AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 1071–1116.
DOI:10.3934/math.2026047
Received: 04 August 2025
Revised: 24 December 2025
Accepted: 07 January 2026
Published: 15 January 2026

Research article

Advancing smart healthcare decision-making: an innovative Fermatean fuzzy N-bipolar soft expert set framework for complex multi-criteria group evaluations

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Abstract: The rapid advancement of smart healthcare systems demands sophisticated mathematical tools to manage uncertainty and conflicting expert opinions in critical decision-making (DM) processes. Multi-criteria group decision making (MCGDM) plays a pivotal role in synthesizing diverse expert evaluations for complex healthcare challenges. However, existing soft set (SS) extensions often struggle to simultaneously capture multinary evaluations, bipolar reasoning, higher-order fuzzy logic, and multi-expert input. To overcome these limitations, we propose the Fermatean fuzzy N-bipolar soft expert set (FFNBSES), which enhances fuzzy representation while integrating multinary, bipolar, and multi-expert evaluations. We formally define the fundamental operations of FFNBSES and demonstrate its algebraic properties. A DM methodology based on FFNBSES is developed and applied to a healthcare case study, showcasing its superior capability to handle expert consensus and disagreement in multi-criteria evaluations. Comparative analysis within the SS theory framework highlights the enhanced flexibility and robustness of FFNBSES for real-world MCGDM problems. This work provides a powerful and comprehensive approach to support smart healthcare transformation through improved group DM under uncertainty.

Keywords: Fermatean fuzzy N-bipolar soft expert sets; multi-criteria group decision-making; smart healthcare; soft expert sets; N-soft sets; Fermatean fuzzy sets
Mathematics Subject Classification: 03E72, 03E75, 90B50

1. Introduction

The digital transformation of healthcare has ushered in the era of smart healthcare systems, where technologies such as artificial intelligence, the Internet of Things, telemedicine, and cloud computing are increasingly integrated into clinical practice to enhance quality, efficiency, and accessibility. These innovations aim to improve diagnostic accuracy, enable proactive care, and mitigate disparities in healthcare delivery—especially in underserved regions [1–3]. As cities pursue sustainability and inclusivity, smart healthcare is becoming a foundational component of modern urban health infrastructure [4–6].

Despite their potential, integrating these technologies into complex healthcare ecosystems presents significant challenges. Decision-makers must evaluate multiple, often conflicting criteria such as cost, patient outcomes, data security, and user acceptance. In this context, multi-criteria decision-making (MCDM) frameworks have proven indispensable for balancing competing demands and conducting structured evaluations of smart health strategies [7–9]. These tools enable administrators and policymakers to incorporate both quantitative metrics and qualitative insights, including expert opinions, to reach informed, consensus-based decisions [10–12].

To better reflect the uncertainty and hesitation inherent in expert judgments, more flexible approaches beyond classical logic have gained attention. Since Zadeh's [13] seminal work on fuzzy sets (FSs), numerous extensions have emerged to enhance the modeling of ambiguity. Atanassov's [14] intuitionistic FSs (IFSs) introduced non membership degrees (NMDs) alongside membership degrees (MDs), enabling the explicit representation of hesitation. Building on this, Pythagorean FSs (PFSs) [15] and Fermatean FSs (FFSs) [16] further relaxed the underlying constraints on MDs and NMDs to accommodate higher levels of uncertainty in DM.

From a modeling perspective, the adoption of FFSs is particularly suitable for expert-driven DM environments that require expert involvement and bipolarity consideration. Compared with earlier FS extensions, including IFSs and PFSs, FFSs provide a broader and more flexible assessment space, enabling experts to express strong supportive and opposing evaluations without imposing overly restrictive conditions on the membership structure. This enhanced capability allows FFSs to handle higher uncertainty and hesitation, which are common in complex healthcare DM scenarios where expert judgments may conflict. Although more complex higher-order FS models can further expand expressive capability, their increased structural complexity may limit interpretability and practical usability in parameterized frameworks involving expert involvement and multinary evaluation. In contrast, FFSs offer a balanced structure that supports deeper uncertainty modeling while remaining conceptually transparent and computationally manageable.

Recent studies have further advanced FFS methodologies and FS-based approaches in MCDM. In particular, new distance measures and divergence metrics for complex FFSs have been developed with applications in MCDM, clustering, post-flood assessment, and pattern classification [17–19]. Moreover, FFSs have demonstrated practical utility across diverse application domains, including system reliability [20], transportation [21], investment analysis [22], and sustainability assessment [23, 24], highlighting their effectiveness in supporting nuanced DM under uncertainty [25]. Other works have demonstrated the versatility of FS-based methods, including spherical FSs for augmented reality evaluation [26], FFSs with Dombi aggregation for blockchain technology selection [27], and interval-valued picture FSs for design concept evaluation [28, 29]. Further contributions include linguistic

Pythagorean hesitant FSs, quasi-rung orthopair FSs, and T-spherical FSs with advanced aggregation strategies [30–32], as well as spherical fuzzy Z-numbers for municipal waste management [33].

While these FS extensions significantly enhance the modeling of vagueness and hesitation, they often lack mechanisms for structured parameterization—an aspect addressed by SS theory since its introduction by Molodtsov [34]. Alkhazaleh and Salleh [35, 36] expanded this theory by incorporating expert perspectives and fuzziness, resulting in soft expert sets (SESs) and fuzzy SESs (FSESs). Broumi and Smarandache [37] further enriched the model by integrating intuitionistic fuzzy logic. Innovations include Pythagorean fuzzy [38], Fermatean fuzzy [39], and hesitant FSESs [40], all aimed at improving model expressiveness in MCGDM contexts. Additionally, neutrosophic-based extensions such as interval-valued [41] and possibility-driven neutrosophic soft expert models [42] offer enhanced flexibility in dealing with uncertain decision environments. Subsequent extensions, including hypersoft sets [43, 44], introduced multi parameter capabilities for greater model flexibility. Fujita and Smarandache [45] generalized these models further through advanced variants like superhypersoft and bi-hypersoft sets.

An essential feature for capturing human reasoning more realistically is the ability to reflect both positive and negative evaluations within a decision framework. The concept of bipolar SSs (BSSs), introduced by Shabir and Naz [46], addresses this by enabling the simultaneous analysis of supportive and opposing information. Dalkilic and Demirtaş [47] extended this notion by introducing bipolar SESs (BSESs), incorporating expert evaluations under bipolarity. Their work inspired hybrid models such as fuzzy BSESs (FBSESs) [48] and m-polar FSESs [49], now applied in healthcare and risk assessment. Musa and Asaad [50] later proposed the bipolar hypersoft set, merging bipolarity with hypersoft logic to handle conflicting assessments more effectively. This was extended to the fuzzy bipolar hypersoft set by Asaad et al. [51] to better manage uncertainty.

All the aforementioned models are based on binary evaluation. To overcome this limitation, Fatimah et al. [52] introduced the concept of N-soft sets (NSSs), providing a multinary evaluation framework through multi parameter structures. Building on this, Ali and Akram [53] proposed (fuzzy) N-soft expert sets ((F)NSESs), which effectively incorporated expert judgment and fuzziness. Akram et al. [54] further enhanced this model using Pythagorean fuzzy logic, thereby improving its capacity to model uncertainty in group DM. Khan et al. [55] later contributed separable NSSs, which are tailored for multinary descriptions involving large-scale parameter sets. In parallel, Riaz et al. [56] introduced an M-parameterized N-soft topology-based TOPSIS approach, which integrated soft topological structures with classical MCDM techniques. Shabir and Fatima [57] contributed by formulating N-bipolar soft sets (NBSSs) for DM applications. Recently, Musa and collaborators have proposed advanced variants including bipolar M-parameterized NSSs [58], N-hypersoft sets [59], and N-bipolar hypersoft sets [60], each aiming to enhance the representation of multidimensional and contradictory information. These efforts culminated in the development of the N-bipolar hypersoft topology [61], which combines bipolarity, hypersoft modeling, and topological structures for robust MCGDM.

Extending this trajectory, the recently introduced N-bipolar soft expert set (NBSES) provides a sophisticated framework capable of integrating multinary, bipolar, and expert-based assessments. Musa et al. [62] presented this model to support complex MCGDM processes under uncertainty. Its practical relevance has been demonstrated in several domains. A fuzzy NBSES (FNBSES) approach was applied to healthcare facility allocation [63], addressing conflicting stakeholder priorities. Further extensions, such as intuitionistic FNBSES (IFNBSES) [64], have been utilized in cybersecurity risk assessments for financial institutions. Likewise, Pythagorean FNBSES (PFNBSES) [65] have supported strategic DM in urban transportation planning. Collectively, these models reflect the adaptability and effectiveness of N-bipolar soft expert-based approaches in addressing complex, real-world MCDM problems.

1.1. Challenges in existing approaches and the need for the proposed model

Classical fuzzy information is inherently limited because it employs a single MD to represent uncertainty. This makes it unsuitable for situations where hesitation, indeterminacy, or conflicting expert views must be explicitly modeled. For example, in medical DM, one expert may strongly support a treatment while another raises serious concerns; classical FSs cannot represent such bipolar and contradictory judgments simultaneously. Moreover, classical fuzzy frameworks are single-valued, lack parameterization, and do not incorporate expert involvement, which restricts their practicality in complex group DM problems. Therefore, more flexible frameworks such as SSs and their extensions have been developed to overcome these drawbacks.

Despite the progress made in SS theory and its numerous extensions, several critical limitations remain. Classical SSs and their fuzzy adaptations typically rely on binary evaluations and do not accommodate bipolar reasoning. Although BSSs addressed this gap by enabling both positive and negative assessments, they remain constrained in evaluation granularity and uncertainty handling.

The introduction of NSSs improved expressiveness through multinary evaluations and parameter richness, yet these models lack bipolar integration. Fuzzy enhancements—such as intuitionistic, Pythagorean, and even Fermatean fuzzy approaches—have added depth in modeling uncertainty and hesitation but fall short in structured, parameterized frameworks capable of handling expert-driven, conflicting information.

Even recent advanced models like PFNBSES may be insufficient in scenarios that demand high-level uncertainty handling, fine-grained evaluation, and expert consensus involving contradictory opinions.

To overcome these limitations, the FFNBSES model is proposed. This new framework:

- enables simultaneous analysis of positive and negative evaluations;
- applies Fermatean fuzzy logic to capture deep uncertainty and hesitation;
- incorporates multinary evaluation mechanisms suitable for nuanced decision contexts;
- supports expert involvement for collective DM in uncertain environments.

1.2. Scope and core contributions of the study

This study aims to construct a comprehensive and adaptive DM model that integrates bipolar reasoning, Fermatean fuzzy logic, multinary evaluation, and multi-expert input. The FFNBSES framework is proposed to address these needs and demonstrate its utility in complex decision environments.

The main contributions of this research are outlined as follows:

- Theoretical foundation of FFNBSES: A complete mathematical formulation of the FFNBSES model is presented, including essential operations such as union, intersection, complement, equality, and others, along with their algebraic properties and examples to ensure clarity and consistency.
- Design of a DM framework: An algorithmic structure is developed to apply the FFNBSES model in real-world MCGDM tasks. The procedure incorporates bipolar Fermatean fuzzy evaluations from multiple experts.
- Healthcare-oriented case study: The model is validated through its application to a smart healthcare scenario. This demonstrates the framework's practical relevance in evaluating alternatives across technological, clinical, and ethical dimensions.
- Comparative evaluation within SS models: A structured comparison with existing SS variants is conducted to highlight FFNBSES's advancements in membership structure, evaluation scale, bipolar handling, and expert integration. Strengths and limitations are discussed to provide a balanced outlook.

1.3. Layout of the paper

The organization of this paper is as follows. Section 2 presents essential notations and background definitions used throughout the paper. It also reviews several foundational models, including SSs, BSSs, and their expert-based and N-extended variants, thereby establishing a comparative basis for the proposed approach. Section 3 introduces the FFNBSES model, defines its key components, and formalizes a range of operations along with their corresponding algebraic properties and examples. Section 4 develops a DM methodology based on the FFNBSES framework and applies it to a healthcare-related scenario. Section 5 evaluates the FFNBSES model in the context of other SS-based approaches, offering a comparative analysis. Finally, Section 6 concludes the study and outlines potential directions for future research.

2. Preliminary concepts

This section introduces the essential definitions and notations adopted throughout the paper. Let $\Bbbk$ be the universal set of alternatives (or objects), and $\wp$ denote the set of attributes (or parameters). The evaluation scale is defined as $\Theta = \{0, 1, \dots, N-1\}$, where $N \in \{2, 3, \dots\}$. Let $\mathcal{E}$ represent the set of experts, and $O = \{0 = \text{disagree}, 1 = \text{agree}\}$ be the set of expert opinions. We define $F = \wp \times \mathcal{E} \times O$, and let $\zeta \subseteq F$.

We begin by reviewing classical FS models, where each alternative is assigned an MD in $[0, 1]$, allowing nuanced modeling of uncertainty. Extensions such as IFS, PFS, and FFS enrich this representation by incorporating additional membership dimensions, supporting multi-valued aggregation and more informed DM. The associated score and accuracy functions then offer a systematic basis for ranking alternatives according to these fuzzy numbers.

Definition 2.1. *Let $\mu^{\oplus}, \mu^{\ominus} : \Bbbk \to [0, 1]$ be the membership and non-membership functions associated with each $\kappa \in \Bbbk$. Then, the collection $\Psi = \{\langle \kappa, \mu^{\oplus}(\kappa), \mu^{\ominus}(\kappa) \rangle : \kappa \in \Bbbk\}$ represents the following:*

i. an FS [13], when $\mu^{\ominus}(\kappa) = 0$ for every $\kappa \in \Bbbk$;

ii. *an IFS [14], if* $0 \leq \mu^{\oplus}(\kappa) + \mu^{\ominus}(\kappa) \leq 1$*;*

iii. *a PFS [15], if* $0 \leq (\mu^{\oplus}(\kappa))^2 + (\mu^{\ominus}(\kappa))^2 \leq 1$*;*

iv. *an FFS [16], if* $0 \leq (\mu^{\oplus}(\kappa))^3 + (\mu^{\ominus}(\kappa))^3 \leq 1$.

Definition 2.2. *[16] Consider* $\psi = (\varphi^{\oplus}, \varphi^{\ominus})$ *as a Fermatean fuzzy number (FFN). Then,*

i. *The score value of* ψ *is given by* $\mathbb{S}(\psi) = (\varphi^{\oplus})^3 - (\varphi^{\ominus})^3$*, where* $\mathbb{S}(\psi) \in [-1, 1]$.

ii. *The accuracy value of* ψ *is defined as* $\mathbb{A}(\psi) = (\varphi^{\oplus})^3 + (\varphi^{\ominus})^3$*, with* $\mathbb{A}(\psi) \in [0, 1]$.

Definition 2.3. *[16] Let* $\psi_1 = (\varphi_1^{\oplus}, \varphi_1^{\ominus})$ *and* $\psi_2 = (\varphi_2^{\oplus}, \varphi_2^{\ominus})$ *be two FFNs, with their respective score values* $\mathbb{S}(\psi_1)$*,* $\mathbb{S}(\psi_2)$ *and accuracy values* $\mathbb{A}(\psi_1)$*,* $\mathbb{A}(\psi_2)$*. Then the comparison is made as follows:*

i. *If* $\mathbb{S}(\psi_1) > \mathbb{S}(\psi_2)$*, then* $\psi_1 \succ \psi_2$.

ii. *If* $\mathbb{S}(\psi_1) = \mathbb{S}(\psi_2)$*, Then,*

- *if* $\mathbb{A}(\psi_1) > \mathbb{A}(\psi_2)$*, then* $\psi_1 \succ \psi_2$*;*
- *if* $\mathbb{A}(\psi_1) < \mathbb{A}(\psi_2)$*, then* $\psi_1 \prec \psi_2$*;*
- *if* $\mathbb{A}(\psi_1) = \mathbb{A}(\psi_2)$*, then* $\psi_1 \simeq \psi_2$.

Building on these FS models, SSs provide a structured framework to represent relationships between objects and parameters, enabling parameterized decision analysis.

Definition 2.4. *[34] An SS refers to an ordered pair* $(\lambda, \wp)$*, where* $\lambda : \wp \to 2^{\Bbbk}$ *and* $2^{\Bbbk}$ *represents the set of all crisp subsets of* $\Bbbk$.

Definition 2.5. *A pair* (λ, ζ) *is described as follows:*

i. *It is called an SES [35] if* $\lambda : \zeta \to 2^{\Bbbk}$*;*

ii. *It is termed an FSES [36] if* $\lambda : \zeta \to \mathbb{F}^{\Bbbk}$*, where* $\mathbb{F}^{\Bbbk}$ *denotes the collection of all FSs over* $\Bbbk$*;*

iii. *It is referred to as an IFSES [37] if* $\lambda : \zeta \to \mathbb{I}^{\Bbbk}$*, where* $\mathbb{I}^{\Bbbk}$ *denotes the collection of all IFSs over* $\Bbbk$*;*

iv. *It is identified as a PFSES [38] if* $\lambda : \zeta \to \mathbb{P}^{\Bbbk}$*, where* $\mathbb{P}^{\Bbbk}$ *denotes the collection of all PFSs over* $\Bbbk$*;*

v. *It is known as an FFSES [39] if* $\lambda : \zeta \to \mathbb{F}_e^{\Bbbk}$*, where* $\mathbb{F}_e^{\Bbbk}$ *denotes the collection of all FFSs over* $\Bbbk$.

To further extend this framework, the NOT operation is introduced to model opposite or negative attributes explicitly.

Definition 2.6. *[35] The NOT set corresponding to a set* ζ*, denoted by* $\neg\zeta$*, is given by* $\neg\zeta = \{\neg\ell \mid \ell \in \zeta\}$*, where for each element* $\ell = (\flat, \upsilon, o)$ *such that* $\flat \in \wp$*,* $\upsilon \in \mathcal{E}$*, and* $o \in \mathcal{O}$*, the negation* $\neg\ell$ *is defined as* $(\neg\flat, \upsilon, o)$*, representing the opposite of the element* ℓ.

While SSs handle single evaluations, BSSs extend this concept by distinguishing between positive and negative assessments, thus capturing both supportive and opposing perspectives.

Definition 2.7. *[46] A BSS is characterized by the triple* $(\lambda, \delta, \wp)$*, where* $\lambda : \wp \to 2^{\Bbbk}$ *and* $\delta : \neg\wp \to 2^{\Bbbk}$*, satisfying the condition that for every* $\flat \in \wp$*, it holds that* $\lambda(\flat) \cap \delta(\neg\flat) = \emptyset$*, with* $\lambda(\flat), \delta(\neg\flat) \subseteq \Bbbk$.

Definition 2.8. *A triple* (λ, δ, ζ) *is characterized as follows:*

i. It is called a BSES [47] if $\lambda : \zeta \to 2^{\Bbbk}$ *and* $\delta : \neg\zeta \to 2^{\Bbbk}$*, satisfying* $\lambda(\ell) \cap \delta(\neg\ell) = \emptyset$ *for all* $\ell \in \zeta$*, where* $\lambda(\ell), \delta(\neg\ell) \subseteq \Bbbk$.

ii. It is referred to as an FBSES [48] if $\lambda : \zeta \to \mathbb{F}^{\Bbbk}$ *and* $\delta : \neg\zeta \to \mathbb{F}^{\Bbbk}$*, such that for each* $\ell \in \zeta$ *and* $\kappa \in \Bbbk$*, the constraint* $0 \leq \lambda(\ell)(\kappa) + \delta(\neg\ell)(\kappa) \leq 1$ *holds, where* $\lambda(\ell)(\kappa), \delta(\neg\ell)(\kappa) \in [0, 1]$.

To accommodate more granular assessments, NSSs generalize SSs using multigraded evaluation scales, while their fuzzy variants integrate membership and non-membership degrees for richer modeling.

Definition 2.9. *[52] An NSS is defined as a triple* $(\eta, \wp, N)$*, where* $\eta : \wp \to 2^{\Bbbk\times\Theta}$*, such that for each* $\flat \in \wp$*, there exists a unique pair* $(\kappa, \vartheta_\flat) \in \Bbbk \times \Theta$ *satisfying* $(\kappa, \vartheta_\flat) \in \eta(\flat)$*, where* $\kappa \in \Bbbk$ *and* $\vartheta_\flat \in \Theta$*. The set* $2^{\Bbbk\times\Theta}$ *denotes all crisp subsets of* $\Bbbk \times \Theta$.

Definition 2.10. *A triple* (η, ζ, N) *is defined as follows:*

i. It is called an NSES [53] if $\eta : \zeta \to 2^{\Bbbk\times\Theta}$ *assigns to each element* $\ell \in \zeta$ *a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk\times\Theta$ *with* $\kappa \in \Bbbk$ *and* $\vartheta_\ell \in \Theta$*, such that* $(\kappa, \vartheta_\ell) \in \eta(\ell)$.

ii. It is called an FNSES [53] if $\eta : \zeta \to \mathbb{F}^{\Bbbk\times\Theta}$*, where for every* $\ell \in \zeta$ *there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\kappa \in \Bbbk$ *and* $\vartheta_\ell \in \Theta$*, satisfying* $\langle(\kappa, \vartheta_\ell), \eta(\kappa, \vartheta_\ell)\rangle \in \eta(\ell)$*. Here,* $\mathbb{F}^{\Bbbk\times\Theta}$ *represents the set of all FSs over* $\Bbbk \times \Theta$.

iii. It is called an IFNSES [54] if $\eta : \zeta \to \mathbb{I}^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$ *there exists a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\kappa \in \Bbbk$ *and* $\vartheta_\ell \in \Theta$ *such that* $\langle(\kappa, \vartheta_\ell), \eta^{\oplus}(\kappa, \vartheta_\ell), \eta^{\ominus}(\kappa, \vartheta_\ell)\rangle \in \eta(\ell)$*, where* $0 \leq \eta^{\oplus}(\kappa, \vartheta_\ell) + \eta^{\ominus}(\kappa, \vartheta_\ell) \leq 1$ *and* $\eta^{\oplus}(\kappa, \vartheta_\ell), \eta^{\ominus}(\kappa, \vartheta_\ell) \in [0, 1]$*. The set* $\mathbb{I}^{\Bbbk\times\Theta}$ *denotes all IFSs on* $\Bbbk \times \Theta$.

iv. It is called a PFNSES [54] if $\eta : \zeta \to \mathbb{P}^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$ *there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\kappa \in \Bbbk$ *and* $\vartheta_\ell \in \Theta$ *satisfying* $\langle(\kappa, \vartheta_\ell), \eta^{\oplus}(\kappa, \vartheta_\ell), \eta^{\ominus}(\kappa, \vartheta_\ell)\rangle \in \eta(\ell)$*, with the condition* $0 \leq (\eta^{\oplus}(\kappa, \vartheta_\ell))^2 + (\eta^{\ominus}(\kappa, \vartheta_\ell))^2 \leq 1$*, where* $\eta^{\oplus}(\kappa, \vartheta_\ell), \eta^{\ominus}(\kappa, \vartheta_\ell) \in [0, 1]$*. The set* $\mathbb{P}^{\Bbbk\times\Theta}$ *represents all PFSs on* $\Bbbk \times \Theta$.

Although NSSs and their fuzzy extensions provide expressive modeling, DM often requires simultaneous consideration of positive and negative evaluations. This motivates NBSSs, which capture dual evaluations for a single expert.

Definition 2.11. *[57] An NBSS is defined as a quadruple* $(\tau, \pi, \wp, N)$ *where* $\tau : \wp \to 2^{\Bbbk\times\Theta}$ *and* $\pi : \neg\wp \to 2^{\Bbbk\times\Theta}$ *satisfy the following conditions: For each* $\flat \in \wp$*, there exists a unique pair* $(\kappa, \vartheta_\flat) \in \Bbbk \times \Theta$ *such that* $(\kappa, \vartheta_\flat) \in \tau(\flat)$*; similarly, for each* $\neg\flat \in \neg\wp$*, there is a unique pair* $(\kappa, \vartheta_{\neg\flat}) \in \Bbbk \times \Theta$ *with* $(\kappa, \vartheta_{\neg\flat}) \in \pi(\neg\flat)$*, and these pairs fulfill the inequality* $\vartheta_\flat + \vartheta_{\neg\flat} \leq N - 1$*, where* $\kappa \in \Bbbk$ *and* $\vartheta_\flat, \vartheta_{\neg\flat} \in \Theta$.

Building on NBSSs, their extensions—NBSES, FNBSES, IFNBSES, and PFNBSES—incorporate multiple experts and various fuzzy membership types, facilitating more nuanced group DM under uncertainty.

Definition 2.12. *A quadruple* (τ, π, ζ, N) *is called*

i. *an NBSES [62], if* $\tau : \zeta \rightarrow 2^{\Bbbk\times\Theta}$ *and* $\pi : \neg\zeta \rightarrow 2^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$*, there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $(\kappa, \vartheta_\ell) \in \tau(\ell)$*, and for each* $\neg\ell \in \neg\zeta$*, there is a unique pair* $(\kappa, \vartheta_{\neg\ell}) \in \Bbbk \times \Theta$ *with* $(\kappa, \vartheta_{\neg\ell}) \in \pi(\neg\ell)$*, satisfying* $\vartheta_\ell + \vartheta_{\neg\ell} \leq N - 1$*, where* $\kappa \in \Bbbk$ *and* $\vartheta_\ell, \vartheta_{\neg\ell} \in \Theta$.

ii. *an FNBSES [63], if* $\tau : \zeta \rightarrow \mathbb{F}^{\Bbbk\times\Theta}$ *and* $\pi : \neg\zeta \rightarrow \mathbb{F}^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$*, there exists a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_\ell), \tau^{\oplus}(\kappa, \vartheta_\ell)\rangle \in \tau(\ell)$*, and for each* $\neg\ell \in \neg\zeta$*, there is a unique pair* $(\kappa, \vartheta_{\neg\ell}) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_{\neg\ell}), \pi^{\oplus}(\kappa, \vartheta_{\neg\ell})\rangle \in \pi(\neg\ell)$*, satisfying* $0 \leq \tau^{\oplus}(\kappa, \vartheta_\ell) + \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}) \leq 1$*, where* $\kappa \in \Bbbk$*,* $\vartheta_\ell, \vartheta_{\neg\ell} \in \Theta$*, and* $\tau^{\oplus}(\kappa, \vartheta_\ell), \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}) \in [0, 1]$.

iii. *an IFNBSES [64], if* $\tau : \zeta \rightarrow \mathbb{I}^{\Bbbk\times\Theta}$ *and* $\pi : \neg\zeta \rightarrow \mathbb{I}^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$*, there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_\ell), \tau^{\oplus}(\kappa, \vartheta_\ell), \tau^{\ominus}(\kappa, \vartheta_\ell)\rangle \in \tau(\ell)$*, and for each* $\neg\ell \in \neg\zeta$*, there is a unique pair* $(\kappa, \vartheta_{\neg\ell}) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_{\neg\ell}), \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}), \pi^{\ominus}(\kappa, \vartheta_{\neg\ell})\rangle \in \pi(\neg\ell)$*, subject to* $0 \leq \tau^{\oplus}(\kappa, \vartheta_\ell) + \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}) \leq 1$ *and* $0 \leq \tau^{\ominus}(\kappa, \vartheta_\ell) + \pi^{\ominus}(\kappa, \vartheta_{\neg\ell}) \leq 1$*, where* $\kappa \in \Bbbk$*,* $\vartheta_\ell, \vartheta_{\neg\ell} \in \Theta$*, and all* $\tau^{\oplus}(\kappa, \vartheta_\ell), \tau^{\ominus}(\kappa, \vartheta_\ell), \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}), \pi^{\ominus}(\kappa, \vartheta_{\neg\ell}) \in [0, 1]$*. Clearly,* $\tau^{\oplus}(\kappa, \vartheta_\ell)$ *and* $\pi^{\oplus}(\kappa, \vartheta_{\neg\ell})$ *are MDs, while* $\tau^{\ominus}(\kappa, \vartheta_\ell)$ *and* $\pi^{\ominus}(\kappa, \vartheta_{\neg\ell})$ *are NMDs.*

iv. *a PFNBSES [65], if* $\tau : \zeta \rightarrow \mathbb{P}^{\Bbbk\times\Theta}$ *and* $\pi : \neg\zeta \rightarrow \mathbb{P}^{\Bbbk\times\Theta}$ *such that for each* $\ell \in \zeta$*, there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_\ell), \tau^{\oplus}(\kappa, \vartheta_\ell), \tau^{\ominus}(\kappa, \vartheta_\ell)\rangle \in \tau(\ell)$*, and for each* $\neg\ell \in \neg\zeta$*, there is a unique pair* $(\kappa, \vartheta_{\neg\ell}) \in \Bbbk \times \Theta$ *with* $\langle(\kappa, \vartheta_{\neg\ell}), \pi^{\oplus}(\kappa, \vartheta_{\neg\ell}), \pi^{\ominus}(\kappa, \vartheta_{\neg\ell})\rangle \in \pi(\neg\ell)$*, subject to* $0 \leq (\tau^{\oplus}(\kappa, \vartheta_\ell))^2 + (\pi^{\oplus}(\kappa, \vartheta_{\neg\ell}))^2 \leq 1$ *and* $0 \leq (\tau^{\ominus}(\kappa, \vartheta_\ell))^2 + (\pi^{\ominus}(\kappa, \vartheta_{\neg\ell}))^2 \leq 1$*, where* $\kappa \in \Bbbk$*,* $\vartheta_\ell, \vartheta_{\neg\ell} \in \Theta$*, and* $\tau^{\oplus}(\kappa, \vartheta_\ell)$*,* $\tau^{\ominus}(\kappa, \vartheta_\ell)$*,* $\pi^{\oplus}(\kappa, \vartheta_{\neg\ell})$*,* $\pi^{\ominus}(\kappa, \vartheta_{\neg\ell}) \in [0, 1]$.

It is important to note that the FNBSES, IFNBSES, and PFNBSES models are fuzzy extensions of the crisp NBSES framework. Consequently, the condition on the evaluation grades, namely $\vartheta_\ell + \vartheta_{\neg\ell} \leq N-1$, is inherited from the underlying crisp model and is assumed to be satisfied for all parameter pairs. Therefore, this condition is not restated in the fuzzy extensions, and the emphasis is placed instead on the additional membership and non-membership constraints that govern the fuzzy evaluations.

3. Fermatean fuzzy N-bipolar soft expert sets

This section presents the innovative FFNBSES model along with its core operations such as the null set, whole set, complement, subset, equality, agreement, disagreement, union, and intersection. Each operation is accompanied by its algebraic properties and clarified through relevant examples.

Definition 3.1. *A quadruple* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *is termed an FFNBSES if* $\overline{\gamma} : \zeta \rightarrow \mathbb{F}_e^{\Bbbk\times\Theta}$ *and* $\underline{\gamma} : \neg\zeta \rightarrow \mathbb{F}_e^{\Bbbk\times\Theta}$*, such that for each* $\ell \in \zeta$*, there is a unique pair* $(\kappa, \vartheta_\ell) \in \Bbbk \times \Theta$ *satisfying* $\langle(\kappa, \vartheta_\ell), \overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell), \overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell)\rangle \in \overline{\gamma}(\ell)$*, and for each* $\neg\ell \in \neg\zeta$*, there is a unique pair* $(\kappa, \vartheta_{\neg\ell}) \in \Bbbk \times \Theta$ *such that* $\langle(\kappa, \vartheta_{\neg\ell}), \underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}), \underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell})\rangle \in \underline{\gamma}(\neg\ell)$*, subject to the following conditions:*

$$0 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 \leq 1,$$

$$0 \leq (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 \leq 1,$$

where $\kappa \in \Bbbk$*,* $\vartheta_\ell, \vartheta_{\neg\ell} \in \Theta$*, and* $\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell), \overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell), \underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}), \underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}) \in [0, 1]$*. Here,* $\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell)$ *and* $\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell})$ *represent the MDs, whereas* $\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell)$ *and* $\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell})$ *denote the NMDs. The set* $\mathbb{F}_e^{\Bbbk\times\Theta}$ *stands for all FFSs on* $\Bbbk \times \Theta$*.*

Remark 3.1. *The following observations, in relation to Definition 3.1, clarify the structural assumptions inherited from earlier models and explain the rationale behind the additional Fermatean fuzzy constraints introduced in the FFNBSES framework.*

1) *As an extension of the previously defined FNBSES models, the FFNBSES framework likewise presumes that the evaluation grades satisfy $\vartheta_\ell + \vartheta_{\neg\ell} \leq N - 1$, as inherited from the crisp NBSES structure. Accordingly, this grading condition is assumed to hold for all admissible evaluations, while the novelty of the FFNBSES model lies in the introduction of Fermatean fuzzy membership constraints that further regulate the interaction between positive and negative evaluations.*

2) *The cubic-sum constraints serve to regulate the joint influence of a parameter and its opposite within the bipolar evaluation structure. Since ℓ and $\neg\ell$ represent two opposing perspectives of the same criterion, their associated membership and non membership values are not treated as independent quantities. The imposed condition ensures that strong support for a parameter is accompanied by correspondingly limited support for its opposite, thereby preventing contradictory extreme assessments. The use of the cubic form, characteristic of Fermatean fuzzy sets, amplifies high membership values and enforces stricter control as evaluations approach their upper bounds. In this way, the constraint captures the intended bipolar relationship in a multi-graded fuzzy context while maintaining logical consistency.*

From now on, we consider the sets $\mathbb{k} = \{\kappa_1, \kappa_2, \ldots, \kappa_n\}$, $\wp = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_m\}$, and $\mathcal{E} = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_t\}$ to be finite, unless otherwise specified. Under this assumption, the FFNBSES $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ can be conveniently expressed in a tabular form. In particular, we use the abbreviations $\overline{\gamma}(\ell_i)(\kappa_j) = \langle \vartheta_{ij_{\ell_i}}, \overline{\gamma}^{\oplus}_{ij}, \overline{\gamma}^{\ominus}_{ij} \rangle$ and $\underline{\gamma}(\neg\ell_i)(\kappa_j) = \langle \vartheta_{ij_{\neg\ell_i}}, \underline{\gamma}^{\oplus}_{ij}, \underline{\gamma}^{\ominus}_{ij} \rangle$ to denote the elements $\langle (\kappa_j, \vartheta_{ij_{\ell_i}}), \overline{\gamma}^{\oplus}(\kappa_j, \vartheta_{ij_{\ell_i}}), \overline{\gamma}^{\ominus}(\kappa_j, \vartheta_{ij_{\ell_i}}) \rangle \in \overline{\gamma}(\ell_i)$ and $\langle (\kappa_j, \vartheta_{ij_{\neg\ell_i}}), \underline{\gamma}^{\oplus}(\kappa_j, \vartheta_{ij_{\neg\ell_i}}), \underline{\gamma}^{\ominus}(\kappa_j, \vartheta_{ij_{\neg\ell_i}}) \rangle \in \underline{\gamma}(\neg\ell_i)$, respectively. The corresponding tabular representation is presented in Table 1.

Table 1. Tabular representation of the FFNBSES $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$.

$(\overline{\gamma}, \underline{\gamma}, \zeta, N)$	κ_1	κ_2	$\cdots$	κ_n
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_1, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_1, 1)(\kappa_n)$
$(\mathfrak{b}_1, \upsilon_2, 1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_2, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\mathfrak{b}_1, \upsilon_t, 1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_t, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_t, 1)(\kappa_n)$
$(\mathfrak{b}_2, \upsilon_1, 1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_1, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_1, 1)(\kappa_n)$
$(\mathfrak{b}_2, \upsilon_2, 1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_2, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\mathfrak{b}_2, \upsilon_t, 1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_t, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_2, \upsilon_t, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\mathfrak{b}_m, \upsilon_1, 1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_1, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_1, 1)(\kappa_n)$
$(\mathfrak{b}_m, \upsilon_2, 1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_2, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_1, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\mathfrak{b}_m, \upsilon_t, 1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_t, 1)(\kappa_1)$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\mathfrak{b}_m, \upsilon_t, 1)(\kappa_n)$

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$(\overline{\gamma}, \underline{\gamma}, \zeta, N)$	κ_1	κ_2	$\cdots$	κ_n
$(\flat_1, \upsilon_1, 0)$	$\overline{\gamma}(\flat_1, \upsilon_1, 0)(\kappa_1)$	$\overline{\gamma}(\flat_1, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_1, \upsilon_1, 0)(\kappa_n)$
$(\flat_1, \upsilon_2, 0)$	$\overline{\gamma}(\flat_1, \upsilon_2, 0)(\kappa_1)$	$\overline{\gamma}(\flat_1, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_1, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_1, \upsilon_t, 0)$	$\overline{\gamma}(\flat_1, \upsilon_t, 0)(\kappa_1)$	$\overline{\gamma}(\flat_1, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_1, \upsilon_t, 0)(\kappa_n)$
$(\flat_2, \upsilon_1, 0)$	$\overline{\gamma}(\flat_2, \upsilon_1, 0)(\kappa_1)$	$\overline{\gamma}(\flat_2, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_2, \upsilon_1, 0)(\kappa_n)$
$(\flat_2, \upsilon_2, 0)$	$\overline{\gamma}(\flat_2, \upsilon_2, 0)(\kappa_1)$	$\overline{\gamma}(\flat_2, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_2, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_2, \upsilon_t, 0)$	$\overline{\gamma}(\flat_2, \upsilon_t, 0)(\kappa_1)$	$\overline{\gamma}(\flat_2, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_2, \upsilon_t, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m, \upsilon_1, 0)$	$\overline{\gamma}(\flat_m, \upsilon_1, 0)(\kappa_1)$	$\overline{\gamma}(\flat_m, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_m, \upsilon_1, 0)(\kappa_n)$
$(\flat_m, \upsilon_2, 0)$	$\overline{\gamma}(\flat_m, \upsilon_2, 0)(\kappa_1)$	$\overline{\gamma}(\flat_m, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_1, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m, \upsilon_t, 0)$	$\overline{\gamma}(\flat_m, \upsilon_t, 0)(\kappa_1)$	$\overline{\gamma}(\flat_m, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\overline{\gamma}(\flat_m, \upsilon_t, 0)(\kappa_n)$
$(\neg\flat_1, \upsilon_1, 1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 1)(\kappa_n)$
$(\neg\flat_1, \upsilon_2, 1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_1, \upsilon_t, 1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 1)(\kappa_n)$
$(\neg\flat_2, \upsilon_1, 1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 1)(\kappa_n)$
$(\neg\flat_2, \upsilon_2, 1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_2, \upsilon_t, 1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m, \upsilon_1, 1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 1)(\kappa_n)$
$(\neg\flat_m, \upsilon_2, 1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 1)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m, \upsilon_t, 1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 1)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 1)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 1)(\kappa_n)$
$(\neg\flat_1, \upsilon_1, 0)$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_1, 0)(\kappa_n)$
$(\neg\flat_1, \upsilon_2, 0)$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_1, \upsilon_t, 0)$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_1, \upsilon_t, 0)(\kappa_n)$
$(\neg\flat_2, \upsilon_1, 0)$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_1, 0)(\kappa_n)$
$(\neg\flat_2, \upsilon_2, 0)$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_2, \upsilon_t, 0)$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_2, \upsilon_t, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

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$(\overline{\gamma}, \underline{\gamma}, \zeta, N)$	κ_1	κ_2	$\cdots$	κ_n
$(\neg\flat_m, \upsilon_1, 0)$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_1, 0)(\kappa_n)$
$(\neg\flat_m, \upsilon_2, 0)$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_2, 0)(\kappa_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m, \upsilon_t, 0)$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 0)(\kappa_1)$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 0)(\kappa_2)$	$\cdots$	$\underline{\gamma}(\neg\flat_m, \upsilon_t, 0)(\kappa_n)$

To showcase the effectiveness and clarity of our proposed model, we present the following illustrative example:

Example 3.1. *Suppose a metropolitan planning authority is tasked with selecting the most suitable city to pilot a new smart public transportation system. The five candidate cities are denoted by* $\mathbb{k} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$. *Two critical evaluation criteria are selected:*

$$\wp = \{\flat_1 = \textit{infrastructure readiness},\ \flat_2 = \textit{technology integration capacity}\},$$

along with their associated negative criteria:

$$\neg\wp = \{\neg\flat_1 = \textit{infrastructure bottlenecks},\ \neg\flat_2 = \textit{technical integration risks}\}.$$

An internal transit committee performs an initial feasibility assessment, which is subsequently reviewed by two national urban mobility experts $\mathcal{E} = \{\upsilon_1, \upsilon_2\}$. *Each expert provides symbolic evaluations using a standardized transportation-readiness scale:*

- *"⋇" denotes critical limitations (e.g., outdated infrastructure or incompatible systems).*
- *"△" denotes minor limitations (e.g., partial readiness or small integration issues).*
- *"△△" denotes moderate readiness (some strengths, but room for improvement).*
- *"△△△" denotes strong readiness (good alignment with smart transit goals).*
- *"△△△△" denotes full readiness (fully equipped for implementation).*

Table 2. Symbolic evaluation of cities for smart transit system deployment by Metropolitan Planning Authority.

$\zeta \setminus \mathbb{k}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	△△△	△△	△△△△	△	⋇
$(\flat_1, \upsilon_2, 1)$	△△	△△	△△	△△△	△
$(\flat_2, \upsilon_1, 1)$	△△	△△	△△△	△	⋇
$(\flat_2, \upsilon_2, 1)$	⋇	△	△△△△	△△	△△
$(\flat_1, \upsilon_1, 0)$	⋇	△△	⋇	⋇	△
$(\flat_1, \upsilon_2, 0)$	⋇	△	△△	△	△
$(\flat_2, \upsilon_1, 0)$	△	⋇	⋇	△△	△△

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$\zeta \setminus \Bbbk$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_2, \upsilon_2, 0)$	$\triangle\triangle\triangle$	$\triangle\triangle$	$\divideontimes$	$\triangle\triangle$	$\triangle\triangle$
$(\neg\flat_1, \upsilon_1, 1)$	$\divideontimes$	$\triangle\triangle$	$\divideontimes$	$\triangle\triangle$	$\divideontimes$
$(\neg\flat_1, \upsilon_2, 1)$	$\triangle$	$\triangle$	$\triangle\triangle$	$\divideontimes$	$\divideontimes$
$(\neg\flat_2, \upsilon_1, 1)$	$\divideontimes$	$\triangle$	$\divideontimes$	$\triangle\triangle$	$\divideontimes$
$(\neg\flat_2, \upsilon_2, 1)$	$\divideontimes$	$\divideontimes$	$\divideontimes$	$\triangle\triangle$	$\triangle$
$(\neg\flat_1, \upsilon_1, 0)$	$\triangle\triangle$	$\triangle$	$\divideontimes$	$\divideontimes$	$\triangle\triangle$
$(\neg\flat_1, \upsilon_2, 0)$	$\triangle\triangle\triangle$	$\triangle\triangle$	$\triangle$	$\triangle\triangle$	$\triangle\triangle$
$(\neg\flat_2, \upsilon_1, 0)$	$\triangle\triangle$	$\divideontimes$	$\triangle\triangle\triangle$	$\triangle$	$\triangle$
$(\neg\flat_2, \upsilon_2, 0)$	$\triangle$	$\triangle\triangle$	$\triangle$	$\triangle$	$\triangle$

Each qualitative grade is mapped to a specific numerical value within the set $\Theta = \{0, 1, 2, 3, 4\}$*, defined as follows:*

- 0 *corresponds to* $\divideontimes$.
- 1 *corresponds to* $\triangle$.
- 2 *corresponds to* $\triangle\triangle$.
- 3 *corresponds to* $\triangle\triangle\triangle$.
- 4 *corresponds to* $\triangle\triangle\triangle\triangle$.

In line with Definition 2.12 (i), the detailed numerical evaluations of the 5BSES $(\tau, \pi, \zeta, 5)$ *are systematically arranged and presented in Table 3.*

Table 3. Symbolic-to-numerical mapping of grades in the Metropolitan Planning Authority evaluation scale.

$(\tau, \pi, \zeta, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	3	2	4	1	0
$(\flat_1, \upsilon_2, 1)$	2	2	2	3	1
$(\flat_2, \upsilon_1, 1)$	2	2	3	1	0
$(\flat_2, \upsilon_2, 1)$	0	1	4	2	2
$(\flat_1, \upsilon_1, 0)$	0	2	0	0	1
$(\flat_1, \upsilon_2, 0)$	0	1	2	1	1
$(\flat_2, \upsilon_1, 0)$	1	0	0	2	2
$(\flat_2, \upsilon_2, 0)$	3	2	0	2	2
$(\neg\flat_1, \upsilon_1, 1)$	0	2	0	2	0
$(\neg\flat_1, \upsilon_2, 1)$	1	1	2	0	0
$(\neg\flat_2, \upsilon_1, 1)$	0	1	0	2	0
$(\neg\flat_2, \upsilon_2, 1)$	0	0	0	2	1
$(\neg\flat_1, \upsilon_1, 0)$	2	1	0	0	2
$(\neg\flat_1, \upsilon_2, 0)$	3	2	1	2	2

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$\zeta \setminus \Bbbk$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\neg\flat_2, \upsilon_1, 0)$	2	0	3	1	1
$(\neg\flat_2, \upsilon_2, 0)$	1	2	1	1	1

To ensure a comprehensive evaluation process, experts provide their judgments in the form of an FF5BSES. Utilizing the grading scheme outlined above, the assignment of MDs and NMDs is conducted according to the framework established by FFSs. The corresponding evaluation criteria and grade boundaries are summarized in Table 4.

Table 4. Evaluation grades and their corresponding criteria for the Metropolitan Planning Authority assessments.

Grade	Criterion
$\vartheta_\ell = 0$	$0.0 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 < 0.2$
$\vartheta_\ell = 1$	$0.2 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 < 0.4$
$\vartheta_\ell = 2$	$0.4 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 < 0.6$
$\vartheta_\ell = 3$	$0.6 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 < 0.8$
$\vartheta_\ell = 4$	$0.8 \leq (\overline{\gamma}^{\oplus}(\kappa, \vartheta_\ell))^3 + (\overline{\gamma}^{\ominus}(\kappa, \vartheta_\ell))^3 \leq 1.0$
Grade	Criterion
$\vartheta_{\neg\ell} = 0$	$0.0 \leq (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 < 0.2$
$\vartheta_{\neg\ell} = 1$	$0.2 \leq (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 < 0.4$
$\vartheta_{\neg\ell} = 2$	$0.4 \leq (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 < 0.6$
$\vartheta_{\neg\ell} = 3$	$0.6 \leq (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 < 0.8$
$\vartheta_{\neg\ell} = 4$	$0.8 \leq (\underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell}))^3 + (\underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell}))^3 \leq 1.0$

It is important to note that the thresholds defining the evaluation grades in Table 4 (i.e., 0.0, 0.2, 0.4, 0.6, 0.8) are based on expert experience and standard practices in the relevant domain. These thresholds guide the mapping of FFNBSES numerical evaluations to qualitative grades, while still allowing flexible adjustment to reflect expert judgment in different scenarios. Although alternative threshold settings may slightly affect the final ranking of alternatives, the FFNBSES framework ensures that the evaluation process remains consistent, interpretable, and adaptable across diverse DM contexts. Additionally, within each cubic-sum range corresponding to a grade, multiple FFN pairs satisfy the defining inequality, and these pairs can be selected based on expert judgment, ensuring reproducibility while providing flexibility to accommodate different applications or contexts.

Based on these thresholds and expert judgments, the finalized FF5BSES evaluations for each alternative are compiled in Table 5.

Table 5. Expert evaluations of Metropolitan Planning Authority planning scenarios in the form of an FF5BSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	$\langle 3, 0.9, 0.2\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 4, 0.8, 0.7\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\flat_1, \upsilon_2, 1)$	$\langle 2, 0.6, 0.6\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 1, 0.5, 0.6\rangle$
$(\flat_2, \upsilon_1, 1)$	$\langle 2, 0.3, 0.8\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 3, 0.4, 0.9\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 0, 0.5, 0.4\rangle$
$(\flat_2, \upsilon_2, 1)$	$\langle 0, 0.3, 0.4\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.3, 0.8\rangle$
$(\flat_1, \upsilon_1, 0)$	$\langle 0, 0.1, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 0, 0.4, 0.4\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$
$(\flat_1, \upsilon_2, 0)$	$\langle 0, 0.2, 0.3\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 1, 0.6, 0.4\rangle$
$(\flat_2, \upsilon_1, 0)$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.5, 0.4\rangle$	$\langle 0, 0.0, 0.1\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.7, 0.6\rangle$
$(\flat_2, \upsilon_2, 0)$	$\langle 3, 0.7, 0.7\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.0, 0.2\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.4, 0.7\rangle$
$(\neg\flat_1, \upsilon_1, 1)$	$\langle 0, 0.0, 0.1\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 0, 0.1, 0.0\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 0, 0.1, 0.1\rangle$
$(\neg\flat_1, \upsilon_2, 1)$	$\langle 1, 0.6, 0.2\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 2, 0.1, 0.8\rangle$	$\langle 0, 0.5, 0.4\rangle$	$\langle 0, 0.2, 0.4\rangle$
$(\neg\flat_2, \upsilon_1, 1)$	$\langle 0, 0.4, 0.4\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.3, 0.3\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.0, 0.5\rangle$
$(\neg\flat_2, \upsilon_2, 1)$	$\langle 0, 0.1, 0.5\rangle$	$\langle 0, 0.5, 0.2\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 1, 0.1, 0.6\rangle$
$(\neg\flat_1, \upsilon_1, 0)$	$\langle 2, 0.6, 0.7\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.1, 0.0\rangle$	$\langle 0, 0.4, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$
$(\neg\flat_1, \upsilon_2, 0)$	$\langle 3, 0.7, 0.7\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 2, 0.4, 0.7\rangle$
$(\neg\flat_2, \upsilon_1, 0)$	$\langle 2, 0.5, 0.7\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 3, 0.2, 0.9\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 1, 0.2, 0.6\rangle$
$(\neg\flat_2, \upsilon_2, 0)$	$\langle 1, 0.6, 0.3\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 1, 0.5, 0.5\rangle$

We proceed by defining essential operations on FFNBSESs and outlining their algebraic properties grounded in the underlying theory. Each operation is accompanied by illustrative examples. The operations covered include the null and whole sets, complement, subset relation, equality, notions of agreement and disagreement, along with union and intersection.

We start with two important definitions, null and whole sets, which represent the lower and upper bounds in the FFNBSES framework.

Definition 3.2. *An FFNBSES* $(\overline{\gamma}^{\mathbb{N}}, \underline{\gamma}^{\mathbb{N}}, \zeta, N)$ *is termed a relative null set if, for every* $\ell \in \zeta$ *and* $\kappa \in \Bbbk$, *it holds that* $\overline{\gamma}^{\mathbb{N}}(\ell)(\kappa) = \langle 0, 0.0, 1.0\rangle$, *and for every* $\neg\ell \in \neg\zeta$ *and* $\kappa \in \Bbbk$, *it holds that* $\underline{\gamma}^{\mathbb{N}}(\neg\ell)(\kappa) = \langle N-1, 1.0, 0.0\rangle$.

Definition 3.3. *An FFNBSES* $(\overline{\gamma}^{\mathbb{W}}, \underline{\gamma}^{\mathbb{W}}, \zeta, N)$ *is termed a relative whole set if, for every* $\ell \in \zeta$ *and* $\kappa \in \Bbbk$, *it holds that* $\overline{\gamma}^{\mathbb{W}}(\ell)(\kappa) = \langle N-1, 1.0, 0.0\rangle$, *and for every* $\neg\ell \in \neg\zeta$ *and* $\kappa \in \Bbbk$, *it holds that* $\underline{\gamma}^{\mathbb{W}}(\neg\ell)(\kappa) = \langle 0, 0.0, 1.0\rangle$.

The following is the definition of subsethood and equality, which establish hierarchical relationships and enable comparison between FFNBSESs.

Definition 3.4. *An FFNBSES* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$ *is considered a subset of* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$, *denoted by* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \,\breve{\subseteq}\, (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$, *if the following criteria hold:*

1) $\zeta_1 \subseteq \zeta_2$;

2) For each $\ell \in \zeta_1$ *and* $\kappa \in \Bbbk$: $\vartheta_{1\ell} \leq \vartheta_{2\ell}$, $\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}) \leq \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})$, *and* $\overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell}) \leq \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})$, *where* $\langle(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ *and* $\langle(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$;

3) *For each* $\neg\ell \in \neg\zeta_1$ *and* $\kappa \in \mathbb{k}$*:* $\vartheta_{2\neg\ell} \leq \vartheta_{1\neg\ell}$, $\underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}) \leq \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell})$, *and* $\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}) \leq \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})$, *where* $\langle(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ *and* $\langle(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$.

Example 3.2. *Building on Example 3.1, consider two FF5BSESs* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ *as presented in Tables 6 and 7, respectively. It is evident that* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5) \check{\subseteq} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ *holds.*

Table 6. Tabular representation of the FF5BSES $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ in Example 3.2.

$(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 1, 0.2, 0.7\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 0, 0.0, 0.5\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 3, 0.2, 0.9\rangle$	$\langle 3, 0.1, 0.9\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 1, 0.2, 0.7\rangle$
$(\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 0, 0.0, 0.3\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 2, 0.4, 0.8\rangle$	$\langle 1, 0.0, 0.7\rangle$	$\langle 1, 0.2, 0.6\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 1, 0.7, 0.2\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 0, 0.5, 0.1\rangle$	$\langle 1, 0.6, 0.2\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.7, 0.1\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.8, 0.0\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 0, 0.4, 0.1\rangle$	$\langle 0, 0.5, 0.1\rangle$	$\langle 1, 0.6, 0.2\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 3, 0.9, 0.4\rangle$

Table 7. Tabular representation of the FF5BSES $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ in Example 3.2.

$(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 2, 0.6, 0.6\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 0, 0.4, 0.1\rangle$
$(\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 3, 0.5, 0.8\rangle$	$\langle 3, 0.2, 0.9\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.6, 0.5\rangle$
$(\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 1, 0.6, 0.3\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 3, 0.8, 0.5\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.5\rangle$	$\langle 1, 0.4, 0.6\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 0, 0.4, 0.4\rangle$	$\langle 0, 0.3, 0.5\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 0, 0.3, 0.5\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 1, 0.6, 0.2\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 0, 0.4, 0.1\rangle$	$\langle 0, 0.1, 0.5\rangle$	$\langle 0, 0.5, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 1, 0.4, 0.6\rangle$

Next, we define positive agree/disagree and negative agree/disagree, which we utilize in our methodology to facilitate computationally efficient handling of expert opinions.

Definition 3.5. *Two FFNBSESs* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$ *are considered equal if both* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\subseteq} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N) \check{\subseteq} (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$ *hold true.*

Definition 3.6. *Let* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *be an FFNBSES. The positive agree part, denoted by* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_1}$*, is a subset of* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *defined as*

$$(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_1} = \{\overline{\gamma}^{\oplus_1}(\ell) \mid \ell \in \wp \times \mathcal{E} \times \{1\}\}.$$

Definition 3.7. *Let* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *be an FFNBSES. The positive disagree part, denoted by* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_0}$*, is a subset of* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *given by*

$$(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_0} = \{\overline{\gamma}^{\oplus_0}(\ell) \mid \ell \in \wp \times \mathcal{E} \times \{0\}\}.$$

Definition 3.8. *Consider an FFNBSES* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$. *The negative agree part, represented as* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_1}$, *is a subset of* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *described by*

$$(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_1} = \{\underline{\gamma}^{\ominus_1}(\neg\ell) \mid \neg\ell \in \neg\wp \times \mathcal{E} \times \{1\}\}.$$

Definition 3.9. *Given an FFNBSES* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$. *The negative disagree part, denoted as* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_0}$, *is a subset of* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *and defined by*

$$(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_0} = \{\underline{\gamma}^{\ominus_0}(\neg\ell) \mid \neg\ell \in \neg\wp \times \mathcal{E} \times \{0\}\}.$$

Example 3.3. *Let us consider the FF5BSES* $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ *displayed in Table 5 from Example 3.1. The corresponding positive agree, positive disagree, negative agree, and negative disagree components are given in Tables 8–11.*

Table 8. Positive agree $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$ derived from the FF5BSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ in Example 3.1.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	$\langle 3, 0.9, 0.2\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 4, 0.8, 0.7\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\flat_1, \upsilon_2, 1)$	$\langle 2, 0.6, 0.6\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 1, 0.5, 0.6\rangle$
$(\flat_2, \upsilon_1, 1)$	$\langle 2, 0.3, 0.8\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 3, 0.4, 0.9\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 0, 0.5, 0.4\rangle$
$(\flat_2, \upsilon_2, 1)$	$\langle 0, 0.3, 0.4\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.3, 0.8\rangle$

Table 9. Positive disagree $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$ derived from the FF5BSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ in Example 3.1.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 0)$	$\langle 0, 0.1, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 0, 0.4, 0.4\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$
$(\flat_1, \upsilon_2, 0)$	$\langle 0, 0.2, 0.3\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 1, 0.6, 0.4\rangle$
$(\flat_2, \upsilon_1, 0)$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.5, 0.4\rangle$	$\langle 0, 0.0, 0.1\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.7, 0.6\rangle$
$(\flat_2, \upsilon_2, 0)$	$\langle 3, 0.7, 0.7\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.0, 0.2\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.4, 0.7\rangle$

Table 10. Negative agree $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$ derived from the FF5BSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ in Example 3.1.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\neg\flat_1, \upsilon_1, 1)$	$\langle 0, 0.0, 0.1\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 0, 0.1, 0.0\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 0, 0.1, 0.1\rangle$
$(\neg\flat_1, \upsilon_2, 1)$	$\langle 1, 0.6, 0.2\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 2, 0.1, 0.8\rangle$	$\langle 0, 0.5, 0.4\rangle$	$\langle 0, 0.2, 0.4\rangle$
$(\neg\flat_2, \upsilon_1, 1)$	$\langle 0, 0.4, 0.4\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.3, 0.3\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.0, 0.5\rangle$
$(\neg\flat_2, \upsilon_2, 1)$	$\langle 0, 0.1, 0.5\rangle$	$\langle 0, 0.5, 0.2\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 1, 0.1, 0.6\rangle$

Table 11. Negative disagree $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$ derived from the FF5BSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ in Example 3.1.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\neg\mathfrak{b}_1, \upsilon_1, 0)$	$\langle 2, 0.6, 0.7\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.1, 0.0\rangle$	$\langle 0, 0.4, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 3, 0.7, 0.7\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 2, 0.4, 0.7\rangle$
$(\neg\mathfrak{b}_2, \upsilon_1, 0)$	$\langle 2, 0.5, 0.7\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 3, 0.2, 0.9\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 1, 0.2, 0.6\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.6, 0.3\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 1, 0.5, 0.5\rangle$

The complementation in the FFNBSES environment is defined as follows:

Definition 3.10. *The complement of* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$, *denoted by* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\check{c}}$, *is defined as* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\check{c}} = (\overline{\gamma}^{\check{c}}, \underline{\gamma}^{\check{c}}, \zeta, N)$, *where for every* $\ell \in \zeta$ *and* $\kappa \in \Bbbk$, *it follows that* $\overline{\gamma}^{\check{c}}(\ell) = \underline{\gamma}(\neg\ell)$, *which implies* $\vartheta_t^{\check{c}} = \vartheta_{\neg\ell}$, $\overline{\gamma}^{\oplus\check{c}}(\kappa, \vartheta_\ell) = \underline{\gamma}^{\oplus}(\kappa, \vartheta_{\neg\ell})$, *and* $\overline{\gamma}^{\ominus\check{c}}(\kappa, \vartheta_\ell) = \underline{\gamma}^{\ominus}(\kappa, \vartheta_{\neg\ell})$. *Likewise, for each* $\neg\ell \in \neg\zeta$ *and* $\kappa \in \Bbbk$, *we have* $\underline{\gamma}^{\check{c}}(\neg\ell) = \overline{\gamma}(\ell)$, *i.e.,* $\vartheta_{\neg\ell}^{\check{c}} = \vartheta_t$, $\underline{\gamma}^{\oplus\check{c}}(\kappa, \vartheta_{\neg\ell}) = \overline{\gamma}^{\oplus}(\kappa, \vartheta_t)$, *and* $\underline{\gamma}^{\ominus\check{c}}(\kappa, \vartheta_{\neg\ell}) = \overline{\gamma}^{\ominus}(\kappa, \vartheta_t)$.

Example 3.4. *Let us refer to the FF5BSES* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ *provided in Table 6 of Example 3.2. The corresponding complement,* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)^{\check{c}}$, *is displayed in Table 12.*

Table 12. The complement $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)^{\check{c}}$ of the FF5BSES $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ in Example 3.2.

$(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)^{\check{c}}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 1, 0.7, 0.2\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 2, 0.8, 0.3\rangle$	$\langle 0, 0.5, 0.1\rangle$	$\langle 1, 0.6, 0.2\rangle$
$(\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.7, 0.1\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.8, 0.0\rangle$
$(\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 0, 0.4, 0.1\rangle$	$\langle 0, 0.5, 0.1\rangle$	$\langle 1, 0.6, 0.2\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 3, 0.9, 0.4\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 1, 0.2, 0.7\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 0, 0.0, 0.5\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 1)$	$\langle 3, 0.2, 0.9\rangle$	$\langle 3, 0.1, 0.9\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.1, 0.6\rangle$	$\langle 1, 0.2, 0.7\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 0)$	$\langle 0, 0.0, 0.3\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 2, 0.4, 0.8\rangle$	$\langle 1, 0.0, 0.7\rangle$	$\langle 1, 0.2, 0.6\rangle$

Proposition 3.1. *Let* $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$ *be an FFNBSES, and let* $(\overline{\gamma}^{\mathbb{N}}, \underline{\gamma}^{\mathbb{N}}, \zeta, N)$ *and* $(\overline{\gamma}^{\mathbb{W}}, \underline{\gamma}^{\mathbb{W}}, \zeta, N)$ *represent the relative null and relative whole, respectively. Then,*

1) $((\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\check{c}})^{\check{c}} = (\overline{\gamma}, \underline{\gamma}, \zeta, N)$;

2) $(\overline{\gamma}^{\mathbb{N}}, \underline{\gamma}^{\mathbb{N}}, \zeta, N)^{\check{c}} = (\overline{\gamma}^{\mathbb{W}}, \underline{\gamma}^{\mathbb{W}}, \zeta, N)$;

3) $(\overline{\gamma}^{\mathbb{W}}, \underline{\gamma}^{\mathbb{W}}, \zeta, N)^{\check{c}} = (\overline{\gamma}^{\mathbb{N}}, \underline{\gamma}^{\mathbb{N}}, \zeta, N)$.

Proof. Straightforward. □

The definitions of extended and restricted unions and intersections generalize classical set-theoretic operations to the FFNBSES environment. These operations retain essential properties such as commutativity and associativity.

Definition 3.11. *The extended union of* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$ *is denoted and defined as* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \,\check{\cup}_{\check{e}}\, (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}, \underline{\gamma}, \zeta_1 \cup \zeta_2, \max(N_1, N_2))$*, where for all* $\ell \in \zeta_1 \cup \zeta_2$*,*

$$\overline{\gamma}(\ell) = \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus \zeta_2, \\ \overline{\gamma}_2(\ell), & \text{if } \ell \in \zeta_2 \setminus \zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \\ \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_1 \cap \zeta_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ *and* $\langle(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cup \neg\zeta_2$*,*

$$\underline{\gamma}(\neg\ell) = \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus \neg\zeta_2, \\ \underline{\gamma}_2(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_2 \setminus \neg\zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \\ \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \\ \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap \neg\zeta_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ *and* $\langle(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$.

Example 3.5. *With reference to Example 3.1, the FF5BSESs* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ *are listed in Tables 13 and 14, respectively. Their extended union outcome is presented in Table 15.*

Table 13. Tabular representation of the FF5BSES $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ in Example 3.5.

$(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 3, 0.5, 0.8\rangle$	$\langle 2, 0.3, 0.8\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 0, 0.1, 0.4\rangle$	$\langle 4, 0.9, 0.6\rangle$
$(\mathfrak{b}_2, \upsilon_1, 0)$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 0, 0.4, 0.5\rangle$	$\langle 1, 0.5, 0.5\rangle$
$(\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 2, 0.1, 0.8\rangle$	$\langle 3, 0.9, 0.2\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.3\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.0, 0.5\rangle$	$\langle 1, 0.5, 0.6\rangle$	$\langle 0, 0.1, 0.3\rangle$
$(\neg\mathfrak{b}_2, \upsilon_1, 0)$	$\langle 2, 0.8, 0.3\rangle$	$\langle 0, 0.5, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 4, 0.9, 0.5\rangle$	$\langle 3, 0.8, 0.5\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.2, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.4, 0.4\rangle$	$\langle 0, 0.5, 0.1\rangle$

Table 14. Tabular representation of the FF5BSES $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ in Example 3.5.

$(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.0, 0.2\rangle$	$\langle 1, 0.4, 0.6\rangle$	$\langle 3, 0.7, 0.7\rangle$	$\langle 1, 0.5, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$
$(\mathfrak{b}_1, \upsilon_2, 1)$	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 3, 0.6, 0.8\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.5, 0.0\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 4, 1.0, 0.0\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 2, 0.5, 0.7\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 2, 0.6, 0.6\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 1)$	$\langle 3, 0.8, 0.6\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 0, 0.3, 0.3\rangle$	$\langle 1, 0.0, 0.7\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.1, 0.6\rangle$	$\langle 3, 0.9, 0.1\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.4, 0.8\rangle$	$\langle 0, 0.2, 0.4\rangle$

Table 15. The extended union $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5) = (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cup \zeta_2, 5)$ in Example 3.5.

$(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cup \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 3, 0.5, 0.2\rangle$	$\langle 2, 0.4, 0.6\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 1, 0.5, 0.4\rangle$	$\langle 4, 0.9, 0.6\rangle$
$(\mathfrak{b}_1, \upsilon_2, 1)$	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\mathfrak{b}_2, \upsilon_1, 0)$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 0, 0.4, 0.5\rangle$	$\langle 1, 0.5, 0.5\rangle$
$(\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 3, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.5\rangle$	$\langle 2, 0.5, 0.0\rangle$	$\langle 3, 0.9, 0.2\rangle$	$\langle 4, 1.0, 0.0\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.7\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.0, 0.6\rangle$	$\langle 1, 0.5, 0.6\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\neg\mathfrak{b}_1, \upsilon_2, 1)$	$\langle 3, 0.8, 0.6\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 0, 0.3, 0.3\rangle$	$\langle 1, 0.0, 0.7\rangle$
$(\neg\mathfrak{b}_2, \upsilon_1, 0)$	$\langle 2, 0.8, 0.3\rangle$	$\langle 0, 0.5, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 4, 0.9, 0.5\rangle$	$\langle 3, 0.8, 0.5\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.1, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.4, 0.8\rangle$	$\langle 0, 0.2, 0.4\rangle$

Proposition 3.2. *Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$, $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$, and $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$ be FFN$_1$BSES, FFN$_2$BSES, and FFN$_3$BSES, respectively. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{e}}\ ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2))\ \check{\cup}_{\check{e}}\ (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$.

Proof. Straightforward. □

Definition 3.12. *The extended intersection of $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$ and $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$ is denoted and defined as $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cap}_{\check{e}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}, \underline{\gamma}, \zeta_1 \cup \zeta_2, \max(N_1, N_2))$, where for all $\ell \in \zeta_1 \cup \zeta_2$,*

$$\overline{\gamma}(\ell) = \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus \zeta_2, \\ \overline{\gamma}_2(\ell), & \text{if } \ell \in \zeta_2 \setminus \zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \min\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \\ \max\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_1 \cap \zeta_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ and $\langle(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cup \neg\zeta_2$,

$$\underline{\gamma}(\neg\ell) = \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus \neg\zeta_2, \\ \underline{\gamma}_2(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_2 \setminus \neg\zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \\ \max\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \\ \min\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap \neg\zeta_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ and $\langle(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$.

Example 3.6. *Consider the FF5BSESs* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ *provided in Tables 13 and 14, respectively. Table 16 presents the extended intersection derived from them.*

Table 16. The extended intersection $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5) = (\overline{\gamma}_4, \underline{\gamma}_4, \zeta_1 \cup \zeta_2, 5)$ in Example 3.6.

$(\overline{\gamma}_4, \underline{\gamma}_4, \zeta_1 \cup \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	$\langle 0, 0.0, 0.8\rangle$	$\langle 1, 0.3, 0.8\rangle$	$\langle 3, 0.7, 0.7\rangle$	$\langle 0, 0.1, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$
$(\flat_1, \upsilon_2, 1)$	$\langle 1, 0.1, 0.6\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\flat_2, \upsilon_1, 0)$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.4, 0.1\rangle$	$\langle 2, 0.8, 0.2\rangle$	$\langle 0, 0.4, 0.5\rangle$	$\langle 1, 0.5, 0.5\rangle$
$(\flat_2, \upsilon_2, 0)$	$\langle 2, 0.6, 0.8\rangle$	$\langle 1, 0.5, 0.7\rangle$	$\langle 0, 0.1, 0.8\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\neg\flat_1, \upsilon_1, 1)$	$\langle 2, 0.5, 0.3\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.3, 0.5\rangle$	$\langle 2, 0.6, 0.6\rangle$	$\langle 0, 0.1, 0.3\rangle$
$(\neg\flat_1, \upsilon_2, 1)$	$\langle 3, 0.8, 0.6\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 3, 0.8, 0.5\rangle$	$\langle 0, 0.3, 0.3\rangle$	$\langle 1, 0.0, 0.7\rangle$
$(\neg\flat_2, \upsilon_1, 0)$	$\langle 2, 0.8, 0.3\rangle$	$\langle 0, 0.5, 0.3\rangle$	$\langle 1, 0.5, 0.5\rangle$	$\langle 4, 0.9, 0.5\rangle$	$\langle 3, 0.8, 0.5\rangle$
$(\neg\flat_2, \upsilon_2, 0)$	$\langle 1, 0.2, 0.6\rangle$	$\langle 3, 0.9, 0.1\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.4, 0.4\rangle$	$\langle 0, 0.5, 0.1\rangle$

Proposition 3.3. *Let* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$, $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$, *and* $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$ *be FFN*$_1$*BSES, FFN*$_2$*BSES, and FFN*$_3$*BSES, respectively. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{\varepsilon}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$.

Proof. Straightforward. □

Definition 3.13. *The restricted union of* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$ *is denoted and defined as* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}, \underline{\gamma}, \zeta_1 \cap \zeta_2, \max(N_1, N_2))$, *where for all* $\ell \in \zeta_1 \cap \zeta_2 \neq \emptyset$,

$$\overline{\gamma}(\ell) = \langle (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\}\rangle,$$

where $\langle (\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ *and* $\langle (\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cap \neg\zeta_2 \neq \emptyset$,

$$\underline{\gamma}(\neg\ell) = \langle (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\}\rangle,$$

where $\langle (\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ *and* $\langle (\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$.

Example 3.7. *Recall the two FF5BSESs* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ *listed in Tables 13 and 14, respectively. Their corresponding restricted union is displayed in Table 17.*

Table 17. The restricted union $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5) = (\overline{\gamma}_5, \underline{\gamma}_5, \zeta_1 \cap \zeta_2, 5)$ in Example 3.7.

$(\overline{\gamma}_5, \underline{\gamma}_5, \zeta_1 \cap \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 3, 0.5, 0.2\rangle$	$\langle 2, 0.4, 0.6\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 1, 0.5, 0.4\rangle$	$\langle 4, 0.9, 0.6\rangle$
$(\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 3, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.5\rangle$	$\langle 2, 0.5, 0.0\rangle$	$\langle 3, 0.9, 0.2\rangle$	$\langle 4, 1.0, 0.0\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.7\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 0, 0.0, 0.6\rangle$	$\langle 1, 0.5, 0.6\rangle$	$\langle 0, 0.1, 0.4\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.1, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$	$\langle 1, 0.6, 0.5\rangle$	$\langle 0, 0.4, 0.8\rangle$	$\langle 0, 0.2, 0.4\rangle$

Proposition 3.4. *Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$, $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$, and $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$ be FFN$_1$BSES, FFN$_2$BSES, and FFN$_3$BSES, respectively. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{r}}\ ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2))\ \check{\cup}_{\check{r}}\ (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$.

Proof. Straightforward. □

Definition 3.14. *The restricted intersection of $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$ and $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$ is denoted and defined as $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)\ \check{\cap}_{\check{r}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}, \underline{\gamma}, \zeta_1 \cap \zeta_2, \max(N_1, N_2))$, where for all $\ell \in \zeta_1 \cap \zeta_2 \neq \emptyset$,*

$$\overline{\gamma}(\ell) = \langle(\kappa, \min\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \min\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \max\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\}\rangle,$$

where $\langle(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ and $\langle(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cap \neg\zeta_2 \neq \emptyset$,

$$\underline{\gamma}(\neg\ell) = \langle(\kappa, \max\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \max\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \min\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\}\rangle,$$

where $\langle(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ and $\langle(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$.

Example 3.8. *Consider once more the FF5BSESs $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)$ and $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5)$ as detailed in Tables 13 and 14. The outcome of their restricted intersection is presented in Table 18.*

Table 18. The restricted intersection $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, 5)\ \check{\cap}_{\check{r}}\ (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, 5) = (\overline{\gamma}_6, \underline{\gamma}_6, \zeta_1 \cap \zeta_2, 5)$ in Example 3.8.

$(\overline{\gamma}_6, \underline{\gamma}_6, \zeta_1 \cap \zeta_2, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 0, 0.0, 0.8\rangle$	$\langle 1, 0.3, 0.8\rangle$	$\langle 3, 0.7, 0.7\rangle$	$\langle 0, 0.1, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$
$(\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 2, 0.6, 0.8\rangle$	$\langle 1, 0.5, 0.7\rangle$	$\langle 0, 0.1, 0.8\rangle$	$\langle 0, 0.1, 0.3\rangle$	$\langle 3, 0.4, 0.9\rangle$
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 2, 0.5, 0.3\rangle$	$\langle 2, 0.7, 0.6\rangle$	$\langle 1, 0.3, 0.5\rangle$	$\langle 2, 0.6, 0.6\rangle$	$\langle 0, 0.1, 0.3\rangle$
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	$\langle 1, 0.2, 0.6\rangle$	$\langle 3, 0.9, 0.1\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.4, 0.4\rangle$	$\langle 0, 0.5, 0.1\rangle$

Proposition 3.5. *Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$, $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$, and $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$ be FFN$_1$BSES, FFN$_2$BSES, and FFN$_3$BSES, respectively. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{r}} (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$.

Proof. Straightforward. □

We now proceed to explore the connections among extended union, extended intersection, restricted union, and restricted intersection in the context of FFNBSESs. These operations satisfy classical algebraic properties, including De Morgan's laws, absorption, and distribution, thereby preserving the familiar set-theoretic structure within the FFNBSES environment.

Proposition 3.6. *Let* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta, N)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta, N)$ *be two FFNBSESs. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta, N) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta, N) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta, N) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta, N)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta, N) \check{\cap}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta, N) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta, N) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta, N)$.

Proof. Straightforward. □

Proposition 3.7. *Let* $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$ *and* $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$ *be two FFNBSESs. Then,*

1) $((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N))^{\check{c}} = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)^{\check{c}} \check{\cap}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)^{\check{c}}$;

2) $((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N))^{\check{c}} = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)^{\check{c}} \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)^{\check{c}}$;

3) $((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N))^{\check{c}} = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)^{\check{c}} \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)^{\check{c}}$;

4) $((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N))^{\check{c}} = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)^{\check{c}} \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)^{\check{c}}$.

Proof. (1) Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N) = (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cup \zeta_2, N)$. Then, $((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N))^{\check{c}} = (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cup \zeta_2, N)^{\check{c}} = (\overline{\gamma}_3^{\check{c}}, \underline{\gamma}_3^{\check{c}}, \zeta_1 \cup \zeta_2, N)$. For all $\ell \in \zeta_1 \cup \zeta_2$,

$$\overline{\gamma}_3(\ell) = \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus \zeta_2, \\ \overline{\gamma}_2(\ell), & \text{if } \ell \in \zeta_2 \setminus \zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \\ \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_1 \cap \zeta_2, \end{cases}$$

where $\langle (\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}) \rangle \in \overline{\gamma}_1(\ell)$ and $\langle (\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell}) \rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cup \neg\zeta_2$,

$$\underline{\gamma}_3(\neg\ell) = \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus \neg\zeta_2, \\ \underline{\gamma}_2(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_2 \setminus \neg\zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \\ \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \\ \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap \neg\zeta_2, \end{cases}$$

where $\langle(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_1^{\oplus}(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_1^{\ominus}(\kappa,\vartheta_{1\neg\ell})\rangle\in\underline{\gamma}_1(\neg\ell)$ and $\langle(\kappa,\vartheta_{2\neg\ell}),\underline{\gamma}_2^{\oplus}(\kappa,\vartheta_{2\neg\ell}),\underline{\gamma}_2^{\ominus}(\kappa,\vartheta_{2\neg\ell})\rangle\in\underline{\gamma}_2(\neg\ell)$.

Then, for all $\ell\in\zeta_1\cup\zeta_2$,

$$\overline{\gamma}_3^{\check{c}}(\ell)=\underline{\gamma}_3(\neg\ell)=\begin{cases}\underline{\gamma}_1(\neg\ell), & \text{if } \ell\in\zeta_1\setminus\zeta_2,\\ \underline{\gamma}_2(\neg\ell), & \text{if } \ell\in\zeta_2\setminus\zeta_1,\\ \left\langle\begin{array}{l}(\kappa,\min\{\vartheta_{1\neg\ell},\vartheta_{2\neg\ell}\}),\\ \min\{\underline{\gamma}_1^{\oplus}(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_2^{\oplus}(\kappa,\vartheta_{2\neg\ell})\},\\ \max\{\underline{\gamma}_1^{\ominus}(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_2^{\ominus}(\kappa,\vartheta_{2\neg\ell})\}\end{array}\right\rangle, & \text{if } \ell\in\zeta_1\cap\zeta_2.\end{cases}$$

Similarly, for all $\neg\ell\in\neg\zeta_1\cup\neg\zeta_2$,

$$\underline{\gamma}_3^{\check{c}}(\neg\ell)=\overline{\gamma}_3(\ell)=\begin{cases}\overline{\gamma}_1(\ell), & \text{if } \neg\ell\in\neg\zeta_1\setminus\neg\zeta_2,\\ \overline{\gamma}_2(\ell), & \text{if } \neg\ell\in\neg\zeta_2\setminus\neg\zeta_1,\\ \left\langle\begin{array}{l}(\kappa,\max\{\vartheta_{1\ell},\vartheta_{2\ell}\}),\\ \max\{\overline{\gamma}_1^{\oplus}(\kappa,\vartheta_{1\ell}),\overline{\gamma}_2^{\oplus}(\kappa,\vartheta_{2\ell})\},\\ \min\{\overline{\gamma}_1^{\ominus}(\kappa,\vartheta_{1\ell}),\overline{\gamma}_2^{\ominus}(\kappa,\vartheta_{2\ell})\}\end{array}\right\rangle, & \text{if } \neg\ell\in\neg\zeta_1\cap\neg\zeta_2.\end{cases}$$

On the other hand, let $(\overline{\gamma}_1,\underline{\gamma}_1,\zeta_1,N)^{\check{c}}\ \check{\cap}_{\check{e}}\ (\overline{\gamma}_2,\underline{\gamma}_2,\zeta_2,N)^{\check{c}}=(\overline{\gamma}_4,\underline{\gamma}_4,\zeta_1\cup\zeta_2,N)$. For all $\ell\in\zeta_1\cup\zeta_2$,

$$\overline{\gamma}_4(\ell)=\begin{cases}\overline{\gamma}_1^{\check{c}}(\ell), & \text{if } \ell\in\zeta_1\setminus\zeta_2,\\ \overline{\gamma}_2^{\check{c}}(\ell), & \text{if } \ell\in\zeta_2\setminus\zeta_1,\\ \left\langle\begin{array}{l}(\kappa,\min\{\vartheta_{1\ell}^{\check{c}},\vartheta_{2\ell}^{\check{c}}\}),\\ \min\left\{\overline{\gamma}_1^{\oplus\check{c}}(\kappa,\vartheta_{1\ell}^{\check{c}}),\overline{\gamma}_2^{\oplus\check{c}}(\kappa,\vartheta_{2\ell}^{\check{c}})\right\},\\ \max\left\{\overline{\gamma}_1^{\ominus\check{c}}(\kappa,\vartheta_{1\ell}^{\check{c}}),\overline{\gamma}_2^{\ominus\check{c}}(\kappa,\vartheta_{2\ell}^{\check{c}})\right\}\end{array}\right\rangle, & \text{if } \ell\in\zeta_1\cap\zeta_2.\end{cases}$$

$$=\begin{cases}\underline{\gamma}_1(\neg\ell), & \text{if } \ell\in\zeta_1\setminus\zeta_2,\\ \underline{\gamma}_2(\neg\ell), & \text{if } \ell\in\zeta_2\setminus\zeta_1,\\ \left\langle\begin{array}{l}(\kappa,\min\{\vartheta_{1\neg\ell},\vartheta_{2\neg\ell}\}),\\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_2^{\oplus}(\kappa,\vartheta_{2\neg\ell})\right\},\\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa,\vartheta_{1\neg\ell}),\underline{\gamma}_2^{\ominus}(\kappa,\vartheta_{2\neg\ell})\right\}\end{array}\right\rangle, & \text{if } \ell\in\zeta_1\cap\zeta_2.\end{cases}$$

Similarly, for all $\neg\ell\in\neg\zeta_1\cup\neg\zeta_2$,

$$\underline{\gamma}_4(\neg\ell)=\begin{cases}\underline{\gamma}_1^{\check{c}}(\neg\ell), & \text{if } \neg\ell\in\neg\zeta_1\setminus\neg\zeta_2,\\ \underline{\gamma}_2^{\check{c}}(\neg\ell), & \text{if } \neg\ell\in\neg\zeta_2\setminus\neg\zeta_1,\\ \left\langle\begin{array}{l}(\kappa,\max\{\vartheta_{1\neg\ell}^{\check{c}},\vartheta_{2\neg\ell}^{\check{c}}\}),\\ \max\left\{\underline{\gamma}_1^{\oplus\check{c}}(\kappa,\vartheta_{1\neg\ell}^{\check{c}}),\underline{\gamma}_2^{\oplus\check{c}}(\kappa,\vartheta_{2\neg\ell}^{\check{c}})\right\},\\ \min\left\{\underline{\gamma}_1^{\ominus\check{c}}(\kappa,\vartheta_{1\neg\ell}^{\check{c}}),\underline{\gamma}_2^{\ominus\check{c}}(\kappa,\vartheta_{2\neg\ell}^{\check{c}})\right\}\end{array}\right\rangle, & \text{if } \neg\ell\in\neg\zeta_1\cap\neg\zeta_2.\end{cases}$$

$$= \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus \neg\zeta_2, \\ \overline{\gamma}_2(\ell), & \text{if } \neg\ell \in \neg\zeta_2 \setminus \neg\zeta_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap \neg\zeta_2. \end{cases}$$

Since $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cup \zeta_2, N)^{\check{c}}$ and $(\overline{\gamma}_4, \underline{\gamma}_4, \zeta_1 \cup \zeta_2, N)$ coincide for every $\ell \in \zeta_1 \cup \zeta_2$ and $\neg\ell \in \neg\zeta_1 \cup \neg\zeta_2$, the result follows directly.
The remaining parts can be established by analogous reasoning. □

Proposition 3.8. *Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$ and $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)$ be two FFNBSESs. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{e}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$;

3) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{r}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$;

4) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{r}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$.

Proof. (1) Suppose that $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N) = (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cap \zeta_2, N)$. Then, for all $\ell \in \zeta_1 \cap \zeta_2$,

$$\overline{\gamma}_3(\ell) = \langle (\kappa, \min\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \min\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\}, \max\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\} \rangle,$$

where $\langle (\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}) \rangle \in \overline{\gamma}_1(\ell)$ and $\langle (\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell}) \rangle \in \overline{\gamma}_2(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cup \neg\zeta_2$,

$$\underline{\gamma}_3(\neg\ell) = \langle (\kappa, \max\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \max\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \min\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\}) \rangle,$$

where $\langle (\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}) \rangle \in \underline{\gamma}_1(\neg\ell)$ and $\langle (\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell}) \rangle \in \underline{\gamma}_2(\neg\ell)$.

Now, let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_1 \cap \zeta_2, N) = (\overline{\gamma}_4, \underline{\gamma}_4, \zeta_1 \cup (\zeta_1 \cap \zeta_2), N) = (\overline{\gamma}_4, \underline{\gamma}_4, \zeta_1, N)$. Then, for all $\ell \in \zeta_1 \cup (\zeta_1 \cap \zeta_2)$,

$$\overline{\gamma}_4(\ell) = \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus (\zeta_1 \cap \zeta_2), \\ \overline{\gamma}_3(\ell), & \text{if } \ell \in (\zeta_1 \cap \zeta_2) \setminus \zeta_1 = \emptyset, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{3\ell}\}), \\ \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_1 \cap (\zeta_1 \cap \zeta_2). \end{cases}$$

$$= \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus (\zeta_1 \cap \zeta_2), \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \min\{\vartheta_{1\ell}, \vartheta_{2\ell}\}\}), \\ \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \min\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\right\}\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \max\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_1 \cap (\zeta_1 \cap \zeta_2). \end{cases}$$

where $\langle(\kappa, \vartheta_{3\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\rangle \in \overline{\gamma}_3(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_1 \cup (\neg\zeta_1 \cap \neg\zeta_2)$,

$$\underline{\gamma}_4(\neg\ell) = \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus (\neg\zeta_1 \cap \neg\zeta_2), \\ \underline{\gamma}_3(\neg\ell), & \text{if } \neg\ell \in (\neg\zeta_1 \cap \neg\zeta_2) \setminus \neg\zeta_1 = \emptyset, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{3\neg\ell}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap (\neg\zeta_1 \cap \neg\zeta_2). \end{cases}$$

$$= \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus (\neg\zeta_1 \cap \neg\zeta_2), \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \max\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \max\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\right\}\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \min\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_1 \cap (\neg\zeta_1 \cap \neg\zeta_2), \end{cases}$$

where $\langle(\kappa, \vartheta_{3\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\rangle \in \underline{\gamma}_3(\neg\ell)$.

Hence,

$$\overline{\gamma}_4(\ell) = \begin{cases} \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \setminus (\zeta_1 \cap \zeta_2) \\ \overline{\gamma}_1(\ell), & \text{if } \ell \in \zeta_1 \cap \zeta_2 \end{cases}$$

and

$$\underline{\gamma}_4(\neg\ell) = \begin{cases} \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \setminus (\neg\zeta_1 \cap \neg\zeta_2) \\ \underline{\gamma}_1(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_1 \cap \neg\zeta_2. \end{cases}$$

Therefore, $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cup}_{\check{e}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N)) = (\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N)$.
The other parts can be demonstrated using similar arguments. □

Proposition 3.9. *Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1)$, $(\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)$, and $(\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)$ be FFN$_1$BSES, FFN$_2$BSES, and FFN$_3$BSES, respectively. Then,*

1) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{e}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cap}_{\check{r}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{e}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$;

2) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{e}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cup}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{e}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cup}_{r} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{e}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$;

3) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{e}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cap}_{\check{e}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$;

4) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cup}_{\check{e}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cup}_{\check{e}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$;

5) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cap}_{\check{r}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$;

6) $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} ((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cup}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2)) \check{\cup}_{\check{r}} ((\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cap}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3))$.

Proof. (3) Suppose that $((\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) \check{\cap}_{\check{\varepsilon}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3)) = (\overline{\gamma}_4, \underline{\gamma}_4, \zeta_2 \cup \zeta_3, \max(N_2, N_3))$. Then, for all $\ell \in \zeta_2 \cup \zeta_3$,

$$\overline{\gamma}_4(\ell) = \begin{cases} \overline{\gamma}_2(\ell), & \text{if } \ell \in \zeta_2 \setminus \zeta_3, \\ \overline{\gamma}_3(\ell), & \text{if } \ell \in \zeta_3 \setminus \zeta_2, \\ (\kappa, \min\{\vartheta_{2\ell}, \vartheta_{3\ell}\}), \\ \left\langle \begin{array}{l} \min\left\{\overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}, \\ \max\left\{\overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \zeta_2 \cap \zeta_3, \end{cases}$$

where $\langle(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\rangle \in \overline{\gamma}_2(\ell)$ and $\langle(\kappa, \vartheta_{3\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\rangle \in \overline{\gamma}_3(\ell)$.

Similarly, for all $\neg\ell \in \neg\zeta_2 \cup \neg\zeta_3$,

$$\underline{\gamma}_4(\neg\ell) = \begin{cases} \underline{\gamma}_2(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_2 \setminus \neg\zeta_3, \\ \underline{\gamma}_3(\neg\ell), & \text{if } \neg\ell \in \neg\zeta_3 \setminus \neg\zeta_2, \\ (\kappa, \max\{\vartheta_{2\neg\ell}, \vartheta_{3\neg\ell}\}), \\ \left\langle \begin{array}{l} \max\left\{\underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}, \\ \min\left\{\underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\zeta_2 \cap \neg\zeta_3, \end{cases}$$

where $\langle(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\rangle \in \underline{\gamma}_2(\neg\ell)$ and $\langle(\kappa, \vartheta_{3\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\rangle \in \underline{\gamma}_3(\neg\ell)$.

Let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_4, \underline{\gamma}_4, \zeta_2 \cup \zeta_3, \max(N_2, N_3)) = (\overline{\gamma}_5, \underline{\gamma}_5, \zeta_1 \cap (\zeta_2 \cup \zeta_3), \max(N_1, \max(N_2, N_3)) = (\overline{\gamma}_5, \underline{\gamma}_5, \xi_1 \cup \xi_2, \max(N_1, N_2, N_3))$, where $\xi_1 = \zeta_1 \cap \zeta_2$ and $\xi_2 = \zeta_1 \cap \zeta_3$. Then, for all $\ell \in \xi_1 \cup \xi_2$,

$$\overline{\gamma}_5(\ell) = \langle(\kappa, \max\{\vartheta_{1\ell}, \vartheta_{4\ell}\}), \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_4^{\oplus}(\kappa, \vartheta_{4t})\}, \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_4^{\ominus}(\kappa, \vartheta_{4\ell})\}\rangle,$$

where $\langle(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell})\rangle \in \overline{\gamma}_1(\ell)$ and $\langle(\kappa, \vartheta_{4\ell}), \overline{\gamma}_4^{\oplus}(\kappa, \vartheta_{4\ell}), \overline{\gamma}_4^{\ominus}(\kappa, \vartheta_{4\ell})\rangle \in \overline{\gamma}_4(\ell)$.

Similarly, for all $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$,

$$\underline{\gamma}_5(\neg\ell) = \langle(\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{4\neg\ell}\}), \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_4^{\oplus}(\kappa, \vartheta_{4\neg\ell})\}, \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_4^{\ominus}(\kappa, \vartheta_{4\neg\ell})\}\rangle,$$

where $\langle(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell})\rangle \in \underline{\gamma}_1(\neg\ell)$ and $\langle(\kappa, \vartheta_{4\neg\ell}), \underline{\gamma}_4^{\oplus}(\kappa, \vartheta_{4\neg\ell}), \underline{\gamma}_4^{\ominus}(\kappa, \vartheta_{4\neg\ell})\rangle \in \underline{\gamma}_4(\neg\ell)$.

Hence, for all $\ell \in \xi_1 \cup \xi_2$,

$$\overline{\gamma}_5(\ell) = \begin{cases} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \left\langle \begin{array}{l} \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_1 \setminus \xi_2, \\ (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{3\ell}\}), \\ \left\langle \begin{array}{l} \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_2 \setminus \xi_1, \\ (\kappa, \max\{\vartheta_{1\ell}, \min\{\vartheta_{2\ell}, \vartheta_{3\ell}\}\}), \\ \left\langle \begin{array}{l} \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \min\left\{\overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \max\left\{\overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_1 \cap \xi_2. \end{cases}$$

Similarly, for all $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$,

$$\underline{\gamma}_5(\neg\ell) = \begin{cases} \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_1 \setminus \neg\xi_2, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{3\neg\ell}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_2 \setminus \neg\xi_1, \\ \left\langle \begin{array}{l} \langle(\kappa, \min\{\vartheta_{1\neg\ell}, \max\{\vartheta_{2\neg\ell}, \vartheta_{3\neg\ell}\}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \max\left\{\underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \min\left\{\underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_1 \cap \neg\xi_2. \end{cases}$$

On the other hand, let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_2, \underline{\gamma}_2, \zeta_2, N_2) = (\overline{\gamma}_6, \underline{\gamma}_6, \zeta_1 \cap \zeta_2, \max(N_1, N_2))$. Then, for all $\ell \in \zeta_1 \cap \zeta_2$,

$$\overline{\gamma}_6(\ell) = \langle(\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2t})\}, \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\}\rangle.$$

Similarly, for all $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$,

$$\underline{\gamma}_6(\neg\ell) = \langle(\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\}, \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\}\rangle.$$

Next, let $(\overline{\gamma}_1, \underline{\gamma}_1, \zeta_1, N_1) \check{\cup}_{\check{r}} (\overline{\gamma}_3, \underline{\gamma}_3, \zeta_3, N_3) = (\overline{\gamma}_7, \underline{\gamma}_7, \zeta_1 \cap \zeta_2, \max(N_1, N_2))$. Then, for all $\ell \in \zeta_1 \cap \zeta_3$,

$$\overline{\gamma}_7(\ell) = \langle(\kappa, \max\{\vartheta_{1\ell}, \vartheta_{3\ell}\}), \max\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3t})\}, \min\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\}\rangle.$$

Similarly, for all $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$,

$$\underline{\gamma}_7(\neg\ell)(\kappa) = \langle(\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{3\neg\ell}\}), \min\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\}, \max\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\}\rangle.$$

Now, suppose that $(\overline{\gamma}_6, \underline{\gamma}_6, \zeta_1 \cap \zeta_2, \max(N_1, N_2)) \check{\cap}_{\check{e}} (\overline{\gamma}_7, \underline{\gamma}_7, \zeta_1 \cap \zeta_3, \max(N_1, N_3)) = (\overline{\gamma}_8, \underline{\gamma}_8, \xi_1 \cup \xi_2, \max(N_1, N_2, N_3))$ where $\xi_1 = \zeta_1 \cap \zeta_2$ and $\xi_2 = \zeta_1 \cap \zeta_3$. Then, for all $\ell \in \xi_1 \cup \xi_2$,

$$\overline{\gamma}_8(\ell) = \begin{cases} \overline{\gamma}_6(\ell), & \text{if } \ell \in \xi_1 \setminus \xi_2, \\ \overline{\gamma}_7(\ell), & \text{if } \ell \in \xi_2 \setminus \xi_1, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{6\ell}, \vartheta_{7\ell}\}), \\ \min\left\{\overline{\gamma}_6^{\oplus}(\kappa, \vartheta_{6\ell}), \overline{\gamma}_7^{\oplus}(\kappa, \vartheta_{7\ell})\right\}, \\ \max\left\{\overline{\gamma}_6^{\ominus}(\kappa, \vartheta_{6\ell}), \overline{\gamma}_7^{\ominus}(\kappa, \vartheta_{7\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_1 \cap \xi_2. \end{cases}$$

$$= \begin{cases} \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}), \\ \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_1 \setminus \xi_2, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{1\ell}, \vartheta_{3\ell}\}), \\ \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}, \\ \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_2 \setminus \xi_1, \\ \left\langle \begin{array}{l} (\kappa, \min\{\max\{\vartheta_{1\ell}, \vartheta_{2\ell}\}, \max\{\vartheta_{1\ell}, \vartheta_{3\ell}\}\}), \\ \min\left\{\max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\ell})\right\}, \max\left\{\overline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\ell})\right\}\right\}, \\ \max\left\{\min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\ell})\right\}, \min\left\{\overline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\ell}), \overline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \ell \in \xi_1 \cap \xi_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{6\iota}), \overline{\gamma}_6^{\oplus}(\kappa, \vartheta_{6\iota}), \overline{\gamma}_6^{\ominus}(\kappa, \vartheta_{6\iota})\rangle \in \overline{\gamma}_6(\ell)$ and $\langle(\kappa, \vartheta_{7\ell}), \overline{\gamma}_7^{\oplus}(\kappa, \vartheta_{7\ell}), \overline{\gamma}_7^{\ominus}(\kappa, \vartheta_{7\ell})\rangle \in \overline{\gamma}_7(\ell)$.

Similarly, for all $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$,

$$\underline{\gamma}_8(\neg\ell) = \begin{cases} \underline{\gamma}_6(\neg\ell), & \text{if } \neg\ell \in \neg\xi_1 \setminus \neg\xi_2, \\ \underline{\gamma}_7(\neg\ell), & \text{if } \neg\ell \in \neg\xi_2 \setminus \neg\xi_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\vartheta_{6\neg\ell}, \vartheta_{7\neg\ell}\}), \\ \max\left\{\underline{\gamma}_6^{\oplus}(\kappa, \vartheta_{6\neg\ell}), \underline{\gamma}_7^{\oplus}(\kappa, \vartheta_{7\neg\ell})\right\}, \\ \min\left\{\underline{\gamma}_6^{\ominus}(\kappa, \vartheta_{6\neg\ell}), \underline{\gamma}_7^{\ominus}(\kappa, \vartheta_{7\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_1 \cap \neg\xi_2. \end{cases}$$

$$= \begin{cases} \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_1 \setminus \neg\xi_2, \\ \left\langle \begin{array}{l} (\kappa, \min\{\vartheta_{1\neg\ell}, \vartheta_{3\neg\ell}\}), \\ \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}, \\ \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_2 \setminus \neg\xi_1, \\ \left\langle \begin{array}{l} (\kappa, \max\{\min\{\vartheta_{1\neg\ell}, \vartheta_{2\neg\ell}\}, \min\{\vartheta_{1\neg\ell}, \vartheta_{3\neg\ell}\}\}), \\ \max\left\{\min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\oplus}(\kappa, \vartheta_{2\neg\ell})\right\}, \min\left\{\underline{\gamma}_1^{\oplus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\oplus}(\kappa, \vartheta_{3\neg\ell})\right\}\right\}, \\ \min\left\{\max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_2^{\ominus}(\kappa, \vartheta_{2\neg\ell})\right\}, \max\left\{\underline{\gamma}_1^{\ominus}(\kappa, \vartheta_{1\neg\ell}), \underline{\gamma}_3^{\ominus}(\kappa, \vartheta_{3\neg\ell})\right\}\right\} \end{array} \right\rangle, & \text{if } \neg\ell \in \neg\xi_1 \cap \neg\xi_2, \end{cases}$$

where $\langle(\kappa, \vartheta_{6\neg\ell}), \underline{\gamma}_6^{\oplus}(\kappa, \vartheta_{6\neg\ell}), \underline{\gamma}_6^{\ominus}(\kappa, \vartheta_{6\neg\ell})\rangle \in \underline{\gamma}_6(\neg\ell)$ and $\langle(\kappa, \vartheta_{7\neg\ell}), \underline{\gamma}_7^{\oplus}(\kappa, \vartheta_{7\neg\ell}), \underline{\gamma}_7^{\ominus}(\kappa, \vartheta_{7\neg\ell})\rangle \in \underline{\gamma}_7(\neg\ell)$.

Since $(\overline{\gamma}_5, \underline{\gamma}_5, \xi_1 \cup \xi_2, \max(N_1, N_2, N_3))$ and $(\overline{\gamma}_8, \underline{\gamma}_8, \xi_1 \cup \xi_2, \max(N_1, N_2, N_3))$ coincide for all $\ell \in \xi_1 \cup \xi_2$ and $\neg\ell \in \neg\xi_1 \cup \neg\xi_2$, the result holds.

The remaining parts can be shown using a similar approach. □

4. FFNBSES-based decision-making methodology and healthcare application

Effective DM in complex scenarios requires models that integrate diverse expert opinions, including both positive and negative assessments under uncertainty. This section presents the FFNBSES-based framework, which systematically aggregates bipolar fuzzy evaluations from multiple experts to identify optimal alternatives.

After outlining the FFNBSES algorithm and its flowchart, the section applies the framework to a smart healthcare case study, demonstrating its capability to support multi-criteria decisions where technological, clinical, and ethical factors must be balanced.

4.1. Decision-making framework and algorithm

This subsection presents the procedural framework for the FFNBSES-based DM approach. The methodology is formalized in Algorithm 1 and visually represented in Figure 1 to enhance clarity and ease of implementation. The process begins with the definition of key components, including the set of alternatives, parameters, expert group, and binary opinion space. The FFNBSES structure serves as the core input, encapsulating complex evaluations from multiple experts.

The procedure systematically categorizes expert opinions into four bipolar evaluation subsets based on agreement and polarity. For each category, FFNs are extracted and their corresponding score values are computed using a predefined scoring function. These scores are then aggregated to derive positive and negative influence measures, which are subsequently combined to obtain a final score for each alternative. The alternative with the highest overall score is selected as the optimal decision.

This approach ensures a rigorous treatment of expert opinions under uncertainty and bipolarity, offering a refined and nuanced tool for complex DM scenarios. The flowchart following the algorithm further illustrates each step for clearer operational understanding.

To simplify the computational procedure, expert evaluations are initially organized as ordered triples—such as $\gamma(\ell_i)(\kappa_j) = \langle \vartheta_{ij_{\ell_i}}, \gamma_{ij}^{\oplus}, \gamma_{ij}^{\ominus} \rangle$ and $\gamma(\neg\ell_i)(\kappa_j) = \langle \vartheta_{ij_{\neg\ell_i}}, \gamma_{ij}^{\oplus}, \gamma_{ij}^{\ominus} \rangle$—and arranged in tabular format. Each triple is subsequently converted into a single numerical score, denoted by s_{ij}, which is computed using the score function $S(\psi)$ from Definition 2.2(i). This transformation streamlines the data, facilitates comparison and aggregation of expert inputs, and reduces the complexity of the decision matrix, thereby supporting the identification of the most appropriate alternative.

Before presenting the DM algorithm, we clarify the rationale behind the final score aggregation. In the FFNBSES framework, positive and negative evaluations are modeled using FFNs subject to identical cubic-sum constraints. Consequently, the resulting score values derived from Definition 2.2 are normalized on a common scale. This guarantees that the positive score ρ_j^+ and the negative score ρ_j^- are directly comparable in magnitude, and their difference $\rho_j = \rho_j^+ - \rho_j^-$ represents a meaningful net-effect measure that balances supportive and opposing assessments.

Algorithm 1 FFNBSES-based DM and optimal alternative selection

Input:

i. $\Bbbk$: the set of possible alternatives;

ii. $\wp$: the collection of decision parameters;

iii. $\mathcal{E}$: the expert panel participating in the evaluation;

iv. $O = \{0 = \text{disagree},\ 1 = \text{agree}\}$: the binary opinion space;

v. FFNBSES structure $(\overline{\gamma}, \underline{\gamma}, \zeta, N)$, where $\zeta \subseteq F$ and $F = \wp \times \mathcal{E} \times O$.

Procedure:

i. Classify the expert evaluations into four bipolar categories:
   - positive agreement: $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_1}$;
   - positive disagreement: $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_0}$;
   - negative agreement: $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_1}$;
   - negative disagreement: $(\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_0}$.

ii. For each category, extract the corresponding FFNs and compute their score values (Definition 2.2).

iii. For each alternative j, compute the sum of scores:

$$\begin{aligned}
\varsigma_j^{+1} &= \sum_i s_{ij} \quad \text{from } (\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_1},\\
\varsigma_j^{+0} &= \sum_i s_{ij} \quad \text{from } (\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\oplus_0},\\
\varsigma_j^{-1} &= \sum_i s_{ij} \quad \text{from } (\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_1},\\
\varsigma_j^{-0} &= \sum_i s_{ij} \quad \text{from } (\overline{\gamma}, \underline{\gamma}, \zeta, N)^{\ominus_0}.
\end{aligned}$$

iv. Compute the positive, negative, and final scores:

$$\rho_j^+ = \varsigma_j^{+1} - \varsigma_j^{+0}, \quad \rho_j^- = \varsigma_j^{-1} - \varsigma_j^{-0}, \quad \rho_j = \rho_j^+ - \rho_j^-.$$

v. Determine the optimal alternative l by solving the following:

$$\text{maximize } \rho_j \quad \text{subject to } j \in \Bbbk.$$

Output: the optimal alternative $\kappa_l \in \Bbbk$ corresponding to the highest score ρ_l.

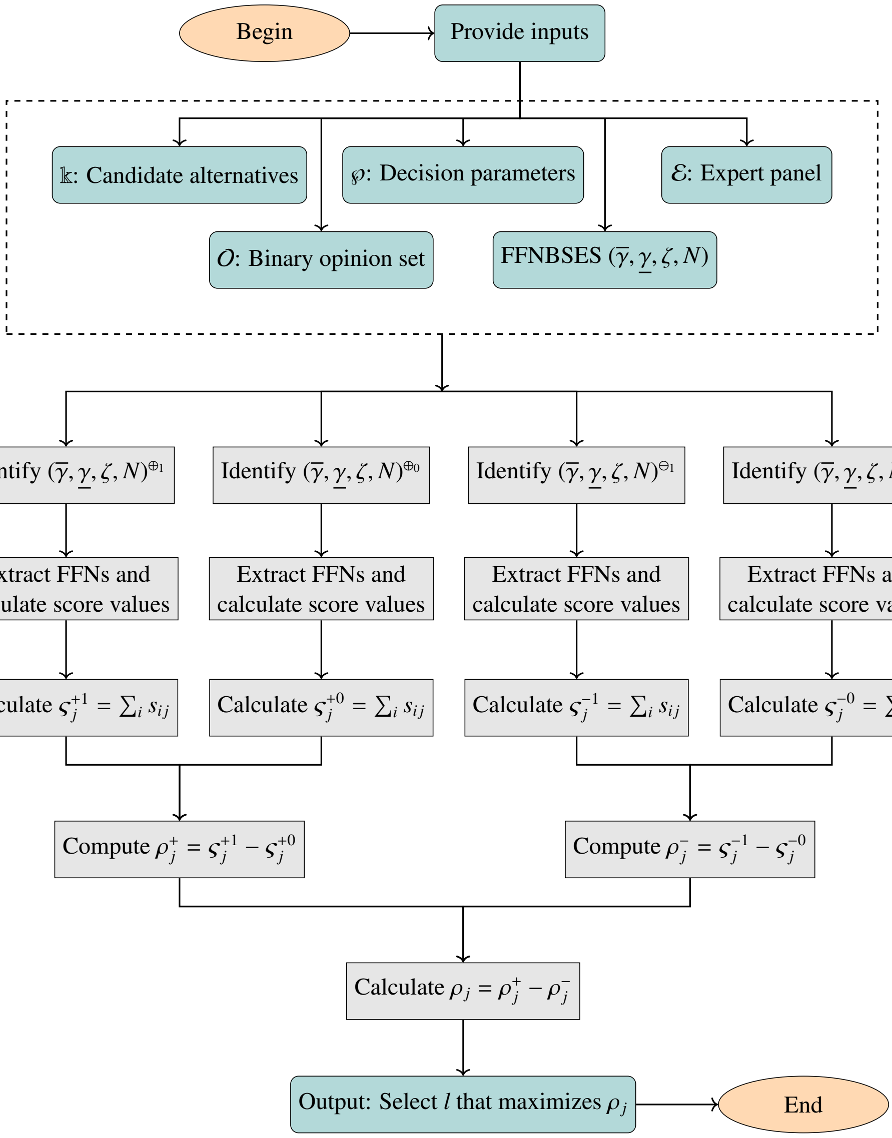

Figure 1. Flowchart of the proposed algorithm (Algorithm 1).

4.2. Application to smart healthcare transformation: a case study

Smart healthcare is an emerging paradigm that integrates advanced digital technologies with traditional medical services to enhance efficiency, accuracy, and accessibility in healthcare delivery. Rising global demand—driven by aging populations, chronic disease prevalence, and urban–rural disparities—has prompted governments and institutions to explore smart technologies for modernizing infrastructure and optimizing patient care.

Implementing smart healthcare systems, however, presents a complex MCDM problem. Stakeholders must evaluate trade-offs among technological feasibility, clinical effectiveness, cost-efficiency, data security, and equitable access. Traditional models, such as classical FSs or SSs, often fail to fully capture these complexities: Some consider only binary or unipolar evaluations, neglecting negative aspects such as cybersecurity risks or implementation challenges; others ignore multinary assessments or lack mechanisms to aggregate opinions from multiple experts, leading to incomplete or biased decision support.

The proposed FFNBSES model addresses these limitations. By supporting multinary evaluations, it enables experts to provide nuanced assessments of each smart healthcare strategy across multiple criteria. Its bipolar structure captures both positive and negative aspects, allowing, for example, simultaneous consideration of clinical effectiveness and potential risks such as AI misdiagnoses or data breaches. Furthermore, FFNBSES incorporates expert aggregation, ensuring that insights from healthcare professionals, biomedical engineers, policy analysts, and data scientists are effectively combined to guide balanced, evidence-informed decisions.

In this case study, we consider five smart healthcare transformation strategies currently evaluated by a national healthcare authority:

$$\Bbbk = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}.$$

Each strategy κ_i corresponds to a targeted initiative:

- κ_1: deploying AI-based diagnostic tools in outpatient clinics;
- κ_2: implementing remote patient monitoring for chronic diseases;
- κ_3: establishing telemedicine infrastructure in rural hospitals;
- κ_4: integrating Electronic Health Record (EHR) systems across departments;
- κ_5: using robotics for surgical assistance and patient care.

The evaluation is based on two main criteria representing the primary goals of smart healthcare transformation:

$$\wp = \{\flat_1, \flat_2\},$$

where

- $\flat_1$: clinical effectiveness—the extent to which the strategy improves diagnostic accuracy, treatment quality, and patient outcomes;
- $\flat_2$: accessibility—the degree to which the strategy ensures equitable healthcare access across geographic and demographic boundaries.

To capture the associated challenges, the negative aspects are denoted as:

$$\neg\wp = \{\neg\flat_1, \neg\flat_2\},$$

where

- $\neg\flat_1$: risk of clinical error—potential issues such as algorithmic bias, data misinterpretation, or system malfunction;
- $\neg\flat_2$: inequity in access—arising from digital illiteracy, infrastructure gaps, or economic barriers.

An expert panel is established to assess the strategies:

$$\mathcal{E} = \{\upsilon_1, \upsilon_2\},$$

comprising a senior hospital administrator and a biomedical informatics researcher. Each expert evaluates all five strategies with respect to each criterion and its bipolar counterpart, using binary opinions from the set:

$$\mathcal{O} = \{0 = \text{disagree}, 1 = \text{agree}\}.$$

To reflect qualitative nuances, the symbolic system adopted is as follows:

- $\divideontimes$ denotes extremely poor performance—a red flag indicating high clinical or ethical risk;
- Multiple $\vartriangle$ symbols denote positive performance levels—with more $\vartriangle$ symbols reflecting greater compliance or success.

These expert evaluations—tabulated and symbolically represented in Table 19—form the foundation for applying our FFNBSES DM framework. Aggregating these assessments allows stakeholders to systematically identify the most effective smart healthcare transformation strategy that maximizes benefits while mitigating potential challenges.

Table 19. Initial expert evaluations of smart healthcare transformation strategies.

$\zeta \setminus \mathbb{k}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	△△△	△△	△△△△	△△	△△
$(\mathfrak{b}_1, \upsilon_2, 1)$	△△	△	△△△	△△△	△
$(\mathfrak{b}_2, \upsilon_1, 1)$	△△△	△△	△△	△△	△
$(\mathfrak{b}_2, \upsilon_2, 1)$	⋇	△△	△△△△	△△	△△
$(\mathfrak{b}_1, \upsilon_1, 0)$	△	△△	⋇	△△	⋇
$(\mathfrak{b}_1, \upsilon_2, 0)$	△	△△△	△	⋇	△△△
$(\mathfrak{b}_2, \upsilon_1, 0)$	⋇	△	⋇	△	△△△
$(\mathfrak{b}_2, \upsilon_2, 0)$	△△△	△△	⋇	△△	△
$(\neg\mathfrak{b}_1, \upsilon_1, 1)$	⋇	△	⋇	△△	△△
$(\neg\mathfrak{b}_1, \upsilon_2, 1)$	△△	△△	△	△	△△
$(\neg\mathfrak{b}_2, \upsilon_1, 1)$	△	△	△	△	△△
$(\neg\mathfrak{b}_2, \upsilon_2, 1)$	△△△△	△△	⋇	△△	△
$(\neg\mathfrak{b}_1, \upsilon_1, 0)$	△△△	△△	⋇	△△	△
$(\neg\mathfrak{b}_1, \upsilon_2, 0)$	△△	△	△△△	△△△	⋇
$(\neg\mathfrak{b}_2, \upsilon_1, 0)$	△△	△△	△△	△△	⋇
$(\neg\mathfrak{b}_2, \upsilon_2, 0)$	⋇	△	△△△	△△	△△

The checkmarks from the evaluations are converted into numerical values ranging from 0 to 4 using the approach from Example 3.1, and interpreted based on the grading intervals in Table 4. The FFNBSES model application is illustrated in Table 20 for the evaluation of smart healthcare strategies.

Table 20. Tabular representation of the FFNBSES $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$ for smart healthcare strategies.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	$\langle 3, 0.6, 0.8\rangle$	$\langle 2, 0.6, 0.6\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 2, 0.8, 0.0\rangle$	$\langle 2, 0.7, 0.5\rangle$
$(\mathfrak{b}_1, \upsilon_2, 1)$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 3, 0.7, 0.7\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 1, 0.6, 0.4\rangle$

(Continued on next page)

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_2, \upsilon_1, 1)$	$\langle 3, 0.8, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 1, 0.1, 0.6\rangle$
$(\flat_2, \upsilon_2, 1)$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 4, 0.9, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.7, 0.4\rangle$
$(\flat_1, \upsilon_1, 0)$	$\langle 1, 0.2, 0.7\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 0, 0.0, 0.5\rangle$
$(\flat_1, \upsilon_2, 0)$	$\langle 1, 0.0, 0.6\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 0, 0.0, 0.3\rangle$	$\langle 3, 0.6, 0.8\rangle$
$(\flat_2, \upsilon_1, 0)$	$\langle 0, 0.0, 0.1\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 0, 0.0, 0.2\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 3, 0.9, 0.3\rangle$
$(\flat_2, \upsilon_2, 0)$	$\langle 3, 0.9, 0.2\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.1, 0.4\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$
$(\neg\flat_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.4\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 0, 0.0, 0.3\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.6\rangle$
$(\neg\flat_1, \upsilon_2, 1)$	$\langle 2, 0.4, 0.8\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 1, 0.6, 0.1\rangle$	$\langle 2, 0.8, 0.2\rangle$
$(\neg\flat_2, \upsilon_1, 1)$	$\langle 1, 0.7, 0.0\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.0, 0.7\rangle$	$\langle 2, 0.8, 0.1\rangle$
$(\neg\flat_2, \upsilon_2, 1)$	$\langle 4, 0.9, 0.6\rangle$	$\langle 2, 0.4, 0.8\rangle$	$\langle 0, 0.0, 0.5\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.6, 0.4\rangle$
$(\neg\flat_1, \upsilon_1, 0)$	$\langle 3, 0.9, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$
$(\neg\flat_1, \upsilon_2, 0)$	$\langle 2, 0.6, 0.6\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 3, 0.9, 0.1\rangle$	$\langle 0, 0.1, 0.2\rangle$
$(\neg\flat_2, \upsilon_1, 0)$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 2, 0.3, 0.8\rangle$	$\langle 2, 0.1, 0.8\rangle$	$\langle 0, 0.0, 0.4\rangle$
$(\neg\flat_2, \upsilon_2, 0)$	$\langle 0, 0.0, 0.5\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 3, 0.8, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.5, 0.7\rangle$

We classify the expert evaluations into four FFNBSES categories: $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$ for strong agreement with positive impact, $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$ for disagreement with positive impact, $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$ for strong agreement with negative impact, and $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$ for disagreement with negative impact. These categories are detailed in Tables 21–24.

Table 21. Tabular form of $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$—positive criterion, strong agreement.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	$\langle 3, 0.6, 0.8\rangle$	$\langle 2, 0.6, 0.6\rangle$	$\langle 4, 1.0, 0.0\rangle$	$\langle 2, 0.8, 0.0\rangle$	$\langle 2, 0.7, 0.5\rangle$
$(\flat_1, \upsilon_2, 1)$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 3, 0.7, 0.7\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 1, 0.6, 0.4\rangle$
$(\flat_2, \upsilon_1, 1)$	$\langle 3, 0.8, 0.5\rangle$	$\langle 2, 0.7, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 1, 0.1, 0.6\rangle$
$(\flat_2, \upsilon_2, 1)$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 4, 0.9, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.7, 0.4\rangle$

Table 22. Tabular form of $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$—positive criterion, disagreement.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 0)$	$\langle 1, 0.2, 0.7\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.2, 0.8\rangle$	$\langle 0, 0.0, 0.5\rangle$
$(\flat_1, \upsilon_2, 0)$	$\langle 1, 0.0, 0.6\rangle$	$\langle 3, 0.5, 0.8\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 0, 0.0, 0.3\rangle$	$\langle 3, 0.6, 0.8\rangle$
$(\flat_2, \upsilon_1, 0)$	$\langle 0, 0.0, 0.1\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 0, 0.0, 0.2\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 3, 0.9, 0.3\rangle$
$(\flat_2, \upsilon_2, 0)$	$\langle 3, 0.9, 0.2\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 0, 0.1, 0.4\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.3, 0.7\rangle$

Table 23. Tabular form of $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$—negative criterion, strong agreement.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\neg\flat_1, \upsilon_1, 1)$	$\langle 0, 0.1, 0.4\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 0, 0.0, 0.3\rangle$	$\langle 2, 0.7, 0.5\rangle$	$\langle 2, 0.6, 0.6\rangle$
$(\neg\flat_1, \upsilon_2, 1)$	$\langle 2, 0.4, 0.8\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$	$\langle 1, 0.6, 0.1\rangle$	$\langle 2, 0.8, 0.2\rangle$
$(\neg\flat_2, \upsilon_1, 1)$	$\langle 1, 0.7, 0.0\rangle$	$\langle 1, 0.2, 0.6\rangle$	$\langle 1, 0.6, 0.3\rangle$	$\langle 1, 0.0, 0.7\rangle$	$\langle 2, 0.8, 0.1\rangle$
$(\neg\flat_2, \upsilon_2, 1)$	$\langle 4, 0.9, 0.6\rangle$	$\langle 2, 0.4, 0.8\rangle$	$\langle 0, 0.0, 0.5\rangle$	$\langle 2, 0.5, 0.7\rangle$	$\langle 1, 0.6, 0.4\rangle$

Table 24. Tabular form of $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$—negative criterion, disagreement.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\neg\flat_1, \upsilon_1, 0)$	$\langle 3, 0.9, 0.4\rangle$	$\langle 2, 0.8, 0.4\rangle$	$\langle 0, 0.0, 0.4\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 1, 0.2, 0.7\rangle$
$(\neg\flat_1, \upsilon_2, 0)$	$\langle 2, 0.6, 0.6\rangle$	$\langle 1, 0.3, 0.6\rangle$	$\langle 3, 0.6, 0.8\rangle$	$\langle 3, 0.9, 0.1\rangle$	$\langle 0, 0.1, 0.2\rangle$
$(\neg\flat_2, \upsilon_1, 0)$	$\langle 2, 0.8, 0.4\rangle$	$\langle 2, 0.4, 0.7\rangle$	$\langle 2, 0.3, 0.8\rangle$	$\langle 2, 0.1, 0.8\rangle$	$\langle 0, 0.0, 0.4\rangle$
$(\neg\flat_2, \upsilon_2, 0)$	$\langle 0, 0.0, 0.5\rangle$	$\langle 1, 0.6, 0.4\rangle$	$\langle 3, 0.8, 0.6\rangle$	$\langle 2, 0.6, 0.7\rangle$	$\langle 2, 0.5, 0.7\rangle$

From these tables, we extract the FFNs corresponding to each strategy and compute their score values as per Definition 2.2. The resulting scores, reflecting performance across all categories, are summarized in Tables 25–28.

Table 25. Score values of FFNs in $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$ and corresponding ς_j^{+1} calculations.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	−0.296	0.000	1.000	0.512	0.218
$(\flat_1, \upsilon_2, 1)$	−0.218	−0.335	0.000	−0.387	0.152
$(\flat_2, \upsilon_1, 1)$	0.387	0.279	0.448	−0.504	−0.215
$(\flat_2, \upsilon_2, 1)$	−0.064	−0.218	0.513	−0.127	0.279
$\varsigma_j^{+1} = \sum_i s_{ij}$	$\varsigma_1^{+1} = -0.191$	$\varsigma_2^{+1} = -0.274$	$\varsigma_3^{+1} = 1.961$	$\varsigma_4^{+1} = -0.506$	$\varsigma_5^{+1} = 0.434$

Table 26. Score values of FFNs in $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$ and corresponding ς_j^{+0} calculations.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\oplus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\flat_1, \upsilon_1, 1)$	−0.335	−0.218	−0.064	−0.504	−0.125
$(\flat_1, \upsilon_2, 1)$	−0.216	−0.387	−0.335	−0.027	−0.296
$(\flat_2, \upsilon_1, 1)$	−0.001	−0.335	−0.008	−0.189	0.702
$(\flat_2, \upsilon_2, 1)$	0.721	0.218	−0.063	−0.218	−0.316
$\varsigma_j^{+0} = \sum_i s_{ij}$	$\varsigma_1^{+0} = 0.169$	$\varsigma_2^{+0} = -0.722$	$\varsigma_3^{+0} = -0.470$	$\varsigma_4^{+0} = -0.938$	$\varsigma_5^{+0} = -0.035$

Table 27. Score values of FFNs in $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$ and corresponding ς_j^{-1} calculations.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_1}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	−0.063	0.152	−0.027	0.218	0.000
$(\mathfrak{b}_1, \upsilon_2, 1)$	−0.448	−0.279	−0.335	0.215	0.504
$(\mathfrak{b}_2, \upsilon_1, 1)$	0.343	−0.208	0.189	−0.343	0.511
$(\mathfrak{b}_2, \upsilon_2, 1)$	0.513	−0.448	−0.125	−0.218	0.152
$\varsigma_j^{-1} = \sum_i s_{ij}$	$\varsigma_1^{-1} = 0.345$	$\varsigma_2^{-1} = -0.783$	$\varsigma_3^{-1} = -0.298$	$\varsigma_4^{-1} = -0.128$	$\varsigma_5^{-1} = 1.167$

Table 28. Score values of FFNs in $(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$ and corresponding ς_j^{-0} calculations.

$(\overline{\gamma}, \underline{\gamma}, \zeta, 5)^{\ominus_0}$	κ_1	κ_2	κ_3	κ_4	κ_5
$(\mathfrak{b}_1, \upsilon_1, 1)$	0.665	0.448	−0.064	−0.279	−0.335
$(\mathfrak{b}_1, \upsilon_2, 1)$	0.000	−0.189	−0.296	0.728	−0.007
$(\mathfrak{b}_2, \upsilon_1, 1)$	0.448	−0.279	−0.485	−0.511	−0.064
$(\mathfrak{b}_2, \upsilon_2, 1)$	−0.125	0.152	0.296	−0.127	−0.218
$\varsigma_j^{-0} = \sum_i s_{ij}$	$\varsigma_1^{-0} = 0.988$	$\varsigma_2^{-0} = 0.132$	$\varsigma_3^{-0} = -0.549$	$\varsigma_4^{-0} = -0.189$	$\varsigma_5^{-0} = -0.624$

Based on the evaluations from individual criteria and experts, Tables 29 and 30 summarize the aggregated positive and negative contributions for each smart healthcare strategy. Specifically, Table 29 aggregates the positive scores from Tables 25 and 26, while Table 30 aggregates the negative scores from Tables 27 and 28. Table 31 consolidates these aggregated values into the final net scores for each strategy. Figures 2–4 provide visual representations of these results, displaying positive, negative, and net scores for all alternatives. From the final scores, the highest net value, i.e., the alternative that maximizes ρ_j, is ρ_3, indicating that strategy κ_3—implementing advanced remote patient monitoring systems—is the most suitable choice under this evaluation framework.

Table 29. Aggregated positive scores (ρ_j^+) for smart healthcare strategies.

$\varsigma_j^{+1} = \sum_i s_{ij}$	$\varsigma_j^{+0} = \sum_i s_{ij}$	$\rho_j^+ = \varsigma_j^{+1} - \varsigma_j^{+0}$
$\varsigma_1^{+1} = -0.191$	$\varsigma_1^{+0} = 0.169$	$\rho_1^+ = -0.360$
$\varsigma_2^{+1} = -0.274$	$\varsigma_2^{+0} = -0.722$	$\rho_2^+ = 0.448$
$\varsigma_3^{+1} = 1.961$	$\varsigma_3^{+0} = -0.470$	$\rho_3^+ = 2.431$
$\varsigma_4^{+1} = -0.506$	$\varsigma_4^{+0} = -0.938$	$\rho_4^+ = 0.432$
$\varsigma_5^{+1} = 0.434$	$\varsigma_5^{+0} = -0.035$	$\rho_5^+ = 0.469$

Table 30. Aggregated negative scores (ρ_j^-) for smart healthcare strategies.

$\varsigma_j^{-1} = \sum_i s_{ij}$	$\varsigma_j^{-0} = \sum_i s_{ij}$	$\rho_j^- = \varsigma_j^{-1} - \varsigma_j^{-0}$
$\varsigma_1^{-1} = 0.345$	$\varsigma_1^{-0} = 0.988$	$\rho_1^- = -0.643$
$\varsigma_2^{-1} = -0.783$	$\varsigma_2^{-0} = 0.132$	$\rho_2^- = -0.915$
$\varsigma_3^{-1} = -0.298$	$\varsigma_3^{-0} = -0.549$	$\rho_3^- = 0.251$
$\varsigma_4^{-1} = -0.128$	$\varsigma_4^{-0} = -0.189$	$\rho_4^- = 0.061$
$\varsigma_5^{-1} = 1.167$	$\varsigma_5^{+0} = -0.624$	$\rho_5^- = 1.791$

Table 31. Final aggregated scores (ρ_j) for smart healthcare strategies.

$\rho_j^+ = \varsigma_j^{+1} - \varsigma_j^{+0}$	$\rho_j^- = \varsigma_j^{-1} - \varsigma_j^{-0}$	$\rho_j = \rho_j^+ - \rho_j^-$
$\rho_1^+ = -0.360$	$\rho_1^- = -0.643$	$\rho_1 = 0.283$
$\rho_2^+ = 0.448$	$\rho_2^- = -0.915$	$\rho_2 = 1.363$
$\rho_3^+ = 2.431$	$\rho_3^- = 0.251$	$\rho_3 = 2.180$
$\rho_4^+ = 0.432$	$\rho_4^- = 0.061$	$\rho_4 = 0.371$
$\rho_5^+ = 0.469$	$\rho_5^- = 1.791$	$\rho_5 = -1.322$

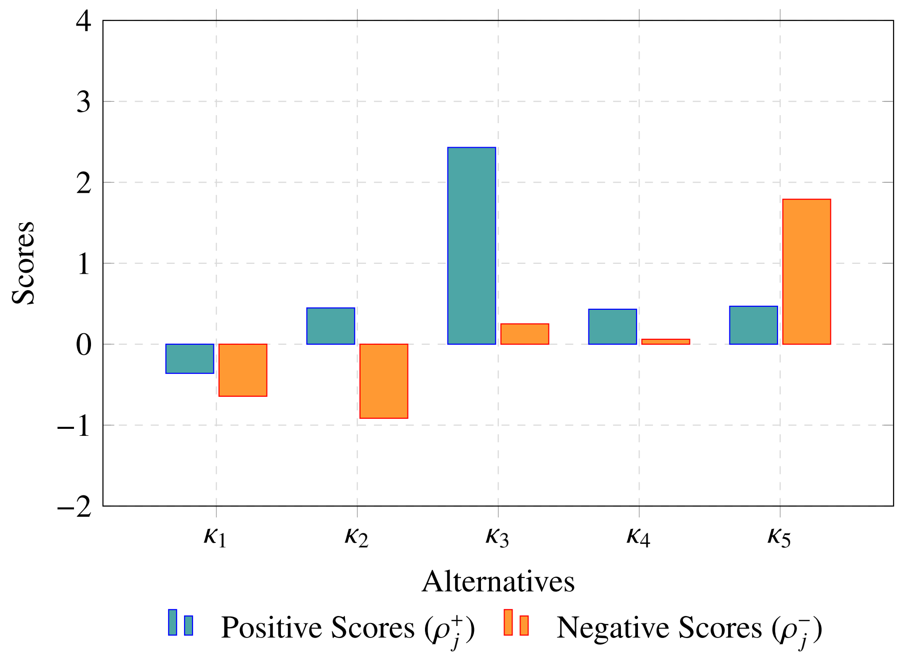

Figure 2. Positive and negative scores for all smart healthcare strategies.

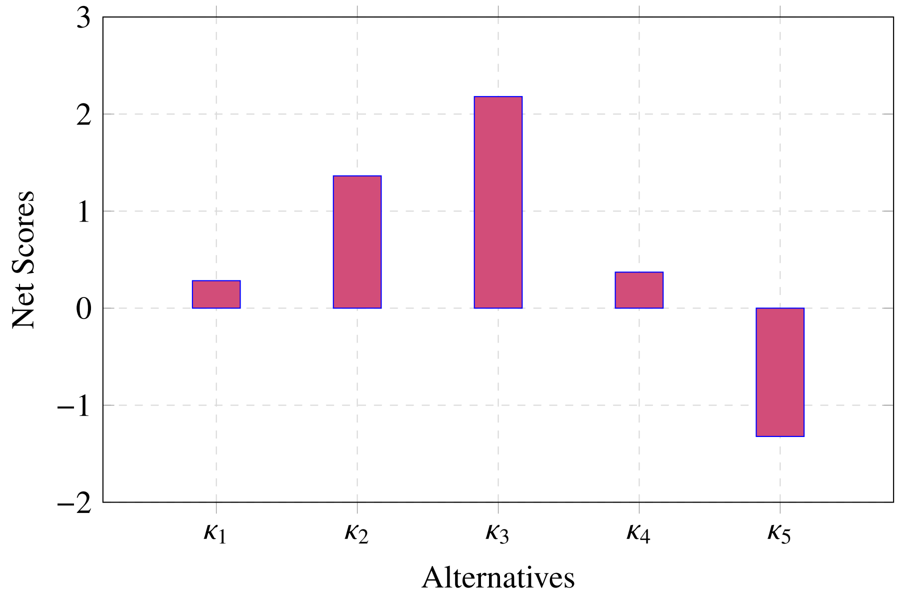

Figure 3. Net scores for all smart healthcare strategies.

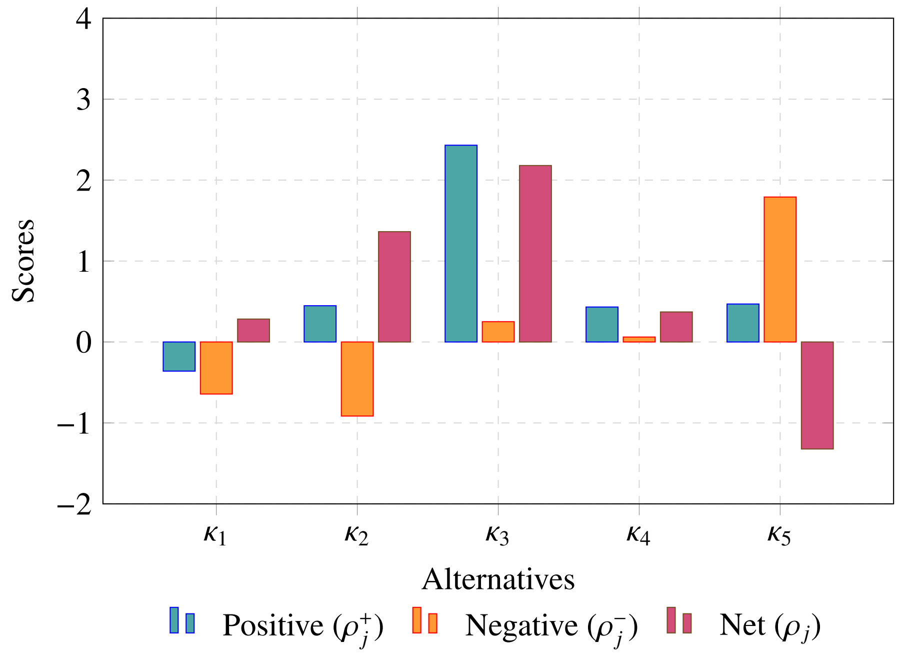

Figure 4. Aggregated positive, negative, and net scores for all smart healthcare strategies.

5. Evaluation of the FFNBSES model within the SS framework

This section evaluates the proposed FFNBSES model in the context of SS theory and its major extensions. We begin with a comparative analysis highlighting how FFNBSES relates to and advances beyond existing models across several dimensions, including membership structure, evaluation type and scale, bipolar handling, parameterization, and expert involvement.

Subsequently, we outline the main advantages that make FFNBSES a robust and expressive DM model. Lastly, we identify its current limitations to offer a balanced assessment and indicate directions for further development.

5.1. Comparative analysis of FFNBSES with existing models

We compare the FFNBSES model with relevant existing models, including Classical Fuzzy Models, Classical Soft Sets, Bipolar Soft Sets, N-Soft Sets, and N-Bipolar Soft Sets. As summarized in Table 32, this comparison considers key aspects such as membership type, membership superiority, parameterization support, evaluation mechanism, bipolarity, and expert involvement.

The FFNBSES model exhibits a comprehensive structure by integrating Fermatean fuzzy logic, multinary evaluation, bipolar reasoning, and multi-expert input, making it well-suited for complex DM problems characterized by uncertainty and subjectivity.

Table 32. Benchmarking FFNBSES against original models and their extensions across SS model groups.

Model Group	Model	Membership Type	Membership Superiority	Parameterization Support	Evaluation Type	Evaluation Scale	Bipolar Capability	Expert Involvement
Classical Fuzzy Models	FS [13]	F	Low	No	Continuous	Single-Valued	No	No
	IFS [14]	IF	Medium	No	Continuous	Single-Valued	No	No
	PFS [15]	PF	High	No	Continuous	Single-Valued	No	No
	FFS [16]	FF	Very High	No	Continuous	Single-Valued	No	No
Soft Sets	SS [34]	None	–	Yes	Discrete	Binary	No	Single
	SES [35]	None	–	Yes	Discrete	Binary	No	Multiple
	FSES [36]	F	Low	Yes	Continuous	Binary	No	Multiple
	IFSES [37]	IF	Medium	Yes	Continuous	Binary	No	Multiple
	PFSES [38]	PF	High	Yes	Continuous	Binary	No	Multiple
	FFSES [39]	FF	Very High	Yes	Continuous	Binary	No	Multiple
Bipolar Soft Sets	BSS [46]	None	–	Yes	Discrete	Binary	Yes	Single
	BSES [47]	None	–	Yes	Discrete	Binary	Yes	Multiple
	FBSES [48]	F	Low	Yes	Continuous	Binary	Yes	Multiple
N-Soft Sets	NSS [52]	None	–	Yes	Discrete	Multinary	No	Single
	NSES [53]	None	–	Yes	Discrete	Multinary	No	Multiple
	FNSES [53]	F	Low	Yes	Continuous	Multinary	No	Multiple
	IFNSES [54]	IF	Medium	Yes	Continuous	Multinary	No	Multiple
	PFNSES [54]	PF	High	Yes	Continuous	Multinary	No	Multiple
N-Bipolar Soft Sets	NBSS [57]	None	–	Yes	Discrete	Multinary	Yes	Single
	NBSES [62]	None	–	Yes	Discrete	Multinary	Yes	Multiple
	FNBSES [63]	F	Low	Yes	Continuous	Multinary	Yes	Multiple
	IFNBSES [64]	IF	Medium	Yes	Continuous	Multinary	Yes	Multiple
	PFNBSES [65]	PF	High	Yes	Continuous	Multinary	Yes	Multiple
	FFNBSES (Proposed)	FF	Very High	Yes	Continuous	Multinary	Yes	Multiple

As observed from Table 32, existing models such as FNBSES, IFNBSES, and PFNBSES are primarily distinguished by their underlying fuzzy membership type, which limits their ability to capture extreme supportive or opposing expert opinions. FNBSES and IFNBSES provide limited membership expressiveness, while PFNBSES, although higher-order, cannot fully accommodate extreme evaluations or complex uncertainty patterns. The proposed FFNBSES model overcomes these limitations by employing Fermatean fuzzy logic while retaining multinary evaluation, bipolar reasoning, and multi-expert input. This enhanced framework allows finer discrimination between closely competing alternatives, ensures more stable decision outcomes under uncertainty, and enables systematic aggregation of diverse expert judgments, demonstrating superiority in both mathematical expressiveness and practical applicability in complex MCDM scenarios. In particular, the qualitative ratings of "Membership Superiority" (Low, Medium, High, Very High) correspond to the relative volume of admissible pairs within each model's fuzzy framework, reflecting the flexibility of the membership space and, indirectly, the maximum permissible hesitation index. This provides a more objective basis for the comparative labels and highlights why FFNBSES allows the largest feasible region for capturing expert judgments.

5.2. Advantages of the FFNBSES model

The FFNBSES model offers key benefits compared with FNBSES, IFNBSES, and PFNBSES:

- Enhanced membership representation: captures higher uncertainty and hesitation, enabling finer discrimination among closely competing alternatives and subtle differences in expert judgments.
- Multinary evaluation support: accommodates multilevel and categorical assessments with higher-order membership expressiveness, improving decision stability and practical applicability.
- Bipolar reasoning: handles both positive and negative aspects of criteria more reliably, reflecting nuanced expert opinions.
- Expert integration: systematically incorporates multiple expert evaluations, capturing diverse judgments and conflicts effectively.
- Unified generalization: generalizes earlier models as special cases, overcoming structural limitations and supporting broader applicability in complex MCDM scenarios.

5.3. Limitations of the FFNBSES model

The FFNBSES model provides a robust framework for MCGDM, yet several limitations should be noted:

- Computational overhead: the FFNBSES framework involves evaluating and aggregating two coupled mappings, $\overline{\gamma}$ and $\underline{\gamma}$, for each parameter-alternative-expert combination, in addition to handling an N-grade scale and enforcing cubic-sum constraints. This multi layered structure significantly increases computational complexity, particularly for problems with many alternatives, parameters, and experts. Consequently, DM time can grow rapidly with the dataset size. Optimization strategies such as parallel processing, sparse data structures, and pre-aggregation remain relevant, but their necessity is directly linked to these FFNBSES-specific operations rather than being generic suggestions.
- Expert dependency: the model relies on accurate expert input, which may be subjective. The DM algorithm (Algorithm 1) classifies agreements and disagreements systematically, ensuring differences are incorporated into decisions. Future extensions could automate detection of inconsistencies or outliers.
- Comparative benchmarking: FFNBSES has not been systematically benchmarked against other fuzzy/bipolar MCDM models. Formal comparisons would help identify performance bottlenecks and guide optimizations.

- Interpretability: complex aggregated outputs may be challenging to interpret. Decision matrices, score aggregation, and ranking help, but advanced visualization techniques could further enhance understanding.
- Software support: limited implementations exist in current DM platforms, requiring custom programming. Providing reusable libraries in Python, R, or MATLAB would facilitate broader adoption.

These limitations indicate areas for methodological refinement and practical improvements.

6. Conclusions

This study proposed the FFNBSES framework as a comprehensive approach for handling uncertainty, multinary evaluations, and conflicting expert opinions in MCGDM, with application to smart healthcare. Integrating Fermatean fuzzy logic with bipolar, multinary, and multi-expert input enhances expressiveness and robustness. The formal definitions, algebraic properties, and healthcare case study demonstrate practical utility and improved DM capabilities compared to existing SS approaches.

6.1. Future studies

Future research will focus on enhancing the FFNBSES framework and addressing its limitations:

- Algorithmic optimization: efficient computational strategies to handle large-scale datasets.
- Expert input automation: automated validation, consistency checks, and outlier detection to improve reliability.
- Software implementation and visualization: user-friendly libraries and advanced visualization tools to enhance interpretability.
- Machine learning integration: using machine learning to estimate expert evaluations, detect patterns in multinary and bipolar data, and support adaptive aggregation of uncertain inputs.
- Cross-domain extensions: exploring applications of the FFNBSES model in broader fields such as financial markets, energy systems, medical imaging, game theory, and neuroscience, thereby reinforcing its generalization potential beyond healthcare.

These directions aim to strengthen FFNBSES's scalability, applicability, and usability, supporting informed and efficient DM under uncertainty. Furthermore, by extending its use to diverse real-world domains, the framework offers a generalizable service to complex DM problems across multiple disciplines.

Author contributions

Baravan Asaad: Formal analysis, Investigation, Validation, Writing–review & editing; Sagvan Musa: Conceptualization, Methodology, Formal analysis, Validation, Writing–original draft, Writing–review & editing; Badr Alharbi: Formal analysis, Investigation, Validation, Funding acquisition, Writing–review & editing; Zanyar Ameen: Methodology, Formal analysis, Data curation, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This research work was funded by Umm Al-Qura University, Saudi Arabia, under grant number 25UQU4281756GSSR01.

Acknowledgments

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia, for funding this research work through grant number 25UQU4281756GSSR01.

Conflict of interest

The authors declare no conflict of interest.

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