Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1219–1238.
DOI: 10.3934/math.2026052
Received: 12 November 2025
Revised: 26 December 2025
Accepted: 08 January 2026
Published: 16 January 2026

Research article

On the Ulam-type stability of higher-order iterative Volterra integro-delay differential equations

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Abstract: In this study, we addressed a higher-order iterative Volterra integro-delay differential equation (HOIVIDDE) involving two variable time delays. Our primary focus was on establishing the uniqueness of solutions and analyzing Ulam-type stability properties of the considered HOIVIDDE. We presented three novel results concerning Ulam–Hyers–Rassias (U-H-R), σ-semi-Ulam–Hyers (σ-semi-U-H), and Ulam–Hyers (U-H) stability for HOIVIDDE, along with uniqueness results for the associated initial value problem (IVP). The analysis was conducted using the properties of iterative functions, the Banach fixed point theorem, and the Bielecki metric. Notably, this was the first study that extended and enhanced these qualitative properties to an nth-order HOIVIDDE. To illustrate the applicability of the results obtained here, we provided an example verifying the requirements of the new theorems.

Keywords: Ulam-type stability; iterative methods; integro-delay differential equation; higher order; Banach fixed point theorem; Bielecki metric
Mathematics Subject Classification: 34K05, 45D05, 45J05, 47H10

1. Introduction

It is well known that a Volterra integral equation (IE) or integro-differential equation (IDE) is said to possess Ulam-type stability if every approximate solution of the equation lies close to an exact solution.

This concept originated from a question posed by S.M. Ulam [30], which subsequently led to the development of various generalized notions of stability, including Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, generalized Ulam–Hyers–Rassias stability, and σ-semi-Ulam–Hyers stability, etc.

Following Ulam's foundational work of Ulam [30], substantial works have addressed Ulam-type stability of ODEs, functional differential equations (FDEs), Volterra IEs, and Volterra IDEs, etc. Some notable contributions in this direction include the works of Abbas and Benchohra [1], Akkouchi [2], Cădariu et al. [3], Castro and Guerra [4], Castro and Ramos ([5,6]), Castro and Simoes [7], Chauhan et al. [9], Gavruta [12], Graef et al. [13], Janfada and Sadeghi [14], Jung ([15–17]), Moroşanu and Petruşel [18], Öğrekçi et al. [19], Petruşel et al. [20], Rassias [21], Shah and Zada [22], Tunç and Tunç [25], Tunç et al. ([27–29]), and their various extensions. In addition to these works, available literature indicates that the Ulam-type stability of first-order iterative FDEs was first investigated by Egri [11]. This line of study was later extended to first-order iterative IDEs by Tunç and Tunç [24], and subsequently by Tunç et al. [26]. In addition, among recent studies on Volterra integral equations, Aourir and Laeli Dastjerd [31] proposed an approximate mesh free algorithm based on radial basis function collocation for solving third kind Volterra integral equations with nonlinear vanishing delays. In a subsequent work, Aourir and Laeli Dastjerd [32] investigated a radial basis function based collocation method for the numerical solution of delayed third kind Volterra integral equations, which arise in the modeling of epidemics, biological systems, and other physical phenomena.

Despite considerable progress in studying Ulam-type stability for various classes of equations, including ODEs, FDEs, Volterra IEs, and Volterra IDEs, only a limited number of works have investigated these concepts for higher-order integro-differential equations (HOIDEs). To the best of our knowledge, only two recent studies have addressed this topic: Those by Castro and Simoes [8] and Simoes et al. [23]. A brief overview of these contributions is provided below.

In 2018, Castro and Simoes [8] investigated the Ulam-type stability of HOIDE as follows:

$$y^{(n)}(x) = f\left(x, y(x), \int_a^x k\left(x, \tau, y(\tau), y(\alpha(\tau))\right) d\tau\right). \tag{1.1}$$

Castro and Simoes [8] derived sufficient requirements under which HOIDE (1.1) admits the Ulam-type stability.

Later, in 2021, Simoes et al. [23] studied the Ulam-type stability of a Volterra HOIDE, given as follows:

$$\phi^{(n)}(x) = F\left(x, \phi(x), \int_a^x G\left(x, t, \phi(t), \phi'(t), ..., \phi^{(n-1)}(t)\right) dt\right). \tag{1.2}$$

Simoes et al. [23] established sufficient requirements under which HOIDE (1.2) possesses the Ulam-type stability.

In this paper, motivated by the works of Castro and Simoes [8], Simoes et al. [23], and the related results in the aforementioned literature, we study the following higher-order iterative Volterra integro-delay differential equation involving two variable time delays:

$$u^{(n)}(x) = H\left(x, u^{[1]}(x), u^{[1]}(p(x)), ..., u^{[m]}(x), u^{[m]}(p(x)),\right.$$

$$\int_a^x Z\left(x,t,u^{[1]}(t),u^{[1]}(r(t)),...,u^{[m]}(t),u^{[m]}(r(t))\right)dt\Bigg), \tag{1.3}$$

with

$$u^{(i)}(a)=0,\ i=0,1,...,n-1, \tag{1.4}$$

where $x \in [a,b]$, $a,\ b \in R$, $n \in N$, $u \in C^n([a,b])$, $p,\ r : [a,b] \to (0,\infty)$ with $p(x) \leq x$, $r(t) \leq t$, $H : [a,b]\times\mathbb{C}^{2m}\times\mathbb{C} \to \mathbb{C}$ and $Z : [a,b]\times[a,b]\times\mathbb{C}^{2m} \to \mathbb{C}$ are continuous functions, $u^{[m]}(t)$ denotes m^{th} iterate of the function u having the property that

$$u^{[1]}(t)=u(t),...,u^{[m]}(t)=\underbrace{u\left(u(...u(u(t)))\right)}_{m-times}. \tag{1.5}$$

We note that HOIVIDDE (1.3) includes variable but finite delays and the notation $\mathbb{C}$ represents the complex numbers.

This study establishes three new and improved outcomes on the Ulam-type stability as well as the uniqueness of solutions of a new mathematical model as described by HOIVIDDE (1.3) with requirements (1.4). To the best of our knowledge, no existing studies have addressed Ulam-type stability and uniqueness of solutions for HOIVIDDEs including multiple delays and m^{th} iterative arguments. Therefore, this study presents the first and novel contributions to these important qualitative properties in this context. It is also observed that HOIDE (1.1), which involves a variable delay without iterative arguments, constitutes a special case of HOIVIDDE (1.3), where more general nonlinear functions, two variable delays, and iterative arguments are incorporated. In particular, the right-hand side of HOIDE (1.1) can be viewed as a specific form of the right-hand side of HOIVIDDE (1.3). Consequently, HOIVIDDE (1.3) generalizes and extends HOIDE (1.1). Moreover, both HOIVIDDE (1.3) and HOIDE (1.2) are of nth-order. However, the right-hand sides of HOIDE (1.2) and HOIVIDDE (1.3) have the following different expressions, respectively:

$$H\left(x,u^{[1]}(x),u^{[1]}(p(x)),...,u^{[m]}(x),u^{[m]}(p(x)),\int_a^x Z\left(x,t,u^{[1]}(t),u^{[1]}(r(t)),...,u^{[m]}(t),u^{[m]}(r(t))\right)dt\right),$$

and

$$F\left(x,\phi(x),\int_a^x G\left(x,t,\phi(t),\phi'(t),...,\phi^{(n-1)}(t)\right)dt\right).$$

As observed, the right-hand side of HOIVIDDE (1.3) contains two variable delays, whereas the right-hand side of HOIDE (1.2) involves no delay terms. In addition, the functional structures appearing on the right-hand sides of HOIVIDDE (1.3) and HOIDE (1.2) are substantially different. These distinctions clarify the differences between the present model and that studied by Simoes et al. [23]. The presence of variable delays together with iterative terms leads to additional analytical difficulties. As a result, the analysis of uniqueness of solutions and Ulam-type stability for HOIVIDDE (1.3) provides a new perspective relative to existing results. The results of this paper extend and complement those of Castro and Simoes [8], and go beyond the setting considered in Simoes et al. [23], and they also constitute original contributions to the existing literature.

Next, in practical applications, it is both necessary and valuable to investigate approximate solutions of mathematical models and to determine whether the approximate solutions remain close to their exact counterparts. This fact motivates the investigation of Ulam-type stability and uniqueness for HOIVIDDEs, which have significant scientific relevance. Therefore, the results obtained herein are original and contribute meaningfully to the qualitative theory of iterative functional-integro-differential equations. Additionally, we provide a specific example to illustrate the verification of the requirements associated with the main results. These observations underscore the novelty and theoretical significance of the present work.

The paper proceeds as follows: Section 2 formulates essential background definitions and a basic fixed point theorem. In Section 3, we present a result on stability in the sense of U-H-R, along with the uniqueness of solutions to HOIVIDDE (1.3). Section 4 contains two results concerning stability in the sense of σ-semi-U-H and U-H, as well as the uniqueness of solutions to HOIVIDDE (1.3). In Section 5, an illustrative example is provided to highlight the main contributions of the paper. To conclude, Section 6 provides a summary of the paper and discusses possible directions for future work and open problems.

2. Preliminaries

In what follows, we introduce the basic definitions of Ulam-type stability that are essential for the subsequent analysis. We also introduce a fundamental fixed point result, the Bielecki metric and two remarks, which play a central role in establishing both the Ulam-type stability as well as uniqueness of solutions for the class of HOIVIDDEs considered in this work.

Throughout the paper, let

$$H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt\right),$$

denote

$$H\left(x, u^{[1]}(x), ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, u^{[1]}(t), ..., u^{[m]}(r(t))\right) dt\right).$$

Definition 2.1. *If $u \in C^n([a, b])$, $n \in N$, satisfying*

$$\left| u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt\right)\right| \le \beta, \tag{2.1}$$

with $x \in [a, b]$, $\beta \ge 0$, HOIVIDDE (1.3) *has a solution u_0 and there exists a real number $C > 0$, independent of u_0 and u, satisfying*

$$|u_0(x) - u(x)| \le C\beta,$$

for each $x \in [a, b]$, then HOIVIDDE (1.3) *is said to be U-H stable.*

Definition 2.2. *Let* $\sigma > 0$ *be a non-decreasing continuous function described in* $[a,b]$. *If* $u \in C^n([a,b])$, $n \in N$, *satisfying*

$$\left| u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt \right) \right| \leq \sigma(x), \tag{2.2}$$

with $x \in [a,b]$, *HOIVIDDE* (1.3) *has a solution* u_0 *and there exists a real number* $C > 0$, *independent of* u_0 *and* u, *satisfying*

$$|u_0(x) - u(x)| \leq C\sigma(x),$$

for each $x \in [a,b]$, *then HOIVIDDE* (1.3) *is said to be U-H-R stable.*

Definition 2.3. *Let* $\sigma > 0$ *be a non-decreasing continuous function described in* $[a,b]$. *If* $u \in C^n([a,b])$, $n \in N$, *satisfying*

$$\left| u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt \right) \right| \leq \beta, \tag{2.3}$$

with $x \in [a,b]$, $\beta \geq 0$, *HOIVIDDE* (1.3) *has a solution* u_0 *and there exists a real number* $C > 0$, *independent of* u_0 *and* u, *having the property that*

$$|u_0(x) - u(x)| \leq C\sigma(x),$$

for each $x \in [a,b]$, *then HOIVIDDE* (1.3) *is said to be* σ*-semi U-H stable.*

Remark 2.1. *Throughout this paper we assume that* $C^n([a,b])$ *denotes the space of n-times continuously differentiable functions on* $[a,b]$. *Next, the initial data* $u^{(i)}(a) = 0$, $i = 0, 1, ..., n-1$, *are consistent with the assumption* $u \in C^n([a,b])$.

Remark 2.2. *In the space* $C^n([a,b])$, *the generalization of the Bielecki metric is described by*

$$d_B(u,v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{2.4}$$

with the requirement that σ *is a nondecreasing function and* $\sigma \in C^n([a,b],(0,\infty))$, *and* $(C^n([a,b]), d_B)$ *is a complete metric space (Cădariu et al. [3], Castro and Simoes [8]).*

Theorem 2.1. *([10]) Let* $(\mathfrak{I}, d)$ *be a generalized complete metric space and let* $P : \mathfrak{I} \to \mathfrak{I}$ *be a strictly contractive operator with a Lipschitz constant* $L_C < 1$. *If there exist a nonnegative integer k having the property that* $d(P^{k+1}\ell, P^k\ell) < \infty$ *for some* $\ell \in \mathfrak{I}$, *then the following propositions hold true:*

(C-1) the sequence $(P^n\ell)_{n \in N}$ *converges to a fixed point* ℓ^* *of P;*
(C-2) ℓ^* *is the unique fixed point of P in* $\mathfrak{I}^* = \left\{\gamma \in \mathfrak{I} : d\left(P^k\ell, \gamma\right) < \infty\right\}$;
(C-3) if $\gamma \in \mathfrak{I}^*$, *then*

$$d(\gamma, \ell^*) \leq \frac{1}{1 - L_C} d(P\gamma, \gamma). \tag{2.5}$$

3. Stability in the sense of U-H-R

The next theorem establishes sufficient requirements for the U-H-R stability of HOIVIDDE (1.3) and the uniqueness of solutions of IVP (1.3), (1.4). The initial new result of the present study is stated in Theorem 3.1.

Theorem 3.1. *Assume that there exist constants $L_{H_k} > 0$, $L_{Z_k} > 0$, $k = 1, 2, ..., m$, $L_H > 0$ and $K > 0$ such that the following requirements are satisfied:*

(A-1) $H : [a, b] \times \mathbb{C}^{2m} \times \mathbb{C} \to \mathbb{C}$ *is a continuous function having the property that*

$$\left|H\left(x, u(x), u(p(x)), ..., u^{[m]}(p(x)), \theta(x)\right) - H\left(x, \upsilon(x), \upsilon(p(x)), ..., \upsilon^{[m]}(p(x)), \gamma(x)\right)\right|$$

$$\leq \sum_{k=1}^{m} L_{H_k}\left(\left|u^{[k]}(x) - \upsilon^{[m]}(x)\right| + \left|u^{[k]}(p(x)) - \upsilon^{[k]}(p(x))\right|\right) + L_H\left|\theta(x) - \gamma(x)\right|;$$

(A-2) $Z : [a, b] \times [a, b] \times \mathbb{C}^{2m} \to \mathbb{C}$ *is a continuous function having the property that*

$$\left|Z\left(x, t, u(t), u(p(t)), ..., u^{[m]}(r(t))\right) - Z\left(x, t, \upsilon(t), \upsilon(r(t)), ..., \upsilon^{[m]}(r(t))\right)\right|$$

$$\leq \sum_{k=1}^{m} L_{Z_k}\left(\left|u^{[k]}(t) - v^{[k]}(t)\right| + \left|u^{[k]}(r(t)) - \upsilon^{[k]}(r(t))\right|\right);$$

(A-3) $\sigma : [a, b] \to (0, \infty)$ *is a positive, non-decreasing, continuous function having the property that*

$$\int_a^x \sigma(t)dt \leq \ell\sigma(x), \ \ell > 0, \ \ell \in R,$$

for each $x \in [a, b]$.

If $u \in C^n([a, b])$ *and this function satisfies the inequality*

$$\left|u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right)dt\right)\right| \leq \sigma(x), \tag{3.1}$$

for each $x \in [a, b]$, *and*

$$2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k} < 1,$$

then HOIVIDDE (1.3) *with* $u^{(i)}(a) = 0, \ i = 0, 1, ..., n-1$, *has a unique solution* $u_0 \in C^n([a, b])$ *and*

$$|u_0(x) - u(x)| \leq \frac{\ell^n}{1 - 2\sum\limits_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k}}\sigma(x), \tag{3.2}$$

for each $x \in [a, b]$.

Proof. Employing the initial conditions $u^{(j)}(a) = 0,\ j = 0, 1, ..., n-1$, and standard properties of integration, we observe that HOIVIDDE (1.3) coincides with the following IE:

$$u(x) = \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\Big(r_1, u^{[1]}(r_1), u^{[1]}(p(r_1)), ..., u^{[m]}(p(r_1)),$$

$$\left.\int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), u^{[1]}(r(t)), ..., u^{[m]}(r(t))\Big)\, dt\right) dr_1 dr_2 ... dr_n.$$

We now describe an operator $T : C^n([a,b]) \to C^n([a,b])$ by

$$(Tu)(x) = \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\Big(r_1, u^{[1]}(r_1), u^{[1]}(p(r_1)), ..., u^{[m]}(p(r_1)),$$

$$\left.\int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), u^{[1]}(r(t)), ..., u^{[m]}(r(t))\Big)\, dt\right) dr_1 dr_2 ... dr_n, \tag{3.3}$$

for each $u \in C^n([a,b])$ and $x \in [a,b]$.

Assume that u is a continuous function. Taking into consideration the definition of the operator T in (3.3), we arrive at

$$|(Tu)(x) - (Tu)(x_0)|$$

$$= \left| \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\left(r_1, u^{[1]}(r_1), ..., u^{[m]}(p(r_1)), \int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), ..., u^{[m]}(r(t))\Big)\, dt\right) dr_1 ... dr_n \right.$$

$$\left. - \int_a^{x_0} \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\left(r_1, u^{[1]}(r_1), ..., u^{[m]}(p(r_1)), \int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), ..., u^{[m]}(r(t))\Big)\, dt\right) dr_1 ... dr_n \right|$$

$$= \left| \int_{x_0}^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\left(r_1, u^{[1]}(r_1), ..., u^{[m]}(p(r_1)), \int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), ..., u^{[m]}(r(t))\Big)\, dt\right) dr_1 dr_2 ... dr_n \right|$$

$$\to 0 \ \text{ when } \ x \to x_0.$$

Based on the requirement (A-3) of Theorem 3.1, it also follows that

$$\int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} \int_a^{r_1} \sigma(t) dt dr_1 dr_2 ... dr_n \le \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} \ell\sigma(r_1) dr_1 dr_2 ... dr_n$$

$$\le \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_3} \ell^2 \sigma(r_2) dr_2 ... dr_n$$

$$\vdots$$

$$\leq \int_a^x \ell^n \sigma(r_n) dr_n$$
$$\leq \ell^{n+1}\sigma(x). \tag{3.4}$$

We now show that the operator T is strictly contractive with respect to the Bielecki metric described by (2.4). Indeed, employing the requirements (A-1), (A-2) and the Bielecki metric, for each $u, v \in C^n([a,b])$, we obtain the following outcome:

$$d_B(Tu, Tv) = \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)}$$

$$= \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \left| \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} H\left(r_1, u^{[1]}(r_1), ..., u^{[m]}(p(r_1)), \right.\right.$$
$$\left.\int_a^{r_1} Z\left(r_1, t, u^{[1]}(t), ..., u^{[m]}(r(t))\right) dt\right) dr_1 dr_2 ... dr_n$$
$$- \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} H\left(r_1, v^{[1]}(r_1), ..., v^{[m]}(p(r_1)), \right.$$
$$\left.\left.\int_a^{r_1} Z\left(r_1, t, v^{[1]}(t), ..., v^{[m]}(r(t))\right) dt\right) dr_1 dr_2 ... dr_n \right|$$

$$\leq \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \sum_{k=1}^m L_{H_k} \left|u^{[k]}(r_1) - v^{[k]}(r_1)\right| dr_1 dr_2 ... dr_n$$
$$+ \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \sum_{k=1}^m L_{H_k} \left|u^{[k]}(p(r_1)) - v^{[k]}(p(r_1))\right| dr_1 dr_2 ... dr_n$$
$$+ \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \int_a^{r_1} \sum_{k=1}^m L_{H_k} L_{Z_k} \left|u^{[k]}(t) - v^{[k]}(t)\right| dt dr_1 dr_2 ... dr_n$$
$$+ \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \int_a^{r_1} \sum_{k=1}^m L_{H_k} L_{Z_k} \left|u^{[k]}(r(t)) - v^{[k]}(r(t))\right| dt dr_1 dr_2 ... dr_n$$
$$= \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \sum_{k=1}^m L_{H_k} \sigma(r_1) \frac{\left|u^{[k]}(r_1) - v^{[k]}(r_1)\right|}{\sigma(r_1)} dr_1 dr_2 ... dr_n$$
$$+ \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \sum_{k=1}^m L_{H_k} \sigma(r_1) \frac{\left|u^{[k]}(p(r_1)) - v^{[k]}(p(r_1))\right|}{\sigma(r_1)} dr_1 dr_2 ... dr_n$$

$$+\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\int_a^{r_1}\sum_{k=1}^{m}L_{H_k}L_{Z_k}\sigma(t)\frac{\left|u^{[k]}(t)-v^{[k]}(t)\right|}{\sigma(t)}dtdr_1dr_2...dr_n$$

$$+\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\int_a^{r_1}\sum_{k=1}^{m}L_{H_k}L_{Z_k}\sigma(t)\frac{\left|u^{[k]}(r(t))-v^{[k]}(r(t))\right|}{\sigma(t)}dtdr_1dr_2...dr_n$$

$$\leq\sum_{k=1}^{m}L_{H_k}\sup_{r_1\in[a,b]}\frac{\left|u^{[k]}(r_1)-v^{[k]}(r_1)\right|}{\sigma(r_1)}\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\sigma(r_1)dr_1dr_2...dr_n$$

$$+\sum_{k=1}^{m}L_{H_k}\sup_{r_1\in[a,b]}\frac{\left|u^{[k]}(p(r_1))-v^{[k]}(p(r_1))\right|}{\sigma(r_1)}\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\sigma(r_1)dr_1dr_2...dr_n$$

$$+\sum_{k=1}^{m}L_{H_k}L_{Z_k}\sup_{t\in[a,b]}\frac{\left|u^{[k]}(t)-v^{[k]}(t)\right|}{\sigma(t)}\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\int_a^{r_1}\sigma(t)dr_1dr_2...dr_n$$

$$+\sum_{k=1}^{m}L_{H_k}L_{Z_k}\sup_{t\in[a,b]}\frac{\left|u^{[k]}(r(t))-v^{[k]}(r(t))\right|}{\sigma(t)}\sup_{x\in[a,b]}\frac{1}{\sigma(x)}\int_a^x\int_a^{r_n}\int_a^{r_{n-1}}...\int_a^{r_2}\int_a^{r_1}\sigma(t)dr_1dr_2...dr_n. \tag{3.5}$$

Subsequently, from (3.4) and (3.5), we derive

$$d_B(Tu,Tv)=\sup_{x\in[a,b]}\frac{|(Tu)(x)-(Tv)(x)|}{\sigma(x)}$$

$$\leq\ell^n\sum_{k=1}^{m}L_{H_k}\left[\sup_{r_1\in[a,b]}\frac{\left|u^{[k]}(r_1)-v^{[k]}(r_1)\right|}{\sigma(r_1)}+\sup_{r_1\in[a,b]}\frac{\left|u^{[k]}(p(r_1))-v^{[k]}(p(r_1))\right|}{\sigma(r_1)}\right]$$

$$+\ell^{n+1}\sum_{k=1}^{m}L_{H_k}L_{Z_k}\left[\sup_{t\in[a,b]}\frac{\left|u^{[k]}(t)-v^{[k]}(t)\right|}{\sigma(t)}+\sup_{t\in[a,b]}\frac{\left|u^{[k]}(r(t))-v^{[k]}(r(t))\right|}{\sigma(t)}\right]. \tag{3.6}$$

Indeed, let $u\in C^n([a,b],[a,b])$ with $|u(t_1)-u(t_2)|\leq K|t_1-t_2|,\ t_1,t_2\in[a,b],\ K>0,\ K\in R$. Thereafter, the coming step of the calculations requires the following relations:

$$d_B(u^{[1]},v^{[1]})=\sup_{t\in[a,b]}\frac{|u(t)-v(t)|}{\sigma(t)}=d_B(u,v),$$

$$d_B(u^{[1]}(p(t)),v^{[1]}(p(t)))=\sup_{t\in[a,b]}\frac{|u(p(t))-v(p(t))|}{\sigma(t)}$$

$$=d_B(u(p(t)),v(p(t))),$$

$$d_B(u^{[2]},v^{[2]})=\sup_{t\in[a,b]}\frac{|u(u(t))-v(v(t))|}{\sigma(t)}$$

$$=\sup_{t\in[a,b]}\frac{|u(u(t))-u(v(t))+u(v(t))-v(v(t))|}{\sigma(t)}$$

$$\leq K \sup_{t\in[a,b]} \frac{|u(t)-v(t)|}{\sigma(t)} + \sup_{t\in[a,b]} \frac{|u(u(t))-u(v(t))|}{\sigma(t)}$$

$$\leq (1+K)\, d_B(u,v) = \sum_{k=0}^{1} K^k d_B(u,v), \; K>0, \; K\in R,$$

$$d_B(u^{[2]}(p(t)), v^{[2]}(p(t))) = \sup_{t\in[a,b]} \frac{|u(u(p(t))) - v(v(p(t)))|}{\sigma(t)}$$

$$\leq K \sup_{t\in[a,b]} \frac{|u(p(t))-v(p(t))|}{\sigma(t)} + \sup_{t\in[a,b]} \frac{|u(u(p(t)))-u(v(p(t)))|}{\sigma(t)}$$

$$\leq (1+K)\, d_B(u,v) = \sum_{k=0}^{1} K^k d_B(u,v),$$

$$d_B(u^{[3]}, v^{[3]}) = \sup_{t\in[a,b]} \frac{\left|u(u^{[2]}(t)) - v(v^{[2]}(t))\right|}{\sigma(t)}$$

$$\leq K \sup_{t\in[a,b]} \frac{\left|u^{[2]}(t)-v^{[2]}(t)\right|}{\sigma(t)} + \sup_{t\in[a,b]} \frac{\left|u(u^{[2]}(t))-u(v^{[2]}(t))\right|}{\sigma(t)}$$

$$\leq \left(1+K+K^2\right) d_B(u,v) = \sum_{k=0}^{2} K^k d_B(u,v),$$

$$d_B(u^{[3]}(p(t))), v^{[3]}(p(t)))) = \sup_{t\in[a,b]} \frac{\left|u(u^{[2]}(p(t))) - v(v^{[2]}(p(t)))\right|}{\sigma(t)}$$

$$\leq K \sup_{t\in[a,b]} \frac{\left|u^{[2]}(p(t))-v^{[2]}(p(t))\right|}{\sigma(t)} + \sup_{t\in[a,b]} \frac{\left|u(u^{[2]}(p(t)))-u(v^{[2]}(p(t)))\right|}{\sigma(t)}$$

$$\leq \left(1+K+K^2\right) d_B(u,v) = \sum_{k=0}^{2} K^k d_B(u,v),$$

$$...$$

$$d_B(u^{[m]}, v^{[m]}) = \sup_{t\in[a,b]} \frac{\left|u(u^{[m]}(t)) - v(v^{[m]}(t))\right|}{\sigma(t)}$$

$$\leq \left(1+K+K^2+...+K^{m-1}\right) d_B(u,v) = \sum_{k=0}^{m-1} K^k d_B(u,v),$$

$$d_B(u^{[m]}(p(t)), v^{[m]}(p(t))) = \sup_{t\in[a,b]} \frac{\left|u(u^{[m]}(p(t))) - v(v^{[m]}(p(t)))\right|}{\sigma(t)}$$

$$\leq \left(1+K+K^2+...+K^{m-1}\right) = \sum_{k=0}^{m-1} K^k d_B(u,v).$$

Similar calculations can be carried out for the remaining terms; for brevity, the details are omitted. Hence, summing up the corresponding relations, we obtain

$$\ell^n \sum_{k=1}^{m} L_{H_k} \left[\sup_{r_1 \in [a,b]} \frac{\left| u^{[k]}(r_1) - v^{[k]}(r_1) \right|}{\sigma(r_1)} + \sup_{r_1 \in [a,b]} \frac{\left| u^{[k]}(p(r_1)) - v^{[k]}(p(r_1)) \right|}{\sigma(r_1)} \right]$$

$$\leq 2\ell^n \sum_{k=1}^{m} L_{H_k} k K^{m-k} d_B(u, v),$$

and

$$\ell^{n+1} \sum_{k=1}^{m} L_{H_k} L_{Z_k} \left[\sup_{t \in [a,b]} \frac{\left| u^{[k]}(t) - v^{[k]}(t) \right|}{\sigma(t)} + \sup_{t \in [a,b]} \frac{\left| u^{[k]}(r(t)) - v^{[k]}(r(t)) \right|}{\sigma(t)} \right]$$

$$\leq 2\ell^{n+1} \sum_{k=1}^{m} L_{H_k} L_{Z_k} k K^{m-k} d_B(u, v).$$

Thereby, by substituting the above results into (3.6), we obtain

$$d_B(Tu, Tv) \leq 2 \sum_{k=1}^{m} L_{H_k} \left(1 + L_{Z_k} \ell\right) \ell^n k K^{m-k} d_B(u, v).$$

On the account of the requirement

$$2 \sum_{k=1}^{m} L_{H_k} \left(1 + L_{Z_k} \ell\right) \ell^n k K^{m-k} < 1,$$

it follows that the operator T is strictly contractive. Thereafter, according to Theorem 2.1, we conclude that IVP (1.3), (1.4) has a unique solution and HOIVIDDE (1.3) possesses the U-H-R stability.

We now provide the remaining part of the proof of Theorem 3.1. Indeed, from the inequality (2.2), it follows that

$$-\sigma(x) \leq u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt \right) \leq \sigma(x), \tag{3.7}$$

for each $x \in [a, b]$. Hence, adopting integration techniques, we see that

$$\left| \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} H\left[r_1, u^{[1]}(r_1), u^{[2]}(r_1), ..., u^{[m]}(r_1) \right., \right.$$

$$\left. \left. \int_a^{r_1} Z\left(r_1, t, u^{[1]}(t), u^{[2]}(t), ..., u^{[m]}(t)\right) \right] dt dr_1 dr_2 ... dr_n - u(x) \right|$$

$$\leq \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} ... \int_a^{r_2} \ell\sigma(r_1) dr_1 dr_2 ... dr_n$$

$$\leq \ell^n \sigma(x). \tag{3.8}$$

Next, employing relations (3.3) and (3.8), we obtain

$$|(Tu)(x) - u(x)| \leq \ell^n \sigma(x). \tag{3.9}$$

In view of the above results and relation (2.5), we deduce that

$$d_B(u_0, u) \leq \frac{1}{1 - 2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k}} d_B(Tu, u). \tag{3.10}$$

Utilizing the definition adopted for the metric d_B, inequalities (3.9) and (3.10), we infer that

$$\sup_{x\in[a,b]} \frac{|u_0(x) - u(x)|}{\sigma(x)} \leq \frac{\ell^n}{1 - 2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k}}. \tag{3.11}$$

Thus, the inequality (3.2) is satisfied. These results mean that, depending upon the requirements (A-1)-(A-3) of Theorem 3.1, HOIVIDDE (1.3) admits the U-H-R stability and IVP (1.3), (1.4) has a unique solution. □

4. Stability in the sense of σ-semi U–H and U–H

In the next two theorems, we establish sufficient requirements for the σ-semi U–H and U–H stability of HOIVIDDE (1.3) and the uniqueness of solutions of IVP (1.3), (1.4). The second and third results of the present study are given in Theorems 4.1 and 4.2, respectively.

Theorem 4.1. *Assume that the requirements (A-1)–(A-3) of Theorem 3.1 are satisfied. If* $u \in C^n([a,b])$ *and this function satisfies the inequality*

$$\left| u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt\right)\right| \leq \beta, \tag{4.1}$$

for each $x \in [a,b]$*, where* $\beta \geq 0,\ \beta \in R$*, and*

$$2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k} < 1,$$

then HOIVIDDE (1.3) *with* $u^{(i)}(a) = 0,\ i = 0, 1, ..., n-1$*, has a unique solution* $u_0 \in C^n([a,b])$ *having the property that*

$$|u_0(x) - u(x)| \leq \frac{(b-a)^n \beta}{1 - 2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k}} \sigma(x), \tag{4.2}$$

for each $x \in [a,b]$.

Proof. We describe an operator $T : C^n([a,b]) \to C^n([a,b])$ by

$$(Tu)(x) = \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} H\Big(r_1, u^{[1]}(r_1), u^{[1]}(p(r_1)), ..., u^{[m]}(p(r_1)),$$

$$\int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), u^{[1]}(r(t)), ..., u^{[m]}(r(t))\Big)\,dt\Big)\,dr_1 dr_2 ... dr_n, \tag{4.3}$$

for each $x \in [a,b]$ and $u \in C^n([a,b])$.

Using the same reasoning as in the proof of Theorem 3.1, it follows that the operator T is strictly contractive with respect to the Bielecki metric, owing to the requirement

$$2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k} < 1.$$

Thereby , we can utilize Theorem 2.1, which guarantees that HOIVIDDE (1.3) possesses the σ-semi U-H stability. Next, requirement (4.1), together with integration, leads to

$$\Bigg|\int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} H\Big[r_1, u^{[1]}(r_1), u^{[2]}(r_1), ..., u^{[m]}(r_1)\,,$$

$$\int_a^{r_1} Z\Big(r_1, t, u^{[1]}(t), u^{[2]}(t), ..., u^{[m]}(t)\Big)\Big]\,dt dr_1 dr_2 ... dr_n - u(x)\Bigg|$$

$$\le \int_a^x \int_a^{r_n} \int_a^{r_{n-1}} \cdots \int_a^{r_2} \beta\sigma(r_1) dr_1 dr_2 ... dr_n$$

$$\le \beta(b-a)^n, \tag{4.4}$$

for each $x \in [a,b]$.

Subsequently, using relations (4.3) and (4.4), we obtain

$$|(Tu)(x) - u(x)| \le \beta(b-a)^n, \tag{4.5}$$

From the definition adopted for the metric d_B, inequalities (4.4) and (4.5), we infer that

$$\sup_{x\in[a,b]} \frac{|u_0(x) - u(x)|}{\sigma(x)} \le \frac{1}{1 - 2\sum_{k=1}^{m} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n k K^{m-k}} \sup_{x\in[a,b]} \frac{\beta(b-a)^n}{\sigma(x)}. \tag{4.6}$$

Thus, the inequality (2.5) is satisfied. These results mean that, depending upon the requirements (A-1)–(A-3) of Theorem 4.1, IVP (1.3), (1.4) possesses a unique solution and HOIVIDDE (1.3) admits the σ-semi U-H stability. □

Theorem 4.2. *We assume that the requirements (A-1)–(A-3) of Theorem 3.1 are satisfied. If* $u \in C^n([a,b])$ *and this function satisfies the inequality*

$$\left| u^{(n)}(x) - H\left(x, ..., u^{[m]}(p(x)), \int_a^x Z\left(x, t, ..., u^{[m]}(r(t))\right) dt \right) \right| \leq \beta, \tag{4.7}$$

for each $x \in [a,b]$, *where* $\beta \geq 0$, $\beta \in R$, *and*

$$2 \sum_{k=1}^{m} L_{H_k} \left(1 + L_{Z_k}\ell\right) \ell^n k K^{m-k} < 1, \tag{4.8}$$

then HOIVIDDE (1.3) *with* $u^{(i)}(a) = 0,\ i = 0, 1, ..., n-1$, *has a unique solution* $u_0 \in C^n([a,b])$ *having the property that*

$$|u_0(x) - u(x)| \leq \frac{(b-a)^n \sigma(b)}{1 - 2 \sum\limits_{k=1}^{m} L_{H_k} \left(1 + L_{Z_k}\ell\right) \ell^n k K^{m-k}}, \tag{4.9}$$

for each $x \in [a,b]$.

Proof. By applying Theorem 4.1, the uniqueness of solutions to IVP (1.3), (1.4) and the U-H stability of HOIVIDDE (1.3) are established. For the sake of brevity, the proof is omitted. □

Remark 4.1. *It should be noted that the proofs of Theorems 3.1, 4.1 and 4.2 rely on the Banach contraction mapping principle, the definitions of U–H–R, σ-semi U–H, and U–H stability of HOIVIDDE* (1.3), *the Bielecki metric, and the properties of iterative functions. These fundamental tools necessitate the sufficient conditions (A-1), (A-2) and (A-3) imposed in Theorems 3.1, 4.1 and 4.2.*

5. Application of results

We illustrate the application of the results through the following specific example.

Example 5.1. *Let* $x \in [0, 0.5]$. *We now focus on the following HOIVIDDE, considered as a specific case of the nonlinear HOIVIDDE* (1.3), *incorporating a variable time delay:*

$$\begin{aligned} u''(x) = &- \frac{1}{500} x^3 u^{[1]}(x) - \frac{16}{500} x^3 u^{[1]}\left(\frac{x}{2}\right) + 12x^2 \\ &+ \frac{1}{500} \int_0^x \left[t^2 u^{[1]}(t) + 16t^2 u^{[1]}\left(2^{-1}t\right) \right] dt, \end{aligned} \tag{5.1}$$

with

$$u^{(i)}(0) = 0,\ i = 0, 1. \tag{5.2}$$

In the next step, we perform a comparison between HOIVIDDE (1.3) *with* (1.4) *and HOIVIDDE* (5.1) *with* (5.2), *which reveals the following outcomes, respectively: ,*

$$H\left(x, u^{[1]}(x), u^{[1]}(p(x)), ..., u^{[m]}(x), u^{[m]}(p(x))\right)$$

$$= H\left(x, u^{[1]}(x), u^{[1]}\left(2^{-1}x\right)\right) = H\left(x, u(x), u\left(2^{-1}x\right)\right)$$

$$= -\frac{1}{500}x^3u^{[1]}(x) - \frac{16}{500}x^3u^{[1]}\left(2^{-1}x\right)$$

$$= \frac{1}{500}x^3u(x) - \frac{16}{500}x^3u\left(2^{-1}x\right),$$

$$Z\left(x, t, u^{[1]}(t), u^{[1]}(r(t)), ..., u^{[m]}(t), u^{[m]}(r(t))\right)$$

$$= Z\left(x, t, u^{[1]}(t), u^{[1]}\left(2^{-1}t\right)\right) = Z\left(x, t, u(t), u\left(2^{-1}t\right)\right)$$

$$= \frac{1}{500}tu^{[1]}(t) + \frac{16}{500}t^2u^{[1]}\left(2^{-1}t\right)$$

$$= \frac{1}{500}tu(t) + \frac{16}{500}t^2u\left(2^{-1}t\right),$$

$$H : [0, 0.5] \times R^2 \to R,$$

$$Z : [0, 0.5] \times [0, 0.5] \times R^2 \to R,$$

$$n = 2, m = 1, [a, b] = [0, 0.5], u^{(i)}(0) = 0,\ i = 0, 1,$$

$$p(x) = \frac{x}{2}, r(t) = \frac{t}{2},$$

$$p(x) = \frac{x}{2} \leq x,\ \textit{for each } x \in [0, 0.5],$$

and

$$r(t) = \frac{t}{2} \leq t\ \textit{for each } t \in [0, 0.5].$$

By virtue of the requirements (A-1) and (A-2) Theorems 3.1, 4.1 and 4.2, it follows that the following inequalities hold:

$$|H(x, u(x), u(p(x))) - H(x, \upsilon(x), \upsilon(p(x)))|$$

$$= \left|H\left(x, u(x), u\left(2^{-1}x\right)\right) - H\left(x, \upsilon(x), \upsilon\left(2^{-1}x\right)\right)\right|$$

$$= \left|\frac{1}{500}x^3u(x) - \frac{16}{500}x^3u\left(2^{-1}x\right) - \frac{1}{500}x^3\upsilon(x) + \frac{16}{500}x^3\upsilon\left(2^{-1}x\right)\right|$$

$$\leq \frac{1}{500}x^3|u(x) - \upsilon(x)| + \frac{16}{500}x^3\left|u\left(2^{-1}x\right) - \upsilon\left(2^{-1}x\right)\right|$$

$$\leq \frac{1}{4000}|u(x) - \upsilon(x)| + \frac{1}{250}\left|u\left(2^{-1}x\right) - \upsilon\left(2^{-1}x\right)\right|$$

$$\leq \frac{1}{250}\left(|u(x) - \upsilon(x)| + \left|u\left(2^{-1}x\right) - \upsilon\left(2^{-1}x\right)\right|\right),$$

where

$$L_{H_1} = \frac{1}{250};$$

$$|Z(x, t, u(t), u(p(t))) - Z(x, t, \upsilon(t), \upsilon(r(t)))|$$

$$= \left|Z\left(x, t, u(t), u\left(2^{-1}t\right)\right) - Z\left(x, t, \upsilon(t), \upsilon\left(2^{-1}t\right)\right)\right|$$

$$= \frac{1}{500}\left|tu(t) + 16t^2u\left(2^{-1}t\right) - t\upsilon(t) - 16t^2\upsilon\left(2^{-1}t\right)\right|$$
$$\leq \frac{t}{500}|u(t) - \upsilon(t)| + \frac{4t^2}{125}\left|u\left(2^{-1}t\right) - \upsilon\left(2^{-1}t\right)\right|$$
$$\leq \frac{1}{1000}|u(t) - \upsilon(t)| + \frac{1}{125}\left|u\left(2^{-1}t\right) - \upsilon\left(2^{-1}t\right)\right|$$
$$\leq \frac{1}{125}\left(|u(t) - \upsilon(t)| + \left|u\left(2^{-1}t\right) - \upsilon\left(2^{-1}t\right)\right|\right),$$

where

$$L_{Z_1} = \frac{1}{125}.$$

Hence, the requirements (A-1) and (A-2) of Theorems 3.1, 4.1 and 4.2 are satisfied.

We also consider a continuous function $\sigma : [0, 0.5] \to (0, \infty)$ *described by*

$$\sigma = 0.1\exp(2.2x),$$

which satisfies the following inequality:

$$\int_0^x 0.1\exp(2.2t)dt \leq 0.1\ell\exp(2.2x),$$

for each $\ell \in [0.4545, \infty)$*. Thus, the requirement (A-3) of Theorems 3.1, 4.1 and 4.2 is also satisfied. For the subsequent step, depending on the values*

$$k\text{=}1,\ m = 1,\ n = 2, L_{H_1} = \frac{1}{250} \text{ and } L_{Z_1} = \frac{1}{125},$$

if $\ell \in [0.4545, 9.699)$*, then it follows that*

$$2\sum_{k=1}^{1} L_{H_k}\left(1 + L_{Z_k}\ell\right)\ell^n kK^{m-k} = 2L_{H_1}\left(1 + L_{Z_1}\ell\right)\ell^2$$
$$\leq 2 \times \frac{1}{250}\left(1 + \frac{\ell}{125}\right)\ell^2$$
$$= \left(\frac{1}{125} + \frac{\ell}{15625}\right)\ell^2 < 1.$$

We choose $u(x) = \frac{1}{2}x^3$ *as an approximate solution of HOIVIDDEs* (5.1)*, where* $x \in [0, 0.5]$*. Then, in the light* (3.1) *and* (5.1)*, it follows that*

$$\left|u''(x) + \frac{1}{500}x^3u^{[1]}(x) + \frac{16}{500}x^3u^{[1]}\left(\frac{x}{2}\right) - 12x^2 - \frac{1}{500}\int_0^x\left[t^2u^{[1]}(t) + 16t^2u^{[1]}\left(2^{-1}t\right)\right]\right|$$
$$= \left|3x - \frac{1}{400}x^6 - 12x^2\right| \leq 0.1\exp(2.2x) = \sigma(x),\ x \in [0, 0.5].$$

Next, it can be provided that $u_0(x) = x^4$ *is exact solution of HOIVIDDE* (5.1)*, Moreover, based on the current data accordingly, it follows that*

$$|u(x) - u_0(x)| = \left|\frac{1}{2}x^3 - x^4\right|,$$

and

$$\begin{aligned}
&\frac{\ell^2}{1 - 2L_{H_1}\left(1 + L_{Z_1}\ell\right)\ell^2}\sigma(x) \\
&= \frac{(0.4545)^2 \times 0.1}{1 - 2 \times \frac{1}{250}\left(1 + \frac{0.4545}{125}\right) \times (0.4545)^2} \exp(2.2x) \\
&\cong \frac{0.02066}{0.9983} \exp(2.2x) \\
&\cong 0.02070 \exp(2.2x).
\end{aligned}$$

Consequently, we observe that

$$|u(x) - u_0(x)| = \left|\frac{1}{2}x^3 - x^4\right| \leq 0.0207 \exp(2.2x), \quad \textit{for each} \;\; x \in [0, 0.5].$$

Then, the inequality of (3.2) *Theorem 3.1 is fulfilled. Moreover, it can be shown that the inequalities* (4.2) *and* (4.9) *of Theorems 4.1 and 4.2 are also fulfilled, respectively. Hence, as a result we conclude that HOIVIDDE* (5.1) *possesses U–H–R, σ-semi-U–H, and U–H stability, along with a unique solution for the associated initial condition* (5.2)*.*

Remark 5.1. *From the above discussion, it is clear that the functions involved in HOIVIDDE* (5.1) *satisfy the conditions (A-1) and (A-2) of Theorems 3.1, 4.1 and 4.2. Moreover, the function* $u_0(x) = x^4$ *is an exact solution of HOIVIDDE* (5.1) *and fulfills the initial condition* $u^{(i)}(0) = 0,\ i = 0, 1$. *We also chose the function* $u(x) = \frac{1}{2}x^3$ *as an approximate solution of HOIVIDDE* (5.1)*. In addition, the function* $\sigma(x) = 0.1 \exp(2.2x)$ *satisfying the condition (A-3) of Theorems 3.1, 4.1 and 4.2 has been identified. Under the remaining assumptions, we have shown that both the functions* $u(x) = \frac{1}{2}x^3$ *and* $u_0(x) = x^4$ *also satisfy the inequality* (3.2)*. These observations confirm the applicability of the conditions of Theorem 3.1. The verification of the remaining requirements of Theorems 4.1 and 4.2 can be carried out in a similar manner and is therefore omitted for brevity.*

6. Conclusions

In this work, we have investigated a class of nonlinear HOIVIDDEs involving variable time delays. Our focus was on establishing sufficient requirements that guarantee various types of Ulam-type stability, including U–H–R, σ-semi-U–H and U–H stability. In addition, we proved the uniqueness of solutions for the associated initial value problem. The analytical approach was based on a combination of tools, including the properties of iterative functions, the Banach fixed point theorem, the Bielecki metric, and other relevant techniques from functional analysis. These tools enabled us to derive rigorous results under general requirements, extending the theory of Ulam-type stability to a broader class of integro-delay differential equations. The findings of this study are novel and contribute significantly to the ongoing development of the qualitative theory of differential equations, IEs and Volterra HOIDE with and without delays. One of the main contributions is the generalization and improvement of Ulam-type stability results to nth-order HOIVIDDEs, which, to the best of our knowledge, has not been previously addressed in the literature. For future work, it would be worthwhile to extend the methods and results of this study to fractional-order integro-delay differential equations. In particular, investigating similar stability properties for equations involving Caputo, Riemann–Liouville, and Hilfer fractional derivatives could open up new avenues in the analysis the qualitative properties of differential equations.

Author contributions

S. Pinelas, C. Tunç, O. Tunç and M. Ş. Oguz: Conceptualization, validation, formal analysis; S. Pinelas, C. Tunç and O. Tunç: Methodology, project administration; S. Pinelas and C. Tunç: Investigation, supervision, funding acquisition; O. Tunç and M. Ş. Oguz: Writing-original draft preparation; C. Tunç and O. Tunç: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by CIDMA under the Portuguese Foundation for Science and Technology (FCT, https://ror.org/00snfqn58) Multi-Annual Financing Program for R&D Units, grants UID/4106/2025 and UID/PRR/4106/2025. The authors sincerely thank the anonymous referees for their careful reading, valuable suggestions, and constructive comments, which have substantially improved the quality and presentation of the results.

Conflict of interest

Prof. Sandra Pinelas is the Guest Editor of special issue "Recent advances in difference and differential equations and applications" for AIMS Mathematics. Prof. Sandra Pinelas was not involved in the editorial review and the decision to publish this article.

The authors declare no conflict of interest.

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