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Research article

New determinant expressions of Bernoulli, Euler & Tribonacci polynomials

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Abstract: In 1875, Glaisher systematically found several interesting determinant expressions of numbers, including Bernoulli, Cauchy, and Euler numbers. In this paper, we identify several determinants that express Euler polynomials. Goy and Shattuck presented several determinantal expressions of some families of Toeplitz–Hessenberg matrices with Tribonacci number entries. However, a determinant expression of Tribonacci numbers has not been studied much. By using a similar form of determinants to Euler's, we also give some determinant representations of generalized Tribonacci numbers.

Keywords: Euler polynomials; Tribonacci numbers; determinants

1. Introduction

Historically, Glaisher [8] appears to have been among the first to systematically derive several interesting determinant expressions for various number sequences, including Bernoulli, Cauchy, and Euler numbers. These are results where these numbers are represented in terms of determinants. We can see the Fibonacci determinants in [28, 29] where the Fibonacci numbers are elements of the determinants. Several results such that Fibonacci numbers are expressed in terms of determinants can be seen in [14, 23].

In [24, Corollary 2], a determinant expression of Cauchy polynomials $c_n(x)$, defined by

$$\frac{t}{(1+t)^x\log(1+t)}=\sum_{n=0}^\infty c_n(x)\frac{t^n}{n!}\,,$$

is given by

$$c_n(x) = n!\begin{vmatrix} r_x(2) & 1 & 0 & \cdots & 0 \\ r_x(3) & r_x(2) & 1 & & \vdots \\ r_x(4) & & \ddots & & 0 \\ \vdots & & & r_x(2) & 1 \\ r_x(n+1) & \cdots & r_x(4) & r_x(3) & r_x(2) \end{vmatrix},$$

where

$$r_x(n) = \frac{d}{dx}\frac{(-1)^{n-1}(x)_n}{n!} = \frac{1}{n!}\sum_{\ell=0}^{n-1}(-1)^{\ell}(\ell+1)\begin{bmatrix} n \\ \ell+1 \end{bmatrix} x^{\ell},$$

$(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial with $(x)_0 = 1$, and $\begin{bmatrix} n \\ m \end{bmatrix}$ is the Stirling number of the first kind defined by

$$(x)_n = \sum_{m=0}^{n}(-1)^{n-m}\begin{bmatrix} n \\ m \end{bmatrix} x^m .$$

When $x = 0$, we have an expression of Cauchy numbers ([8, p.50]):

$$c_n = n!\begin{vmatrix} \frac{1}{2} & 1 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & & \vdots \\ \frac{1}{4} & & \ddots & & 0 \\ \vdots & & & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \cdots & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix}.$$

Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} . \tag{1.1}$$

Euler numbers $E_n = 2^n E_n(1/2)$ are given by

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} . \tag{1.2}$$

A determinantal expression of Euler numbers is given by

$$E_{2n} = (-1)^n n!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \vdots \\ \frac{1}{6!} & & \ddots & & 0 \\ \vdots & & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \cdots & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}$$

([8, p.52]). It is generalized in terms of hypergeometric Euler numbers ([27]). Notice that a determinantal expression of Euler numbers of the second kind $\widehat{E}_n$, defined by

$$\frac{t}{\sinh t} = \sum_{n=0}^{\infty} \widehat{E}_n \frac{t^n}{n!}, \tag{1.3}$$

is given by

$$\widehat{E}_{2n} = (-1)^n n! \begin{vmatrix} \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & 1 & & \vdots \\ \frac{1}{7!} & & \ddots & & 0 \\ \vdots & & & \frac{1}{3!} & 1 \\ \frac{1}{(2n+1)!} & \cdots & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix} \tag{1.4}$$

([22, Corollary 2.2]).

In this paper, we give a determinantal expression of Euler polynomials, which is given as in Theorem 1. From the relationship with Euler polynomials, we can also provide a determinant expression for Bernoulli polynomials.

On the other hand, determinants related to Fibonacci numbers are also fascinating. There are many references to matrices or determinants that have Fibonacci numbers as elements (see, e.g., [1, 2, 10, 12, 15, 17, 19, 20], [28, Ch. 33], [29, Ch. 39], [32]). Nevertheless, many of the results so far have dealt with cases involving Fibonacci numbers or numbers related to them as elements, and there have been relatively few results directly giving Fibonacci numbers themselves. In [25], several determinants expressing Fibonacci and related polynomials are given, motivated by the results in [14,23]. The results related to Tribonacci are even scarcer (see, e.g., [3, 5–7, 9, 13, 16, 18, 21, 29–31]), but we have found a determinant representation that gives generalized Tribonacci polynomials.

The determinants discussed in this paper are of a type of Hessenberg matrices, and this type of lower triangular matrices not only allows for the transformation between the outer and inner elements using the inversion formula ([26]), but also has the advantage of being able to express the elements using a summation formula. See propositions in Section 2.

2. Euler determinants

Theorem 1.

$$E_n(x) = n! \begin{vmatrix} \frac{x+(x-1)}{2\cdot 1!} & 1 & 0 & \cdots & 0 \\ \frac{x^2+(x-1)^2}{2\cdot 2!} & \frac{x+(x-1)}{2\cdot 1!} & 1 & & \vdots \\ \frac{x^3+(x-1)^3}{2\cdot 3!} & & \ddots & & 0 \\ \vdots & & & \frac{x+(x-1)}{2\cdot 1!} & 1 \\ \frac{x^n+(x-1)^n}{2\cdot n!} & \cdots & \frac{x^3+(x-1)^3}{2\cdot 3!} & \frac{x^2+(x-1)^2}{2\cdot 2!} & \frac{x+(x-1)}{2\cdot 1!} \end{vmatrix}.$$

When $x = 1/2$, that of Euler numbers can be reduced.

Corollary 1.

$$E_{2n} = 2^n n! \begin{vmatrix} 0 & 1 & 0 & & & \cdots & 0 \\ \frac{1}{2!}\left(\frac{1}{2}\right)^2 & 0 & 1 & & & & \vdots \\ 0 & \frac{1}{2!}\left(\frac{1}{2}\right)^2 & 0 & & & & \\ \frac{1}{4!}\left(\frac{1}{2}\right)^4 & 0 & \frac{1}{2!}\left(\frac{1}{2}\right)^2 & \ddots & & & \vdots \\ 0 & & & & \ddots & & 0 \\ \vdots & & & & \frac{1}{2!}\left(\frac{1}{2}\right)^2 & 0 & 1 \\ \frac{1}{2\cdot n!}\left(\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n\right) & \cdots & 0 & \frac{1}{4!}\left(\frac{1}{2}\right)^4 & 0 & \frac{1}{2!}\left(\frac{1}{2}\right)^2 & 0 \end{vmatrix}.$$

Note that $E_n = 0$ when n is odd.

In order to prove this result, there are several methods to find the determinantal representations. One of them is due to the generating function method presented in [14, 23]. This method is due to Cameron's operator [4].

Lemma 1. *We have*

$$\sum_{n=0}^{\infty} h_n t^n = \frac{c}{1 - z_1 t + z_2 t^2 - z_3 t^3 + \cdots}$$
$$\Longleftrightarrow$$
$$h_n = c \begin{vmatrix} z_1 & 1 & 0 & \cdots & 0 \\ z_2 & z_1 & 1 & \ddots & \vdots \\ z_3 & z_2 & & & 0 \\ \vdots & & \ddots & z_1 & 1 \\ z_n & \cdots & z_3 & z_2 & z_1 \end{vmatrix},$$

where c is the constant independent of h_n, n, and t.

Proof of Theorem 1. By the generating function in (1.1),

$$\begin{aligned} \left(\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\right)^{-1} &= \frac{e^t \cdot e^{-xt} + e^{-xt}}{2} \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\right)\left(\sum_{\ell=0}^{\infty}\frac{(-x)^\ell t^\ell}{\ell!}\right) + \sum_{\ell=0}^{\infty}\frac{(-x)^\ell t^\ell}{\ell!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}(-x)^\ell + (-x)^n\right)\frac{t^n}{n!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{(x-1)^n + x^n}{n!}\right)t^n. \end{aligned}$$

Hence, by Lemma 1, we have the desired determinant. □

A different determinantal expression from (1.4) of Euler numbers of the second kind given in Corollary 2, below. To this end, we consider the representation of Bernoulli polynomials. Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\,. \tag{2.1}$$

A determinant expression of Bernoulli polynomials is given as follows.

Theorem 2.

$$B_n(x)=n!\begin{vmatrix}\frac{x^2-(x-1)^2}{2!} & 1 & 0 & \cdots & 0\\ \frac{x^3-(x-1)^3}{3!} & \frac{x^2-(x-1)^2}{2!} & 1 & & \vdots\\ \frac{x^4-(x-1)^4}{4!} & & \ddots & & 0\\ \vdots & & & \frac{x^2-(x-1)^2}{2!} & 1\\ \frac{x^{n+1}-(x-1)^{n+1}}{(n+1)!} & \cdots & \frac{x^4-(x-1)^4}{4!} & \frac{x^3-(x-1)^3}{3!} & \frac{x^2-(x-1)^2}{2!}\end{vmatrix}.$$

Euler numbers of the second kind are given by $\widehat{E_n}=2^nB_n(1/2)$. When $x=1/2$ in Theorem 2, a different determinant expression from (1.4) of Euler numbers of the second kind can be reduced. Note that $\widehat{E_n}=0$ when n is odd.

Corollary 2.

$$\widehat{E_n}=2^n n!\begin{vmatrix}0 & 1 & 0 & & & \cdots & 0\\ \frac{2}{3!}\left(\frac{1}{2}\right)^3 & 0 & 1 & & & & \vdots\\ 0 & \frac{2}{3!}\left(\frac{1}{2}\right)^2 & 0 & & & & \\ \frac{2}{5!}\left(\frac{1}{2}\right)^5 & 0 & \frac{2}{3!}\left(\frac{1}{2}\right)^3 & \ddots & & & \vdots\\ 0 & & & & \ddots & & 0\\ \vdots & & & & \frac{2}{3!}\left(\frac{1}{2}\right)^3 & 0 & 1\\ \frac{1}{(n+1)!}\left(\left(\frac{1}{2}\right)^{n+1}+\left(-\frac{1}{2}\right)^{n+1}\right) & \cdots & 0 & \frac{2}{5!}\left(\frac{1}{2}\right)^5 & 0 & \frac{2}{3!}\left(\frac{1}{2}\right)^3 & 0\end{vmatrix}.$$

Proof of Theorem 2. By the generating function in (2.1),

$$\begin{aligned}\left(\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\right)^{-1}&=\frac{e^t-1}{t}\cdot e^{-xt}\\ &=\left(\sum_{k=0}^\infty\frac{t^k}{(k+1)!}\right)\left(\sum_{\ell=0}^\infty\frac{(-x)^\ell t^\ell}{\ell!}\right)\\ &=\sum_{n=0}^\infty\left(\sum_{\ell=0}^n\frac{(-x)^\ell}{(n+1-\ell)!\ell!}\right)t^n\\ &=\sum_{n=0}^\infty\left(\sum_{\ell=0}^{n+1}\frac{(-x)^\ell}{(n+1-\ell)!\ell!}-\frac{(-x)^{n+1}}{(n+1)!}\right)t^n\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!} t^n + \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{(n+1)!} t^n$$
$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{n+1} - (x-1)^{n+1}}{(n+1)!} \right) t^n .$$

Hence, by Lemma 1, we get the desired result. □

2.1. More identities

By applying the inversion relation (see, e.g., [26]) to Theorem 2, we can have the determinant value whose elements are Bernoulli polynomials.

Proposition 1. *We have*

$$\frac{x^{n+1} - (x-1)^{n+1}}{(n+1)!} = \begin{vmatrix} \frac{B_1(x)}{1!} & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{B_n(x)}{n!} & \cdots & \cdots & \frac{B_1(x)}{1!} \end{vmatrix}$$
$$= \sum_{\substack{t_1 + 2t_2 \cdots + nt_n = n \\ t_1, \dots, t_n \ge 0}} \binom{t_1 + \cdots + t_n}{t_1, \dots, t_n} (-1)^{n - t_1 - \cdots - t_n} \left(\frac{B_1(x)}{1!} \right)^{t_1} \left(\frac{B_2(x)}{2!} \right)^{t_2} \cdots \left(\frac{B_n(x)}{n!} \right)^{t_n}$$
$$= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \dots, i_k \ge 1}} \frac{B_{i_1}(x)}{i_1!} \cdots \frac{B_{i_k}(x)}{i_k!} ,$$

where $\binom{t_1 + \cdots + t_n}{t_1, \dots, t_n} = \frac{(t_1 + \cdots + t_n)!}{t_1! \cdots t_n!}$ *is the multinomial coefficient.*

Proof. By using the inversion relation

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & 0 \\ R(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ R(n) & \cdots & R(2) & R(1) \end{vmatrix} \iff R(n) = \begin{vmatrix} \alpha_1 & 1 & & 0 \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix} ,$$

we obtain the desired result. □

Proposition 2. *We have*

$$B_n(x) = n! \sum_{t_1 + 2t_2 + \cdots + nt_n = n} \binom{t_1 + \cdots + t_n}{t_1, \dots, t_n} (-1)^{t_1 + \cdots + t_n}$$
$$\times \left(\frac{x^2 - (x-1)^2}{2!} \right)^{t_1} \cdots \left(\frac{x^{n+1} - (x-1)^{n+1}}{(n+1)!} \right)^{t_n}$$
$$= n! \sum_{j=1}^{n} (-1)^j \sum_{\substack{i_1 + \cdots + i_j = n \\ i_1, \dots, i_j \ge 1}} \frac{x^{i_1+1} - (x-1)^{i_1+1}}{(i_1+1)!} \cdots \frac{x^{i_j+1} - (x-1)^{i_j+1}}{(i_j+1)!} .$$

From Corollary 2, we also have the following.

Proposition 3. *We have*

$$\frac{1}{(n+1)!}\left(\left(\frac{1}{2}\right)^{n+1}+\left(-\frac{1}{2}\right)^{n+1}\right)=\begin{vmatrix}\frac{\widehat{E}_1}{2^1\cdot 1!} & 1 & & 0\\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1\\ \frac{\widehat{E}_n}{2^n\cdot n!} & \cdots & \cdots & \frac{\widehat{E}_1}{2^1\cdot 1!}\end{vmatrix}$$
$$=\sum_{\substack{t_1+2t_2\cdots+nt_n=n\\ t_1,\dots,t_n\ge 0}}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}\left(\frac{\widehat{E}_1}{2^1\cdot 1!}\right)^{t_1}\left(\frac{\widehat{E}_2}{2^2\cdot 2!}\right)^{t_2}\cdots\left(\frac{\widehat{E}_n}{2^n\cdot n!}\right)^{t_n}$$
$$=\sum_{k=1}^{n}(-1)^k\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}\frac{\widehat{E}_{i_1}}{2^{i_1}\cdot i_1!}\cdots\frac{\widehat{E}_{i_k}}{2^{i_k}\cdot i_k!}\,.$$

Proposition 4. *We have*

$$\widehat{E}_n=2^n\cdot n!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{t_1+\cdots+t_n}$$
$$\times\left(\frac{1}{2!}\left(\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2\right)\right)^{t_1}\cdots\left(\frac{1}{(n+1)!}\left(\left(\frac{1}{2}\right)^{n+1}+\left(-\frac{1}{2}\right)^{n+1}\right)\right)^{t_n}$$
$$=2^n\cdot n!\sum_{j=1}^{n}(-1)^j\sum_{\substack{i_1+\cdots+i_j=n\\ i_1,\dots,i_j\ge 1}}\frac{1}{(i_1+1)!}\left(\left(\frac{1}{2}\right)^{i_1+1}+\left(-\frac{1}{2}\right)^{i_1+1}\right)$$
$$\cdots\frac{1}{(i_j+1)!}\left(\left(\frac{1}{2}\right)^{i_j+1}+\left(-\frac{1}{2}\right)^{i_j+1}\right).$$

3. Tribonacci determinants

In [9], several determinantal expressions of some families of Toeplitz–Hessenberg matrices are presented with Tribonacci number entries. These determinant formulas may also be expressed equivalently as identities that involve sums of products of multinomial coefficients and Tribonacci numbers. However, as was the case in their paper, there are not many studies expressing Tribonacci numbers in other papers either. Here, Tribonacci numbers T_n ([29, Ch. 49]) are defined by

$$T_n=T_{n-1}+T_{n-2}+T_{n-3}\quad(n\ge 3),\quad T_0=0,\quad T_1=T_2=1\,. \tag{3.1}$$

Note that some more different Tribonacci numbers are defined with different initial values. One of the simplest expressions is

$$T_{n+1}=\begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0\\ -1 & 1 & 1 & & & \vdots\\ 1 & -1 & & & & \vdots\\ 0 & & \ddots & & 1 & 0\\ \vdots & & & -1 & 1 & 1\\ 0 & \cdots & 0 & 1 & -1 & 1 \end{vmatrix}. \tag{3.2}$$

Notice that Tribonacci polynomials $t_n(x)$ ([11], [29, Sec. 49.3]) are defined by

$$t_n(x)=x^2t_{n-1}(x)+xt_{n-2}(x)+t_{n-3}(x)\quad (n\ge 3),$$
$$t_0(x)=0,\quad t_1(x)=1,\quad t_2(x)=x^2\,.$$

For fixed real numbers a,b,c with $(a,b,c)\ne(0,0,0)$, consider more general Tribonacci numbers $\mathcal{T}_n=\mathcal{T}_n(a,b,c)$, defined by the recurrence relation

$$\mathcal{T}_n=a\mathcal{T}_{n-1}+b\mathcal{T}_{n-2}+c\mathcal{T}_{n-3}\quad (n\ge 3),\quad \mathcal{T}_0=0,\quad \mathcal{T}_1=\mathcal{T}_2=1\,. \tag{3.3}$$

Then we have the following determinant expression.

Theorem 3. *For $n\ge 3$,*

$$\mathcal{T}_{n+1}=\begin{vmatrix} 1 & 1 & 0 & \cdots & & \cdots & 0\\ 1-a-b & 1 & 1 & & & & \vdots\\ D & 1-a-b & 1 & & & & \\ (1-a)D & D & & \ddots & & & \vdots\\ (1-a)^2D & & \ddots & & 1 & 1 & 0\\ \vdots & & & D & 1-a-b & 1 & 1\\ (1-a)^{n-3}D & \cdots & (1-a)^2D & (1-a)D & D & 1-a-b & 1 \end{vmatrix},$$

where $D=(a-1)(a+b-1)+c$.

Remark. When $a=b=c=1$ in Theorem 3, the determinant expression of (3.2) is reduced.

Proof of Theorem 3. By the definition of (3.3), we get the generating function

$$\sum_{n=0}^\infty \mathcal{T}_{n+1}t^n=\frac{1-(a-1)t}{1-at-bt^2-ct^3}$$
$$=\frac{1}{1-t-(a+b-1)t^2-Dt^3-(a-1)Dt^4-(a-1)^2Dt^5-\cdots}\,.$$

By Lemma 1, we have the desired result. □

We shall introduce the inversion relation, which is to interchange the outer and inner elements of the determinant. A simple version can be seen in [26].

Lemma 2. *For two sequences $\{\alpha_n\}_{n\ge0}$ and $\{\beta_n\}_{n\ge0}$, we have for $n \ge 3$*

$$\alpha_n = \begin{vmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & c_1 & 1 & & \vdots \\ \beta_3 & & \ddots & & 0 \\ \vdots & & & c_1 & 1 \\ \beta_n & \cdots & \beta_3 & c_2 & c_1 \end{vmatrix}$$

$$\iff \quad \alpha_n - c_1\alpha_{n-1} + c_2\alpha_{n-2} + \sum_{j=3}^{n-3}(-1)^j\beta_j\alpha_{n-3} + (-1)^{n-2}\beta_{n-2}(c_1^2 - c_2) + (-1)^{n-1}\beta_{n-1}c_1 + (-1)^n\beta_n = 0$$

$$\iff \quad \beta_n = \begin{vmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_1^2 - c_2 & c_1 & 1 & & \vdots \\ \alpha_3 & & \ddots & & 0 \\ \vdots & & & c_1 & 1 \\ \alpha_n & \cdots & \alpha_3 & c_1^2 - c_2 & c_1 \end{vmatrix}.$$

By using the inversion relation of Lemma 2, we have a determinantal expression with Tribonacci number entries. Note that the simplest case where $a = b = c = 1$ is shown in [9, (3.4)]*.

Theorem 4. *For $n \ge 3$,*

$$(1-a)^{n-3}((a-1)(a+b-1)+c) = \begin{vmatrix} \mathcal{T}_2 & 1 & 0 & \cdots & 0 \\ \mathcal{T}_3 & \mathcal{T}_2 & & & \vdots \\ \mathcal{T}_4 & & \ddots & & 0 \\ \vdots & & & \mathcal{T}_2 & 1 \\ \mathcal{T}_{n+1} & \cdots & \mathcal{T}_4 & \mathcal{T}_3 & \mathcal{T}_2 \end{vmatrix}.$$

Proof. Applying Lemma 2 with $c_1 = 1 = \mathcal{T}_2$, $c_2 = 1 - a - b$, $\alpha_n = (1-a)^{n-3}D$ (with $D = (a-1)(a+b-1)+c$) and $\beta_n = \mathcal{T}_{n+1}$. By $c_1^2 - c_2 = a + b = \mathcal{T}_3$, we get the desired result. □

3.1. More determinantal representations of Tribonacci numbers

Theorem 5.

$$\mathcal{T}_{2n+2} = \begin{vmatrix} \mathcal{T}_4 & 1 & 0 & \cdots & & \cdots & 0 \\ U_1 & \mathcal{T}_4 & 1 & & & & \vdots \\ U_2 & U_1 & \mathcal{T}_4 & & & & \\ (ab-b+c)U_2 & U_2 & & & & & \vdots \\ (ab-b+c)^2U_2 & & & & \ddots & 1 & 0 \\ \vdots & & & U_2 & U_1 & \mathcal{T}_4 & 1 \\ (ab-b+c)^{n-3}U_2 & \cdots & (ab-b+c)^2U_2 & (ab-b+c)U_2 & U_2 & U_1 & \mathcal{T}_4 \end{vmatrix},$$

*Since our definition and theirs are off by one number, their result in [9, (3.4)] is not a special case of ours.

where $\mathcal{T}_4 = a^2 + ab + b + c$, $U_1 = (ab + c)(a^2 + ab + c) - a(ab + 2c) = \mathcal{T}_4^2 - \mathcal{T}_6$, *and* $U_2 = (ab + c)^2((a - 1)(a + b - 1) + c)$.

When $a = b = c = 1$ in Theorem 5, we have the following.

Corollary 3.

$$T_{2n+2} = \begin{vmatrix} 4 & 1 & 0 & \cdots & 0 \\ 3 & 4 & 1 & & \vdots \\ 4 & 3 & \ddots & & 0 \\ \vdots & & & 4 & 1 \\ 4 & \cdots & 4 & 3 & 4 \end{vmatrix} .$$

In order to prove Theorem 5, we need the following relation.

Lemma 3. *For* $n \geq 0$,

$$\mathcal{T}_{n+6} = (a^2 + 2b)\mathcal{T}_{n+4} + (2ac - b^2)\mathcal{T}_{n+2} + c^2\mathcal{T}_n .$$

In particular, when $a = b = c = 1$,

$$T_{n+6} = 3T_{n+4} + T_{n+2} + T_n .$$

Proof. By the definition (3.3), we have

$$\begin{aligned} \mathcal{T}_{n+6} &= a\mathcal{T}_{n+5} + b\mathcal{T}_{n+4} + c\mathcal{T}_{n+3} \\ &= a(a\mathcal{T}_{n+4} + b\mathcal{T}_{n+3} + c\mathcal{T}_{n+2}) + b\mathcal{T}_{n+4} + c(a\mathcal{T}_{n+2} + b\mathcal{T}_{n+1} + c\mathcal{T}_n) \\ &= (a^2 + b)b\mathcal{T}_{n+3} + (a\mathcal{T}_{n+3} + b\mathcal{T}_{n+2} + c\mathcal{T}_{n+1}) + (2ac - b^2)\mathcal{T}_{n+2} + c^2\mathcal{T}_n \\ &= (a^2 + 2b)\mathcal{T}_{n+4} + (2ac - b^2)\mathcal{T}_{n+2} + c^2\mathcal{T}_n . \end{aligned}$$

□

Proof of Theorem 5. It is clear that the identity is valid for $n = 1$. For convenience, put $\alpha = ab - b + c$. We shall prove that for $n \geq 2$,

$$\mathcal{T}_{2n+2} = \mathcal{T}_4\mathcal{T}_{2n} - U_1\mathcal{T}_{2n-2} + U_2\sum_{i=0}^{n-2}(-\alpha)^i\mathcal{T}_{2n-2i-4} . \tag{3.4}$$

When $n = 2$, by $U_1 = \mathcal{T}_4^2 - \mathcal{T}_6$, (3.4) is valid.

Assume that the relation (3.4) is valid for some n. By Lemma 3, with the fact that $\mathcal{T}_4 - \alpha = a^2 + 2b$, $\alpha\mathcal{T}_4 - U_1 = 2ac - b^2$, and $U_2 - \alpha U_1 = c^2$, we get

$$\mathcal{T}_{2n+4} = (\mathcal{T}_4 - \alpha)\mathcal{T}_{2n+2} + (\alpha\mathcal{T}_4 - U_1)\mathcal{T}_{2n} + (U_2 - \alpha U_1)\mathcal{T}_{2n-2} . \tag{3.5}$$

By using the assumption of mathematical induction and the relation (3.5), we have

$$U_2\sum_{i=0}^{n-1}(-\alpha)^i\mathcal{T}_{2n-2i-2} = U_2\left(\mathcal{T}_{2n-2} - \alpha\sum_{i=0}^{n-2}(-\alpha)^i\mathcal{T}_{2n-2i-4}\right)$$

$$= U_2\mathcal{T}_{2n-2} - \alpha(\mathcal{T}_{2n+2} - \mathcal{T}_4\mathcal{T}_{2n} + U_1\mathcal{T}_{2n-2})$$
$$= \mathcal{T}_{2n+4} - \mathcal{T}_4\mathcal{T}_{2n+2} + U_1\mathcal{T}_{2n}\,.$$

Hence, (3.4) is also valid for $n+1$.

Now, expanding the determinant in Theorem 5 along the first row repeatedly, we get

$$\mathcal{T}_4\mathcal{T}_{2n} - \begin{vmatrix} U_1 & 1 & 0 & \cdots & & & 0 \\ U_2 & \mathcal{T}_4 & 1 & & & & \\ \alpha U_2 & U_1 & \ddots & & & & \\ \alpha^2 U_2 & U_2 & & \ddots & & & \vdots \\ \vdots & & & & \ddots & 1 & 0 \\ \alpha^{n-4}U_2 & & & & U_1 & \mathcal{T}_4 & 1 \\ \alpha^{n-3}U_2 & \alpha^{n-5}U_2 & \cdots & \alpha U_2 & U_2 & U_1 & \mathcal{T}_4 \end{vmatrix}$$

$$= \mathcal{T}_4\mathcal{T}_{2n} - U_1\mathcal{T}_{2n-2} + \begin{vmatrix} U_2 & 1 & 0 & \cdots & & 0 \\ \alpha U_2 & \mathcal{T}_4 & 1 & & & \\ \alpha^2 U_2 & U_1 & \ddots & & & \vdots \\ \vdots & & & \ddots & 1 & 0 \\ \alpha^{n-4}U_2 & & & U_1 & \mathcal{T}_4 & 1 \\ \alpha^{n-3}U_2 & \alpha^{n-6}U_2 & \cdots & U_2 & U_1 & \mathcal{T}_4 \end{vmatrix}$$

$$= \mathcal{T}_4\mathcal{T}_{2n} - U_1\mathcal{T}_{2n-2} + U_2\mathcal{T}_{2n-4} - \alpha U_2\mathcal{T}_{2n-6} + \alpha^2 U_2\mathcal{T}_{2n-8} - \cdots$$
$$+ (-1)^{n-4}\begin{vmatrix} \alpha^{n-4}U_2 & 1 \\ \alpha^{n-3}U_2 & \mathcal{T}_4 \end{vmatrix}$$
$$= \mathcal{T}_4\mathcal{T}_{2n} - U_1\mathcal{T}_{2n-2} + U_2\sum_{i=0}^{n-2}(-\alpha)^i\mathcal{T}_{2n-2i-4} = \mathcal{T}_{2n+2}\,.$$

We used the relation (3.4) in the final part. □

By using the inversion formula, we have a determinantal expression with Tribonacci entries.

Proposition 5. *For $n \geq 3$,*

$$(ab-b+c)^{n-3}U_2 = \begin{vmatrix} \mathcal{T}_4 & 1 & 0 & \cdots & 0 \\ \mathcal{T}_6 & \mathcal{T}_4 & 1 & & \vdots \\ \mathcal{T}_8 & \mathcal{T}_6 & \ddots & & 0 \\ \vdots & & & \mathcal{T}_4 & 1 \\ \mathcal{T}_{2n+2} & \cdots & \mathcal{T}_8 & \mathcal{T}_6 & \mathcal{T}_4 \end{vmatrix}.$$

When $a = b = c = 1$ in Proposition 5, we have the following. See [9, Theorem 3.2][†].

[†]Note that our definition of Tribonacci numbers and theirs are off by one number.

Corollary 4.

$$\begin{vmatrix} T_4 & 1 & 0 & \cdots & 0 \\ T_6 & T_4 & 1 & & \vdots \\ T_8 & T_6 & \ddots & & 0 \\ \vdots & & & T_4 & 1 \\ T_{2n+2} & \cdots & T_8 & T_6 & T_4 \end{vmatrix} = 4\,.$$

3.2. A determinantal expression of $\mathcal{T}_{2n+1}$

The results mentioned above may seem to be straightforward and can be applied easily to any other cases. It is a completely different problem whether an elegant or clean expression can be obtained.

In [9, Theorem 3.2], the following is presented for $n \ge 3$. However, its generalization has not been found yet.

$$\begin{vmatrix} T_3 & 1 & 0 & \cdots & 0 \\ T_5 & T_3 & 1 & & \vdots \\ T_7 & T_5 & \ddots & & 0 \\ \vdots & & & T_3 & 1 \\ T_{2n+1} & \cdots & T_7 & T_5 & T_3 \end{vmatrix} = 4(-1)^{n-1}\,.$$

By Lemma 2, its inverse relation is given by

$$T_{2n+1} = \begin{vmatrix} 2 & 1 & 0 & \cdots & & 0 \\ -3 & 2 & 1 & & & \vdots \\ 4 & -3 & \ddots & & & \vdots \\ -4 & & & \ddots & & 0 \\ \vdots & & & & 2 & 1 \\ 4(-1)^{n-1} & \cdots & -4 & 4 & -3 & 2 \end{vmatrix}\,.$$

By brute force, we can still obtain the determinantal representation by using the generating function method ([14, 23]), which is presented to find the determinantal representations.

By the recurrence relation in Lemma 3, we have

$$\begin{aligned}\sum_{n=0}^\infty \mathcal{T}_{2n+1}t^n &= \frac{\mathcal{T}_1 + (\mathcal{T}_3 - (a^2+2b)\mathcal{T}_1)t + (\mathcal{T}_5 - (a^2+2b)\mathcal{T}_3 - (2ac-b^2)\mathcal{T}_1)t^2}{1-(a^2+2b)t-(2ac-b^2)t^2-c^2t^3}\\ &= \frac{1-(a^2-a+b)t-c(a-1)t^2}{1-(a^2+2b)t-(2ac-b^2)t^2-c^2t^3}\,.\end{aligned}$$

Hence, we obtain that

$$\left(\sum_{n=0}^\infty \mathcal{T}_{2n+1}t^n\right)^{-1} = 1 - z_1 t + z_2 t^2 - z_3 t^3 + z_4 t^4 - \cdots\,,$$

where

$$z_1 = a + b\,,$$

$$
\begin{aligned}
z_2 &= a^2(a+b-1)+(a+1)c\,,\\
z_3 &= -(a^2(a+b-1)(a^2-a+b)+a(a+2)(a-1)+2ab+c)c)\,,\\
z_4 &= a^2(a+b-1)(a^2-a+b)^2\\
&\qquad + c(a(a+2)(a-1)^2+(2ab+c)(2a^2-a+b-1))\,,\\
&\cdots.
\end{aligned}
$$

Then, by Lemma 1, we get

$$
\mathcal{T}_{2n+1} = c\begin{vmatrix} z_1 & 1 & 0 & \cdots & 0\\ z_2 & a+b & 1 & \ddots & \vdots\\ z_3 & z_2 & & & 0\\ \vdots & & \ddots & a+b & 1\\ \cdots & \cdots & z_3 & z_2 & z_1 \end{vmatrix}.
$$

However, this is not an elegant expression at all different from that of $\mathcal{T}_{2n+2}$.

4. Concluding remarks

The results presented in this paper contribute to the broader field of matrix theory, particularly those associated with specialized number sequences and their determinant representation.

Author contributions

Takao Komatsu: Writing – original draft; Fatih Yilmaz: Writing – review and editing; All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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