\psi(t))^\beta - \psi^\beta(t)\|
\end{aligned}$$

for $t \in (t_{n-1}, t_n]$. Therefor, we can deduce that

$$\widetilde{y}_\tau(t) \to y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t) \quad \text{in } W, \quad \text{as } \tau \to 0, \tag{3.38}$$

for all $t \in I$. Using (36), we conclude

$$\widetilde{y}_\tau(t) \rightharpoonup y_0 + I_{0,t}^{\beta;\psi}u(t) \quad \text{in } W, \quad \text{as } \tau \to 0, \tag{3.39}$$

for $t \in I$. Furthermore, according to the compactness of M, we have

$$M(\widetilde{y}_\tau(t)) \to M(y_0 + I_{0,t}^{\beta;\psi}u(t)) \quad \text{in } Y, \quad \text{as } \tau \to 0, \tag{3.40}$$

for $t \in I$. Meanwhile, we obtain

$$\begin{aligned}
\|\widetilde{y}_\tau(t) - \widehat{y}_\tau(t)\| &= \frac{\tau^\beta}{\Gamma(\beta+1)}\Big\|\sum_{j=1}^{n}\Big((\psi'(\xi_{j-1}))^\beta(n-j+1)^\beta - (\psi'(\xi_j))^\beta(n-j)^\beta\Big)u_\tau^j \\
&\quad - \sum_{j=1}^{n-1}\Big((\psi'(\xi_j))^\beta(n-j)^\beta - (\psi'(\xi_{j+1}))^\beta(n-j-1)^\beta\Big)u_\tau^j\Big\| \\
&= \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}\sum_{j=1}^{n-1}\Big|(n-j+1)^\beta - 2(n-j)^\beta + (n-j-1)^\beta\Big|\,\|u_\tau^j\| \\
&\quad + \frac{\tau^\beta {M_0}^\beta}{\Gamma(\beta+1)}\|u_\tau^n\| \\
&\le \frac{\tau^\beta {M_0}^\beta C_1}{\Gamma(\beta+1)}\Big(\sum_{j=1}^{n-1}\Big|(n-j+1)^\beta - 2(n-j)^\beta + (n-j-1)^\beta\Big| + 1\Big)
\end{aligned}$$

$$\leq \frac{\tau^{\beta} M_0{}^{\beta} C_1}{\Gamma(\beta+1)}(1+n^{\beta}-(n-1)^{\beta}) \to 0, \quad \text{as } \tau \to 0, \tag{3.41}$$

for $t \in (t_{n-1}, t_n]$. Based on (3.41) and the compactness of M, it is true that

$$M(\widehat{y_\tau}(t)) \to M(y_0 + I_{0,t}^{\beta;\psi} u(t)) \quad \text{in } Y, \quad \text{as } \tau \to 0, \tag{3.42}$$

for $t \in (t_{n-1}, t_n]$. Since $u \in L^p(0,T;W)$, it is obvious that $M(y(t)) \in L^2(0,T;W)$. Lemma 3.1 implies that there is a unique solution $x \in (0,T;X)$ that satisfies

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} G(s, x(s), M(y(s)))\mathrm{d}s + x_0. \tag{3.43}$$

Using condition (H_G), we have

$$\begin{aligned}
\|x_\tau(t) - x(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \|G(s, x_\tau(s), M(\widehat{y_\tau}(s))) - G(s, x(s), M(y(s)))\|\mathrm{d}s \\
&\leq \frac{L_G}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \Big(\|x_\tau(s) - x(s)\| + \|M(\widehat{y_\tau}(s)) + M(y(s))\|\Big)\mathrm{d}s \\
&\leq \frac{L_G}{\Gamma(\beta)} \Big(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\mathrm{d}s\Big) \|M\widehat{y_\tau} - My\|_{C(0,T;Y)} \\
&\quad + \frac{L_G}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \|x_\tau(s) - x(s)\|\mathrm{d}s \\
&\leq \frac{L_G \psi^{\beta}(T)}{\Gamma(\beta+1)} \|M\widehat{y_\tau} - My\|_{C(0,T;Y)} + \frac{L_G}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \|x_\tau(s) - x(s)\|\mathrm{d}s \\
&= \delta(\tau) + \frac{L_G}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \|x_\tau(s) - x(s)\|\mathrm{d}s
\end{aligned}$$

where $\delta(\tau) = \frac{L_G \psi^{\beta}(T)}{\Gamma(\beta+1)} \|M\widehat{y_\tau} - My\|_{C(0,T;Y)}$. Using Gronwall inequality, we have

$$\|x_\tau(t) - x(t)\| \leq C_6 \delta(\tau), \quad \forall t \in I.$$

From (3.42), we obtain

$$\|x_\tau(t) - x(t)\| \leq C_6 \delta(\tau) \to 0, \quad \text{as } \tau \to 0.$$

So, we have $x_\tau \to x$ in $C(I;X)$ as $\tau \to 0$. On the other hand, based on the boundedness of sequence $\{\eta_\tau\}$, it can be concluded that sequence $\{\eta_\tau\}$ has a convergent subsequence (still denoted as $\{\eta_\tau\}$), which implies there exists $\eta \in Y^*$, such that

$$\eta_\tau \rightharpoonup \eta \quad \text{in } Y^* \quad \text{as} \quad \tau \to 0. \tag{3.44}$$

By applying the conclusion of [3, Lemma 12], the fact that

$$\eta_\tau^k \in \partial J(x_\tau(t_k), M(y_\tau^k)) \quad \text{for} \quad k = 1, 2, ..., N,$$

and combined with $x_\tau \to x$ in $C(0,T;X)$, (3.42) and (3.44), then utilizing [3, Theorem 3.13], we can derive that

$$\eta(t) \in \partial J(x(t), M(y_0 + I_{0,t}^{\beta;\psi} u(t)))$$

for a.e. $t \in (0,T)$. From boundedness of sequence $\{\widetilde{u}_\tau\}$ and the hypothesis (H_h) , it is ensure that

$$\begin{aligned}
&\left\| \int_0^t h(t,s)\widetilde{u}_\tau(s)\mathrm{d}s - \int_0^{t_k} h(t_k,s)\widetilde{u}_\tau(s)\mathrm{d}s \right\| \\
&= \left\| \int_0^t h(t,s)\widetilde{u}_\tau(s)ds - \int_0^t h(t_k,s)\widetilde{u}_\tau(s)\mathrm{d}s - \int_t^{t_k} h(t_k,s)\widetilde{u}_\tau(s)\mathrm{d}s \right\| \\
&\leq \int_t^{t_k} \|h(t_k,s)\|\|\widetilde{u}_\tau(s)\|ds + \int_0^t \|h(t,s) - h(t_k,s)\|\|\widetilde{u}_\tau(s)\|\mathrm{d}s \\
&\leq \tau m_h C_5 + \tau L_h C_5 \to 0(\tau \to 0) \quad \text{for a.e. } t \in [t_{k-1}, t_k].
\end{aligned} \tag{3.45}$$

Furthermore, we introduce the Nemytskii operator $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{W} \to \mathcal{W}^*$ by

$$(\mathcal{G}_1 u)(t) = \mathcal{S}(\int_0^t h(t,s)u(s)\mathrm{d}s) \text{ and } (\mathcal{G}_2 u)(t) = \mathcal{S}(u(t))$$

for all $u \in Y$ and a.e. $t \in I$. According to (3.36), we have

$$\lim_{\tau\to 0} \langle \mathcal{G}_1 \widetilde{u}_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} = \langle \mathcal{G}_1 u, v\rangle_{\mathcal{W}^*\times\mathcal{W}}$$

for all $v \in \mathcal{W}$. Let

$$\omega_\tau(t) = \epsilon_{\mathcal{R}} + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} h(t_k,s)u_\tau^j \mathrm{d}s, \quad t \in (t_{j-1}, t_j].$$

According to $(H_{\mathcal{R}})$, (H_h) and (3.45), we have

$$\begin{aligned}
&\mathcal{G}_2(\omega_\tau - \epsilon_{\mathcal{R}}) - \mathcal{G}_1(\widetilde{u}_\tau) \\
&= \mathcal{S}(\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} h(t_k,s)u_\tau^j \mathrm{d}s) - \mathcal{S}(\int_0^t h(t,s)\widetilde{u}_\tau(t)\mathrm{d}s) \to 0 \text{ strongly in } \mathcal{W}^*, \text{ as } \tau \to 0,
\end{aligned}$$

which implies

$$\begin{aligned}
&\lim_{\tau\to 0}\langle \mathcal{G}_2 \omega_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} \\
&= \lim_{\tau\to 0} \left(\langle \mathcal{G}_2(\omega_\tau - \epsilon_{\mathcal{R}}) - \mathcal{G}_1(\widetilde{u}_\tau), v\rangle_{\mathcal{W}^*\times\mathcal{W}} + \langle \mathcal{G}_1(\widetilde{u}_\tau), v\rangle_{\mathcal{Y}^*\times\mathcal{Y}} + \langle \mathcal{G}_2(\epsilon_{\mathcal{R}}), v\rangle_{\mathcal{W}^*\times\mathcal{W}} \right) \\
&= \langle \mathcal{G}_1(u), v\rangle_{\mathcal{W}^*\times\mathcal{W}} + \langle \mathcal{G}_2(\epsilon_{\mathcal{R}}), v\rangle_{\mathcal{W}^*\times\mathcal{W}}.
\end{aligned} \tag{3.46}$$

Next, we prove that $(x,u) \in C(0,T;X) \times \mathcal{W}$ is the solution to Problem 5. To this end, we define the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ by

$$(\mathcal{A}v)(t) = A(v(t)), \quad (\mathcal{B}v)(t) = B(y_0 + I_{0,t}^{\beta;\psi} v(t)) \quad \text{and} \quad (\mathcal{M}v)(t) = M(v(t))$$

for $v \in \mathcal{W}$, a.e. $t \in (0, T)$, respectively. According to (3.19) and $A \in \mathcal{L}(W, W^*)$, we have

$$\mathcal{A}\widetilde{u}_\tau \rightharpoonup \mathcal{A}u \quad \text{in} \quad \mathcal{W}^* \quad \text{as} \quad \tau \to 0. \tag{3.47}$$

Under (3.20) and (H_B), we obtain

$$B(y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t)) \rightharpoonup B(y_0 + I_{0,t}^{\beta;\psi}u(t)) \quad \text{in} \quad \mathcal{W}^*, \quad \text{as} \quad \tau \to 0,$$

for all $t \in I$. Moreover, we have

$$\begin{aligned}
\langle \mathcal{B}\widetilde{y}_\tau, v\rangle &= \langle B(y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t)), v(t)\rangle \\
&\leq \|B(y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t))\|\|v(t)\| \\
&\leq \left(\frac{2\psi^\beta(T)\|B\|C_5}{\Gamma(\beta+1)} + T\|B\|\|y_0\|\right)\|v(t)\|.
\end{aligned}$$

Furthermore, by utilizing above inequality and Lebesgue-dominated convergence theorem, we can obtain

$$\begin{aligned}
\lim_{\tau\to 0}\langle \mathcal{B}\widetilde{y}_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} &= \lim_{\tau\to 0}\int_0^T \langle B(y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t)), v(t)\rangle \mathrm{d}t \\
&= \int_0^T \lim_{\tau\to 0}\langle B(y_0 + I_{0,t}^{\beta;\psi}\widetilde{u}_\tau(t)), v(t)\rangle \mathrm{d}t \\
&= \int_0^T \langle B(y_0 + I_{0,t}^{\beta;\psi}u(t)), v(t)\rangle \mathrm{d}t \\
&= \langle \mathcal{B}y, v\rangle_{\mathcal{W}^*\times\mathcal{W}}.
\end{aligned} \tag{3.48}$$

By using the compactness of the Nemytskii operator $\mathcal{M}$, we find that

$$\lim_{\tau\to 0}\langle \eta_\tau, \mathcal{M}v\rangle = \langle \eta, \mathcal{M}v\rangle \tag{3.49}$$

for all $v \in \mathcal{W}$. Moreover, according to [24, lemma 3.3], we know that

$$f_\tau \to f \text{ strongly in } \mathcal{W}^*, \text{ as } \tau \to 0. \tag{3.50}$$

From (3.46)–(3.50), we obtain the following result

$$\begin{aligned}
&\limsup_{\tau\to 0}\langle \mathcal{A}\widetilde{u}_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} + \limsup_{\tau\to 0}\langle \mathcal{B}\widetilde{y}_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} + \limsup_{\tau\to 0}\langle \mathcal{G}_2\omega_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} \\
&\quad + \limsup_{\tau\to 0}\langle \eta_\tau, \mathcal{M}v\rangle_{\mathcal{W}^*\times\mathcal{W}} - \liminf_{\tau\to 0}\langle f_\tau, v\rangle_{\mathcal{W}^*\times\mathcal{W}} \geq 0,
\end{aligned}$$

for all $v \in \mathcal{W}$. Thus, we have

$$\langle \mathcal{A}u + \mathcal{B}y + \mathcal{S}u + \mathcal{M}^*\eta, v\rangle_{\mathcal{W}^*\times\mathcal{W}} \geq \langle f, v\rangle_{\mathcal{W}^*\times\mathcal{W}},$$

where $\eta(t) \in \partial J(x(t), M(y_0 + I_{0,t}^{\beta;\psi}u(t)))$ for a.e. $t \in (0, T)$. This implies $(u, x) \in \mathcal{W} \times C(0, T; X)$ is a solution of Problem 5, which finishes the proof of the theorem. □

4. The mechanical model

In this section, we shall consider a class of history-dependent viscoelastic frictional contact problem. We use Ω to represent the open, bounded subset of $\mathbb{R}^d(d = 2, 3)$ occupied by a viscoelastic body. $\mathbb{S}^d$ stands for the second order symmetric $d \times d$ matrices. The boundary $\partial\Omega$ is assumed to be composed of three sets: Γ_D, Γ_N and Γ_C, with meas$(\Gamma_D) > 0$. We assume that the evolutionary process of the body belongs to time interval $t \in I$ with $T > 0$. $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t, \boldsymbol{x})$ and $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ represent the stress field and the displacement field, respectively. Consider the following two inner products and norms:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = \sqrt{(\boldsymbol{v} \cdot \boldsymbol{v})}, \quad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = \sqrt{(\boldsymbol{\tau} \cdot \boldsymbol{\tau})}, \quad \forall\, \boldsymbol{\sigma} \in (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

We provide the tangential and normal components of the vector as

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \quad \vartheta_\nu = \boldsymbol{\vartheta} \cdot \boldsymbol{\nu}, \quad \boldsymbol{\vartheta}_\tau = \boldsymbol{\vartheta} - \vartheta_\nu \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ represents the outward unit normal at Γ. The linearized strain tensors $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are expressed by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad (\varepsilon_{ij}(\boldsymbol{u})) = \frac{1}{2}(u_{i,j} + u_{j,i})(i, j = 1, \cdots, d),$$

where $u_{i,j} = \partial u_i / \partial x_j$. We denote that $\mathcal{D} = I \times \Omega, \mathcal{T}_D = I \times \Gamma_D, \mathcal{T}_C = I \times \Gamma_C$ and $\mathcal{T}_N = I \times \Gamma_N$.

Problem 7. *Find a displacement field* $\boldsymbol{u} : \mathcal{D} \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \mathcal{D} \to \mathbb{S}^d$ *and a bonding field* $\theta : \mathcal{T}_C \to [0, 1]$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi}\boldsymbol{u}(t)) + \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi}\boldsymbol{u}(t))ds \quad \text{in } \mathcal{D}, \tag{4.1}$$
$$\text{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \mathcal{D}, \tag{4.2}$$
$$\boldsymbol{u}(t) = 0 \quad \text{on } \mathcal{T}_D, \tag{4.3}$$
$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on } \mathcal{T}_N, \tag{4.4}$$
$$-\sigma_\nu(t) \in \partial j_\nu(\theta(t), u_\nu(t)) \quad \text{on } \mathcal{T}_C, \tag{4.5}$$
$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(\theta(t), \boldsymbol{u}_\tau(t)) \quad \text{on } \mathcal{T}_C, \tag{4.6}$$
$${}^C D_{0,t}^{\alpha;\psi}\theta(t) = Q(t, \theta(t), \boldsymbol{u}(t)) \quad \text{on } \mathcal{T}_C, \tag{4.7}$$
$$\theta(0) = \theta_0 \quad \text{on } \Gamma_C, \tag{4.8}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{4.9}$$

For the convenience of readers, we give a brief mechanical explanation for the equations and conditions in Problem 7. A general viscoelastic constitutive law is of the form

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))ds + \int_0^t \mathscr{D}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(t))ds. \tag{4.10}$$

Here, $\mathscr{A}$ represents the viscosity operator, $\mathscr{B}$ is the elasticity operator, and $\mathscr{C}$, $\mathscr{D}$ represent relaxation tensors. Consequently, the viscoelastic constitutive law (4.10) describes a nonhomogeneous material. Note that (4.10) illustrates the fact that the current value of the stress depends on the current value of the strain and strain rate, as well as on their history. Particular cases can be obtained, for instance, when $\mathscr{C} = \mathscr{D} \equiv 0$. Then, Eq (4.10) reduces to the so-called viscoelastic constitutive law with short memory

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))). \tag{4.11}$$

A second important particular case is obtained from (4.10) in the case when $\mathscr{A} = \mathscr{C} \equiv 0$. The corresponding constitutive law is the so-called viscoelastic constitutive law with long memory, i.e.

$$\boldsymbol{\sigma}(t) = \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{D}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(t))ds. \tag{4.12}$$

A third important particular case is obtained from (4.10) in the case when $\mathscr{D} \equiv 0$. The corresponding constitutive law is

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))ds. \tag{4.13}$$

Such constitutive laws have been used in the literature in order to model the behavior of real materials like rubbers, rocks, metals, pastes, and polymers. In particular, Eq (4.13) was employed in [25, 26] in order to model the hysteresis damping in elastomers. Introducing fractional calculus into friction contact problems is mainly to more accurately describe the memory, path dependence, and nonlinear characteristics of complex mechanical behaviors, thereby making up for the limitations of traditional integer-order models. Based on this, we study the (4.1) fractional viscoelastic constitutive relations

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi}\boldsymbol{u}(t))) + \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi}\boldsymbol{u}(t))ds.$$

Equation (4.2) represents the equation of motion. Here $\boldsymbol{f}_0$ represents the density of volume forces, and Div is the divergence operator. We assume that the body is held fixed on Γ_D, and therefore the displacement boundary condition satisfies condition (4.3). Equation (4.4) stands for the traction boundary condition and $\boldsymbol{f}_N$ represents the surface tractions on Γ_N. The normal contact condition (4.5) and the friction condition (4.6) are modeled by the Clarke subdifferential of a nonconvex potential j_ν and j_τ, respectively. Here, j_ν and j_τ depend on the adhesion $\theta(t)$. For a more detailed explanation of (4.5) and (4.6), please refer to [3, 27]. The function θ is the adhesion field which governed by a ψ-fractional ordinary differential equation (4.7) depending on the displacement. In (4.8) and (4.9), $\theta(0) = \theta_0$ and $\boldsymbol{u}(0) = \boldsymbol{u}_0$ denote the initial adhesion field and displacement field, respectively.

To obtain the variational formulation of problem 7, we provide the following function space $\mathcal{X}, \mathcal{Y}$, $\mathcal{Z}$, $\mathcal{V}$ and $\mathcal{H}$ defined by

$$\begin{aligned}
&\mathcal{X} = L^2(\Omega; \mathbb{S}^d), \ \ \mathcal{Y} = \{\boldsymbol{\vartheta} \in H^1(\Omega; \mathbb{R}^d) : \boldsymbol{\vartheta} = \boldsymbol{0} \text{ on } \Gamma_D\},\\
&\mathcal{Z} = L^2((\Gamma_C; \mathbb{R}^d), \ \ \mathcal{V} = L^2(\Omega; \mathbb{R}^d), \ \text{ and } \ \mathcal{H} = L^2(\Gamma_C),\\
&Q_\infty = \{\boldsymbol{\varepsilon} = (\varepsilon_{ijkl}) : \varepsilon_{ijkl} = \varepsilon_{jikl} = \varepsilon_{klij} \in L^\infty(\Omega), 1 \le i, j, k, l \le d\}.
\end{aligned}$$

It is obvious that $\mathcal{X}$ is Hilbert space with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_X = \int_\Omega \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})d\boldsymbol{x} \ \text{ for all } \ \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{X}$$

and the associated norm $\|\cdot\|_{\mathcal{X}}$. On the space $\mathcal{Y}$, we define the inner product by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{Y}} = \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{X}} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{Y}.$$

Moreover, we obtain that $\|\boldsymbol{\vartheta}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le C_k\|\widetilde{\gamma}\|\|\boldsymbol{\vartheta}\|_{\mathcal{Y}}$ for $\boldsymbol{\vartheta} \in \mathcal{Y}$, where $C_k > 0$ is the Korn constant and the trace operator is $\widetilde{\gamma} : \mathcal{Y} \to L^2(\Gamma_C; \mathbb{R}^d)$. Next, we give the hypotheses on the data $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\boldsymbol{Q}$, j_ν, j_τ, $\boldsymbol{f}_D$ and $\boldsymbol{f}_N$ as follows:
$(H_{\mathscr{A}})$: $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

$$\begin{cases} \text{(i) } \mathscr{A} = (a_{ijkl}) \in Q_\infty, \quad 0 \le i, j, k, l \le d; \\ \text{(ii) there exists } L_{\mathscr{A}} > 0 \text{ such that } \mathscr{A}\boldsymbol{\tau} : \boldsymbol{\tau} \ge L_{\mathscr{A}}\|\boldsymbol{\tau}\|^2_{\mathbb{S}^d} \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d. \end{cases}$$

$(H_{\mathscr{B}})$: $\mathscr{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

$$\begin{cases} \text{(i) } \mathscr{B} = (b_{ijkl}) \in Q_\infty, \quad 0 \le i, j, k, l \le d; \\ \text{(ii) } \mathscr{B}\boldsymbol{\tau} : \boldsymbol{\tau} \ge \boldsymbol{0} \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d. \end{cases}$$

$(H_{\mathscr{C}}) : \Omega \times (0, T) \times \mathbb{S}^d \to \mathbb{S}^d, \mathscr{C} \in C(0, T; Q_\infty)$ is such that

$$\begin{cases} \text{(i) } \mathscr{C} = (c_{ijkl}) \in Q_\infty, \quad 0 \le i, j, k, l \le d; \\ \text{(ii) } \mathscr{C}\boldsymbol{\tau} : \boldsymbol{\tau} \ge \boldsymbol{0} \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d; \\ \text{(iiI) } \mathscr{C} \text{ is Lipschitz continuous with Lipschitz constant } L_{\mathscr{C}} > 0. \end{cases}$$

$(H_{j_\nu}) : j_\nu : \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following conditions

$$\begin{cases} \text{(i) } j_\nu(\cdot, \theta, \mu) \text{ is measurable on } \Gamma_C \text{ for all } \mu \in \mathbb{R}, \theta \in \mathbb{R}; \\ \text{(ii) } j_\nu(\boldsymbol{x}, \theta, \cdot) \text{ is locally Lipschitz a.e. } \boldsymbol{x} \in \Gamma_C, \theta \in \mathbb{R}; \\ \text{(iii) } j_\nu(\boldsymbol{x}, \theta, \cdot) \text{ or } -j_\nu(\boldsymbol{x}, \theta, \cdot) \text{ is regular a.c. } \boldsymbol{x} \in \Gamma_C, \theta \in \mathbb{R}; \\ \text{(iv) there exists } c_\nu > 0 \text{ such that } |\partial j_\nu(\boldsymbol{x}, \theta, \mu)| \le c_\nu(1 + |\mu|) \text{ for all } \mu \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Sigma_C; \\ \text{(v) there exists } m_\nu > 0 \text{ such that } (\eta_1 - \eta_2)(\mu_1 - \mu_2) \ge -m_\nu|\mu_1 - \mu_2|^2 \\ \quad \text{for all } \eta_i \in \partial j_\nu(\boldsymbol{x}, \theta_i, \mu_i), \theta_i, \mu_i \in \mathbb{R}, i = 1, 2 \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \end{cases}$$

$(H_{j_\tau}) : j_\tau : \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{cases} \text{(i) } j_\tau(\cdot, \theta, \boldsymbol{\mu}) \text{ is measurable on } \Sigma_C, \text{ for all } \boldsymbol{\mu} \in \mathbb{R}^d, \theta \in \mathbb{R}; \\ \text{(ii) } j_\tau(\boldsymbol{x}, \theta, \cdot) \text{ is locally Lipschitz a.e. } \boldsymbol{x} \in \Gamma_C, \theta \in \mathbb{R}; \\ \text{(iii) } j_\tau(\boldsymbol{x}, \theta, \cdot) \text{ or } -j_\tau(\boldsymbol{x}, \theta, \cdot) \text{ is regular a.e. } \boldsymbol{x} \in \Gamma_C, \theta \in \mathbb{R}; \\ \text{(iv) there exists } c_\tau > 0 \text{ such that } |\partial j_\tau(\boldsymbol{x}, \theta, \boldsymbol{\mu})| \le c_\tau(1 + \|\boldsymbol{\mu}\|_{\mathbb{R}^d}) \text{ for all } \boldsymbol{\mu} \in \mathbb{R}^d \text{ a.e. } \boldsymbol{x} \in \Gamma_C; \\ \text{(v) there exists } m_\tau > 0 \text{ such that } (\boldsymbol{\eta_1} - \boldsymbol{\eta_2})(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \ge -m_\tau\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2_{\mathbb{R}^d} \\ \quad \text{for all } \boldsymbol{\eta_i} \in \partial j_\tau(\boldsymbol{x}, \theta_i, \boldsymbol{\mu}_i), \boldsymbol{\mu}_i \in \mathbb{R}^d, \theta_i \in \mathbb{R}\ (i = 1, 2) \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \end{cases}$$

$(H_Q): Q : \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies that

$$\begin{cases} \text{(i) } Q(\cdot, \kappa, \boldsymbol{\iota}) \text{ is measurable on } \Gamma_C \text{ for all } (\kappa, \boldsymbol{\iota}) \in \mathbb{R} \times \mathbb{R}^d; \\ \text{(ii) } |Q(\boldsymbol{x}, \kappa_1, \boldsymbol{\iota}_1) - Q(\boldsymbol{x}, \kappa_2, \boldsymbol{\iota}_2)| \leq L_Q(|\kappa_1 - \kappa_2| + \|\boldsymbol{\iota}_1 - \boldsymbol{\iota}_2\|) \\ \quad \text{for all } \kappa_1, \kappa_2 \in \mathbb{R}, \boldsymbol{\iota}_1, \boldsymbol{\iota}_2 \in \mathbb{R}^d \text{ and } a.e.\ \boldsymbol{x} \in \Gamma_C \text{ with } L_Q > 0; \\ \text{(iii) } Q(\boldsymbol{x}, 0, \boldsymbol{\iota}) = 0, Q(\boldsymbol{x}, \kappa, \boldsymbol{\iota}) \geq 0 \text{ for } \kappa \leq 0, \text{ and } Q(\boldsymbol{x}, \kappa, \boldsymbol{\iota}) \leq 0 \text{ for } \kappa \geq 1 \\ \quad \text{for all } \boldsymbol{\iota} \in \mathbb{R}^d,\ a.e.\ \boldsymbol{x} \in \Gamma_C. \end{cases}$$

(H_2) : $\boldsymbol{f}_0 \in L^2(I; L^2(\Omega; \mathbb{R}^d)), \boldsymbol{f}_N \in L^2(I; L^2(\Gamma_N; \mathbb{R}^d)), \boldsymbol{u}_0 \in \mathcal{Y}$ and $\theta_0 \in L^2(\Gamma_C)$.

By multiplying $\boldsymbol{\vartheta} \in \mathcal{Y}$ on both sides of Eq (4.2), we can obtain

$$\int_\Omega \mathrm{Div}\boldsymbol{\sigma} \cdot \boldsymbol{\vartheta} \mathrm{d}\boldsymbol{x} = -\langle \boldsymbol{f}_0(t), \boldsymbol{\vartheta} \rangle_{\mathcal{V}}.$$

Furthermore, by virtue of following Green formula

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{Div}\boldsymbol{\sigma} \cdot \boldsymbol{\vartheta} \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\vartheta} \mathrm{d}\Gamma,$$

we obtain

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \rangle_{\mathcal{X}} = \langle \boldsymbol{f}_0(t), \boldsymbol{\vartheta} \rangle_{\mathcal{V} \times \mathcal{Y}} + \int_{\Gamma_D} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\vartheta} \mathrm{d}\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\vartheta} \mathrm{d}\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\vartheta} \mathrm{d}\Gamma.$$

Applying the (4.3) and (4.4), we have

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \rangle_{\mathcal{X}} = \langle \boldsymbol{f}_0(t), \boldsymbol{\vartheta} \rangle_{\mathcal{V} \times \mathcal{Y}} + \langle \boldsymbol{f}_N(t), \boldsymbol{\vartheta} \rangle_{L^2(\Gamma_N; \mathbb{R}^d) \times \mathcal{Y}} + \int_{\Gamma_C} (\sigma_\nu(t) \vartheta_\nu + \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{\vartheta}_\tau) \mathrm{d}\Gamma. \tag{4.14}$$

Furthermore, by the definition of the subgradient, (4.5) and (4.6), we have

$$\sigma_\nu(t) \vartheta_\nu \leq -j_\nu^0(\theta(t), u_\nu(t); \vartheta_\nu), \quad \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{\vartheta}_\tau \leq -j_\nu^0(\theta(t), \boldsymbol{u}_\nu(t); \boldsymbol{\vartheta}_\nu). \tag{4.15}$$

By utilizing the Riesz representation principle, we know that there exists an element $\boldsymbol{f} \in \mathcal{Y}^*$ such that

$$\langle \boldsymbol{f}(t), \boldsymbol{\vartheta} \rangle_{\mathcal{Y}^* \times \mathcal{Y}} = \langle \boldsymbol{f}_0(t), \boldsymbol{\vartheta} \rangle_{\mathcal{V}} + \langle \boldsymbol{f}_N(t), \boldsymbol{\vartheta} \rangle_{L^2(\Gamma_N; \mathbb{R}^d)} \tag{4.16}$$

for all $\boldsymbol{\vartheta} \in \mathcal{Y}$, a.e. $t \in I$, where $\mathcal{Y}^*$ denotes the dual space of $\mathcal{Y}$. By substituting inequality (4.15) into (4.14) and combining (4.1) and (4.16), we can obtain

$$\begin{aligned} &\langle \mathscr{A}(\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \rangle_{\mathcal{X}} + \langle \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \rangle_{\mathcal{X}} + \langle \int_0^t \mathscr{C}(t-s) \boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t)) \mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}) \rangle \\ &+ \int_{\Gamma_C} j_\nu^0(\theta(t), u_\nu(t); \vartheta_\nu) + j_\tau^0(\theta(t), \boldsymbol{u}_\tau(t); \boldsymbol{\vartheta}_\tau) \mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{\vartheta} \rangle_{\mathcal{Y}^* \times \mathcal{Y}} \end{aligned}$$

for a.e. $t \in (0, T)$. Combining the last inequality and (4.7)–(4.9), we obtain the variational formulation of Problem 7.

Problem 8. *Find $\theta \in C(I, X), \boldsymbol{u} \in L^1(I; Y)$ such that*

$$
\begin{cases}
\langle \mathscr{A}(\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle_X + \langle \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle_X + \langle \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}({}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t))\mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle \\
\quad + \int_{\Gamma_C} j_\nu^0(\theta(t), u_\nu(t); \vartheta_\nu) + j_\tau^0(x(t), \boldsymbol{u}_\tau(t); \boldsymbol{\vartheta}_\tau)\mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} \\
\quad \text{for all } \boldsymbol{\vartheta} \in \mathcal{Y}, \text{ a.e. t} \in (0, \mathrm{T}), \\
{}^C D_{0,t}^{\alpha;\psi} \theta(t) = Q(t, \theta(t), \boldsymbol{u}(t)) \quad \text{on } \mathcal{T}_C, \\
\theta(0) = \theta_0 \quad \text{on } \Gamma_C, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega.
\end{cases}
\tag{4.17}
$$

Theorem 4.1. *Assume that $(H_{\mathscr{A}})$, $(H_{\mathscr{B}})$, $(H_{\mathscr{C}})$, (H_{j_ν}), (H_{j_τ}), (H_Q) and (H_2) hold. Then, Problem 8 has at last one solution $(\boldsymbol{u}, \theta) \in L^1(I; Y) \times C(I, X)$.*

Proof. The proof based on Theorem 3.2. To this end, we define operators $A : \mathcal{Y} \to \mathcal{Y}^*$, $B : \mathcal{Y} \to \mathcal{Y}^*$, $J : \mathcal{Z} \times \mathcal{H} \to \mathbb{R}$ and $\mathcal{R} : C(0, T; \mathcal{Y}) \to C(0, T; \mathcal{Y}^*)$ by

$$\langle A\boldsymbol{u}, \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} = \langle \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle_X \ \text{ for } \ \boldsymbol{u}, \boldsymbol{\vartheta} \in \mathcal{Y}, \tag{4.18}$$

$$\langle B\boldsymbol{u}, \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} = \langle \mathscr{B}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle_X \ \text{ for } \ \boldsymbol{u}, \boldsymbol{\vartheta} \in \mathcal{Y}, \tag{4.19}$$

$$J(\theta, \boldsymbol{v}) = \int_{\Gamma_C} (j_\nu^0(\theta(t), v_\nu(t)) + j_\tau^0(\theta(t), \boldsymbol{v}_\tau(t))\mathrm{d}\Gamma \ \text{ for } \ \boldsymbol{v} \in \mathcal{H}, \ \theta \in \mathcal{Z}, \tag{4.20}$$

$$\langle (\mathcal{R}\boldsymbol{u})(t), \boldsymbol{\vartheta}\rangle = \langle \int_0^t \mathscr{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\rangle_X \ \text{ for } \ \boldsymbol{u}, \boldsymbol{\vartheta} \in \mathcal{Y}. \tag{4.21}$$

Also, we consider the trace operator $\gamma : \mathcal{Y} \to \mathcal{Z}$, let $M = \gamma$ and $G : (0, T) \times \mathcal{Z} \times \mathcal{H}$ be defined by

$$G(t, \theta, \boldsymbol{v})(\boldsymbol{\rho}) = Q(\boldsymbol{\rho}, t, \theta(\boldsymbol{\rho}), \boldsymbol{v}(\boldsymbol{\rho})) \ \text{ for } \ \boldsymbol{v} \in \mathcal{H}, \ \theta \in \mathcal{Z} \text{ a.e. } \boldsymbol{\rho} \in \Gamma_{\mathrm{C}}. \tag{4.22}$$

According to (4.18)–(4.22), Problem 8 can be transformed into the following abstract ψ-fractional differential hemivariational inequality: find $\theta \in C(I, X), \boldsymbol{u} \in L^1(I; Y)$ such that

$$
\begin{cases}
\langle A({}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t)), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \langle B(\boldsymbol{u}(t)), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \langle (\mathcal{R}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u})(t), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} \\
\quad + J^0(\theta(t), M\boldsymbol{u}(t); M\boldsymbol{\vartheta}) \geq \langle \boldsymbol{f}(t), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^* \times \mathcal{Y}} \\
\quad \text{for all } \boldsymbol{\vartheta} \in \mathcal{Y}, \text{ a.e. t} \in (0, \mathrm{T}), \\
{}^C D_{0,t}^{\alpha;\psi} \theta(t) = G(t, \theta(t), \boldsymbol{u}(t)) \quad \text{for a.e. } t \in (0, T), \\
\theta(0) = \theta_0, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0.
\end{cases}
\tag{4.23}
$$

Moreover, we denote $\boldsymbol{\zeta}(t) ={}^C D_{0,t}^{\alpha;\psi} \boldsymbol{u}(t)$ for a.e. $t \in (0, T)$. Thus, we have

$$\boldsymbol{u}(t) = I_{0,t}^{\alpha;\psi} \boldsymbol{\zeta}(t) + \boldsymbol{u}_0 \text{ for a.e. t} \in (0, \mathrm{T}). \tag{4.24}$$

Then, (70) can be rewritten as follows. Find $\boldsymbol{\zeta} \in L^1(0, T; \mathcal{Y})$, and $\theta \in C(0, T; X)$ such that

$$\begin{cases} \langle A(\boldsymbol{\zeta}(t)), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^*\times\mathcal{Y}} + \langle B(I_{0,t}^{\alpha;\psi}\boldsymbol{\zeta}(t) + \boldsymbol{u}_0), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^*\times\mathcal{Y}} + \langle(\mathcal{R}\boldsymbol{\zeta})(t), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^*\times\mathcal{Y}} \\ \quad + J^0(\theta(t), M(I_{0,t}^{\alpha;\psi}\boldsymbol{\zeta}(t) + \boldsymbol{u}_0(t)); M\boldsymbol{\vartheta}) \geq \langle \boldsymbol{f}(t), \boldsymbol{\vartheta}\rangle_{\mathcal{Y}^*\times\mathcal{Y}} \\ \quad \text{for all } \boldsymbol{\vartheta} \in \mathcal{Y}, \text{ a.e. t} \in (0, \text{T}), \\ {}^C D_{0,t}^{\alpha;\psi}\theta(t) = G(t, \theta(t), M(I_{0,t}^{\alpha;\psi}\boldsymbol{\zeta}(t) + \boldsymbol{u}_0)) \quad \text{for a.e. } t \in (0, T), \\ x(0) = x_0. \end{cases} \tag{4.25}$$

Now, we verify that the operators A, B, J, and $\mathcal{R}$ defined by (4.18)–(4.21) satisfy the assumptions (H_A), (H_B), (H_J) and $(H_{\mathcal{R}})$, respectively.

$$\begin{aligned} \langle(\mathcal{R}\boldsymbol{u}_1)(t) - (\mathcal{R}\boldsymbol{u}_2)(t), \boldsymbol{\vartheta}\rangle &\leq \int_\Omega \int_0^t \|\mathscr{C}(t-s)\|_{Q_\infty} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))\|_{\mathbb{S}^d} \mathrm{d}s \|\boldsymbol{\varepsilon}(\boldsymbol{\vartheta})\|_{\mathbb{S}^d} \mathrm{d}\boldsymbol{x} \\ &\leq \max_{t\in[0,T]} \|\mathscr{C}(t)\|_{Q_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_1(s)\|_V \mathrm{d}s \|\boldsymbol{\vartheta}\|_V \end{aligned}$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C(0, T; V)$ and $\boldsymbol{\vartheta} \in V$. Thus, we have

$$\|(\mathcal{R}\boldsymbol{u}_1)(t) - (\mathcal{R}\boldsymbol{u}_2)(t)\|_{V^*} \leq \max_{t\in[0,T]} \|\mathscr{C}(t)\|_{Q_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_1(s)\|_V \mathrm{d}s.$$

This means that $\mathcal{R}$ satisfies $(H_{\mathcal{R}})$ with $L_{\mathcal{R}} = \max_{t\in[0,T]} \|\mathscr{C}(t)\|_{Q_\infty}$. Under the assumption $(H_{\mathscr{A}})$, operator A given by (65) satisfies hypothesis (H_A). Since operator $\mathscr{B}$ satisfies properties $(H_{\mathscr{B}})$, this yields that operator B satisfies (H_B). Based on assumptions (H_{j_ν}), (H_{j_τ}) and [3, Corollary 4.15], we can conclude that the conditions $(H_J)(i)$ and (ii) are satisfied and $L_J = \max\{\sqrt{3\text{meas}(\Gamma_C)}, 1\}(c_\nu + c_\tau)$. The upper semicontinuous of the function $(\theta, \boldsymbol{u}) \mapsto J^0(\theta, \boldsymbol{u}; \boldsymbol{\vartheta})$ can be derived from the upper semicontinuous of j_ν, j_τ and Fatou's lemma, this is condition $(H_J)(iii)$ is satisfied. According to [4, Theorem 3.9.34], we conclude that the trace operator γ satisfies the (H_M). Finally, by using hypothesis (H_Q), we know that operator G defined by (4.22) satisfies condition (H_G). □

5. Conclusions

In this paper, we investigate a class of ψ-Caputo fractional differential hemivariational inequalities with history-dependent operators. As an application, a class of history-dependent viscoelastic friction contact problems that account for adhesion phenomena is investigated. Finally, the solvability of the solution for this friction contact model is established.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of Shanxi (No.202303021221168) and the National Natural Science Foundation of Datong (No.2025080).

Conflict of interest

The author declares that he has no conflict of interest.

References

1. G. W. S. Blair, The role of psychophysics in rheology, *J. Colloid Sci.*, **2** (1947), 21–32. https://doi.org/10.1016/0095-8522(47)90007-X
2. P. D. Panagiotopoulos, Nonconvex energy functions. Hemivariational inequalities and substationarity principles, *Acta. Mechanica*, **48** (1983), 111–130. https://doi.org/10.1007/BF01170410
3. S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*, New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-4232-5
4. Z. H. Liu, S. D. Zeng, D. Motreanu, Partial differential hemivariational inequalities, *Adv. Nonlinear Anal.*, **7** (2018), 571–586. https://doi.org/10.1515/anona-2016-0102
5. Z. H. Liu. Existence results for quasilinear parabolic hemivariational inequalities, *J. Diff. Equ.*, **244** (2008), 1395–1409. https://doi.org/10.1016/j.jde.2007.09.001
6. S. D. Zeng, S. Migórski, Z. H. Liu, Well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities, *SIAM J. Optim.*, **31** (2021), 2829–2862. https://doi.org/10.1137/20M1351436
7. Y. Liu, Z. H. Liu, N. S. Papageorgiou, Sensitivity analysis of optimal control problems driven by dynamic history-dependent variational-hemivariational inequalities, *J. Diff. Equ.*, **342** (2023), 559–595. https://doi.org/10.1016/j.jde.2022.10.009
8. L. Lu, Z. H. Liu, Existence and controllability results for stochastic fractional evolution hemivariational inequalities, *Appl. Math. Comput.*, **268** (2015), 1164–1176. https://doi.org/10.1016/j.amc.2015.07.023
9. J. F. Han, S. Migórski, H. D. Zeng, Weak solvability of a fractional viscoelastic frictionless contact problem, *Appl. Math. Comput.*, **303** (2017), 1–18. https://doi.org/10.1016/j.amc.2017.01.009
10. J. W. Hao, J. R. Wang, L. Lu, Coupled system of fractional hemivariational inequalities with applications, *Optimization*, **73** (2024), 969–994. https://doi.org/10.1080/02331934.2022.2133544
11. B. Zeng, S. H. Wang, Existence for fractional evolutionary inclusions involving nonlinear weakly continuous operators with applications, *Chaos Solitons Fract.*, **185** (2024), 115178. https://doi.org/10.1016/j.chaos.2024.115178
12. C. P. Li, Z. Q. Li, The finite-time blow-up for semilinear fractional diffusion equations with time ψ-Caputo derivative, *J. Nonlinear Sci.*, **32** (2022), 82. https://doi.org/10.1007/s00332-022-09841-6
13. C. P. Li, Z. Q. Li, Stability and ψ-algebraic decay of the solution to ψ-fractional differential system, *Int. J. Nonlinear. Sci.*, **24** (2023), 695–733. https://doi.org/10.1515/ijnsns-2021-0189
14. N. N'Gbo, C. P. Li, M. Cai, Chaos detection in generalized ψ-fractional differential systems, *J. Comput. Nonlinear. Dynam.*, **20** (2025), 031002. https://doi.org/10.1115/1.4067471

15. S. D. Zeng, S. Migórski, A class of time-fractional hemivariational inequalities with application to frictional contact problem, *Commun. Nonlinear Sci. Numer. Simul.*, **56** (2018), 34–48. https://doi.org/10.1016/j.cnsns.2017.07.016
16. J. X. Cen, Y. J. Liu, V. T. Nguen, S. D. Zeng, Existence of solutions for fractional evolution inclusion with application to mechanical contact problems, *Fractals*, **2021** (2021), 2140036. https://doi.org/10.1142/S0218348X21400363
17. J. F. Han, C. P. Li, S. D. Zeng, Applications of generalized fractional hemivariational inequalities in solid viscoelastic contact mechanics, *Commun. Nonlinear Sci. Numer. Simul.*, **115** (2022), 106718. https://doi.org/10.1016/j.cnsns.2022.106718
18. M. Raous, L. Cangémi, M. Cocu, A consistent model coupling adhesion, friction, and unilateral contact, *Comput. Methods. Appl. Mech. Eng.*, **177** (1999), 383–399. https://doi.org/10.1016/S0045-7825(98)00389-2
19. J. F. Han, Y. Li, S. Migórski, Analysis of an adhesive contact problem for viscoelastic materials with long memory, *J. Math. Anal. Appl.*, **427** (2015), 646–668. https://doi.org/10.1016/j.jmaa.2015.02.055
20. D. Paczka, Adhesive contact problem for viscoplastic materials, *Nonlinear Anal. Real World Appl.*, **51** (2020), 102982. https://doi.org/10.1016/j.nonrwa.2019.102982
21. S. Migórski, S. D. Zeng, Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model, *Nonlinear Anal. Real World Appl.*, **43** (2018), 121–143. https://doi.org/10.1016/j.nonrwa.2018.02.008
22. Z. Denkowski, S. Migórski, N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, New York: Kluwer Academic/Plenum Publishers, 2003.
23. S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, Numerical techniques for the variable order time fractional diffusion equation, *Appl. Math. Comput.*, **218** (2012), 10861–10870. https://doi.org/10.1016/j.amc.2012.04.047
24. C. Carstensen, J. Gwinner, A theory of discretization for nonlinear evolution inequalities applied to parabolic signorini problems, *Ann. Mat. Pura. Appl.*, **177** (1999), 363–394. https://doi.org/10.1007/BF02505918
25. H. T. Banks, G. A. Pinter, L. K. Potter, J. M. Gaitens, L. C. Yanyo, Modeling of quasistatic and dynamic load responses of fflled viscoelastic materials, In: *Mathematical Modeling: Case Studies from Industry, E. Cumberbatch and A. Fitt, Eds., Cambridge University Press*, 2011, 229–252.
26. H. T. Banks, G. A. Pinter, L. K. Potter, B. C. Munoz, L. C. Yanyo, Estimation and control related issues in smart material structure and ffuids, In: *Optimization Techniques and Applications, L. Caccetta et al., Eds., Curtain University Press*, 1998, 19–34.
27. J. T. Oden, J. A. C. Martins, Models and computational methods for dynamic friction phenomena, *Comput. Methods Appl. Mech. Eng.*, **52** (1985), 527–634. https://doi.org/10.1016/0045-7825(85)90009-X