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Research article

Characterizations of slant-type submanifolds in (α, p)-golden geometry

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Abstract: Motivated by the growing interest in geometric structures defined through polynomial tensor identities, we investigated slant and semi-slant submanifolds of almost (α, p)-golden Riemannian manifolds a generalized class encompassing golden, and complex golden structures. We introduced the notions of (α, p)-slant and semi-slant submanifolds, established their characterizations and integrability conditions, and constructed illustrative examples. The results extend and unify existing theories in golden and complex golden geometry within a common framework.

Keywords: (α, p)-structure; (α, p)-golden manifold; slant submanifold; semi-slant submanifold
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1. Introduction

Slant submanifolds have attracted continuous interest in differential geometry since their introduction by Chen in the early 1990s [1]. They generalize both invariant and anti-invariant submanifolds by permitting the angle between the structure tensor field and the tangent space to be constant but nontrivial that is, neither 0 nor $\frac{\pi}{2}$. This generalization provides greater flexibility in submanifold theory while preserving essential structural symmetries.

Over the past few decades, slant and semi-slant submanifolds have been extensively investigated across diverse geometric contexts. Early studies concentrated on almost Hermitian and Sasakian manifolds [2, 3], where the underlying structure tensors naturally induce nontrivial angles between tangent and image spaces. Subsequent work extended these ideas to Kenmotsu [4], para-Hermitian [5], and cosymplectic manifolds [6], demonstrating that the slant condition provides an effective tool for analyzing both the geometry of the submanifolds and the properties of their ambient spaces. The versatility of the slant concept has further inspired numerous generalizations, including pointwise slant and semi-slant submanifolds, as well as lightlike and warped product constructions in both Riemannian and semi-Riemannian settings.

To accommodate a broader spectrum of curvature behaviors and to generalize the interactions between structure tensors and submanifolds, several new types of structures have been introduced. Among the most prominent are golden and metallic structures, which arise from polynomial relations imposed on the associated $(1,1)$-tensor fields. A golden structure is defined by the identity

$$\Phi^2 = \Phi + I,$$

where Φ is a $(1,1)$-tensor field and I is the identity map [7]. This relation mirrors the defining equation of the classical golden ratio and serves as a starting point for modeling geometric patterns that exhibit repetitive or structured behavior. Metallic structures extend this idea by adopting the more general identity

$$\Phi^2 = a\Phi + bI,$$

where $a, b \in \mathbb{Z}^+$, thus incorporating a wider class of deformation parameters [8, 9]. These structures have been effectively employed to define submanifolds with distinct curvature behavior and to develop new geometric interpretations of classic curvature constraints. From a geometric perspective, such polynomial structures provide a natural framework for controlling the interaction between the tangent and normal components of submanifolds. This makes them particularly suitable for the study of slant-type geometries, where the behavior of projection operators plays a central role.

Roughly speaking, a slant submanifold is characterized by the fact that the angle between the image of a tangent vector under the structure tensor and the tangent space itself remains constant. Semi-slant and hemi-slant submanifolds arise as natural mixed configurations, where invariant, anti-invariant, and slant directions coexist through orthogonal distributions. These distributions describe how the tangent bundle decomposes into geometrically meaningful components. More recently, further generalizations have been introduced, such as the (α, p)-golden structure, defined by the polynomial identity

$$\Phi_{\alpha,p}^2 = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4} p^2 I,$$

have been introduced to provide a more flexible framework for geometric modeling [10]. This two-parameter structure allows interpolation between various well-known configurations, including the classical golden case. The additional freedom offered by α and p makes this framework suitable for investigating new types of slant and semi-slant submanifolds that may be difficult to characterize under more restrictive structures. More recently, Hreţcanu and Druţă-Romaniuc [11] conducted a detailed study of semi-invariant submanifolds in almost (α, p)-golden Riemannian manifolds, establishing integrability conditions and mixed totally geodesic characterizations that broaden the geometric understanding of these structures.

Slant and semi-slant submanifolds in golden and metallic manifolds have been widely studied, with substantial contributions from several researchers. Blaga and Hreţcanu [12] examined their integrability and curvature properties, while Gök et al. [13] focused on their characterizations in the golden setting. Further investigations into warped product constructions [14], lightlike geometries [15], and semi-invariant warped product submanifolds [16] have demonstrated the depth and versatility of this field. Prior to the formalization of almost (α, p)-golden manifolds in [10], slant-type geometries had already been explored in extended contexts, including hemi-slant [17] and bi-slant configurations [18]. These earlier developments provided a rich foundation that naturally motivated the study of such structures within the (α, p) framework, where both theoretical and structural aspects have since been investigated.

Recent studies have explored lightlike and screen semi-invariant submanifolds under golden-type metrics [19, 20], highlighting the adaptability of these structures in pseudo-Riemannian contexts. Other investigations have introduced CR-warped product submanifolds [21], examined Clairaut-type Riemannian maps [22], and analyzed pointwise slant [23], pointwise bi-slant [24], and Casorati-type curvature properties [25]. Collectively, this line of research reflects the ongoing development of slant geometry in connection with broader areas of modern differential geometry. In contrast to these studies, which primarily address invariant, semi-invariant, or warped product configurations, the present paper is devoted to the systematic investigation of slant and semi-slant submanifolds in the setting of almost (α, p)-golden Riemannian manifolds. Specifically, we establish new characterization results formulated directly in terms of the tangential and normal components of the (α, p)-golden, and we present explicit examples in $\mathbb{R}^4$ to illustrate the theoretical results.

Despite these advances, explicit examples of slant and semi-slant submanifolds in almost (α, p)-golden Riemannian manifolds remain relatively scarce. While many theoretical results have been established, there is still a need for concrete constructions that illustrate how these structures operate in practice. The explicit construction of such submanifolds not only validates the theoretical conditions but also provides valuable insight into the geometric behavior of the manifold under the influence of the (α, p)-tensor.

In this paper, we aim to address this gap by constructing detailed examples of slant and semi-slant submanifolds within almost (α, p)-golden Riemannian manifolds. We define specific immersions in $\mathbb{R}^4$ equipped with the appropriate tensor field and verify the slant conditions directly. In doing so, we illustrate how the interaction between the algebraic identity

$$\Phi_{\alpha,p}^2 = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4}p^2 I$$

and the differential structure of M determines the geometry of the submanifold. These constructions not only extend known results from classical golden geometry but also highlight the enhanced flexibility offered by the (α, p) framework.

2. Preliminaries

The basic identities, decompositions, and properties of the (α, p)-golden structure presented below are adapted from [10], the standard reference that first introduced this generalization. They are included here for completeness and clarity without reproducing detailed proofs.

Let $(\bar{M}, g)$ be a Riemannian manifold and let $\Phi_{\alpha,p}$ be a (1,1)-tensor field on an even-dimensional manifold $\bar{M}$. The triple $(\bar{M}, \Phi_{\alpha,p}, g)$ is called an *almost* (α, p)*-golden Riemannian manifold* if the structure tensor satisfies the polynomial identity

$$\Phi_{\alpha,p}^2 = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4}p^2 I, \tag{2.1}$$

and the Riemannian metric g on $\bar{M}$, satisfies the following equalities for any $\chi, \Upsilon \in \Gamma(T\bar{M})$:

$$g\left(\Phi_{\alpha,p}\chi, \Upsilon\right) = \alpha g\left(\chi, \Phi_{\alpha,p}\Upsilon\right) + \frac{p}{2}(1 - \alpha)g\big(\chi, \Upsilon\big), \tag{2.2}$$

$$g\left(\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\Upsilon\right) = \frac{p}{2}\left(g\left(\Phi_{\alpha,p}\chi, \Upsilon\right) + g\left(\chi, \Phi_{\alpha,p}\Upsilon\right)\right) + p^2 g\big(\chi, \Upsilon\big), \tag{2.3}$$

where $\alpha \in \{-1, 1\}$, $p \in \mathbb{R} \setminus \{0\}$, and I is the identity transformation.

If the structure tensor $\Phi_{\alpha,p}$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$, i.e.,

$$\bar{\nabla}\Phi_{\alpha,p} = 0,$$

then the manifold is called a *locally decomposable almost* (α, p)*-golden Riemannian manifold*.

Let $M \subset \bar{M}$ be an isometrically immersed submanifold with the induced metric and Levi-Civita connection ∇. For each $\chi \in \Gamma(TM)$, the tensor field $\Phi_{\alpha,p}$ can be decomposed as a sum of tangential and normal components, i.e.,

$$\Phi_{\alpha,p}\chi = T\chi + \mathcal{N}\chi, \tag{2.4}$$

where $T\chi \in \Gamma(TM)$ and $\mathcal{N}\chi \in \Gamma(T^{\perp}M)$. Similarly, we can write each $U \in \Gamma(T^{\perp}M)$ as

$$\Phi_{\alpha,p}U = \tau U + \eta U,$$

where $\tau U \in \Gamma(TM)$ and $\eta U \in \Gamma(T^{\perp}M)$. These decompositions induce four linear operators $T, \mathcal{N}, \tau, \eta$, satisfying the following metric compatibility relations:

$$\begin{aligned} g(T\chi, \Upsilon) &= \alpha g(\chi, T\Upsilon) + \frac{p(1-\alpha)}{2} g(\chi, \Upsilon), \\ g(\eta U, V) &= \alpha g(U, \eta V) + \frac{p(1-\alpha)}{2} g(U, V), \\ g(\mathcal{N}\chi, U) &= \alpha g(\chi, \tau U). \end{aligned}$$

Moreover, the standard Gauss and Weingarten formulas hold:

$$\bar{\nabla}_{\chi}\Upsilon = \nabla_{\chi}\Upsilon + h(\chi, \Upsilon), \tag{2.5}$$

$$\bar{\nabla}_{\chi}U = -A_U\chi + \nabla^{\perp}_{\chi}U. \tag{2.6}$$

Here, h is the second fundamental form, A_U is the shape operator associated with $U \in \Gamma(T^{\perp}M)$, and $\nabla^{\perp}$ is the induced normal connection. Together, they satisfy the equation

$$g(h(\chi, \Upsilon), U) = g(A_U\chi, \Upsilon).$$

We now summarize a set of structural identities satisfied by the induced operators.

Proposition 2.1. *Let* $M \subset \bar{M}$ *be a submanifold of an almost* (α, p)*-golden Riemannian manifold. Then, the following relations hold for all* $\chi \in \Gamma(TM)$, $U \in \Gamma(T^{\perp}M)$*:*

$$\begin{aligned} T^2\chi &= pT\chi + \frac{5\alpha - 1}{4} p^2\chi - \tau(\mathcal{N}\chi), \\ \mathcal{N}(T\chi) &= p\mathcal{N}\chi - \eta(\mathcal{N}\chi), \\ \eta^2 U &= p\eta U + \frac{5\alpha - 1}{4} p^2 U - \mathcal{N}(\tau U), \\ \tau(\eta U) &= p\tau U - T(\tau U). \end{aligned}$$

Corollary 2.1. *If the condition*

$$\tau(\mathcal{N}\chi) = 0$$

holds for all $\chi \in \Gamma(TM)$*, then the tangential operator* T *satisfies the golden identity*

$$T^2 = pT + \frac{5\alpha - 1}{4} p^2 I.$$

3. Slant submanifolds

Definition 3.1. *A submanifold M of an almost (α, p)-golden Riemannian manifold is called a* slant submanifold *if there exists a constant angle $\theta \in [0, \pi/2]$ such that*

$$\|T\chi\| = \cos\theta \, \|\Phi_{\alpha,p}\chi\|$$

for every nonzero $\chi \in \Gamma(TM)$. Equivalently, M is slant if and only if there exists a constant

$$\lambda = \cos^2\theta$$

such that

$$g(T\chi, T\chi) = \lambda g\left(\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\chi\right). \tag{3.1}$$

To clarify how the (α, p)-golden structure influences the geometry of a submanifold, we begin with a basic characterization of slant submanifolds in this setting. The following result establishes a concise algebraic condition that captures the essence of the slant angle in terms of the structure tensor.

Proposition 3.1. *Let M be a slant submanifold of an almost (α, p)-golden Riemannian manifold $(\bar{M}, \Phi_{\alpha,p}, g)$ with slant angle θ. Then, for any $\chi, \Upsilon \in \Gamma(TM)$, we have the following:*

- *For $\alpha = 1$*

$$g(T\chi, T\Upsilon) = \cos^2\theta\left[pg(\chi, T\Upsilon) + p^2 g(\chi, \Upsilon)\right], \tag{3.2}$$

$$g(\mathcal{N}\chi, \mathcal{N}\Upsilon) = \sin^2\theta\left[pg(\chi, T\Upsilon) + p^2 g(\chi, \Upsilon)\right]. \tag{3.3}$$

- *For $\alpha = -1$*

$$g(T\chi, T\Upsilon) = \cos^2\theta \, \frac{3}{2} p^2 g(\chi, \Upsilon), \tag{3.4}$$

$$g(\mathcal{N}\chi, \mathcal{N}\Upsilon) = \sin^2\theta \, \frac{3}{2} p^2 g(\chi, \Upsilon). \tag{3.5}$$

Proof. Since M is a slant submanifold with slant angle θ, substituting $\chi + \Upsilon$ for χ in (3.1) yields

$$\cos^2\theta \, g\left(\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\Upsilon\right) = g\big(T\chi, T\Upsilon\big). \tag{3.6}$$

From Eqs (2.3) and (3.6), we obtain

$$g(T\chi, T\Upsilon) = \cos^2\theta\left(\frac{p}{2}\left(g\left(\Phi_{\alpha,p}\chi, \Upsilon\right) + g\left(\chi, \Phi_{\alpha,p}\Upsilon\right)\right) + p^2 g\big(\chi, \Upsilon\big)\right).$$

Next, combining the last equation with (2.2), we get

$$g(T\chi, T\Upsilon) = \cos^2\theta\left(\frac{p}{2}(\alpha + 1) g\big(\chi, T\Upsilon\big) + \frac{5 - \alpha}{4} p^2 g\big(\chi, \Upsilon\big)\right). \tag{3.7}$$

In Eq (3.7), the case $\alpha = 1$ yields Eq (3.2), while $\alpha = -1$ yields Eq (3.4).

On the other hand, using (2.4), we can write

$$g(\mathcal{N}\chi, \mathcal{N}\Upsilon) = \sin^2\theta \left(\frac{p}{2}\left(g\left(\Phi_{\alpha,p}\chi, \Upsilon\right) + g\left(\chi, \Phi_{\alpha,p}\Upsilon\right)\right) + p^2 g\big(\chi, \Upsilon\big)\right).$$

Combining this with (2.2) leads to

$$g(\mathcal{N}\chi, \mathcal{N}\Upsilon) = \sin^2\theta \left(\frac{p}{2}(\alpha+1)g\big(\chi, T\Upsilon\big) + \frac{5-\alpha}{4}p^2 g\big(\chi, \Upsilon\big)\right). \tag{3.8}$$

Similarly, in Eq (3.8), the case $\alpha = 1$ yields Eq (3.3), and $\alpha = -1$ yields Eq (3.5).

This completes the proof. □

Theorem 3.1. *Let M be a submanifold of an almost (α, p)-golden Riemannian manifold $(\bar{M}, \Phi_{\alpha,p}, g)$. Then M is a slant submanifold with slant angle θ, if and only if there exists $\lambda \in [0, 1]$ such that*

- *for $\alpha = 1$*

$$T^2 = \lambda\left(pT + p^2 I\right), \tag{3.9}$$

- *for $\alpha = -1$*

$$T^2 = pT - \frac{3}{2}\lambda I. \tag{3.10}$$

Proof. Since M is a slant submanifold, it follows from Eqs (2.2) and (3.2) that

$$\begin{aligned} g\big(T^2\chi, \Upsilon\big) &= \alpha g\big(T\chi, T\Upsilon\big) + \frac{p}{2}(1-\alpha)pg\big(T\chi, \Upsilon\big) \\ &= \alpha\cos^2\theta \left(\frac{p}{2}(1+\alpha)g\big(\chi, T\Upsilon\big) + \frac{5-\alpha}{4}p^2 g\big(\chi, \Upsilon\big)\right) + \frac{p}{2}(1-\alpha)g\big(T\chi, \Upsilon\big) \end{aligned}$$

for any $\chi, \Upsilon \in \Gamma(TM)$. This implies that the tangential operator T satisfies the identity

$$T^2 = \frac{p}{2}\left(\cos^2\theta\,(1+\alpha) + (1-\alpha)\right)T + \frac{5-\alpha}{4}p^2\cos^2\theta\, I.$$

For $\alpha \in \{-1, 1\}$, setting

$$\lambda = \cos^2\theta$$

yields Eqs (3.9) and (3.10).

Conversely, assume that there exists a constant that satisfies (3.9). Then for any $\chi \in \Gamma(TM)$, we have

$$g(T^2\chi,\chi) = \lambda\left(pg(T\chi,\chi) + p^2 g(\chi,\chi)\right).$$

Utilizing Eqs (3.2) and (2.3), we obtain

$$g(T\chi, T\chi) = \lambda g\left(\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\chi\right).$$

Hence, we have

$$\|T\chi\|^2 = \lambda\|\Phi_{\alpha,p}\chi\|^2,$$

which implies

$$\|T\chi\| = \cos\theta\,\|\Phi_{\alpha,p}\chi\|$$

for the constant

$$\theta = \cos^{-1}(\sqrt{\lambda}).$$

Consequently, M is a slant submanifold. The converse direction for $\alpha = -1$ is proved by repeating the steps above, substituting (3.10) for (3.9). □

This theorem offers a straightforward criterion for determining whether a submanifold is slant by analyzing how the square of the projection interacts with the (α, p)-golden tensor. The constant λ serves as a direct algebraic encoding of the slant angle, rendering the result both elegant and computationally practical.

We now turn to a structural property associated with the slant distribution. The following theorem establishes a symmetry condition involving the covariant derivative of the tangential operator, which in turn yields a geometric consequence for the behavior of the structure tensor along the tangent bundle.

Theorem 3.2. *Let M be a slant submanifold of an almost (α, p)-golden Riemannian manifold $(\bar{M}, \Phi_{\alpha,p}, g)$. If the tangential operator T satisfies*

$$(\nabla_\chi T)(\Upsilon) = (\nabla_\Upsilon T)(\chi),$$

then,

$$\mathcal{N}([\chi, \Upsilon]) = 0$$

for all $\chi, \Upsilon \in \Gamma(TM)$. That is, the normal component of $\Phi_{\alpha,p}([\chi, \Upsilon])$ vanishes.

Proof. Since

$$\overline{\nabla}\Phi_{\alpha,p} = 0,$$

for all $\chi, \Upsilon \in \Gamma(TM)$, we have

$$0 = \left(\overline{\nabla}_\chi \Phi_{\alpha,p}\right)(\Upsilon) = \overline{\nabla}_\chi \left(\Phi_{\alpha,p}\Upsilon\right) - \Phi_{\alpha,p}\left(\overline{\nabla}_\chi \Upsilon\right).$$

By using the Eqs (2.4)–(2.6), considering only the tangential components, we obtain

$$\left(\nabla_\chi T\right)(\Upsilon) = A_{\mathcal{N}\Upsilon}\chi + (\Phi_{\alpha,p}h(\chi, \Upsilon))^\top.$$

Interchanging χ and Υ, and utilizing the symmetry of the second fundamental form, the subtraction of these expressions yields

$$(\nabla_\chi T)(\Upsilon) - (\nabla_\Upsilon T)(\chi) = A_{\mathcal{N}\Upsilon}\chi - A_{\mathcal{N}\chi}\Upsilon.$$

Consequently, the assumption that ∇T is symmetric implies

$$A_{\mathcal{N}\Upsilon}\chi = A_{\mathcal{N}\chi}\Upsilon$$

for all $\chi, \Upsilon \in \Gamma(TM)$. Finally, since $[\chi, \Upsilon] \in \Gamma(TM)$, the decomposition

$$\Phi_{\alpha,p}([\chi, \Upsilon]) = T([\chi, \Upsilon]) + \mathcal{N}([\chi, \Upsilon])$$

indicates that the normal component of $\Phi_{\alpha,p}([\chi, \Upsilon])$ is precisely $\mathcal{N}([\chi, \Upsilon])$. This leads to

$$\mathcal{N}([\chi, \Upsilon]) = 0.$$

This completes the proof. □

This result establishes a direct connection between the algebraic properties of the (α, p)-golden structure and the covariant behavior of the tangential operator T. As a consequence, the symmetry of ∇T imposes a geometric restriction on the normal component of $\Phi_{\alpha,p}([\chi, \Upsilon])$.

4. Semi-slant submanifolds

Definition 4.1. *Let M be a submanifold of an almost (α, p)-golden Riemannian manifold $(\bar{M}, \Phi_{\alpha,p}, g)$. We say that M is a* semi-slant submanifold *if its tangent bundle TM admits an orthogonal decomposition*

$$TM = D \oplus D_\theta,$$

where D is an invariant distribution under $\Phi_{\alpha,p}$, and D_θ is a slant distribution such that the angle $\theta \in [0, \pi/2]$ between $\Phi_{\alpha,p}\chi$ and the tangent space TM is constant for all nonzero $\chi \in D_\theta$. This constant angle θ is referred to as the slant angle *of M.*

We now present a detailed characterization of the geometric behavior of semi-slant submanifolds in the (α, p)-golden setting. The following theorem describes the interaction of the structure tensor with both the slant and invariant distributions.

Theorem 4.1. *Suppose M is a semi-slant submanifold of an almost (α, p)-golden Riemannian manifold $(\bar{M}, \Phi_{\alpha,p}, g)$, with an orthogonal decomposition of the tangent bundle*

$$TM = D \oplus D_\theta,$$

where D is an invariant distribution, i.e., $\Phi_{\alpha,p}(D) \subset D$, and D_θ is a proper slant distribution with constant slant angle $\theta \in (0, \pi/2)$. Then, the following properties hold:

(1) For any vector field $\chi \in \Gamma(D_\theta)$, we have

$$(PT)^2\chi = \lambda \Phi_{\alpha,p}^2 \chi,$$

where P is the orthogonal projection on D_θ, and

$$\lambda = \cos^2\theta.$$

(2) For any vector field $\Upsilon \in \Gamma(D)$, we have

$$T\Upsilon = \Phi_{\alpha,p}\Upsilon$$

and

$$\mathcal{N}\Upsilon = 0.$$

Proof. Since M is a semi-slant submanifold, its tangent bundle can be decomposed orthogonally as

$$TM = D \oplus D_\theta,$$

where D is invariant under $\Phi_{\alpha,p}$ and D_θ is a slant distribution with slant angle $\theta \in (0, \pi/2)$. With this in mind, let us prove the first property. For any $\chi \in \Gamma(D_\theta)$, the constant slant condition immediately establishes the relationship between the norms:

$$g(PT\chi, PT\chi) = \cos^2\theta \, g\left(\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\chi\right).$$

As a direct consequence of the slant distribution definition, the structure tensor T on D_θ satisfies the algebraic identity

$$(PT)^2\chi = \lambda\Phi^2_{\alpha,p}\chi\,,$$

where

$$\lambda = \cos^2\theta.$$

Next, we prove the second property. Since D is invariant under $\Phi_{\alpha,p}$, we have $\Phi_{\alpha,p}$, we have $\Phi_{\alpha,p}\Upsilon \in \Gamma(D) \subset \Gamma(TM)$ for $\Upsilon \in \Gamma(D)$. As $\Phi_{\alpha,p}\Upsilon$ is tangent to M, the decomposition

$$\Phi_{\alpha,p}\Upsilon = T\Upsilon + \mathcal{N}\Upsilon$$

requires

$$\mathcal{N}\Upsilon = 0 \quad \text{and} \quad T\Upsilon = \Phi_{\alpha,p}\Upsilon.$$

This completes the proof. □

Theorem 4.2. *Let M be a semi-slant submanifold of an almost (α, p)-golden Riemannian manifold with tangent bundle decomposition*

$$TM = D \oplus D_\theta.$$

Then, we have the following

(1) The distribution D is integrable if and only if

$$[\chi, \Upsilon] \in \Gamma(D)$$

for all $\chi, \Upsilon \in \Gamma(D)$. Equivalently,

$$\mathcal{N}([\chi, \Upsilon]) = 0.$$

(2) The distribution D_θ is integrable if and only if

$$(\nabla_\chi T)(\Upsilon) = (\nabla_\Upsilon T)(\chi) \tag{4.1}$$

for all $\chi, \Upsilon \in \Gamma(D_\theta)$.

Proof. (1) Since D is invariant, we have $\Phi_{\alpha,p}\chi, \Phi_{\alpha,p}\Upsilon \in \Gamma(D)$ for any $\chi, \Upsilon \in \Gamma(D)$. If $[\chi, \Upsilon] \in \Gamma(D)$, then D is integrable. Moreover, since

$$\Phi_{\alpha,p}Z = TZ$$

for any $Z \in \Gamma(D)$, it follows that $\mathcal{N}Z = 0$. Thus, the implication

$$\Phi_{\alpha,p}([\chi, \Upsilon]) = T([\chi, \Upsilon]) \Rightarrow \mathcal{N}([\chi, \Upsilon]) = 0$$

provides the equivalent condition.

(2) For D_θ, the argument parallels the slant case. Using the characterization of integrability via the tangential component T, we consider the condition:

$$(\nabla_\chi T)(\Upsilon) = (\nabla_\Upsilon T)(\chi) \Rightarrow T([\chi, \Upsilon]) \in \Gamma(TM).$$

This equality implies that $T([\chi, \Upsilon]) \in \Gamma(TM)$. So, $[\chi, \Upsilon] \in \Gamma(TM)$. Since these hold for any $\chi, \Upsilon \in \Gamma(D_\theta)$, it establishes the integrability of D_θ. □

Theorem 4.3. *Let M be a semi-slant submanifold of an almost (α, p)-golden Riemannian manifold such that the invariant distribution D is totally geodesic in M. Then,*

$$h(\chi, \Upsilon) = 0$$

for all $\chi, \Upsilon \in \Gamma(D)$. Furthermore, if D_θ is also integrable and totally geodesic, then M is totally geodesic in $\bar{M}$.

Proof. If D is totally geodesic in M, then $\nabla_\chi \Upsilon \in \Gamma(D)$ and

$$h(\chi, \Upsilon) = \left(\bar{\nabla}_\chi \Upsilon\right)^\perp = 0$$

for all $\chi, \Upsilon \in \Gamma(D)$. Similarly, if D_θ is also totally geodesic, then any $\chi, \Upsilon \in \Gamma(TM)$ can be decomposed as

$$\chi = \chi_1 + \chi_2$$

and

$$\Upsilon = \Upsilon_1 + \Upsilon_2,$$

where $\chi_1, \Upsilon_1 \in \Gamma(D)$ and $\chi_2, \Upsilon_2 \in \Gamma(D_\theta)$. Using the bilinearity of the second fundamental form and the fact that the components vanish on the respective distributions, we have

$$h(\chi, \Upsilon) = h(\chi_1, \Upsilon_1) + h(\chi_1, \Upsilon_2) + h(\chi_2, \Upsilon_1) + h(\chi_2, \Upsilon_2) = 0.$$

It follows that M is totally geodesic in $\bar{M}$. □

5. Examples

In the examples, the parameter α is retained in symbolic form to emphasize consistency with the general (α, p)-golden tensor structure. However, all explicit numerical computations are carried out under the choice $\alpha = 1$.

5.1. Slant Submanifold

Let

$$\bar{M} = \mathbb{R}^4$$

be the Euclidean 4-space equipped with the standard Riemannian metric g and the diagonal tensor field

$$\Phi_{\alpha,p}(x_1, x_2, x_3, x_4) = \left(p\frac{1+\sqrt{5\alpha}}{2}x_1,\ p\frac{1+\sqrt{5\alpha}}{2}x_2,\ p\frac{1-\sqrt{5\alpha}}{2}x_3,\ p\frac{1-\sqrt{5\alpha}}{2}x_4\right),$$

which satisfies the (α, p)-golden condition

$$\Phi^2_{\alpha,p} = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4}p^2 I.$$

Define the **2**-dimensional submanifold $M \subset \mathbb{R}^4$ by the immersion:

$$r : U \subset \mathbb{R}^2 \to \mathbb{R}^4; \quad r(u, v) = (u, 0, u, v),$$

where u and v are the local coordinates on M. So, the tangent vectors are

$$\chi = \frac{\partial r}{\partial u} = (1,0,1,0) \quad \text{and} \quad \Upsilon = \frac{\partial r}{\partial v} = (0,0,0,1).$$

Then,

$$\Phi_{\alpha,p}(\chi) = \left(p\frac{1+\sqrt{5\alpha}}{2}, 0, p\frac{1-\sqrt{5\alpha}}{2}, 0\right),$$

and the projection of $\Phi_{\alpha,p}(\chi)$ onto TM is given by

$$T\chi = \frac{\langle \Phi_{\alpha,p}(\chi), \chi\rangle}{\|\chi\|^2}\chi = \frac{p}{2}\chi,$$

since

$$\frac{1+\sqrt{5\alpha}}{2} + \frac{1-\sqrt{5\alpha}}{2} = 1.$$

Therefore,

$$\cos\theta = \frac{\|T\chi\|}{\|\Phi_{\alpha,p}\chi\|} = \frac{p/\sqrt{2}}{p\sqrt{\left(\frac{1+\sqrt{5\alpha}}{2}\right)^2 + \left(\frac{1-\sqrt{5\alpha}}{2}\right)^2}}.$$

Since the angle θ is constant at all points on M, it follows that M is indeed a proper slant submanifold.

5.2. Semi-slant submanifold

Let

$$\bar{M} = \mathbb{R}^4$$

carry the same (α, p)-golden structure as in the previous example, defined by

$$\Phi_{\alpha,p}(x_1, x_2, x_3, x_4) = \left(p\frac{1+\sqrt{5\alpha}}{2}x_1,\ p\frac{1+\sqrt{5\alpha}}{2}x_2,\ p\frac{1-\sqrt{5\alpha}}{2}x_3,\ p\frac{1-\sqrt{5\alpha}}{2}x_4\right).$$

Consider the **2**-dimensional submanifold $M \subset \mathbb{R}^4$ defined by the immersion:

$$r : U \subset \mathbb{R}^2 \to \mathbb{R}^4; \quad r(u,v) = (u, v, 0, v),$$

where u and v are the local coordinates on M. Then, the tangent space of the submanifold M is spanned by the vectors

$$\chi = \frac{\partial r}{\partial u} = (1,0,0,0) \quad \text{and} \quad \Upsilon = \frac{\partial r}{\partial v} = (0,1,0,1).$$

Since

$$\Phi_{\alpha,p}(\chi) = \left(p\frac{1+\sqrt{5\alpha}}{2}, 0, 0, 0\right) \in TM,$$

D is an invariant distribution. On the other hand,

$$\Phi_{\alpha,p}(\Upsilon) = \left(0, p\frac{1+\sqrt{5\alpha}}{2}, 0, p\frac{1-\sqrt{5\alpha}}{2}\right) \notin TM$$

implies D_θ is a slant distribution. By projecting $\Phi_{\alpha,p}(\Upsilon)$ onto TM, we obtain the tangential component:

$$T\Upsilon = \mathrm{proj}_{TM}(\Phi_{\alpha,p}(\Upsilon)) = \frac{p}{2}(0, 1, 0, 1).$$

We then compute the slant angle θ. Using the norms

$$\|T\Upsilon\| = \frac{p}{\sqrt{2}} \quad \text{and} \quad \|\Phi_{\alpha,p}\Upsilon\| = p\sqrt{\left(\frac{1+\sqrt{5\alpha}}{2}\right)^2 + \left(\frac{1-\sqrt{5\alpha}}{2}\right)^2},$$

we find that

$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{\left(\frac{1+\sqrt{5\alpha}}{2}\right)^2 + \left(\frac{1-\sqrt{5\alpha}}{2}\right)^2}}.$$

This confirms that M is a proper semi-slant submanifold embedded in $\bar{M}$.

6. Conclusions

In this paper, we investigated slant and semi-slant submanifolds within the framework of (α, p)-golden Riemannian manifolds, which generalize classical golden and complex golden structures through a two-parameter polynomial identity. By analyzing the tangential and normal components of the structure tensor, we derived characterization results for slant and semi-slant submanifolds that extend and unify known identities in the literature.

In particular, we derived explicit algebraic conditions involving the tangential operator T that describe the slant geometry in terms of the constant

$$\lambda = \cos^2\theta,$$

and we clarified how these conditions depend on the choice of $\alpha \in \{-1, 1\}$. We also analyzed integrability related properties and demonstrated how symmetry conditions on the covariant derivative of T influence the geometric behavior of the associated distributions.

To complement the theoretical results, we constructed explicit examples of slant and semi-slant submanifolds in Euclidean spaces equipped with suitable (α, p)-golden structures. These examples demonstrate the applicability of the derived conditions and confirm that the slant angle remains constant along the corresponding distributions.

The results presented here provide a systematic extension of slant-type submanifold theory to the (α, p)-golden setting and may serve as a foundation for further investigations within this generalized geometric framework.

Author contributions

Ayşe Torun: conceptualization, investigation, methodology, validation, writing-original draft, review and editing, funding acquisition; Mustafa Özkan: conceptualization, investigation, methodology, validation, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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