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Research article

On the Lucas-Leonardo numbers in complex and dual-complex number systems

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Abstract: This study aimed to introduce the Lucas-Leonardo numbers in 2-dimensional real algebra and 4-dimensional real Clifford algebra, namely, complex and dual-complex Lucas-Leonardo numbers, respectively. In this sense, basic algebraic properties of these numbers were presented as well as some Fibonacci-type identities such as Cassini, Catalan, and d'Ocagne. The generating function and Binet formula were constructed for the complex and dual-complex forms of Lucas-Leonardo numbers. Some relations between these numbers and other well-known integer sequences were proven. Moreover, some formulas related to the sums of the terms of these sequences were established.

Keywords: complex numbers; complex Lucas-Leonardo numbers; dual-complex numbers; dual-complex Lucas-Leonardo numbers
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1. Introduction

The impulse to study complex numbers as a subject in itself first arose in the 16th century with the discovery of algebraic solutions for the roots of cubic and fourth-degree polynomials by Italian mathematicians Tartaglia and Cardano (1501–1576). Even if only real solutions were of interest, it was soon proved that these formulas sometimes required the treatment of square roots of negative numbers. In fact, it was later proved that the use of complex numbers is inevitable when all three roots are real and distinct. Gerolamo Cardano tried to solve the simpler form of the general cubic equation and found an expression $a + \sqrt{-b}$, but had some misgivings about it. After Cardano, Rafael Bombelli (1526–1572) was the first to explicitly consider paradoxical solutions of cubic equations in this regard and the solution of these problems was developed in complex arithmetic. He defined a notation for $\sqrt{-1}$ and called it "piu'di meno". In the 17th century, Leonard Euler used the notation $i = \sqrt{-1}$ and visualized the complex numbers as points in rectangular coordinates. As Paul J. Nahin described in his work, An imaginary tale [1], the unit $\sqrt{-1}$ has long labored under a narrative of "unfathomable mystery." However, this mystery eventually evolved into a profound appreciation for the "power and beauty" of complex functions. Nahin notes that while mathematicians were once perceived as gazing from "high mountains whose summits are lost in the clouds," the study of these systems has brought this "thin air" down to sea-level pressure, making it both accessible and indispensable to modern science. Since then, various people have modified the original definition of the product of complex numbers. In the 19th century, the English geometer Clifford (1862–1930) added still another variant to the complex products and invented a new number system by using the notation $\varepsilon^2 = 0$, $\varepsilon \neq 0$. This number system was called the dual number system. The ordinary, dual number is a particular member of a two-parameter family of complex number systems often called a binary number or generalized complex number, which is a two-component number of the form $z = x + \varepsilon y$, where $(x, y) \in \mathbb{R}^2$ and ε is a nilpotent number, i.e., $\varepsilon^2 = 0$ and $\varepsilon \neq 0$ [2]. Thus, the dual numbers form a 2-dimensional real algebra:

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \left\{ z = x + \varepsilon y \mid (x, y) \in \mathbb{R}^2, \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0 \right\}.$$

Subsequently, Kotelnikov (1895) and Study (1903) generalized the first applications of dual numbers to mechanics (see [3, 4]). Besides mechanics, this nice concept has a lot of applications in many fields of fundamental sciences, such as algebraic geometry, Riemannian geometry, quantum mechanics, astrophysics, kinematics, and quaternionic formulation of motion in the theory of relativity, see [5–7]. On the other hand, Majernik introduced multicomponent number systems in [8]. There are three types of four-component number systems that were constructed by joining the complex, binary, and two-component numbers. In light of Majernik's study, in [9] Messelmi defined the concept of dual-complex numbers and their holomorphic functions.

A dual-complex number w is an ordered pair of complex numbers (z, t) associated with the complex unit 1 and dual unit ε, where ε is a nilpotent number, i.e., $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. A dual-complex number is typically denoted as $w = z + \varepsilon t$ and the set of dual-complex numbers $\mathbb{DC}$ is defined as

$$\mathbb{DC} = \left\{ w = z + \varepsilon t \mid z, t \in \mathbb{C}, \text{ where } \varepsilon^2 = 0, \varepsilon \neq 0, \text{ and } \varepsilon^0 = 1 \right\}.$$

Here, z and t are called the complex and dual parts, respectively, of the dual-complex number w. If $z = x_1 + ix_2$ and $t = y_1 + iy_2$, then the dual-complex number w can be written as

$$w = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2,$$

where $i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $i\varepsilon = \varepsilon i$.

Dual-complex numbers form a commutative ring with characteristic 0. Moreover, the inherited multiplication of these numbers gives dual-complex numbers the structure of 2-dimensional Clifford algebra and 4-dimensional real Clifford algebra. The base elements of the dual-complex numbers satisfy the following commutative multiplication scheme (Table 1):

Table 1. Multiplication scheme of dual-complex numbers.

$\times$	1	i	ε	$i\varepsilon$
1	1	i	ε	$i\varepsilon$
i	i	-1	$i\varepsilon$	$-\varepsilon$
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

On the other hand, the idea of investigating number systems by writing the coefficients as elements of complex numbers, dual numbers, hyperbolic numbers, etc. is a fascinating area for researchers. Consequently, it is natural to study some well-known versions of integer sequences of the above-mentioned type of numbers. In this sense, Horadam considered the complex-type Fibonacci and Lucas numbers and proved some basic properties of these numbers in [10]. Karataş defined the complex Leonardo numbers, see [11]. Additionally, there are many applications for the theory of dual-complex numbers. In 2017, the dual-complex Fibonacci numbers and dual-complex Lucas numbers were defined and some of their algebraic properties were given by Güngör and Azak [12]. Then in [13], Aydın defined the dual-complex k-Fibonacci numbers and extended the algebraic properties of them to dual-complex k-Fibonacci numbers. In 2024, Yılmaz and Saçlı examined the Leonardo sequence with dual-generalized complex coefficients, see [14].

Motivated by the above-mentioned studies, we have defined complex and dual-complex Lucas-Leonardo numbers. Then, Binet's formula, D'Ocagne's, Cassini's, and Catalan's identities, generating functions, and some basic algebraic identities have been obtained for the Lucas-Leonardo sequence in the complex and dual-complex number systems. Furthermore, some characteristic identities, the addition, subtraction, and multiplication operations, and the definition of the j-modulus have been presented for the dual-complex form of the Lucas-Leonardo numbers.

In this sense, the present paper is organized as follows. We begin by recalling some facts about Fibonacci, Lucas, Leonardo, and Lucas-Leonardo sequences. Sections 3 and 4 are devoted to the complex and dual-complex Lucas-Leonardo numbers, respectively. In these sections, complex Lucas-Leonardo and dual-complex Lucas-Leonardo numbers are defined. Their basic algebraic properties, Fibonacci-type identities, and some summation formulas are proven. Binet's formula, negative extensions, and generating functions are presented for the Lucas-Leonardo numbers in complex and dual-complex number systems. Since the Lucas-Leonardo numbers are related to the well-known integer sequences, the most famous identities such as Catalan, Cassini, and d'Ocagne are constructed for complex and dual-complex Lucas-Leonardo numbers. In addition, conjugations which are crucial for the algebraic and geometric properties of dual-complex numbers, operations, and the norm notion are defined for dual-complex Lucas-Leonardo numbers. Moreover, some relations between Lucas-Leonardo, Fibonacci, and Lucas numbers in the dual-complex number system are presented.

2. Background

Some of the sequences that have been extensively studied are Fibonacci and Lucas. Along with these works, Leonardo numbers have a large place in literature. In this section, some basic notations and results will be recalled related to Fibonacci, Lucas, Leonardo, and Lucas-Leonardo numbers, and the complex and dual-complex forms of Fibonacci, Lucas, and Leonardo numbers.

The Fibonacci and Lucas sequences are constructed with the same second-order homogeneous linear recurrence relation but their initial values differ. That is, the Fibonacci sequence satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\,,\; n \geq 2 \text{ with } F_0 = 0,\, F_1 = 1, \tag{2.1}$$

whereas the Lucas sequence satisfies the recurrence relation

$$L_n = L_{n-1} + L_{n-2}\,,\; n \geq 2 \text{ with } L_0 = 2,\, L_1 = 1 \tag{2.2}$$

(see [15]). Another set of numbers related to Fibonacci and Lucas numbers is the Leonardo sequence, which is one of several non-homogeneous extensions of the Fibonacci recurrence relation. These numbers were first studied in 1981 by Dijkstra [16], who used these numbers as an integral part of his smoothsort algorithm [17] and also analyzed them in some detail. Catarino and Borges defined the Leonardo sequence in [18] by the following recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1\ ,\ n \geq 2 \text{ with } Le_0 = Le_1 = 1. \tag{2.3}$$

On the other hand, a companion sequence of the Leonardo sequence, in Lucas-like form, is called the Lucas-Leonardo sequence. Analogously, the Lucas-Leonardo sequence is defined by changing the initial values in the Leonardo sequence and is written by

$$\Re_n = \Re_{n-1} + \Re_{n-2} + 1\ ,\ n \geq 2 \text{ with } \Re_0 = 3, \Re_1 = 1 \tag{2.4}$$

(see [19]). In addition, the nth Lucas-Leonardo number is defined by the following homogeneous recurrence relation for $n \geq 3$:

$$\Re_n = 2\Re_{n-1} - \Re_{n-3}, \tag{2.5}$$

where $\Re_0 = 3$, $\Re_1 = 1$, $\Re_2 = 5$ (see [20]). Moreover, it has been proven by induction that

$$\Re_n = 2L_n - 1. \tag{2.6}$$

The Fibonacci, Lucas, Leonardo, and Lucas-Leonardo numbers verify the following properties (see [15, 18, 20], respectively):

Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \tag{2.7}$$

$$Le_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1, \qquad \Re_n = 2\alpha^n + 2\beta^n - 1, \tag{2.8}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Negative extensions:

$$\begin{aligned} F_{-n} &= (-1)^{n+1} F_n, & L_{-n} &= (-1)^n L_n, \\ Le_{-n} &= (-1)^n (Le_{n-2} + 1) - 1, & \Re_{-n} &= (-1)^n (\Re_n + 1) - 1. \end{aligned} \tag{2.9}$$

Summation formulas:

$$\begin{cases} \sum_{i=0}^{n} F_i = F_{n+2} - 1, \\ \sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1, \\ \sum_{i=0}^{n} F_{2i-1} = F_{2n} + 1, \end{cases} \quad \begin{cases} \sum_{i=0}^{n} L_i = L_{n+2} - 1, \\ \sum_{i=0}^{n} L_{2i} = L_{2n+1} + 1, \\ \sum_{i=0}^{n} L_{2i-1} = L_{2n} - 3, \end{cases} \tag{2.10}$$

$$\begin{cases} \sum_{i=0}^{n} Le_i = Le_{n+2} - (n+2), \\ \sum_{i=0}^{n} Le_{2i} = Le_{2n+1} - n, \\ \sum_{i=0}^{n} Le_{2i+1} = Le_{2n+2} - (n+2), \end{cases} \quad \begin{cases} \sum_{i=0}^{n} \Re_i = \Re_{n+2} - (n+2), \\ \sum_{i=0}^{n} \Re_{2i} = \Re_{2n+1} - (n-2), \\ \sum_{i=0}^{n} \Re_{2i+1} = \Re_{2n+2} - 2 - (n+2). \end{cases} \tag{2.11}$$

Some special relations:

Some relations between the Lucas and Lucas-Leonardo numbers are as indicated below:

$$L_{n+r} + L_{n-r} = \begin{cases} L_n L_r, \ r = 2k, \\ 5F_n F_r, \ r = 2k+1, \end{cases} \tag{2.12}$$

$$L_{n+r} - L_{n-r} = \begin{cases} 5F_n F_r, r = 2k, \\ L_n L_r, r = 2k+1, \end{cases} \tag{2.13}$$

$$\Re_{n+m} + \Re_{n-m} = \begin{cases} \frac{1}{2}(\Re_n + 1)(\Re_m + 1) - 2, \ m = 2k, \\ \frac{1}{10}(\Re_n + 3)(\Re_m + 3) - 2, \ m = 2k+1, \end{cases} \tag{2.14}$$

$$\Re_{n+m} - \Re_{n-m} = \begin{cases} 10F_n F_m, \ m = 2k, \\ 2L_n L_m, \ m = 2k+1, \end{cases} \tag{2.15}$$

$$\Re_{n+m} + (-1)^m \Re_{n-m} = 2L_m L_n - (1 + (-1)^m), \tag{2.16}$$

$$\Re_{n+m} - (-1)^m \Re_{n-m} = 10F_m F_n - (1 - (-1)^m), \tag{2.17}$$

$$\Re_n^2 + \Re_{n+1}^2 = 20F_{2n+1} - 4(F_{n+3} + F_{n+1}) + 2, \tag{2.18}$$

$$\Re_n^2 - \Re_{n+1}^2 = 4L_{n-1}(1 - L_{n+2}), \tag{2.19}$$

$$\Re_n \Re_{n+1} = 4L_n L_{n+1} - 2L_{n+2} + 1, \tag{2.20}$$

$$\Re_n \Re_{n+2} + \Re_{n+1}\Re_{n+3} = 20F_{2n+3} - 10F_{n+3} + 2, \tag{2.21}$$

$$\Re_n \Re_{n+3} - \Re_{n+1}\Re_{n+2} = 20(-1)^n - 2(F_{n+1} + F_{n-1}), \tag{2.22}$$

$$\Re_{n-k}\Re_{n+k} - \Re_n^2 = (-1)^{n+k}(\Re_k + 1)^2 - 16(-1)^n + 2\Re_n - \Re_{n-k} - \Re_{n+k}, \tag{2.23}$$

$$\Re_m \Re_{n+1} - \Re_{m+1}\Re_n = \Re_{m-1} - \Re_{n-1} - 2(-1)^n [\Re_{m-n+1} + \Re_{m-n-1} + 2]. \tag{2.24}$$

Two-dimensional number systems have been considered by many researchers over the last century. One of these systems is the complex number system. In this manner, the complex forms of Fibonacci and Lucas numbers are defined as follows:

$$\begin{aligned} CF_{n+1} &= CF_n + CF_{n-1} \text{ where } CF_n = F_n + iF_{n+1}, \\ CL_{n+1} &= CL_n + CL_{n-1} \text{ where } CL_n = L_n + iL_{n+1}, \end{aligned} \tag{2.25}$$

with $n \geq 0$ (see [10]). In addition, complex Leonardo numbers are defined as

$$CLe_{n+1} = CLe_n + CLe_{n-1} + (1+i) \ \text{ where } CLe_n = Le_n + iLe_{n+1}, \tag{2.26}$$

with $n \geq 0$ (see [11]). Similarly, the complex form of the Francois numbers can defined as

$$C\mathcal{F}_{n+1} = C\mathcal{F}_n + C\mathcal{F}_{n-1} + (1+i) \ \text{ where } C\mathcal{F}_n = \mathcal{F}_n + i\mathcal{F}_{n+1},$$

with $n \geq 0$.

On the other hand, by utilizing complex and dual numbers, it is natural to study Fibonacci and Lucas's versions of dual-complex numbers. In this sense, the dual-complex Fibonacci and Lucas numbers were defined and studied by Güngör and Azak (see [12]). The nth dual-complex Fibonacci and Lucas numbers are of the forms

$$\mathbb{DC}F_n = F_n + iF_{n+1} + \varepsilon F_{n+2} + i\varepsilon F_{n+3} \text{ and } \mathbb{DC}L_n = L_n + iL_{n+1} + \varepsilon L_{n+2} + i\varepsilon L_{n+3} \quad (2.27)$$

and satisfy the recurrence relations

$$\mathbb{DC}F_n = \mathbb{DC}F_{n-1} + \mathbb{DC}F_{n-2} \text{ and } \mathbb{DC}L_n = \mathbb{DC}L_{n-1} + \mathbb{DC}L_{n-2},$$

respectively. The Binet formula for these numbers is given by:

$$\mathbb{DC}F_n = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta} \text{ and } \mathbb{DC}L_n = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n, \quad (2.28)$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2\varepsilon + \alpha^3 i\varepsilon$ and $\overline{\beta} = 1 + \beta i + \beta^2\varepsilon + \beta^3 i\varepsilon$. Some of the algebraic properties of dual-complex Lucas numbers are as follows:

$$\mathbb{DC}L_m\mathbb{DC}L_{n+1} - \mathbb{DC}L_{m+1}\mathbb{DC}L_n = \left[5(-1)^{n+1} F_{m-n}\right](2 + i + 6\varepsilon + 3i\varepsilon), \quad (2.29)$$

$$\mathbb{DC}L_{-n} = (-1)^n\mathbb{DC}L_n - 5F_n(-1)^n(i + \varepsilon + 2i\varepsilon), \quad (2.30)$$

$$\mathbb{DC}L_n^2 - \mathbb{DC}L_{n+k}\mathbb{DC}L_{n-k} = 5(-1)^{n-k+1} F_k^2(2 + i + 6\varepsilon + 3i\varepsilon). \quad (2.31)$$

Then, dual-complex k-Fibonacci numbers were defined and Fibonacci-type identities were obtained by Aydın in [13]. In 2022, the conclusions about dual-complex Fibonacci and Lucas numbers extended to dual-generalized complex numbers (see [21]). Finally, Yılmaz and Saçlı defined the dual-complex Leonardo sequence in [14] as follows:

$$\mathbb{DC}Le_n = Le_n + iLe_{n+1} + \varepsilon Le_{n+2} + i\varepsilon Le_{n+3}.$$

Dual-complex Leonardo numbers satisfy the second-order non-homogeneous relation $\mathbb{DC}Le_n = \mathbb{DC}Le_{n-1} + \mathbb{DC}Le_{n-2} + A$, where $A = 1 + i + \varepsilon + i\varepsilon$. Some of the sum identities of these numbers necessary for this study are given as follows:

$$\sum_{j=0}^{n} \mathbb{DC}F_{k,s} = \frac{1}{k}\left(\mathbb{DC}F_{k,n+1} + \mathbb{DC}F_{k,n} - \mathbb{DC}F_{k,1} - \mathbb{DC}F_{k,0}\right), \quad (2.32)$$

$$\sum_{j=0}^{n} \mathbb{DC}L_j = \mathbb{DC}L_{n+2} - \mathbb{DC}L_1, \quad (2.33)$$

$$\sum_{j=0}^{n} \mathbb{DC}Le_j = \mathbb{DC}Le_{n+2} - (n + 2) + i(n + 3) + \varepsilon(n + 4) + i\varepsilon(n + 5) - (i + 2\varepsilon + 5i\varepsilon). \quad (2.34)$$

3. The complex Lucas-Leonardo sequence

This section is devoted to the construction of the complex Lucas-Leonardo sequence. In this sense, we will constitute Binet's formula, the generating function, summation formulas, and some basic algebraic and well-known properties of integer sequences such as Catalan, Cassini, and d'Ocagne for the complex Lucas-Leonardo sequence and the properties defining the relationship between the complex forms of well-known sequences and the Lucas-Leonardo sequence. The starting point of our analysis is to define complex Lucas-Leonardo numbers.

Definition 3.1. *For $n \geq 1$, the nth complex Lucas-Leonardo number is defined by*

$$C\Re_n = \Re_n + i\Re_{n+1},$$

where $\Re_n$ is the nth Lucas-Leonardo number.

Note that, throughout this paper, we denote the nth complex Lucas-Leonardo number by $C\Re_n$.

On the other hand, a non-homogeneous recurrence relation can be obtained from the recurrence relation of Lucas-Leonardo numbers and the definition of complex Lucas-Leonardo numbers as follows:

$$C\Re_n = \Re_n + i\Re_{n+1} = (\Re_{n-1} + \Re_{n-2} + 1) + i(\Re_n + \Re_{n-1} + 1) = C\Re_{n-1} + C\Re_{n-2} + \epsilon, \tag{3.1}$$

where $n \geq 2$, $\epsilon = 1 + i$, and also with the initial conditions $C\Re_0 = 3 + i$ and $C\Re_1 = 1 + 5i$. Throughout this paper, $(1 + i)$ will be denoted by ϵ.

By considering the recurrence relation $\Re_n = 2\Re_{n-1} - \Re_{n-3}$ and Definition 3.1, we obtain a different recurrence relation for complex Lucas-Leonardo numbers. That is

$$C\Re_n = (2\Re_{n-1} - \Re_{n-3}) + i(2\Re_n - \Re_{n-2}) = 2C\Re_{n-1} - C\Re_{n-3}, \tag{3.2}$$

for $n \geq 2$ with initial conditions $C\Re_0 = 3 + i$ and $C\Re_1 = 1 + 5i$.

According to these definitions, some of the first complex Lucas-Leonardo numbers are

$$3 + i, 1 + 5i, 5 + 7i, 7 + 13i, 13 + 21i, 21 + 35i, 35 + 57i, 57 + 93i, 93 + 151i, ...$$

Furthermore, the complex Lucas-Leonardo numbers for negative indices can be expressed as follows.

Proposition 3.2. *For $n \geq 1$, the negative subscript of the nth complex Lucas-Leonardo number is given as follows:*

$$C\Re_{-n} = (-1)^n (\Re_n - i\Re_{n-1} - \epsilon + 2) - \epsilon,$$

where $\Re_n$ is the nth Lucas-Leonardo number.

Proof. Considering Definition 3.1 and Eq (2.9), we obtain that

$$\begin{aligned} C\Re_n &= \Re_n + i\Re_{n+1} \\ \Rightarrow \quad C\Re_{-n} &= \Re_{-n} + i\Re_{-n+1} \\ &= \left[(-1)^n (\Re_n + 1) - 1\right] + i\left[(-1)^{n-1} (\Re_{n-1} + 1) - 1\right] \\ &= (-1)^n \left[(\Re_n + 1) - i(\Re_{n-1} + 1) - (1 + i)\right] \\ &= (-1)^n (\Re_n - i\Re_{n-1} - \epsilon + 2) - \epsilon. \end{aligned}$$

□

Now, we will give the generating function for complex Lucas-Leonardo numbers.

Theorem 3.3. *The generating function for complex Lucas-Leonardo numbers is*

$$g(t) = \frac{(3 + i) + t(-5 + 3i) + t^2(3 - 3i)}{1 - 2t + t^3}.$$

Proof. From the formal power series representation of the generating function for $\{C\Re_n\}_{n=0}^{\infty}$, we write $g(t) = \sum_{n=0}^{\infty} C\Re_n t^n$. That is,

$$g(t) = C\Re_0 + C\Re_1 t + C\Re_2 t^2 + ... + C\Re_k t^k + ...$$

So,

$$2tg(t) = 2C\Re_0 t + 2C\Re_1 t^2 + 2C\Re_2 t^3 + ... + 2C\Re_k t^{k+1} + ...,$$
$$-t^3 g(t) = -t^3 C\Re_0 - C\Re_1 t^4 - C\Re_2 t^5 - ... - C\Re_k t^{k+3} - ....$$

Considering the above equations,

$$\begin{aligned}\left(1 - 2t + t^3\right) g(t) &= C\Re_0 + t(C\Re_1 - 2C\Re_0) + t^2(C\Re_2 - 2C\Re_1) + \\ &\quad t^3(C\Re_3 - 2C\Re_2 + C\Re_0) + ... + t^{k+1}(C\Re_{k+1} - 2C\Re_k + C\Re_{k-2}) + ...\end{aligned}$$

is written. In this step, by using Eq (3.2) and the initial conditions, we have

$$g(t) = \frac{(3+i) + t(-5+3i) + t^2(3-3i)}{1 - 2t + t^3}.$$

□

Theorem 3.4. *For any integer $n \geq 0$,*

$$C\Re_n = 2CL_n - \epsilon,$$

where CL_n is the nth complex Lucas number.

Proof. Considering Eq (2.6), Relation 2.26, and Definition 3.1, we obtain that

$$\begin{aligned}C\Re_n &= \Re_n + i\Re_{n+1} = (2L_n - 1) + i(2L_{n+1} - 1) \\ &= 2(L_n + iL_{n+1}) - (1+i) = 2CL_n - \epsilon.\end{aligned}$$

□

Now, we will give Binet's formula complex Lucas-Leonardo numbers.

Theorem 3.5. *For any integer $n \geq 0$, Binet's formula for complex Lucas-Leonardo numbers $C\Re_n$ is*

$$C\Re_n = 2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon,$$

where α and β are roots of the characteristic equations of Lucas-Leonardo numbers, $\underline{\alpha} = 1 + i\alpha$ and $\underline{\beta} = 1 + i\beta$.

Proof. It is known that Binet's formula of Lucas-Leonardo numbers is

$$\Re_n = 2\alpha^n + 2\beta^n - 1,$$

where $\Re_n$ is the nth Lucas-Leonardo number in Eq (2.9). Considering Definition 3.1, the result is obtained as follows:

$$\begin{aligned}C\Re_n &= \Re_n + i\Re_{n+1} = (2\alpha^n + 2\beta^n - 1) + i\left(2\alpha^{n+1} + 2\beta^{n+1} - 1\right) \\ &= 2\alpha^n(1 + i\alpha) + 2\beta^n(1 + i\beta) - (1+i) = 2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon.\end{aligned}$$

□

The next propositions will establish the complex Lucas-Leonardo sequence in terms of the known integer sequences in complex number systems. Some basic connections between the complex Lucas-Leonardo and complex Fibonacci, complex Lucas, complex Leonardo, and complex Francois sequences are as follows.

Proposition 3.6. *The complex Lucas-Leonardo numbers satisfy the following identities:*

(i) $C\Re_n = 2CF_n + 4CF_{n-1} - \epsilon$,
(ii) $C\Re_n = C\mathcal{F}_n + CF_{n-1}$,
(iii) $C\Re_n = 2CLe_n - CLe_{n-1}$,
(iv) $C\Re_n = 2C\mathcal{F}_n - CLe_n$,
(v) $C\Re_n = CL_n - CF_n + CLe_n$,
(vi) $C\Re_n = CLe_n + C\mathcal{F}_n - CF_{n+2} + \epsilon$,

where $n \in \mathbb{Z}$ and CF_n is the nth complex Fibonacci number.

Proof. To prove these identities, some known relations for Fibonacci, Lucas, Leonardo, and the complex forms of these sequences will be used.

i. Considering Theorem 3.4, the fact $F_{n+1} + F_{n-1} = L_n$, and Eq (2.25),

$$\begin{aligned} C\Re_n &= 2CL_n - \epsilon = 2\left(L_n + iL_{n+1}\right) - \epsilon \\ &= 2\left[\left(F_{n+1} + F_{n-1}\right) + i\left(F_{n+2} + F_n\right)\right] - \epsilon \\ &= 2\left(F_{n-1} + iF_n\right) + 2\left(F_{n+1} + iF_{n+2}\right) - \epsilon \\ &= 2CF_{n-1} + 2CF_{n+1} - \epsilon \\ &= 2CF_n + 4CF_{n-1} - \epsilon \end{aligned}$$

is obtained. So the result follows.

ii. The conclusion can be deduced from the recurrence relation given with Definition 3.1 and the equation $\Re_n = \mathcal{F}_n + F_{n-1}$.

iii. The result can be readily observed by using Definition 3.1 and the fact that $\Re_n = 2Le_n - Le_{n-1}$.

iv. Considering Definition 3.1 and the fact that $\Re_n = 2\mathcal{F}_n - Le_n$, the claim is clear.

v. From Definition 3.1 and the fact that $\Re_n = L_n - F_n + Le_n$, we have

$$\begin{aligned} C\Re_n &= \Re_n + i\Re_{n+1} \\ &= \left(L_n - F_n + Le_n\right) + i\left(L_{n+1} - F_{n+1} + Le_{n+1}\right) \\ &= \left(L_n + iL_{n+1}\right) - \left(F_n + iF_{n+1}\right) + \left(Le_n + iLe_{n+1}\right) \\ &= CL_n - CF_n + CLe_n. \end{aligned}$$

vi. The result can be derived from Definition 3.1 and the fact that $\Re_n = Le_n + \mathcal{F}_n - F_{n+2} + 1$ as follows:

$$\begin{aligned} C\Re_n &= \Re_n + i\Re_{n+1} \\ &= \left(Le_n + \mathcal{F}_n - F_{n+2} + 1\right) + i\left(Le_{n+1} + \mathcal{F}_{n+1} - F_{n+3} + 1\right) \\ &= \left(Le_n + iLe_{n+1}\right) + \left(\mathcal{F}_n + i\mathcal{F}_{n+1}\right) - \left(F_{n+2} + F_{n+3}\right) + (1 + i) \\ &= CLe_n + C\mathcal{F}_n - CF_{n+2} + \epsilon. \end{aligned}$$

□

These properties can be extended by utilizing the relations between sequences that were previously introduced.

Some of the most famous identities for well-known integer sequences (Fibonacci, Lucas, Leonardo, etc.) are Catalan's, Cassini's, and d'Ocagne's identities. Since Lucas-Leonardo numbers are related to these sequences, these identities also exist for Lucas-Leonardo numbers. So, we can obtain these identities for complex Lucas-Leonardo numbers as follows.

Theorem 3.7. *(Catalan's Identity) For positive integers n and k, with $n \geq k$, the following identity holds:*

$$C\Re_n^2 - C\Re_{n-k}C\Re_{n+k} = (8+4i)(-1)^n\left[2-(-1)^k L_{2k}\right] + \epsilon\left[C\Re_{n-k} + C\Re_{n+k} - 2C\Re_n\right],$$

where L_n and $\Re_n$ are the nth Lucas and Lucas-Leonardo numbers, respectively.

Proof. Considering the definitions $\underline{\alpha} = (1+i\alpha), \underline{\beta} = (1+i\beta)$, and the fact that $\alpha+\beta = 1$, we have

$$\underline{\alpha}\underline{\beta} = (1+i\alpha)(1+i\beta) = 1 + i\alpha + i\beta - \alpha\beta = 2+i,$$

and from the Binet formula for Lucas numbers, we get

$$\alpha^{-k}\beta^k + \alpha^k\beta^{-k} = \left(\alpha^{-1}\right)^k\beta^k + \alpha^k\left(\beta^{-1}\right)^k = (-\beta)^k\beta^k + \alpha^k(-\alpha)^k = (-1)^k\left(\beta^{2k}+\alpha^{2k}\right) = (-1)^k L_{2k}.$$

By employing the results obtained above along with the Binet formula for complex Lucas-Leonardo numbers, and through some algebraic manipulations,

$$\begin{aligned} LHS &= \left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right)\left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right) - \left(2\underline{\alpha}\alpha^{n-k} + 2\underline{\beta}\beta^{n-k} - \epsilon\right)\left(2\underline{\alpha}\alpha^{n+k} + 2\underline{\beta}\beta^{n+k} - \epsilon\right) \\ &= 4\underline{\alpha}\underline{\beta}(-1)^n\left[2-\left(\alpha^{-k}\beta^k + \alpha^k\beta^{-k}\right)\right] \\ &\quad + \epsilon\left[\left(2\underline{\alpha}\alpha^{n-k} + 2\underline{\beta}\beta^{n-k} - \epsilon\right) + \left(2\underline{\alpha}\alpha^{n+k} + 2\underline{\beta}\beta^{n+k} - \epsilon\right) - \left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right) - \left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right)\right] \\ &= 4(2+i)(-1)^n\left[2-(-1)^k L_{2k}\right] + \epsilon\left[C\Re_{n-k} + C\Re_{n+k} - 2C\Re_n\right] \\ &= (8+4i)(-1)^n\left[2-(-1)^k L_{2k}\right] + \epsilon\left[C\Re_{n-k} + C\Re_{n+k} - 2C\Re_n\right] \end{aligned}$$

is obtained. So the result follows.

□

Taking $k = 1$ in Proposition 3.7, we get Cassini's identity for complex Lucas-Leonardo numbers with the following corollary.

Corollary 3.8. *(Cassini's Identity) For positive integers $n \geq 2$, the following identity holds:*

$$C\Re_n^2 - C\Re_{n-1}C\Re_{n+1} = (40+20i)(-1)^n + \epsilon\left[C\Re_{n-k} + C\Re_{n+k} - 2C\Re_n\right],$$

where $\Re_n$ is the nth Lucas-Leonardo number.

Another well-known identity for special integer sequences is d'Ocagne's identity. Now we will prove d'Ocagne's identity for complex Lucas-Leonardo numbers.

Theorem 3.9. *(d'Ocagne's Identity) For positive integers m and n, with $m > n$ and $n \geq 1$, the following identity holds:*

$$C\Re_m C\Re_{n+1} - C\Re_m C\Re_n = (40+20i)(-1)^{n+1} F_{m-n} + \epsilon\left[C\Re_n + C\Re_{m+1} - C\Re_{n+1} - C\Re_m\right],$$

where F_n and $\Re_n$ are the nth Fibonacci and Lucas-Leonardo numbers, respectively.

Proof. From the Binet formula for complex Lucas-Leonardo numbers and some algebra, we find that:

$$\begin{aligned}
LHS &= \left(2\underline{\alpha}\alpha^m + 2\underline{\beta}\beta^m - \epsilon\right)\left(2\underline{\alpha}\alpha^{n+1} + 2\underline{\beta}\beta^{n+1} - \epsilon\right) - \left(2\underline{\alpha}\alpha^m + 2\underline{\beta}\beta^m - \epsilon\right)\left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right) \\
&= 4\underline{\alpha\beta}\left(\alpha^m\beta^{n+1} - \alpha^{m+1}\beta^n + \alpha^{n+1}\beta^m - \alpha^n\beta^{m+1}\right) \\
&\quad + \epsilon\left[\left(2\underline{\alpha}\alpha^n + 2\underline{\beta}\beta^n - \epsilon\right) + \left(2\underline{\alpha}\alpha^{m+1} + 2\underline{\beta}\beta^{m+1} - \epsilon\right) - \left(2\underline{\alpha}\alpha^{n+1} + 2\underline{\beta}\beta^{n+1} - \epsilon\right) - \left(2\underline{\alpha}\alpha^m + 2\underline{\beta}\beta^m - \epsilon\right)\right] \\
&= (8+4i)\left[\alpha^m\beta^n(\beta-\alpha) + \alpha^n\beta^m(\alpha-\beta)\right] + \epsilon\left[C\Re_n + C\Re_{m+1} - C\Re_{n+1} - C\Re_m\right] \\
&= (8+4i)\left[\sqrt{5}\left(\alpha^n\beta^m - \alpha^m\beta^n\right)\right] + \epsilon\left[C\Re_n + C\Re_{m+1} - C\Re_{n+1} - C\Re_m\right] \\
&= (8+4i)\left[\sqrt{5}\left(\alpha\beta\right)^n\left(\beta^{m-n} - \alpha^{m-n}\right)\right] + \epsilon\left[C\Re_n + C\Re_{m+1} - C\Re_{n+1} - C\Re_m\right] \\
&= (8+4i)\left[-5(-1)^n F_{m-n}\right] + \epsilon\left[C\Re_n + C\Re_{m+1} - C\Re_{n+1} - C\Re_m\right]
\end{aligned}$$

is obtained for the left-hand side and the result follows. In this derivation, besides the Binet formula for complex Lucas-Leonardo numbers, the following steps are performed: the third equality is obtained by substituting $\underline{\alpha\beta} = 2 + i$, the fourth by using $\alpha - \beta = \sqrt{5}$, and the fifth by employing both $\alpha\beta = -1$ and the Binet formula for Fibonacci numbers. □

Now we will give some results concerning sums of terms of the complex Lucas-Leonardo sequence by using some sums of Lucas-Leonardo numbers.

Theorem 3.10. *For $n \geq 0$, summation formulas of complex Lucas-Leonardo numbers are*

$$\begin{aligned}
\sum_{k=0}^{n} C\Re_k &= C\Re_{n+2} - n\epsilon - (2+6i), \\
\sum_{k=0}^{n} C\Re_{2k} &= C\Re_{2n+1} - n\epsilon + (2-4i), \\
\sum_{k=0}^{n} C\Re_{2k+1} &= C\Re_{2n+2} - n\epsilon - (4+2i).
\end{aligned}$$

Also, for $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^{k-1} C\Re_k = \begin{cases} C\Re_{n-1} + 3\epsilon - 8, & n \text{ is odd}, \\ -C\Re_{n-1} + 2\epsilon - 8, & n \text{ is even}. \end{cases}$$

Proof. For the first summation formula, using the recurrence relation of complex Lucas-Leonardo numbers, we have

$$\sum_{k=0}^{n} C\Re_k = \sum_{k=0}^{n} \Re_k + i\sum_{k=0}^{n} \Re_{k+1}.$$

If we consider the first summation formula given with Eq (2.11) for Lucas-Leonardo numbers, the result follows:

$$\begin{aligned}\sum_{k=0}^{n} \Re_k + i \sum_{k=0}^{n} \Re_{k+1} &= [\Re_{n+2} - (n+2)] + i\,[\Re_{n+3} - (n+6)] \\ &= C\Re_{n+2} - (n+2) - i\,(n+6) = C\Re_{n+2} - n\epsilon - (2+6i)\,.\end{aligned}$$

For the second one, from Definition 3.1 and the second and third summation formulas given in Eq (2.11) for Lucas-Leonardo numbers, the proof can be seen as follows:

$$\begin{aligned}\sum_{k=0}^{n} C\Re_{2k} &= \sum_{k=0}^{n} \Re_{2k} + i \sum_{k=0}^{n} \Re_{2k+1} \\ &= [\Re_{2n+1} - (n-2)] + i\,[\Re_{2n+2} - (n+4)] = C\Re_{2n+1} - n\epsilon + (2-4i)\,.\end{aligned}$$

The third one can be shown similarly to the other items, considering Definition 3.1 and the suitable summation formulas given with Eq (2.11) for Lucas-Leonardo numbers.

Finally, we will prove the last summation formula. Using Definitions 2.6 and 3.1, we have

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{k-1}\, C\Re_k &= \sum_{k=0}^{n} (-1)^{k-1}\, \Re_k + i \sum_{k=0}^{n} (-1)^{k-1}\, \Re_{k+1} \\ &= 2 \sum_{k=0}^{n} (-1)^{k-1}\, L_k + (1+i) \sum_{k=0}^{n} (-1)^{k} + 2i \sum_{k=0}^{n} (-1)^{k-1}\, L_{k+1}.\end{aligned}$$

Herein, it should be taken into account whether n is an odd or even number:

- n is an odd number.
  If we consider the second and third relations of (2.10) related to Lucas numbers and Definition 2.4,

$$\begin{aligned}& 2 \sum_{k=0}^{n} (-1)^{k-1}\, L_k + (1+i) \sum_{k=0}^{n} (-1)^{k} + 2i \sum_{k=0}^{n} (-1)^{k-1}\, L_{k+1} \\ = \;& 2\,[-(L_0 + L_n - 1) + (L_{n+1} - 2)] + (1+i)\,0 + 2i\,[-(L_{n+1} - 2) + (L_{n+2} - 1)] \\ = \;& [-(2L_n - 1) - 3 + (2L_{n+1} - 1) - 3] + [-i\,(2L_{n+1} - 1) + 3i + i\,(2L_{n+2} - 1) - i] \\ = \;& (-\Re_n + \Re_{n+1} - 6) + (-i\Re_{n+1} + i\Re_{n+2} + 2i) \\ = \;& -(\Re_n + i\Re_{n+1}) + (\Re_{n+1} + i\Re_{n+2}) - 6 + 2i \\ = \;& -C\Re_n + C\Re_{n+1} - 6 + 2i\end{aligned}$$

  is obtained. In the last equation, by using Eq (3.1), the result follows.

- n is an even number.
  Considering the same facts used in the case of n being an odd number, we get

$$\begin{aligned}& 2 \sum_{k=0}^{n} (-1)^{k-1}\, L_k + (1+i) \sum_{k=0}^{n} (-1)^{k} + 2i \sum_{k=0}^{n} (-1)^{k-1}\, L_{k+1} \\ = \;& 2\,[-(L_0 + L_{n+1} - 1) + (L_n - 2)] + (1+i)\,1 + 2i\,[-(L_{n+2} - 2) + (L_{n+1} - 1)]\end{aligned}$$

$$
\begin{aligned}
&= [-(2L_{n+1}-1)-3+(2L_n-1)-3]+[-i(2L_{n+2}-1)+3i+i(2L_{n+1}-1)-i] \\
&= (-\Re_{n+1}+\Re_n-6)+(-i\Re_{n+2}+i\Re_{n+1}+2i) \\
&= -(\Re_{n+1}+i\Re_{n+2})+(\Re_n+i\Re_{n+1})-5+3i \\
&= -C\Re_{n+1}+C\Re_n-5+3i \\
&= -(C\Re_n+C\Re_{n-1}+\epsilon)+C\Re_n-5+3i \\
&= -C\Re_{n-1}+2\epsilon-8.
\end{aligned}
$$

So, the proof is complete. □

Theorem 3.11. *For $n \geq 0$, the following summation formulas hold true:*

(i) $\sum_{k=0}^{n}(CF_k+C\Re_k)=CF_{n+2}+C\Re_{n+2}-(n+4)\epsilon-2i$,

(ii) $\sum_{k=0}^{n}(CL_k+C\Re_k)=CL_{n+2}+C\Re_{n+2}-(n+3)\epsilon-6i$,

(iii) $\sum_{k=0}^{n}(CLe_k+C\Re_k)=CLe_{n+2}+C\Re_{n+2}-2(n+2)\epsilon-6i$,

where CF_k, CL_k, CLe_k, and $C\Re_k$ are the nth complex Fibonacci, complex Lucas, complex Leonardo, and complex Lucas-Leonardo numbers, respectively.

Proof. To prove the first item, we will use the definition of complex Fibonacci numbers given in Eq (2.25), Definition 3.1, and summation formulas in Eqs (2.10) and (2.11), for Fibonacci and Lucas-Leonardo numbers, respectively, in the following way:

$$
\begin{aligned}
\sum_{k=0}^{n}(CF_k+C\Re_k) &= \sum_{k=0}^{n}(F_k+iF_{k+1})+\sum_{k=0}^{n}(\Re_k+i\Re_{k+1}) \\
&= (F_{n+2}-1)+i(F_{n+3}-1)+[\Re_{n+2}-(n+2)]+i[\Re_{n+3}-(n+2)-4] \\
&= (F_{n+2}+iF_{n+3})-(1+i)+[\Re_{n+2}+i\Re_{n+3}-(n+2)-i(n+2)-4i] \\
&= CF_{n+2}+C\Re_{n+2}-(n+3)\epsilon-4i.
\end{aligned}
$$

The other identities can be proven using similar reasoning. □

Proposition 3.12. *For $n \geq 1$, the following identity holds:*

$$C\Re_{n+1}+C\Re_{n-1}=10CF_n-2\epsilon,$$

where CF_n is the nth complex Fibonacci number.

Proof. Using Theorem 3.4, Relation 2.25 for complex Lucas numbers, and Eq (2.12) for $r=1$,

$$
\begin{aligned}
C\Re_{n+1}+C\Re_{n-1} &= (2CL_{n+1}-\epsilon)+(2CL_{n-1}-\epsilon) \\
&= 2[(L_{n+1}+L_{n-1})+i(L_{n+2}+L_n)]-2\epsilon \\
&= 2(5F_n+i5F_{n+1})-2\epsilon=10CF_n-2\epsilon
\end{aligned}
$$

is obtained. So, the result is obvious. □

Proposition 3.13. *For positive integers n and m, with $n \geq m$, the following identites are true:*

$$C\Re_{n+m} + (-1)^m C\Re_{n-m} = 2L_m CL_n - \epsilon (1 + (-1)^m), \tag{3.3}$$

$$C\Re_{n+m} - (-1)^m C\Re_{n-m} = 10F_m CF_n - \epsilon (1 - (-1)^m), \tag{3.4}$$

where F_n is the nth Fibonacci number, L_n is the nth Lucas number, and CF_n, CL_n, and $C\Re_n$ are the nth complex Fibonacci, complex Lucas, and complex Lucas-Leonardo numbers, respectively.

Proof. For the proof of (3.3), using the recurrence relation given with Definition 3.1, and Eqs (2.25) and (2.16) on the left-hand side, we obtain

$$\begin{aligned} LHS &= [\Re_{n+m} + i\Re_{n+m+1}] + (-1)^m [\Re_{n-m} + i\Re_{n-m+1}] \\ &= [\Re_{n+m} + (-1)^m \Re_{n-m}] + i [\Re_{n+m+1} + (-1)^m \Re_{n-m+1}] \\ &= [2L_m L_n - (1 + (-1)^m)] + i [2L_m L_{n+1} - (1 + (-1)^m)] \\ &= [2L_m L_n + i2L_m L_{n+1}] - (1 + (-1)^m) [1 + i] \\ &= 2L_m CL_n - \epsilon (1 + (-1)^m). \end{aligned}$$

The proof of (3.4) can be seen similarly. □

4. Dual-complex Lucas-Leonardo numbers

The focus of this section is to define the Lucas-Leonardo numbers in the dual complex number system and to conduct a subsequent investigation of their properties. Following the presentation of dual-complex Lucas-Leonardo numbers and their alternative definitions, the definitions of their j-modules and conjugates will be provided. This section also presents several new identities for dual-complex Lucas-Leonardo numbers, which are established by utilizing their classical Lucas-Leonardo counterparts.

Definition 4.1. *The dual-complex Lucas-Leonardo numbers are defined by using the basis $\{1, i, \varepsilon, i\varepsilon\}$, where $\Re_n$ is the nth Lucas-Leonardo number and*

$$\begin{aligned} i &\rightarrow \text{ imaginary unit}, \\ \varepsilon &\rightarrow \text{ pure dual unit}, \\ i\varepsilon &\rightarrow \text{ imaginary dual unit}, \end{aligned}$$

satisfy the conditions $i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, and $i\varepsilon = \varepsilon i$ as follows:

$$\mathbb{DC}\Re_n = (\Re_n + i\Re_{n+1}) + \varepsilon (\Re_{n+2} + i\Re_{n+3}) = \Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3}.$$

In here, $\Re_n + i\Re_{n+1}$ and $\Re_{n+2} + i\Re_{n+3}$ are, respectively, the complex part and the dual part of the $\mathbb{DC}\Re_n$ dual-complex Lucas-Leonardo number. Besides, the real part of the dual-complex Lucas-Leonardo number $\mathbb{DC}\Re_n$ is *real* $(\mathbb{DC}\Re_n) = \Re_n$.

Lemma 4.2. *Let $\mathbb{DC}\Re_n$ be the nth dual-complex Lucas-Leonardo number. Then, for $n \geq 2$, the recurrence relation for dual-complex Lucas-Leonardo numbers is*

$$\mathbb{DC}\Re_n = \mathbb{DC}\Re_{n-1} + \mathbb{DC}\Re_{n-2} + (1 + i + \varepsilon + i\varepsilon),$$

with the initial conditions $\mathbb{DC}\Re_0 = 3 + i + 5\varepsilon + 7i\varepsilon$ *and* $\mathbb{DC}\Re_1 = 1 + 5i + 7\varepsilon + 13i\varepsilon$.

Proof. From Definition 4.1 and Eq (2.4),

$$\begin{aligned}
\mathbb{DC}\Re_n &= \Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3} \\
&= (\Re_{n-1} + \Re_{n-2} + 1) + i(\Re_n + \Re_{n-1} + 1) \\
&\quad + \varepsilon(\Re_{n+1} + \Re_n + 1) + i\varepsilon(\Re_{n+2} + \Re_{n+1} + 1) \\
&= (\Re_{n-1} + i\Re_n + \varepsilon\Re_{n+1} + i\varepsilon\Re_{n+2}) \\
&\quad + (\Re_{n-2} + i\Re_{n-1} + \varepsilon\Re_n + i\varepsilon\Re_{n+1}) + (1 + i + \varepsilon + i\varepsilon) \\
&= \mathbb{DC}\Re_{n-1} + \mathbb{DC}\Re_{n-2} + (1 + i + \varepsilon + i\varepsilon)
\end{aligned}$$

is obtained. □

Additionally, taking into account Definitions 3.1 and 4.1, we have the following recurrence relation for dual-complex Lucas-Leonardo numbers in terms of complex Lucas-Leonardo numbers as follows:

$$\mathbb{DC}\Re_n = C\Re_n + \varepsilon C\Re_{n+2}, \tag{4.1}$$

where $n \geq 0$.

Now, we will present a homogeneous recurrence relation for dual-complex Lucas-Leonardo numbers with the following lemma.

Lemma 4.3. *Let $\mathbb{DC}\Re_n$ be the nth dual-complex Lucas-Leonardo number. Then, for $n \geq 3$,*

$$\begin{cases} \mathbb{DC}\Re_n = 2\mathbb{DC}\Re_{n-1} - \mathbb{DC}\Re_{n-3}, \\ \mathbb{DC}\Re_0 = 3 + i + 5\varepsilon + 7i\varepsilon,\ \mathbb{DC}\Re_1 = 1 + 5i + 7\varepsilon + 13i\varepsilon,\ \mathbb{DC}\Re_2 = 5 + 7i + 13\varepsilon + 21i\varepsilon. \end{cases}$$

Proof. Using Definition 4.1 and Eq (2.5), we have

$$\begin{aligned}
\mathbb{DC}\Re_n &= \Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3} \\
&= 2(\Re_{n-1} + i\Re_n + \varepsilon\Re_{n+1} + i\varepsilon\Re_{n+2}) - (\Re_{n-3} + i\Re_{n-2} + \varepsilon\Re_{n-1} + i\varepsilon\Re_n) \\
&= 2\mathbb{DC}\Re_{n-1} - \mathbb{DC}\Re_{n-3}.
\end{aligned}$$

□

According to these definitions, the first few terms of dual-complex Lucas-Leonardo numbers are

$3 + i + 5\varepsilon + 7i\varepsilon,\ 1 + 5i + 7\varepsilon + 13i\varepsilon,\ 5 + 7i + 13\varepsilon + 21i\varepsilon,\ 7 + 13i + 21\varepsilon + 35i\varepsilon,\ 13 + 21i + 35\varepsilon + 57i\varepsilon, \ldots.$

On the other hand, by considering the operations that were defined for dual generalized complex numbers:

- The equality of any two dual-complex Lucas-Leonardo numbers is given by

$$\mathbb{DC}\Re_n = \mathbb{DC}\Re_m \Leftrightarrow n = m.$$

- The addition and subtraction of any two dual-complex Lucas-Leonardo numbers are given by

$$\begin{aligned}\mathbb{DC}\Re_n \pm \mathbb{DC}\Re_m &= (\Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3}) \\ &\quad \pm (\Re_m + i\Re_{m+1} + \varepsilon\Re_{m+2} + i\varepsilon\Re_{m+3}) \\ &= (\Re_n \pm \Re_m) + i(\Re_{n+1} \pm \Re_{m+1}) \\ &\quad + \varepsilon(\Re_{n+2} \pm \Re_{m+2}) + i\varepsilon(\Re_{n+3} \pm \Re_{m+3}).\end{aligned}$$

- The multiplication of any dual-complex Lucas-Leonardo number by a real λ scalar is given by

$$\lambda\mathbb{DC}\Re_n = \lambda\Re_n + i(\lambda\Re_{n+1}) + \varepsilon(\lambda\Re_{n+2}) + i\varepsilon(\lambda\Re_{n+3}).$$

- The multiplication of any two dual-complex Lucas-Leonardo numbers is given

$$\begin{aligned}\mathbb{DC}\Re_n \times \mathbb{DC}\Re_m &= (\Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3}) \\ &\quad \times (\Re_m + i\Re_{m+1} + \varepsilon\Re_{m+2} + i\varepsilon\Re_{m+3}) \\ &= (\Re_n\Re_m - \Re_{n+1}\Re_{m+1}) + i(\Re_n\Re_{m+1} + \Re_{n+1}\Re_m) \\ &\quad + \varepsilon(\Re_n\Re_{m+2} - \Re_{n+1}\Re_{m+3} + \Re_{n+2}\Re_m - \Re_{n+3}\Re_{m+1}) \\ &\quad + i\varepsilon(\Re_n\Re_{m+3} + \Re_{n+1}\Re_{m+2} + \Re_{n+2}\Re_{m+1} + \Re_{n+3}\Re_m) \\ &= \mathbb{DC}\Re_m \times \mathbb{DC}\Re_n.\end{aligned}$$

Complex and dual conjugations have an important role in the algebraic and geometric properties of dual-complex numbers. The conjugation of dual-complex Lucas-Leonardo numbers can be defined in five different ways since five kinds of conjugations were defined for dual-complex numbers (see [9]):

Complex conjugation. $\mathbb{DC}\Re_n^{\dagger_1} = (\Re_n - i\Re_{n+1}) + \varepsilon(\Re_{n+2} - i\Re_{n+3})$.

Dual conjugation. $\mathbb{DC}\Re_n^{\dagger_2} = (\Re_n + i\Re_{n+1}) - \varepsilon(\Re_{n+2} + i\Re_{n+3})$.

Coupled conjugation. $\mathbb{DC}\Re_n^{\dagger_3} = (\Re_n - i\Re_{n+1}) - \varepsilon(\Re_{n+2} - i\Re_{n+3})$.

Dual-complex conjugation. $\mathbb{DC}\Re_n^{\dagger_4} = (\Re_n - i\Re_{n+1})\left(1 - \varepsilon\frac{\Re_{n+2}+i\Re_{n+3}}{\Re_n+i\Re_{n+1}}\right)$.

Anti-dual conjugation. $\mathbb{DC}\Re_n^{\dagger_5} = (\Re_{n+2} + i\Re_{n+3}) - \varepsilon(\Re_n + i\Re_{n+1})$.

In the following proposition, some properties related to the conjugations of dual-complex Lucas-Leonardo numbers will be given.

Proposition 4.4. *Let $\mathbb{DC}\Re_n$ and $\Re_n$ be, respectively, the nth dual-complex and complex Lucas-Leonardo numbers. Then the following properties hold:*

(i) $\mathbb{DC}\Re_n + \mathbb{DC}\Re_n^{\dagger_1} = 2(\Re_n + \varepsilon\Re_{n+2}) \in \mathbb{D}\Re$,
(ii) $\mathbb{DC}\Re_n \times \mathbb{DC}\Re_n^{\dagger_1} = \Re_n^2 + \Re_{n+1}^2 + 2\varepsilon(\Re_n\Re_{n+2} + \Re_{n+1}\Re_{n+3}) \in \mathbb{D}\Re$,

(iii) $\mathbb{DC}\mathfrak{R}_n + \mathbb{DC}\mathfrak{R}_n^{\dagger_2} = 2(\mathfrak{R}_n + i\mathfrak{R}_{n+1}) \in \mathbb{C}\mathfrak{R}$,
(iv) $\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_2} = \mathfrak{R}_n^2 - \mathfrak{R}_{n+1}^2 + 2i(\mathfrak{R}_n\mathfrak{R}_{n+1}) \in \mathbb{C}\mathfrak{R}$,
(v) $\mathbb{DC}\mathfrak{R}_n + \mathbb{DC}\mathfrak{R}_n^{\dagger_3} = 2(\mathfrak{R}_n + i\varepsilon\mathfrak{R}_{n+3}) \in \mathbb{DC}\mathfrak{R}$,
(vi) $\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_3} = \mathfrak{R}_n^2 + \mathfrak{R}_{n+1}^2 + 2i\varepsilon(\mathfrak{R}_n\mathfrak{R}_{n+3} - \mathfrak{R}_{n+1}\mathfrak{R}_{n+2}) \in \mathbb{DC}\mathfrak{R}$,
(vii) $(\mathfrak{R}_n + i\mathfrak{R}_{n+1}) \times \mathbb{DC}\mathfrak{R}_n^{\dagger_4} = \mathfrak{R}_n^2 + \mathfrak{R}_{n+1}^2 - \varepsilon(\mathfrak{R}_n\mathfrak{R}_{n+2} + \mathfrak{R}_{n+1}\mathfrak{R}_{n+3})$
$+ i\varepsilon(\mathfrak{R}_{n+1}\mathfrak{R}_{n+2} - \mathfrak{R}_n\mathfrak{R}_{n+3}) \in \mathbb{DC}\mathfrak{R}$,
(viii) $\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_4} = \mathfrak{R}_n^2 + \mathfrak{R}_{n+1}^2 \in \mathbb{R}$,
(ix) $\mathbb{DC}\mathfrak{R}_n - \varepsilon\mathbb{DC}\mathfrak{R}_n^{\dagger_5} = \mathfrak{R}_n + i\mathfrak{R}_{n+1} \in \mathbb{C}\mathfrak{R}$,
(x) $\varepsilon\mathbb{DC}\mathfrak{R}_n + \mathbb{DC}\mathfrak{R}_n^{\dagger_5} = \mathfrak{R}_{n+2} + i\mathfrak{R}_{n+3} \in \mathbb{C}\mathfrak{R}$,
(xi) $\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_5} = (\mathfrak{R}_n\mathfrak{R}_{n+2} - \mathfrak{R}_{n+1}\mathfrak{R}_{n+3}) + i(\mathfrak{R}_n\mathfrak{R}_{n+3} + i\mathfrak{R}_{n+1}\mathfrak{R}_{n+2}) + \varepsilon\left(\mathfrak{R}_{n+1}^2 + \mathfrak{R}_{n+2}^2 - \mathfrak{R}_n^2 - \mathfrak{R}_{n+3}^2\right) +$
$2i\varepsilon(\mathfrak{R}_{n+2}\mathfrak{R}_{n+3} - \mathfrak{R}_n\mathfrak{R}_{n+1}) \in \mathbb{DC}\mathfrak{R}$,

where $\mathbb{DC}\mathfrak{R}_n^{\dagger_1}, \mathbb{DC}\mathfrak{R}_n^{\dagger_2}, \mathbb{DC}\mathfrak{R}_n^{\dagger_3}, \mathbb{DC}\mathfrak{R}_n^{\dagger_4}$, and $\mathbb{DC}\mathfrak{R}_n^{\dagger_5}$ are five kinds of conjugations of the dual-complex Lucas-Leonardo number and $\mathbb{R}$, $\mathbb{C}\mathfrak{R}$, $\mathbb{D}\mathfrak{R}$, and $\mathbb{DC}\mathfrak{R}$ show the real, complex Lucas-Leonardo, dual Lucas-Leonardo, and dual-complex Lucas-Leonardo numbers, respectively.

Proof. Taking into account the five types of conjugations:

(i) For the first item, using the definitions of the $\mathbb{DC}\mathfrak{R}_n$ dual-complex Lucas-Leonardo numbers and their complex conjugation, and the addition operation of any two dual-complex Lucas-Leonardo numbers, the conclusion can be seen easily.

(ii) From the definitions of $\mathbb{DC}\mathfrak{R}_n$ and its complex conjugation, the multiplication table for dual-complex numbers, and the multiplication of any two dual-complex Lucas-Leonardo numbers, we have

$$\begin{aligned}\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_1} &= (\mathfrak{R}_n + i\mathfrak{R}_{n+1} + \varepsilon\mathfrak{R}_{n+2} + i\varepsilon\mathfrak{R}_{n+3}) \\ &\quad \times (\mathfrak{R}_n - i\mathfrak{R}_{n+1} + \varepsilon\mathfrak{R}_{n+2} - i\varepsilon\mathfrak{R}_{n+3}) \\ &= \mathfrak{R}_n^2 + \mathfrak{R}_{n+1}^2 \\ &\quad + \varepsilon(\mathfrak{R}_n\mathfrak{R}_{n+2} + \mathfrak{R}_{n+1}\mathfrak{R}_{n+3} + \mathfrak{R}_{n+2}\mathfrak{R}_n + \mathfrak{R}_{n+3}\mathfrak{R}_{n+1}) \\ &= \mathfrak{R}_n^2 + \mathfrak{R}_{n+1}^2 + 2\varepsilon(\mathfrak{R}_n\mathfrak{R}_{n+2} + \mathfrak{R}_{n+1}\mathfrak{R}_{n+3}).\end{aligned}$$

Moreover, if Eqs (2.18) and (2.21) are considered an equivalent relation, this item can be given as

$$\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_1} = [20F_{2n+1} - 4(F_{n+3} + F_{n+1}) + 2] + \varepsilon(40F_{2n+3} - 20F_{n+3} + 4) \in \mathbb{D}F,$$

where $\mathbb{D}F$ shows the dual Fibonacci numbers.

(iii) By applying the addition operation of any two dual-complex Lucas-Leonardo numbers to $\mathbb{DC}\mathfrak{R}_n$ and $\mathbb{DC}\mathfrak{R}_n^{\dagger_2}$, the result follows.

(iv)

$$\begin{aligned}\mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_2} &= (\mathfrak{R}_n + i\mathfrak{R}_{n+1} + \varepsilon\mathfrak{R}_{n+2} + i\varepsilon\mathfrak{R}_{n+3}) \\ &\quad \times (\mathfrak{R}_n + i\mathfrak{R}_{n+1} - \varepsilon\mathfrak{R}_{n+2} - i\varepsilon\mathfrak{R}_{n+3}) \\ &= \mathfrak{R}_n^2 + i\mathfrak{R}_n\mathfrak{R}_{n+1} - \varepsilon\mathfrak{R}_n\mathfrak{R}_{n+2} - i\varepsilon\mathfrak{R}_n\mathfrak{R}_{n+3} + i\mathfrak{R}_n\mathfrak{R}_{n+1} \\ &\quad + i^2\mathfrak{R}_{n+1}^2 - i\varepsilon\mathfrak{R}_{n+1}\mathfrak{R}_{n+2} - i^2\varepsilon\mathfrak{R}_{n+1}\mathfrak{R}_{n+3} + \varepsilon\mathfrak{R}_n\mathfrak{R}_{n+2}\end{aligned}$$

$$
\begin{aligned}
&+ i\varepsilon\Re_{n+1}\Re_{n+2} - \varepsilon^2\Re_{n+2}^2 - i\varepsilon^2\Re_{n+2}\Re_{n+3} + i\varepsilon\Re_n\Re_{n+3} \\
&+ i^2\varepsilon\Re_{n+1}\Re_{n+3} - i\varepsilon^2\Re_{n+2}\Re_{n+3} - i^2\varepsilon^2\Re_{n+3}^2 \\
= \;& \Re_n^2 - \Re_{n+1}^2 + 2i\,(\Re_n\Re_{n+1})
\end{aligned}
$$

is obtained considering the definitions of $\mathbb{DC}\Re_n$ and its dual conjugation and the multiplication operation for any two dual-complex Lucas-Leonardo numbers. Furthermore, from Eqs (2.19) and (2.20), an equivalent relation is

$\mathbb{DC}\Re_n \times \mathbb{DC}\Re_n^{\dagger_2} = \Re_n^2 - \Re_{n+1}^2 + 2i\,(\Re_n\Re_{n+1}) = [4L_{n-1}\,(1 - L_{n+2})] + i\,[8L_nL_{n+1} - 4L_{n+2} + 2] \in \mathbb{C}L.$

(v) It can be seen similarly to options *(i)* and *(iii)*.

(vi) Considering the definitions of $\mathbb{DC}\Re_n$ and its coupled conjugation, the multiplication scheme, and the multiplication operation for any two dual-complex Lucas-Leonardo numbers, the result follows. On the other hand, by using Eqs (2.18) and (2.22), the following equivalent relation is obtained:

$\mathbb{DC}\Re_n \times \mathbb{DC}\Re_n^{\dagger_3} = [20F_{2n+1} - 4\,(F_{n+3} + F_{n+1}) + 2] + 2i\varepsilon\,[20\,(-1)^n - 2\,(F_{n+1} + F_{n-1})] \in \mathbb{DC}F,$

where $\mathbb{DC}F$ shows the dual-complex Fibonacci numbers.

(vii)

$$
\begin{aligned}
LHS &= (\Re_n + i\Re_{n+1}) \times (\Re_n - i\Re_{n+1})\left(1 - \varepsilon\frac{\Re_{n+2} + i\Re_{n+3}}{\Re_n + i\Re_{n+1}}\right) \\
&= (\Re_n - i\Re_{n+1})\,(\Re_n + i\Re_{n+1} - \varepsilon\Re_{n+2} - i\varepsilon\Re_{n+3}) \\
&= \Re_n^2 + \Re_{n+1}^2 - \varepsilon\,(\Re_n\Re_{n+2} + \Re_{n+1}\Re_{n+3}) + i\varepsilon\,(\Re_{n+1}\Re_{n+2} - \Re_{n+3}\Re_n)
\end{aligned}
$$

is obtained from the multiplication operation and the definition of dual-complex conjugation for dual-complex Lucas-Leonardo numbers. Moreover, in terms of Fibonacci numbers, the following relation is obtained for this item by considering Eqs (2.18), (2.21), and (2.22):

$$
\begin{aligned}
(\Re_n + i\Re_{n+1}) \times \mathbb{DC}\Re_n^{\dagger_4} &= [20F_{2n+1} - 4\,(F_{n+3} + F_{n+1}) + 2] - \varepsilon\,[20F_{2n+3} - 10F_{n+3} + 2] \\
&\quad - i\varepsilon\,[20\,(-1)^n - 2\,(F_{n+1} + F_{n-1})] \in \mathbb{DC}F.
\end{aligned}
$$

(viii) Let z, t be $z := \Re_n + i\Re_{n+1}$ and $t := \Re_{n+2} + i\Re_{n+3}$. From the definitions of dual-complex Lucas-Leonardo numbers, their dual-complex conjugations, and the multiplication operation,

$$
\begin{aligned}
\mathbb{DC}\Re_n \times \mathbb{DC}\Re_n^{\dagger_4} &= (z + \varepsilon t)\,\overline{z}\left(1 - \varepsilon\frac{t}{z}\right) \\
&= \left(z - \varepsilon t + \varepsilon t - \varepsilon^2\frac{t^2}{z}\right)\overline{z} \\
&= z\overline{z} = \Re_n^2 + \Re_{n+1}^2,
\end{aligned}
$$

or equivalently from Eq (2.18),

$$
\mathbb{DC}\Re_n \times \mathbb{DC}\Re_n^{\dagger_4} = 20F_{2n+1} - 4\,(F_{n+3} + F_{n+1}) + 2 \in F,
$$

where F_n is the nth Fibonacci number.

(ix–xi) By considering Definition 3.1, its anti-dual conjugation, and the operations subtraction, addition, and multiplication, respectively, the results follow. □

Here, the equivalents of the above given properties can be found in different number systems, such as dual-complex Lucas numbers, using some identities related to Lucas-Leonardo numbers.

The definition of the module for dual-complex Lucas-Leonardo numbers can be composed analogously to the definition of the module in the standard complex space, too.

Definition 4.5. *The j-modulus of $\mathbb{DC}\mathfrak{R}_n$ dual-complex Lucas-Leonardo numbers is defined as follows, with different types of conjugations:*

- $|\mathbb{DC}\mathfrak{R}_n|^2_{\dagger_1} = \mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_1}$,
- $|\mathbb{DC}\mathfrak{R}_n|^2_{\dagger_2} = \mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_2}$,
- $|\mathbb{DC}\mathfrak{R}_n|^2_{\dagger_3} = \mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_3}$,
- $|\mathbb{DC}\mathfrak{R}_n|^2_{\dagger_4} = \mathbb{DC}\mathfrak{R}_n \times \mathbb{DC}\mathfrak{R}_n^{\dagger_4}$.

Now, we will prove some properties of dual-complex Lucas-Leonardo numbers with the following theorem.

Theorem 4.6. *Let $\mathbb{DC}\mathfrak{R}_n$ be the nth dual-complex Lucas-Leonardo number. Then, for $n, m \geq 0$, the following relations hold:*

(i) $\mathbb{DC}\mathfrak{R}_n + \mathbb{DC}\mathfrak{R}_{n+1} = \mathbb{DC}\mathfrak{R}_{n+2} - A$,

(ii) $\mathbb{DC}\mathfrak{R}_{n-1} + \mathbb{DC}\mathfrak{R}_{n+1} = 10\mathbb{DC}F_n - 2A$,

(iii) $\mathbb{DC}\mathfrak{R}_{n+2} - \mathbb{DC}\mathfrak{R}_{n-2} = 10\mathbb{DC}F_n$,

(iv) $\mathbb{DC}\mathfrak{R}_{n+1} - \mathbb{DC}\mathfrak{R}_{n-1} = 2\mathbb{DC}L_n$,

(v) $\mathbb{DC}\mathfrak{R}_{n+m} + \mathbb{DC}\mathfrak{R}_{n-m} = \begin{cases} 2L_m\mathbb{DC}L_n - 2A, & m = 2k, \\ 10F_m\mathbb{DC}F_n - 2A, & m = 2k+1, \end{cases}$

(vi) $\mathbb{DC}\mathfrak{R}_{n+m} - \mathbb{DC}\mathfrak{R}_{n-m} = \begin{cases} 10F_m\mathbb{DC}F_n, & m = 2k, \\ 2L_m\mathbb{DC}L_n, & m = 2k+1, \end{cases}$

(vii) $\mathbb{DC}\mathfrak{R}^2_{n+1} - \mathbb{DC}\mathfrak{R}^2_{n-1} = 4\mathbb{DC}L_n\left[5\mathbb{DC}F_n - A\right]$,

(viii) $\mathbb{DC}\mathfrak{R}_n - i\mathbb{DC}\mathfrak{R}^{\dagger_3}_{n+1} - \varepsilon\mathbb{DC}\mathfrak{R}_{n+2} - i\varepsilon\mathbb{DC}\mathfrak{R}_{n+3} = -(\mathfrak{R}_{n+1} + 1) + 2\varepsilon\mathfrak{R}_{n+4} \in \mathbb{D}\mathfrak{R}$,

where $A = 1 + i + \varepsilon + i\varepsilon$.

Proof. (i) Considering Definitions 2.4 and 4.1, and the addition operation of any two dual-complex Lucas-Leonardo numbers, the result is obtained as follows:

$$\begin{aligned} LHS &= (\mathfrak{R}_n + \mathfrak{R}_{n+1}) + i(\mathfrak{R}_{n+1} + \mathfrak{R}_{n+2}) + \varepsilon(\mathfrak{R}_{n+2} + \mathfrak{R}_{n+3}) + i\varepsilon(\mathfrak{R}_{n+3} + \mathfrak{R}_{n+4}) \\ &= (\mathfrak{R}_{n+2} - 1) + i(\mathfrak{R}_{n+3} - 1) + \varepsilon(\mathfrak{R}_{n+4} - 1) + i\varepsilon(\mathfrak{R}_{n+5} - 1) \\ &= (\mathfrak{R}_{n+2} + i\mathfrak{R}_{n+3} + \varepsilon\mathfrak{R}_{n+4} + i\varepsilon\mathfrak{R}_{n+5}) - (1 + i + \varepsilon + i\varepsilon) \\ &= \mathbb{DC}\mathfrak{R}_{n+2} - A. \end{aligned}$$

(ii) $\mathbb{DC}\mathfrak{R}_{n-1} + \mathbb{DC}\mathfrak{R}_{n+1} = (\mathfrak{R}_{n-1} + \mathfrak{R}_{n+1}) + i(\mathfrak{R}_n + \mathfrak{R}_{n+2}) + \varepsilon(\mathfrak{R}_{n+1} + \mathfrak{R}_{n+3}) + i\varepsilon(\mathfrak{R}_{n+2} + \mathfrak{R}_{n+4})$ is written from Definition 4.1 and the addition operation for them. Here, by using Eq (2.14) for $m = 1$, for the right-hand side of the above equation,

$$\begin{aligned} RHS &= (10F_n - 2) + i(10F_{n+1} - 2) + \varepsilon(10F_{n+2} - 2) + i\varepsilon(10F_{n+3} - 2) \\ &= 10(F_n + iF_{n+1} + \varepsilon F_{n+2} + i\varepsilon F_{n+3}) - 2(1 + i + \varepsilon + i\varepsilon) \\ &= 10\mathbb{DC}F_n - 2A \end{aligned}$$

is obtained. So, the result follows.

(iii) By applying the subtraction operation to any two dual-complex Lucas-Leonardo numbers and considering Eq (2.15),

$$\begin{aligned} LHS &= (\Re_{n+2} - \Re_{n-2}) + i(\Re_{n+3} - \Re_{n-1}) + \varepsilon(\Re_{n+4} - \Re_n) + i\varepsilon(\Re_{n+5} - \Re_{n+1}) \\ &= 10(F_n + iF_{n+1} + \varepsilon F_{n+2} + i\varepsilon F_{n+3}) = 10\mathbb{DC}F_n \end{aligned}$$

is obtained.

(iv) Using Definition 4.1, applying the subtraction operation for dual-complex Lucas-Leonardo numbers, and from Eq (2.15) in the case where $m = 1$, we get the conclusion similar to item (i).

(v) From Definition 4.1, the addition operation for dual-complex Lucas-Leonardo numbers, and Eq (2.14), the result follows as:

$$\begin{aligned} \mathbb{DC}\Re_{n+m} + \mathbb{DC}\Re_{n-m} &= (\Re_{n+m} + \Re_{n-m}) + i(\Re_{n+m+1} + \Re_{n-m+1}) \\ &\quad + \varepsilon(\Re_{n+m+2} + \Re_{n-m+2}) + i\varepsilon(\Re_{n+m+3} + \Re_{n-m+3}) \end{aligned}$$

$$= \begin{cases} 2(L_nL_m - 1) + 2i(L_{n+1}L_m - 1) + 2\varepsilon(L_{n+2}L_m - 1) + 2i\varepsilon(L_{n+3}L_m - 1), & m = 2k, \\ (10F_nF_m - 2) + i(10F_{n+1}F_m - 2) + \varepsilon(10F_{n+2}F_m - 2) + i\varepsilon(10F_{n+3}F_m - 2), & m = 2k+1, \end{cases}$$

$$= \begin{cases} 2L_m\mathbb{DC}L_n - 2A, & m = 2k, \\ 10F_m\mathbb{DC}F_n - 2A, & m = 2k+1. \end{cases}$$

(vi) This item can be proven similarly to item *(v)*, considering the definition of dual-complex Lucas-Leonardo numbers, the subtraction operation for them, and Eq (2.15).

(vii) By using the items (ii), (iv), and considering the difference of two squares, the conclusion can be seen.

(viii) From Definition 4.1, the subtraction operation for them, coupled with conjugation of these numbers, considering the multiplication scheme (Table 1) and Eq (2.4), we have

$$\begin{aligned} LHS &= (\Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3} - i\Re_{n+1} - \Re_{n+2} + i\varepsilon\Re_{n+3} + \varepsilon\Re_{n+4} - \varepsilon\Re_{n+2} - i\varepsilon\Re_{n+3}) \\ &- \varepsilon^2\Re_{n+4} - i\varepsilon^2\Re_{n+5} - i\varepsilon\Re_{n+3} + \varepsilon\Re_{n+4} - i\varepsilon^2\Re_{n+5} - i^2\varepsilon^2\Re_{n+6} \\ &= \Re_n - \Re_{n+2} + 2\varepsilon\Re_{n+4} = -(\Re_{n+1} + 1) + 2\varepsilon\Re_{n+4}. \end{aligned}$$

For equivalent conclusions in terms of Fibonacci and Lucas numbers, the relations between Lucas-Leonardo and Lucas and Fibonacci numbers can be used.

□

Now, we will prove some interrelations related to the dual-complex forms of Lucas-Leonardo, Lucas, Fibonacci, and Leonardo numbers.

Theorem 4.7. *Let $\mathbb{DC}\Re_n$, $\mathbb{DC}F_n$, and $\mathbb{DC}L_n$ be the dual-complex Lucas-Leonardo, dual-complex Fibonacci, and dual-complex Lucas numbers, respectively. Then for all $n \in \mathbb{Z}$, the following identities hold:*

(i) $\mathbb{DC}\Re_n = 2\mathbb{DC}L_n - A$,

(ii) $\mathbb{DC}\Re_n = 3\mathbb{DC}F_{n+1} - \mathbb{DC}F_{n-2} - A$,

(iii) $\mathbb{DC}\Re_n = 2\mathbb{DC}F_{n-1} + 2\mathbb{DC}F_{n+1} - A$,

(iv) $\mathbb{DC}\Re_n = 6\mathbb{DC}F_{n-1} - 2\mathbb{DC}F_{n-2} - A$,

(v) $\mathbb{DC}\Re_n = 4\mathbb{DC}F_{n+2} - 6\mathbb{DC}F_n - A$,
(vi) $\mathbb{DC}\Re_n = 2\mathbb{DC}Le_n - \mathbb{DC}Le_{n-1}$,
(vii) $\mathbb{DC}\Re_n = \mathbb{DC}L_n - \mathbb{DC}F_n + \mathbb{DC}Le_n$,

where $A = 1 + i + \varepsilon + i\varepsilon$.

Proof. We will prove only a few options. For the first one,

$$\begin{aligned}\mathbb{DC}\Re_n &= \Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3} \\ &= 2\left(L_n + iL_{n+1} + \varepsilon L_{n+2} + i\varepsilon L_{n+3}\right) - (1 + i + \varepsilon + i\varepsilon) \\ &= 2\mathbb{DC}L_n - A\end{aligned}$$

is obtained from Eqs (2.4), (2.6), and (2.27).

For the second item, considering Definition 4.1 and the fact that $\Re_n = 3F_{n+1} - F_{n-2} - 1$, we have

$$\begin{aligned}\mathbb{DC}\Re_n &= \Re_n + i\Re_{n+1} + \varepsilon\Re_{n+2} + i\varepsilon\Re_{n+3} \\ &= (3F_{n+1} - F_{n-2} - 1) + i\,(3F_{n+2} - F_{n-1} - 1) \\ &\quad + \varepsilon\,(3F_{n+3} - F_n - 1) + i\varepsilon\,(3F_{n+4} - F_{n+1} - 1) \\ &= 3\left(F_{n+1} + iF_{n+2} + \varepsilon F_{n+3} + i\varepsilon F_{n+4}\right) \\ &\quad - \left(F_{n-2} + iF_{n-1} + \varepsilon F_n + i\varepsilon F_{n+1}\right) - (1 + i + \varepsilon + i\varepsilon) \\ &= 3\mathbb{DC}F_{n+1} - \mathbb{DC}F_{n-2} - A.\end{aligned}$$

To prove the sixth option, let us consider Eq (4.1) and the third option of Proposition 3.6. Then

$$\begin{aligned}\mathbb{DC}\Re_n &= C\Re_n + \varepsilon C\Re_{n+2} \\ &= (2CLe_n - CLe_{n-1}) + \varepsilon\,(2CLe_{n+2} - CLe_{n+1}) \\ &= 2\left(CLe_n + \varepsilon CLe_{n+2}\right) - \left(CLe_{n-1} + \varepsilon CLe_{n+1}\right)\end{aligned}$$

is written. Here, since an analogue of Eq (4.1) also holds for the dual-complex Leonardo numbers, the result is evident.

Using Eq (4.1) and item (v) of Proposition 3.6, the seventh property can be established like the sixth case.

The proofs of the third, fourth, and fifth items follow analogously to the first and second by applying Definition 4.1 and the relationships between the Lucas-Leonardo and Fibonacci sequences. □

Now, some important identities given for well-known integer sequences will be discussed based on the fundamental theorem, which is given above. First, d'Ocagne's identity, which is an example of determinantial identities for Lucas-Leonardo numbers, will be given with respect to dual-complex numbers in the following theorem.

Theorem 4.8. *(d'Ocagne Identity) Let* $\mathbb{DC}\Re_n$ *and* $\mathbb{DC}\Re_m$ *be two dual-complex Lucas-Leonardo numbers. Then the d'Ocagne identity is given by*

$$\begin{aligned}\mathbb{DC}\Re_m\mathbb{DC}\Re_{n+1} - \mathbb{DC}\Re_{m+1}\mathbb{DC}\Re_n &= (-1)^n\left[(\Re_{m-n+1} + \Re_{m-n-1} + 2)(-4 - 2i - 12\varepsilon - 6i\varepsilon)\right] \\ &\quad + A\left(\mathbb{DC}\Re_{m-1} - \mathbb{DC}\Re_{n-1}\right),\end{aligned}$$

where $m > n$, $n \geq 1$.

Proof. Considering Theorem 4.7, after some algebra, the following equation is obtained:

$$\begin{aligned} LHS &= (2\mathbb{DC}L_m - A)(2\mathbb{DC}L_{n+1} - A) - (2\mathbb{DC}L_{m+1} - A)(2\mathbb{DC}L_n - A) \\ &= 4(\mathbb{DC}L_m\mathbb{DC}L_{n+1} - \mathbb{DC}L_{m+1}\mathbb{DC}L_n) + A(2\mathbb{DC}L_{m+1} + 2\mathbb{DC}L_n - 2\mathbb{DC}L_{n+1} - 2\mathbb{DC}L_m). \end{aligned}$$

Here, by considering d'Ocagne's identity for dual-complex Lucas-Leonardo numbers given in Eq (2.29), we get

$$\begin{aligned} LHS &= 4\left[\left(5(-1)^{n+1}F_{m-n}\right)(2 + i + 6\varepsilon + 3i\varepsilon)\right] \\ &\quad + A[2\mathbb{DC}L_{m+1} + 2\mathbb{DC}L_n - 2\mathbb{DC}L_{n+1} - 2\mathbb{DC}L_m]. \end{aligned}$$

Then, using Fact 2.12 and Theorem 4.7,

$$LHS = 4\left[\left((-1)^{n+1}(L_{m-n+1} + L_{m-n-1})\right)(2 + i + 6\varepsilon + 3i\varepsilon)\right] + A[\mathbb{DC}\mathfrak{R}_{m+1} + \mathbb{DC}\mathfrak{R}_n - \mathbb{DC}\mathfrak{R}_{n+1} - \mathbb{DC}\mathfrak{R}_m]$$

is written. Taking into account Eq (2.6) and Lemma 4.2, the following equation is obtained in the last stage:

$$\begin{aligned} LHS &= 4\left[\left((-1)^{n+1}\left(\frac{\mathfrak{R}_{m-n+1} + \mathfrak{R}_{m-n-1} + 2}{2}\right)\right)(2 + i + 6\varepsilon + 3i\varepsilon)\right] \\ &\quad + A[\mathbb{DC}\mathfrak{R}_m + \mathbb{DC}\mathfrak{R}_{m-1} + A + \mathbb{DC}\mathfrak{R}_n - \mathbb{DC}\mathfrak{R}_n - \mathbb{DC}\mathfrak{R}_{n-1} - A - \mathbb{DC}\mathfrak{R}_m] \\ &= (-1)^n[(\mathfrak{R}_{m-n+1} + \mathfrak{R}_{m-n-1} + 2)(-4 - 2i - 12\varepsilon - 6i\varepsilon)] + A(\mathbb{DC}\mathfrak{R}_{m-1} - \mathbb{DC}\mathfrak{R}_{n-1}). \end{aligned}$$

So, the result follows. □

Relation 2.9 has been presented for nega-Lucas-Leonardo numbers, i.e., the Lucas-Leonardo numbers with negative indices. Similarly, the nega-dual-complex Lucas-Leonardo numbers, i.e., the complex Lucas-Leonardo numbers with negative indices, are given with the next theorem.

Theorem 4.9. *The negative subscript of the dual-complex Lucas-Leonardo numbers is given as follows:*

$$\mathbb{DC}\mathfrak{R}_{-n} = (-1)^n[\mathbb{DC}\mathfrak{R}_n - (\mathfrak{R}_{n+2} - \mathfrak{R}_{n-2})(i + \varepsilon + 2i\varepsilon) + A] - A,$$

where $\mathfrak{R}_n$ is the nth Lucas-Leonardo number.

Proof. Considering item (i) of Theorem 4.7 and Eq (2.30), we have

$$\begin{aligned} \mathbb{DC}\mathfrak{R}_{-n} &= 2\mathbb{DC}L_{-n} - A \\ &= 2[(-1)^n\mathbb{DC}L_n - 5F_n(-1)^n(i + \varepsilon + 2i\varepsilon)] - A \\ &= 2[(-1)^n(\mathbb{DC}L_n - 5F_n(i + \varepsilon + 2i\varepsilon))] - A. \end{aligned}$$

Here, using additionally the Facts 2.6 and 2.13, the result follows as

$$\begin{aligned} \mathbb{DC}\mathfrak{R}_{-n} &= 2(-1)^n\left[\frac{\mathbb{DC}\mathfrak{R}_n + A}{2} - (L_{n+2} - L_{n-2})(i + \varepsilon + 2i\varepsilon)\right] - A \\ &= 2(-1)^n\left[\frac{\mathbb{DC}\mathfrak{R}_n + A - (\mathfrak{R}_{n+2} - \mathfrak{R}_{n-2} + 1 - 1)(i + \varepsilon + 2i\varepsilon)}{2}\right] - A \\ &= (-1)^n[\mathbb{DC}\mathfrak{R}_n - (\mathfrak{R}_{n+2} - \mathfrak{R}_{n-2})(i + \varepsilon + 2i\varepsilon) + A] - A. \end{aligned}$$

□

On the other hand, an equivalent equation for nega-dual-complex Lucas-Leonardo numbers can be presented with the following equation by using Eq (2.15) in the case where $m = 2$:

$$\mathbb{DC}\mathfrak{R}_{-n} = (-1)^n \left[\mathbb{DC}\mathfrak{R}_n - 10F_n (i + \varepsilon + 2i\varepsilon) + A\right] - A.$$

Now, Binet's formula, Catalan's identity, and Cassini's identity will be proven for dual-complex Lucas-Leonardo numbers.

Theorem 4.10. *(Binet's Formula) For $n \geq 1$, Binet's formula for the dual-complex Lucas-Leonardo numbers is stated as follows:*

$$\mathbb{DC}\mathfrak{R}_n = 2\overline{\alpha}\alpha^n + 2\overline{\beta}\beta^n - A,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\overline{\alpha} = 1 + \alpha i + \alpha^2\varepsilon + \alpha^3 i\varepsilon$, and $\overline{\beta} = 1 + \beta i + \beta^2\varepsilon + \beta^3 i\varepsilon$.

Proof. From item *(i)* of Theorem 4.7 and considering Binet's formula for dual-complex Lucas numbers given with Eq (2.28),

$$\mathbb{DC}\mathfrak{R}_n = 2\mathbb{DC}L_n - A = 2\left(\overline{\alpha}\alpha^n + \overline{\beta}\beta^n\right) - A$$

is obtained. □

Theorem 4.11. *(Catalan's Identity) Let $\mathbb{DC}\mathfrak{R}_n$ be the nth dual-complex Lucas-Leonardo number. Then, for $k \geq 1$ and $n \geq k$, the Catalan identity is as follows:*

$$\begin{aligned}\mathbb{DC}\mathfrak{R}_n^2 - \mathbb{DC}\mathfrak{R}_{n+k}\mathbb{DC}\mathfrak{R}_{n-k} &= \left[16(-1)^n - (-1)^{n-k}(\mathfrak{R}_k + 1)^2\right](2 + i + 6\varepsilon + 3i\varepsilon) \\ &\quad + A\left(\mathbb{DC}\mathfrak{R}_{n+k} + \mathbb{DC}\mathfrak{R}_{n-k} - 2\mathbb{DC}\mathfrak{R}_n\right),\end{aligned}$$

where $\mathfrak{R}_n$ is the nth Lucas-Leonardo number.

Proof. If item (i) of Theorem 4.7 is used, we get the following result for the left-hand side of the equation:

$$\begin{aligned}LHS &= (2\mathbb{DC}L_n - A)^2 - (2\mathbb{DC}L_{n+k} - A)(2\mathbb{DC}L_{n-k} - A) \\ &= 4\left(\mathbb{DC}L_n^2 - \mathbb{DC}L_{n+k}\mathbb{DC}L_{n-k}\right) + A\left(2\mathbb{DC}L_{n+k} + 2\mathbb{DC}L_{n-k} - 4\mathbb{DC}L_n\right).\end{aligned}$$

Here,

$$\begin{aligned}LHS &= 4\left[5(-1)^{n-k+1}F_k^2(2 + i + 6\varepsilon + 3i\varepsilon)\right] + A\left[(2\mathbb{DC}L_{n+k} - A) + (2\mathbb{DC}L_{n-k} - A) - 2(2\mathbb{DC}L_n - A)\right] \\ &= 4\left[(-1)^{n-k+1}\left(L_k^2 - 4(-1)^k\right)(2 + i + 6\varepsilon + 3i\varepsilon)\right] + A\left(\mathbb{DC}\mathfrak{R}_{n+k} + \mathbb{DC}\mathfrak{R}_{n-k} - 2\mathbb{DC}\mathfrak{R}_n\right) \\ &= 4\left[(-1)^{n-k+1}L_k^2 - 16(-1)^{n+1}\right](2 + i + 6\varepsilon + 3i\varepsilon) + A\left(\mathbb{DC}\mathfrak{R}_{n+k} + \mathbb{DC}\mathfrak{R}_{n-k} - 2\mathbb{DC}\mathfrak{R}_n\right)\end{aligned}$$

is written using [15, Theorem 5.8] and the Catalan identity given in Eq (2.31). After this step, by considering Fact 2.6 with some algebra, we have

$$\begin{aligned}LHS &= 4\left[(-1)^{n-k+1}\left(\frac{\mathfrak{R}_k + 1}{2}\right)^2 - 16(-1)^{n+1}\right](2 + i + 6\varepsilon + 3i\varepsilon) + A\left(\mathbb{DC}\mathfrak{R}_{n+k} + \mathbb{DC}\mathfrak{R}_{n-k} - 2\mathbb{DC}\mathfrak{R}_n\right) \\ &= \left[16(-1)^n - (-1)^{n-k}(\mathfrak{R}_k + 1)^2\right](2 + i + 6\varepsilon + 3i\varepsilon) + A\left(\mathbb{DC}\mathfrak{R}_{n+k} + \mathbb{DC}\mathfrak{R}_{n-k} - 2\mathbb{DC}\mathfrak{R}_n\right).\end{aligned}$$

So the result follows. □

On the other hand, by taking $k = 1$ in Catalan's identity, the following corollary is obtained for dual-complex Lucas-Leonardo numbers.

Corollary 4.12. *(Cassini's Identity) Let $\mathbb{DC}\mathfrak{R}_n$ be a dual-complex Lucas-Leonardo number. Then, for $n \geq 1$, the following identity holds:*

$$\mathbb{DC}\mathfrak{R}_n^2 - \mathbb{DC}\mathfrak{R}_{n+1}\mathbb{DC}\mathfrak{R}_{n-1} = 20\,(-1)^n\,(2 + i + 6\varepsilon + 3i\varepsilon) + A\,(\mathbb{DC}\mathfrak{R}_{n-1} - \mathbb{DC}\mathfrak{R}_{n-2})\,.$$

Now, we will give the generating function for dual-complex Lucas-Leonardo numbers. From the definition of the generating function of a sequence, the generating associated function $g\mathbb{DC}\mathfrak{R}\,(t)$ is defined by $g\mathbb{DC}\mathfrak{R}\,(t) = \sum\limits_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n$.

Theorem 4.13. *(Generating Function) The generating function for dual-complex Lucas-Leonardo numbers is given by*

$$g\mathbb{DC}\mathfrak{R}\,(t) = \frac{\mathbb{DC}\mathfrak{R}_0 + t\,(-5 + 3i - 3\varepsilon - i\varepsilon) + t^2\,(3 - 3i - \varepsilon - 5i\varepsilon)}{1 - 2t + t^3},$$

where $1 - 2t + t^3 \neq 0$.

Proof. By using the definition of a generating function and Lemma 4.3, the result follows as:

$$\begin{aligned}
g\mathbb{DC}\mathfrak{R}\,(t) &= \sum_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n = \mathbb{DC}\mathfrak{R}_0 t^0 + \mathbb{DC}\mathfrak{R}_1 t^1 + \mathbb{DC}\mathfrak{R}_2 t^2 + \sum_{n=3}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n \\
&= (3 + i + 5\varepsilon + 7i\varepsilon) + (1 + 5i + 7\varepsilon + 13i\varepsilon)\,t + (5 + 7i + 13\varepsilon + 21i\varepsilon)\,t^2 \\
&\quad + \sum_{n=3}^{\infty} (2\mathbb{DC}\mathfrak{R}_{n-1} - \mathbb{DC}\mathfrak{R}_{n-3})\,t^n \\
&= (3 + i + 5\varepsilon + 7i\varepsilon) + (1 + 5i + 7\varepsilon + 13i\varepsilon)\,t + (5 + 7i + 13\varepsilon + 21i\varepsilon)\,t^2 \\
&\quad + 2t \sum_{n=3}^{\infty} \mathbb{DC}\mathfrak{R}_{n-1} t^{n-1} - t^3 \sum_{n=3}^{\infty} \mathbb{DC}\mathfrak{R}_{n-3} t^{n-3} \\
&= (3 + i + 5\varepsilon + 7i\varepsilon) + (1 + 5i + 7\varepsilon + 13i\varepsilon)\,t + (5 + 7i + 13\varepsilon + 21i\varepsilon)\,t^2 \\
&\quad + 2t \left(\sum_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n - \mathbb{DC}\mathfrak{R}_1 t - \mathbb{DC}\mathfrak{R}_0 \right) - t^3 \sum_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n \\
&= (3 + i + 5\varepsilon + 7i\varepsilon) + (1 + 5i + 7\varepsilon + 13i\varepsilon)\,t + (5 + 7i + 13\varepsilon + 21i\varepsilon)\,t^2 \\
&\quad + 2t \left(\sum_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n \right) - 2t^2\,(1 + 5i + 7\varepsilon + 13i\varepsilon) - 2t\,(3 + i + 5\varepsilon + 7i\varepsilon) - t^3 \sum_{n=0}^{\infty} \mathbb{DC}\mathfrak{R}_n t^n \\
\Rightarrow \quad g\mathbb{DC}\mathfrak{R}\,(t) &= (3 + i + 5\varepsilon + 7i\varepsilon) + (-5 + 3i - 3\varepsilon - i\varepsilon)\,t \\
&\quad + (3 - 3i - \varepsilon - 5i\varepsilon)\,t^2 + \left(2t - t^3\right) g\mathbb{DC}\mathfrak{R}\,(t) \\
\Rightarrow \quad g\mathbb{DC}\mathfrak{R}\,(t) &= \frac{\mathbb{DC}\mathfrak{R}_0 + t\,(-5 + 3i - 3\varepsilon - i\varepsilon) + t^2\,(3 - 3i - \varepsilon - 5i\varepsilon)}{1 - 2t + t^3}.
\end{aligned}$$

□

Now, we will present the summation formulas for dual-complex Lucas-Leonardo numbers.

Theorem 4.14. *Let $\mathbb{DC}\Re_n$ be a dual-complex Lucas-Leonardo number. Then the following identities hold:*

(i) $\sum_{k=0}^{n} \mathbb{DC}\Re_k = \mathbb{DC}\Re_{n+2} - (n+2)A - (4i + 6\varepsilon + 12i\varepsilon)$,

(ii) $\sum_{k=0}^{n} \mathbb{DC}\Re_{2k} = \mathbb{DC}\Re_{2n+1} - nA - (4i + 2\varepsilon + 6i\varepsilon - 2)$,

(iii) $\sum_{k=0}^{n} \mathbb{DC}\Re_{2k+1} = \mathbb{DC}\Re_{2n+2} - (n+2)A - (2 + 4\varepsilon + 6i\varepsilon)$.

Proof. These identities can be seen by using the definition of dual-complex Lucas-Leonardo numbers and summation formulas for Lucas-Leonardo numbers. We will only prove item (ii). So,

$$\begin{aligned}
\sum_{k=0}^{n} \mathbb{DC}\Re_{2k} &= \sum_{k=0}^{n} (\Re_{2k} + i\Re_{2k+1} + \varepsilon\Re_{2k+2} + i\varepsilon\Re_{2k+3}) \\
&= \sum_{k=0}^{n} \Re_{2k} + i\sum_{k=0}^{n} \Re_{2k+1} + \varepsilon\sum_{k=0}^{n} \Re_{2k+2} + i\varepsilon\sum_{k=0}^{n} \Re_{2k+3} \\
&= [\Re_{2n+1} - (n-2)] + i[\Re_{2n+2} - (n+2) - 2] + \varepsilon[\Re_{2n+3} - (n+2)] + i\varepsilon[\Re_{2n+4} - (n+6)] \\
&= [\Re_{2n+1} + i\Re_{2n+2} + \varepsilon\Re_{2n+3} + i\varepsilon\Re_{2n+4}] - n(1 + i + \varepsilon + i\varepsilon) - (4i + 2\varepsilon + 6i\varepsilon - 2) \\
&= \mathbb{DC}\Re_{2n+1} - nA - (-2 + 4i + 2\varepsilon + 6i\varepsilon).
\end{aligned}$$

□

Theorem 4.15. *Let $\mathbb{DC}\Re_n$, $\mathbb{DC}F_n$, $\mathbb{DC}L_n$, and $\mathbb{DC}Le_n$ be dual-complex Lucas-Leonardo, dual-complex Fibonacci, dual-complex Lucas, and dual-complex Leonardo numbers, respectively. Then, the following identities hold:*

(i) $\sum_{k=0}^{n} (\mathbb{DC}\Re_k + \mathbb{DC}F_k) = \mathbb{DC}\Re_{n+2} + \mathbb{DC}F_{n+2} - (n+3)A - (4i + 7\varepsilon + 14i\varepsilon)$,

(ii) $\sum_{k=0}^{n} (\mathbb{DC}\Re_k + \mathbb{DC}L_k) = \mathbb{DC}\Re_{n+2} + \mathbb{DC}L_{n+2} - (n+3)A - (6i + 9\varepsilon + 18i\varepsilon)$,

(iii) $\sum_{k=0}^{n} (\mathbb{DC}\Re_k + \mathbb{DC}L_k) = \frac{3\mathbb{DC}\Re_{n+2}+1}{2} - (n+3)A - (6i + 9\varepsilon + 18i\varepsilon)$,

(iv) $\sum_{k=0}^{n} (\mathbb{DC}\Re_k + \mathbb{DC}Le_k) = \mathbb{DC}\Re_{n+2} + \mathbb{DC}Le_{n+2} - (2n+4)A - (6i + 10\varepsilon + 20i\varepsilon)$.

Proof. To prove the first item, we will use item (i) of Theorem 4.14, Eq (2.32) in the case where $k = 1$, and the recurrence relation of dual-complex Fibonacci numbers. Then,

$$\begin{aligned}
\sum_{k=0}^{n} (\mathbb{DC}\Re_k + \mathbb{DC}F_k) &= \sum_{k=0}^{n} \mathbb{DC}\Re_k + \sum_{k=0}^{n} \mathbb{DC}F_k \\
&= [\mathbb{DC}\Re_{n+2} - (n+2)A - (4i + 6\varepsilon + 12i\varepsilon)] + [\mathbb{DC}F_{n+1} + \mathbb{DC}F_n - \mathbb{DC}F_1 - \mathbb{DC}F_0] \\
&= \mathbb{DC}\Re_{n+2} + \mathbb{DC}F_{n+2} - (n+3)A - (4i + 7\varepsilon + 14i\varepsilon).
\end{aligned}$$

The other conclusions can be seen similarly by using Theorem 4.14 and Eqs (2.33) and (2.34). □

5. Conclusions

This study extends the existing literature by constructing complex and dual-complex Lucas-Leonardo numbers, building upon the established definitions of complex numbers, dual-complex numbers, and Lucas-Leonardo sequences. Accordingly, the paper presents several significant results concerning these numerical structures. Beyond theoretical interest, these systems offer robust modeling tools for diverse applied sciences, including geometry, quantum physics, applied mathematics, quantum mechanics, Lie groups, kinematics, and differential equations.

Specifically, in robotics and mechanics, dual-complex Lucas-Leonardo numbers provide a discrete parameterization for rigid body motions within the framework of screw theory [22]. In this context, the R_n sequence can represent incremental changes in a robot arm's joint motion; for instance, the specific rotation or extension at each n-th step can be encoded using the sequence's coefficients, facilitating automatic differential kinematics for serial manipulators through dual-number algebras [23].

Furthermore, in the study of quasicrystals, these sequences can be utilized to model atomic arrangements in non-periodic but long-range ordered structures. The recursive properties of Lucas-Leonardo numbers assist in characterizing hidden dimensions and the multifractal nature found in such Fibonacci-type crystalline systems [24]. From a geometrical perspective, the dual-complex framework serves as a compact and efficient algebraic tool for modeling 2D rigid transformations within the $SE(2)$ group [25], describing discrete spiral growth patterns in complex planes. Moreover, the derived generating functions provide analytical solutions for non-homogeneous linear differential and difference equations in dynamical systems and serve as transfer functions for designing recursive digital filters in signal processing.

As another significant application, the dual-complex Lucas-Leonardo framework provides a compact and efficient algebraic tool for modeling rigid body motions within the $SE(2)$ group. Unlike standard homogeneous sequences, the non-homogeneous recurrence of Lucas-Leonardo numbers—characterized by their constant shift factor—offers a unique advantage in analyzing trajectories subject to fixed external bias or mechanical offsets. The integration of these sequences into dual-algebraic frameworks supports advanced programming with dual numbers for mechanism design, enabling the systematic synthesis of complex motion trajectories [26]. The derived Binet formulas and algebraic identities, such as Cassini's and d'Ocagne's, enable the direct calculation of positional estimation and geometric stability in robotic systems. Consequently, this framework reduces computational overhead by bypassing heavy matrix transformations, providing a robust analytical bridge between discrete number theory and practical kinematic analysis.

In conclusion, the integration of Lucas-Leonardo sequences with complex and dual-complex algebras provides a versatile mathematical toolkit, transforming pure number theory into a practical modeling framework for non-homogeneous physical systems. This work lays a comprehensive mathematical foundation for implementing Lucas-Leonardo variants in computational engineering and materials science.

Author contributions

Tuba Çakmak Katırcı: Conceptualization, methodology, investigation, resources, writing–original draft preparation, writing—review and editing, supervision; Can Alçelik: Methodology, investigation, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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