Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 907–914.
DOI: 10.3934/math.2026039
Received: 01 December 2025
Revised: 04 January 2026
Accepted: 08 January 2026
Published: 13 January 2026

Research article

The existence and multiplicity of solutions of Kirchhoff type linearly coupled systems with critical exponents

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Abstract: We consider the critical problems of Kirchhoff type linearly coupled systems. In applying variational methods, we establish both the existence and multiplicity of solutions in the four-dimensional case.

Keywords: Kirchhoff equation; linearly coupled system; Sobolev critical exponent; variational method
Mathematics Subject Classification: 35J20, 35J60

1. Introduction

We consider the following: Kirchhoff-type system with critical exponents,

$$\begin{cases} -b_1(1+\|(u,v)\|^2)\Delta u + a_1 u = u^3 + \lambda v & in\ \Omega, \\ -b_2(1+\|(u,v)\|^2)\Delta v + a_2 v = v^3 + \lambda u & in\ \Omega, \\ u = v = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

with $\|(u,v)\|^2 = b_1\|u\|^2 + b_2\|v\|^2$ and $\|u\|^2 = \int_\Omega |\nabla u|^2$, where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain, $a_i \in \mathbb{R}, i = 1,2, b_i, \lambda > 0$ are constants. Such type equations were first considered by Kirchhoff [1]. The Kirchhoff term arises as a consequence of accounting for the variation in tension of a vibrating string induced by the changes in its length.

The critical problems of Kirchhoff equations for four-dimensional case have already been investigated in [2–4]. In particular, in [3,4], the authors analyzed the case $\lambda = 0$ of (1.1),

$$\begin{cases} -(1+b\|u\|^2)\Delta u + au = u^3, & x \in \Omega, \\ u = 0 \ \ on \ \ \partial\Omega, \end{cases} \tag{1.2}$$

which can be regarded as a Brezis-Nirenberg problem with a Kirchhoff-type perturbation. They proved the existence and multiplicity of solutions for $a < -\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

We remark that the behaviors of (PS) sequences of (1.1) in dimension $N = 3$ or $N \geq 5$ is drastically different from the four-dimensional case. Consequently, the treatments for the cases $N = 3$ or $N \geq 5$ must be different from those adopted in this paper. Motivated by the previous works, we now turn to the investigation of the existence and multiplicity of solutions of (1.1). Here,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^4)\setminus\{0\}} \frac{\int_{\mathbb{R}^4} |\nabla u|^2}{(\int_{\mathbb{R}^4} u^4)^{1/2}},$$

$\{\lambda_k(\Omega)\}(k \in \mathbf{Z}^+)$ is the eigenvalue sequence of $(-\Delta, H_0^1(\Omega))$. It is customary to list the eigenvalue sequence as a strictly increasing sequence

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) < \cdots.$$

Theorem 1.1. *Assume that* $\min\{b_1, b_2\} > S^{-1}$.

(i) If $a_i + b_i\lambda_1(\Omega) > 0, i = 1, 2$, *then system* (1.1) *has a global minimizer solution for* $\lambda > \sqrt{(a_1 + b_1\lambda_1(\Omega))(a_2 + b_2\lambda_1(\Omega))}$ *and has no solution for* $0 < \lambda \leq \sqrt{(a_1 + b_1\lambda_1(\Omega))(a_2 + b_2\lambda_1(\Omega))}$.

(ii) If $a_1 + b_1\lambda_1(\Omega) \leq 0$ *or* $a_2 + b_2\lambda_1(\Omega) \leq 0$, *then system* (1.1) *has a global minimizer solution for all* $\lambda > 0$.

(iii) If $a_i + b_i\lambda_k(\Omega) > 0, i = 1, 2$, *then system* (1.1) *has at least k pairs of distinct solutions for* $\lambda > \sqrt{(a_1 + b_1\lambda_k(\Omega))(a_2 + b_2\lambda_k(\Omega))}$.

(iv) If $a_1 + b_1\lambda_k(\Omega) \leq 0$ *or* $a_2 + b_2\lambda_k(\Omega) \leq 0$, *then system* (1.1) *has at least k pairs of distinct solutions for all* $\lambda > 0$.

System (1.1) is often referred to as a non-local problem. Consequently, some classical estimates and methods commonly used in the study of critical semilinear equations are not directly applicable. For instance, given any (PS) sequences $\{(u_n, v_n)\}$, if $(u_n, v_n) \rightharpoonup (u, v)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$, we do not know whether there hold

$$\|(u_n, v_n)\|^2 \int_\Omega \nabla u_n \nabla \varphi dx \to \|(u, v)\|^2 \int_\Omega \nabla u \nabla \varphi dx, \quad \varphi \in H_0^1(\Omega).$$

In (1.1), the exponent of t in the Kirchhoff term $t^4\|(u, v)\|^4$ coincides with that in $t^4 \int_\Omega (u^4 + v^4)$. The (PS) sequences lack the variant Ambrosetti-Rabinowitz condition, which is often employed in Kirchhoff problems [5]. Hence, it is necessary to avoid the possibility of that (PS) sequences not only concentrate but also be unbounded. To tackle these problems, we introduce a constant S_b similar to the Sobolev constant and an eigenfunction corresponding to the eigenvalue $\lambda_k(\Omega)$. Under the assumptions of parameters of (1.1), we prove the (PS) condition, and derive estimates for the associated energy functional. By combining these results with the arguments in [3, 4], we prove the existence and multiplicity of solutions.

The paper is organized as follows: Section 2, presents the necessary preliminaries, while Section 3, contains the proof of Theorem 1.1.

2. Preliminaries

We denote the completion of $C_c^\infty(\mathbb{R}^4)$ under the norm $\|\cdot\|_{1,2} := (\int_{\mathbb{R}^4} |\nabla \cdot|^2)^{1/2}$ by $D^{1,2}(\mathbb{R}^4)$, and $H := H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm $\|(u,v)\| := (b_1\|u\|^2 + b_2\|v\|^2)^{1/2}$, where $\|\cdot\| = (\int_\Omega |\nabla \cdot|^2)^{1/2}$ is the standard norm of $H_0^1(\Omega)$.

It is well known that solutions of (1.1) correspond to the critical points of the functional $I : H \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{2}[\|(u,v)\|^2 + \int_\Omega (a_1u^2 + a_2v^2 - 2\lambda uv)] + \frac{1}{4}[\|(u,v)\|^4 - \int_\Omega (u^4 + v^4)]. \tag{2.1}$$

We say that $(u,v) \in H$ is a weak solution of (1.1), if and only if (u,v) satisfies

$$\begin{aligned}\langle I'(u,v),(\varphi,\psi)\rangle = b_1(1 + \|(u,v)\|^2)\int_\Omega \nabla u\nabla\varphi + b_2(1 + \|(u,v)\|^2)\int_\Omega \nabla v\nabla\psi \\ + \int_\Omega (a_1u\varphi + a_2v\psi - \lambda v\varphi - \lambda u\psi) - \int_\Omega (u^3\varphi + v^3\psi) = 0\end{aligned}$$

for all $(\varphi,\psi) \in H$.

Define

$$S_b := \inf_{(u,v)\in D^{1,2}(\mathbb{R}^4)\times D^{1,2}(\mathbb{R}^4)\setminus\{(0,0)\}} \frac{b_1\int_{\mathbb{R}^4} |\nabla u|^2 + b_2\int_{\mathbb{R}^4} |\nabla v|^2}{(\int_{\mathbb{R}^4} u^4 + \int_{\mathbb{R}^4} v^4)^{1/2}}. \tag{2.2}$$

We get the estimation of S_b.

Lemma 2.1. *Assume that $b_i > 0, i = 1,2$, then $S_b = \min\{b_1,b_2\}S$.*

Proof. Assume that $b_1 \ge b_2$. Define $f(\tau) := \frac{b_1\tau^2+b_2}{\sqrt{\tau^4+1}}, \tau \ge 0$. Then $f'(\tau) \ge 0$ in $[0, \sqrt{\frac{b_1}{b_2}}]$ and $f'(\tau) < 0$ in $(\sqrt{\frac{b_1}{b_2}}, \infty)$. Thus, $\min_{\tau\ge0} f(\tau) = b_2$. For $\epsilon > 0$, define $U(x) := \frac{2\sqrt{2}}{1+|x|^2}$ and $U_\epsilon(x) := \epsilon^{-1}U(\frac{x}{\epsilon})$. Then U_ϵ achieves the best Sobolev constant S. Choosing $u = \tau U_\epsilon(x)$, $v = U_\epsilon(x)$, we get

$$S_b \le \min_{\tau\ge0} f(\tau)S = b_2S.$$

Now suppose that $\{(u_n,v_n)\}$ is a minimizing sequence for S_b. Choose τ_n such that $\|u_n\|_{L^4} = \tau_n\|v_n\|_{L^4}$. Let z_n satisfy $u_n = \tau_nz_n$. Then $\int_{\mathbb{R}^4} v_n^4 = \int_{\mathbb{R}^4} z_n^4$ and

$$S_b + o_n(1) = \frac{b_1\tau_n^2}{\sqrt{\tau_n^4+1}}\frac{\int_{\mathbb{R}^4} |\nabla z_n|^2}{(\int_{\mathbb{R}^4} z_n^4)^{1/2}} + \frac{b_2}{\sqrt{\tau_n^4+1}}\frac{\int_{\mathbb{R}^4} |\nabla v_n|^2}{(\int_{\mathbb{R}^4} v_n^4)^{1/2}} \ge \min_{\tau\ge0} f(\tau)S = b_2S,$$

where $o_n(1) \to 0$ as $n \to \infty$. Hence, $S_b = \min\{b_1,b_2\}S$. □

The constant S_b will play an important role in our compactness result.

Proposition 2.2. *If $\min\{b_1,b_2\} > S^{-1}$, then every bounded sequence $\{(u_n,v_n)\}$ in H such that $I'(u_n,v_n) \to 0$ has a strongly convergent subsequence.*

Proof. Since $I'(u_n, v_n) \to 0$, we have

$$\begin{aligned} &b_1(1+\|(u_n,v_n)\|^2)\int_\Omega \nabla u_n \nabla\varphi + b_2(1+\|(u_n,v_n)\|^2)\int_\Omega \nabla v_n\nabla\psi \\ &+\int_\Omega (a_1 u_n\varphi + a_2 v_n\psi - \lambda v_n\varphi - \lambda u_n\psi) - \int_\Omega (u_n^3\varphi + v_n^3\psi) = o(\|(\varphi,\psi)\|) \end{aligned} \tag{2.3}$$

for all $(\varphi,\psi)\in H$. Passing to a renamed subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ in H, $\|(u_n - u, v_n - v)\|^2 \to \mu$,

$$\|(u_n, v_n)\|^2 \to \|(u, v)\|^2 + \mu =: \eta,$$

$u_n \to u$ and $v_n \to v$ strongly in $L^2(\Omega)$ and a.e. on Ω. So taking $(\varphi,\psi) = (u_n, v_n)$ in (2.3) gives

$$(1+\eta)[\|(u,v)\|^2+\mu] + \int_\Omega (a_1u^2 + a_2v^2 - 2\lambda uv) - \int_\Omega (u_n^4 + v_n^4) = o(1), \tag{2.4}$$

while taking $(\varphi,\psi) = (u, v)$ and passing to the limit gives

$$(1+\eta)\|(u,v)\|^2 + \int_\Omega (a_1u^2 + a_2v^2 - 2\lambda uv) - \int_\Omega (u^4 + v^4) = 0. \tag{2.5}$$

By Brézis-Lieb Lemma [6, Lemma 1.32], one has

$$\int_\Omega u_n^4 - \int_\Omega u^4 = \int_\Omega |u_n-u|^4 + o(1), \quad \int_\Omega v_n^4 - \int_\Omega v^4 = \int_\Omega |v_n - v|^4 + o(1).$$

By subtracting (2.5) from (2.4) and using the extension by zero ($H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^4)$) together with (2.2), we obtain

$$\mu(1+\eta) = \int_\Omega (|u_n-u|^4 + |v_n-v|^4) + o(1) \le S_b^{-2}\|(u_n-u, v_n-v)\|^4 + o(1).$$

Suppose $\mu > 0$; then passing to the limit and noting that $\eta \ge \mu$ gives $(S_b^{-2} - 1)\mu \ge 1$. It follows from Lemma 2.1 that $S_b^{-2} - 1 < 0$. This contradicts $\mu > 0$. Hence, $\mu = 0$ and $(u_n, v_n) \to (u, v)$ in H. □

In order to obtain the multiplicity of solutions, we need the following lemma [7, Theorem 5.2.23].

Lemma 2.3. *Let X be a Banach space and $g \in C^1(X,\mathbb{R})$ be an even function satisfying the (PS) condition. Assume $\alpha < \gamma$ and either $g(\theta) < \alpha$ or $g(\theta) > \gamma$. If further,*

(1) there are an m-dimensional linear subspace E and $\rho > 0$ such that $\sup_{x\in E\cap\partial B_\rho(\theta)} g(x) \le \gamma$,

(2) there are a j-dimensional linear subspace F such that $\inf_{x\in F^\perp} g(x) > \alpha$, where $F^\perp$ is a complementary space of F,

(3) $m > j$,

then g has at least $m - j$ pairs of distinct critical points.

3. Proof of Theorem 1.1

Existence : Let $\min\{b_1, b_2\} > S^{-1}$. For $(u, v) \in H$, we deduce that

$$\begin{aligned} I(u,v) &= \frac{1}{2}[\|(u,v)\|^2 + \int_\Omega (a_1u^2 + a_2v^2 - 2\lambda uv)] + \frac{1}{4}[\|(u,v)\|^4 - \int_\Omega (u^4+v^4)] \\ &\geq \frac{1}{2}\|(u,v)\|^2 - C\|(u,v)\|^2 + \frac{1-S_b^{-2}}{4}\|(u,v)\|^4 \\ &= \frac{1}{2}\|(u,v)\|^2 - C\|(u,v)\|^2 + \frac{\min\{b_1^2,b_2^2\}S^2 - 1}{4\min\{b_1^2,b_2^2\}S^2}\|(u,v)\|^4, \end{aligned} \tag{3.1}$$

where C depends only on a_1, a_2, λ, and $\lambda_1(\Omega)$. It follows that I is coercive and bounded from below in H. Hence, any (PS) sequence has bounded energy and is bounded in H.

Let $(u_{t,s}, v_t) = t(s\phi_1, \phi_1), t, s \in \mathbb{R}$, where ϕ_1 is the eigenfunction corresponding to the eigenvalue $\lambda_1(\Omega)$. We have

$$\begin{aligned} lI(u_{t,s}, v_t) &= \frac{(b_1s^2+b_2)t^2}{2}\|\phi_1\|^2 + \frac{(a_1s^2+a_2-2\lambda s)t^2}{2}\int_\Omega \phi_1^2 \\ &+ \frac{t^4}{4}\|(s\phi_1,\phi_1)\|^4 - \frac{t^4}{4}\int_\Omega (s^4\phi_1^4 + \phi_1^4) \\ &= \frac{[a_1+b_1\lambda_1(\Omega)]s^2 - 2\lambda s + a_2 + b_2\lambda_1(\Omega)}{2\lambda_1(\Omega)}\|\phi_1\|^2t^2 \\ &+ \frac{\|(s\phi_1,\phi_1)\|^4 - \int_\Omega (s^4\phi_1^4+\phi_1^4)}{4}t^4. \end{aligned} \tag{3.2}$$

Since $\min\{b_1, b_2\} > S^{-1}$, we get

$$\|(s\phi_1,\phi_1)\|^4 - \int_\Omega (s^4\phi_1^4 + \phi_1^4) \geq (1 - S_b^{-2})\|(s\phi_1,\phi_1)\|^4 > 0. \tag{3.3}$$

Now we prove that there exists $\bar{s} \in \mathbb{R}$ such that

$$[a_1 + b_1\lambda_1(\Omega)]s^2 - 2\lambda s + a_2 + b_2\lambda_1(\Omega) < 0. \tag{3.4}$$

There are two cases,

(i) assume that $a_i + b_i\lambda_1(\Omega) > 0, i = 1, 2$, then there exists $\bar{s} \in \mathbb{R}$ such that inequality (3.4) holds for $\lambda > \sqrt{(a_1 + b_1\lambda_1(\Omega))(a_2 + b_2\lambda_1(\Omega))}$;

(ii) assume that $a_1 + b_1\lambda_1(\Omega) \leq 0$, then there exists $\bar{s} \in \mathbb{R}$ such that inequality (3.4) holds for all $\lambda > 0$.

It follows from (3.2)–(3.4) that $I(u_{t,\bar{s}}, v_t) < 0$ for $|t|$ small. Thus $m := \inf_{(u,v)\in H} I < 0$. According to Proposition 2.2 and [8, Theorem 4.4], there exists $(u_0, v_0) \in H$ such that $I(u_0, v_0) = m < 0$. Since system (1.1) admits no semi-trivial solutions (namely, $(u_0, 0)$ or $(0, v_0)$) for $\lambda > 0$, we conclude that (u_0, v_0) with $u_0, v_0 \neq 0$ is a global minimizer solution of (1.1). □

Nonexistence : Assume that $a_i + b_i\lambda_1(\Omega) > 0, i = 1, 2$ and $0 < \lambda \leq \sqrt{(a_1 + b_1\lambda_1(\Omega))(a_2 + b_2\lambda_1(\Omega))}$.

Suppose that $(u_0, v_0) \in H$ is a solution of (1.1), then

$$\begin{aligned}0 &= b_1\|u_0\|^2 + b_2\|v_0\|^2 + \int_\Omega (a_1u_0^2 + a_2v_0^2 - 2\lambda u_0v_0) + \|(u_0, v_0)\|^4 - \int_\Omega (u_0^4 + v_0^4)\\ &\geq b_1\|u_0\|^2 + b_2\|v_0\|^2 - \int_\Omega (b_1\lambda_1(\Omega)u_0^2 + b_2\lambda_1(\Omega)v_0^2) + (1 - S_b^{-2})\|(u_0, v_0)\|^4\\ &\geq \frac{\min\{b_1^2, b_2^2\}S^2 - 1}{\min\{b_1^2, b_2^2\}S^2}\|(u_0, v_0)\|^4.\end{aligned} \tag{3.5}$$

Since $\min\{b_1, b_2\} > S^{-1}$, we get $(u_0, v_0) = (0, 0)$. □

Multiplicity : To prove multiplicity, it suffices to show that the energy functional I satisfies the conditions of Lemma 2.3. Let $X = H$ and $w = (u, v) \in X$. Then we obtain $I(-w) = I(w)$, which implies that I is an even function. In view of (3.1), I is coercive and bounded from below in X for $\min\{b_1, b_2\} > S^{-1}$. Hence, any (PS) sequence has bounded energy and is bounded in X. By Proposition 2.2, the energy functional I satisfies the (PS) condition in X.

Now we prove that I satisfies condition (1) of Lemma 2.3. We claim that there exists $\tilde{s} \in \mathbb{R}$ such that

$$[a_1 + b_1\lambda_k(\Omega)]s^2 - 2\lambda s + a_2 + b_2\lambda_k(\Omega) < 0. \tag{3.6}$$

In fact, there are two cases,

(i) assume that $a_i + b_i\lambda_k(\Omega) > 0$; then there exists $\tilde{s} \in \mathbb{R}$ such that inequality (3.6) holds for $\lambda > \sqrt{(a_1 + b_1\lambda_k(\Omega))(a_2 + b_2\lambda_k(\Omega))}$;

(ii) assume that $a_1 + b_1\lambda_k(\Omega) \leq 0$, then there exists $\tilde{s} \in \mathbb{R}$ such that inequality (3.6) holds for all $\lambda > 0$.

Let ϕ_k be the normalized eigenfunction corresponding to the eigenvalue $\lambda_k(\Omega)$. Define

$$E = span\{(\tilde{s}\phi_1, \phi_1), (\tilde{s}\phi_2, \phi_2), \cdots, (\tilde{s}\phi_k, \phi_k)\},$$

then $dim\ E = k$. Since

$$a_1\tilde{s}^2 - 2\lambda\tilde{s} + a_2 \leq [a_1 + b_1\lambda_k(\Omega)]\tilde{s}^2 - 2\lambda\tilde{s} + a_2 + b_2\lambda_k(\Omega) < 0,$$

and $\|u\|^2 \leq \lambda_k(\Omega)\|u\|^2_{L^2(\Omega)}$, $\|v\|^2 \leq \lambda_k(\Omega)\|v\|^2_{L^2(\Omega)}$ for $(u, v) \in E$, one has

$$\begin{aligned}I(u, v) &= \frac{1}{2}\|(u, v)\|^2 + \frac{1}{2}\int_\Omega (a_1u^2 + a_2v^2 - 2\lambda uv) + \frac{1}{4}\|(u, v)\|^4 - \frac{1}{4}\int_\Omega (u^4 + v^4)\\ &\leq \frac{b_1\tilde{s}^2 + b_2}{2}\|v\|^2 + \frac{a_1\tilde{s}^2 - 2\lambda\tilde{s} + a_2}{2}\int_\Omega v^2 + \frac{1}{4}\|(u, v)\|^4\\ &\leq \frac{b_1\tilde{s}^2 + b_2}{2}\|v\|^2 + \frac{a_1\tilde{s}^2 - 2\lambda\tilde{s} + a_2}{2\lambda_k(\Omega)}\|v\|^2 + \frac{1}{4}\|(u, v)\|^4\\ &= \frac{[a_1 + b_1\lambda_k(\Omega)]\tilde{s}^2 - 2\lambda\tilde{s} + a_2 + b_2\lambda_k(\Omega)}{2\lambda_k(\Omega)(b_1\tilde{s}^2 + b_2)}\|(u, v)\|^2 + \frac{1}{4}\|(u, v)\|^4.\end{aligned}$$

In view of (3.6), there exists $\rho > 0$ such that

$$\sup_{(u,v)\in E\cap\partial B_\rho(0,0)} I(u, v) \leq \gamma < 0 = I(0, 0),$$

where $\partial B_\rho(0,0) = \{(u,v) \in X : \|(u,v)\| = \rho\}$.

Noting that $m = \inf_{(u,v)\in X} I < 0$ follows from the proof of existence, we take $F = \emptyset$ and $\alpha = m - 1$. This choice yields $F^\perp = X$ and $\alpha < \gamma$, and satisfies

$$\inf_{(u,v)\in F^\perp} I(u,v) > \alpha.$$

Hence, I satisfies all conditions of Lemma 2.3. It follows that system (1.1) possesses at least k pairs of distinct solutions. □

Author contributions

Xiaofan Wu, Xueliang Duan and Qingquan Yang: Conceptualization, Methodology, Validation, Writing-original draft, and Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

X. Duan is supported by the Natural Science Foundation of Henan Province (Nos. 252300423535), the Science and Technology Project of Henan Province (Nos. 252102320324), and the International Science and Technology Cooperation Project of Henan Province (Nos. 252102520033). X. Wu and X. Duan are supported by the Young Key Teachers Cultivation Program of Zhengzhou Normal University (Nos. QNGG-242805).

Conflict of interest

The authors declare that there is no conflict of interest.

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