Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 857–880.
DOI: 10.3934/math.2026037
Received: 06 November 2025
Revised: 25 December 2025
Accepted: 05 January 2026
Published: 12 January 2026

Research article

Dynamics on traveling wave solutions for Broer-Kaup equation with distributed delay

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Abstract: This paper investigates the traveling wave solutions of the Broer-Kaup equation with distributed delay. Using geometric singular perturbation theory and the Melnikov function method, a qualitative analysis on the traveling wave equation is carried out. By restricting the system to a locally invariant manifold, the delayed traveling wave equation is reduced to a near-Hamiltonian system. A translation transformation is applied to simplify the near-Hamiltonian planar system. Through involution mapping criterion, the monotonicity of the ratio of two Abelian integrals is established, which ensures that the Melnikov function possesses a unique simple zero. Based on Poincaré and heteroclinic bifurcation theory, the sufficient conditions on the persistence of periodic and kink (anti-kink) wave solutions are derived. Moreover, we present numeric simulations to illustrate the given results.

Keywords: delayed Broer-Kaup equation; bifurcation theory; involution mapping criterion; Melnikov function; traveling wave solution
Mathematics Subject Classification: 34C25, 34C60, 37C27

1. Introduction and main results

Nonlinear equations are essential to capturing complex phenomena in fields, including fluid mechanics, optics, plasma physics, and geochemistry [1], with wave processes such as dispersion and dissipation playing a key role. Among these, shallow water wave equations are of particular importance, valued for their dynamic richness and practical utility. A fundamental concept in this area is the soliton, a solitary wave packet that maintains its profile while traveling at a constant velocity [2, 3].

A popular dispersive equation is the Broer-Kaup equation [4], which has the form

$$\begin{cases} U_t + V_x + \frac{1}{2}(U^2)_x = 0, \\ V_t + U_x + (UV)_x + U_{xxx} = 0. \end{cases} \tag{1.1}$$

Equation (1.1) was first derived by Broer in 1975 to describe the bi-directional propagation of long waves in an infinitely long, narrow channel of constant finite depth. The Broer-Kaup Eq (1.1) serves as a fundamental model in nonlinear physics [5, 6], with $U(x,t)$ representing the horizontal velocity and $V(x,t)$ denoting the water surface displacement. U_t represents horizontal velocity. V_x represents the restoring force driven by the slope of the water surface, and $(U^2)_x$ represents the convection term. V_t describes the rate of change in water surface height. U_x is the divergence of speed. $(UV)_x$ describes the nonlinear coupling and transport effects between wave and current while U_{xxx} is the dispersion effect. Its applicability to fields from plasma physics to mathematical biology has driven scholars to extensively study a wide spectrum of solutions for this equation and its generalized forms. Over the years, the Broer-Kaup equation and its variants have been the subject of considerable investigation through a diverse set of analytical and algebraic techniques. A substantial portion of the literature has relied on various expansion methods, including the extended Painlevé expansion [7], the (G/G')-expansion method [8] and its improved version [9], as well as the tanh-expansion and Kudryashov methods [10]. In parallel, symmetry-based frameworks, such as Lie group analysis and Lie symmetry analysis, have been effectively applied to obtain exact and invariant solutions [11–13]. Furthermore, bifurcation theory from dynamical systems offers another powerful avenue for deriving traveling wave solutions [14]; notably, Meng et al. later employed the same methodology to construct solutions with a profile distinct from those reported earlier [15]. Very recently, the homogeneous balance method and Bäcklund transformation have also been applied to investigate solitary wave solutions [16].

The study of traveling wave persistence primarily focuses on nonlinear effects, including diffusion, dissipation, dispersion, and delay, under common dynamical perturbations such as external forcing and singular (or regular) terms. In practical scenarios, some small perturbations are often unavoidable. This is particularly true for shallow water wave equations, where wave propagation is invariably subject to influences from past states. The response at the current time t is jointly determined by its historical states over a past period, and the influence from different historical moments varies, which is always weighted by the delay kernel function. Therefore, it is necessary to incorporate distributed time-delay perturbations into the model, especially in the convection term. However, research on traveling wave solutions in the perturbed Broer-Kaup equation with distributed delay is notably absent, leaving the question of their persistence open. Consequently, in this paper, we consider the delayed Broer-Kaup equation as the form

$$\begin{cases} U_t + V_x + a((f * U)U)_x = 0, \\ V_t + U_x + b(UV)_x + u_{xxx} = 0, \end{cases} \tag{1.2}$$

where a, b are nonzero real numbers. The a represents the strength of delayed convection term, and b represents the intensity of nonlinear interaction term in the entire dynamical system. The kernel function

$$f = e^{-\frac{t}{\tau}}/\tau$$

with $0 < \tau \ll 1$ is the kernel function, satisfying $f\colon [0,\infty) \to [0,\infty)$ and the normalization condition

$$\int_0^{\infty} f(t)dt = 1, \quad tf(t) \in L^1((0,\infty),\mathbb{R}),$$

then $f * u$, represents a convolution that

$$(f * U)(x,t) = \int_{-\infty}^{t} f(t-s)U(x,s)ds.$$

We aim to investigate the traveling wave solutions of Eq (1.2) with $0 < \tau \ll 1$. While the introduction of convolution enriches the dynamics of shallow water wave equations, it also breaks integrability and introduces analytical challenges. For the resulting singularly perturbed system, such as Eq (1.2), geometric singular perturbation theory (GSPT) provides a powerful framework. This approach establishes a locally invariant manifold, thereby reducing the singular problem to a regularly perturbed one on the manifold. The efficacy of GSPT is evidenced by its successful application in proving the existence of traveling waves for numerous perturbed nonlinear equations, including Korteweg-de Vries type equations [17–19], Fisher equations [20], Benjamin-Bona-Mahoney equations [21–23], FitzHugh-Nagumo equations [24, 25], Kadomtsev-Petviashvili modified equal width equations [26, 27], Camassa-Holm equations [28–30], and jerk equations [31]. Partially, Chebyshev criterion [21, 23], the Picard-Fuchs equation method [32–34], and involution mapping criterion [19, 27] are employed to establish the monotonicity of the ratio of Abelian integrals, which possesses a unique zero of the Melnikov function. In this paper, we utilize involution mapping criterion to prove the uniqueness of zero of Melnikov function and then obtain the uniqueness of the periodic wave solution of Eq (1.2).

For Eq (1.2), making traveling wave transformations

$$U(x,t) = u(x-ct) = u(\xi), \quad V(x,t) = v(x-ct) = v(\xi),$$

where c is the wave speed, has following traveling wave equation:

$$\begin{cases} -cu' + v' + a(\eta u)' = 0, \\ -cv' + u' + b(uv)' + u''' = 0, \end{cases} \tag{1.3}$$

where $'$ is the derivative with respect to ξ and

$$\eta(\xi) := \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} u(\xi + ct)dt \tag{1.4}$$

satisfying that

$$\tau c\eta' = \eta - u.$$

Integrating (1.3) twice to obtain

$$\begin{cases} -cu + v + a(\eta u) = g_1, \\ -cv + u + buv + u'' = g_2, \end{cases} \tag{1.5}$$

where g_1 and g_2 are integral constants, we have the following statements.

Theorem 1.1. *When $a > 8b/7 > 0$, $g_1 = 0$ and*

$$g_2 = -c(a+b)(c^2 - 1 - 2c^2(a+b)/9ab)/3ab,$$

the following statements hold for Eq (1.2)*:*

(i) *For any given parameter $c^* \in (0, c_1)$, there exists a wave speed c defined by*

$$c = c^* + O(\tau),$$

such that Eq (1.2) *admits a unique periodic wave solution of the form*

$$u(x-ct) = m\mu((x-ct)/n) + c(a+b)/(3ab).$$

Here,

$$\mu(z) = \mu((x-ct)/n)$$

is a periodic function. The solution has the following asymptotic properties:

$$\lim_{(c^*,\tau)\to(0,0)} u(x-ct) = 0,$$
$$\lim_{(c^*,\tau)\to(c_1,0)} u(x-ct) = m\mu_{\pm}^{kink}\left(\frac{x-ct}{n}\right) + \frac{c(a+b)}{3ab}$$

with

$$\mu_{\pm}^{kink}\left(\frac{x-ct}{n}\right) = \frac{e^{\pm\sqrt{2}\left(\frac{x-ct}{n}\right)} - 1}{e^{\pm\sqrt{2}\left(\frac{x-ct}{n}\right)} + 1}.$$

Further, the amplitude of this solution converges to $2m$ as $(c^*, \tau) \to (c_1, 0)$ and diminishes to 0 as $(c^*, \tau) \to (0, 0)$.

The parameters in the above expressions are defined as follows:

$$c_1 = \sqrt{\frac{9ab}{7a^2 - 8b^2 + 8ab}}, \quad m = \sqrt{\frac{1}{ab}\left(1 - c^2 + \frac{c^2(a+b)^2}{3ab}\right)}, \quad n = \sqrt{\frac{3ab}{3ab - 3abc^2 + c^2(a+b)^2}}.$$

Note that m and n depend on the wave speed c, which itself is a function of c^* and τ.

(ii) *For*

$$c = c_1 + O(\tau),$$

Eq (1.2) *admits a pair of kink and anti-kink wave solutions given by*

$$u_{1,2}(x-ct) = m\mu_{1,2}\left(\frac{x-ct}{n}\right) + \frac{c(a+b)}{3ab},$$

which satisfies

$$\lim_{\tau\to 0} u_{1,2}(x-ct) = m\mu_{\pm}^{kink}\left(\frac{x-ct}{n}\right) + \frac{c(a+b)}{3ab}.$$

Here, $\mu_{\pm}^{kink}(\cdot)$ is defined in (i).

To the best of our knowledge, the existence of traveling wave solutions for the BK equation incorporating distributed delays remains unexplored in the existing literature. Therefore, this work presents a novel investigation into this problem.

This paper is structured as follows: Some preliminaries including geometric singular perturbation theory and near-Hamiltonian systems, are introduced in Section 2. Section 3 establishes the proof of Theorem 1.1. The system is firstly reduced via GSPT and a translation transformation. Subsequently, involution mapping criterion is employed to demonstrate the strict monotonicity of a ratio of Abelian integrals, from which the uniqueness of a simple zero for the Melnikov function is concluded. The persistence of periodic and kink (anti-kink) wave solutions is then established through Poincaré and heteroclinic bifurcation theory, respectively. Finally, Section 4 provides a brief conclusion and presents numerical simulations.

2. Preliminaries

In this section, we introduce the relevant definitions and lemmas of geometric singular perturbation theory as well as near-Hamiltonian systems.

2.1. Geometric singular perturbation theory

Consider a system

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{2.1}$$

where

$$\prime = \frac{d}{dt}, \quad 0 < \varepsilon \ll 1$$

is a real and small parameter and

$$x = (x_1, x_2, \ldots, x_k)^T \in R^k, \quad y = (y_1, y_2, \ldots, y_l)^T \in R^l.$$

Both f and g are C^∞-smooth on $U \times I$ with $U \subset R^{k+l}$ open and I being an open interval containing 0.

With a change of time scaling $z = \varepsilon t$, system (2.1) can be written as

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = g(x, y, \varepsilon), \end{cases} \tag{2.2}$$

where

$$\cdot = \frac{d}{dz}.$$

The time scale z is slow, and t is fast since $0 < \varepsilon \ll 1$. For non-zero ε, the systems (2.1) and (2.2) are equivalent. This leads to the terminology where (2.1) is the so-called fast system and (2.2) is the slow one. Letting $\varepsilon \to 0$ in (2.1) and it obtains the layer system

$$\begin{cases} x'(t) = f(x, y, 0), \\ y'(t) = 0. \end{cases} \tag{2.3}$$

In this context, x and y are referred to as the fast and slow variables, respectively. Let $\varepsilon \to 0$ in (2.2), the limit system is given by

$$\begin{cases} f(x, y, 0) = 0, \\ \dot{y} = g(x, y, 0). \end{cases} \tag{2.4}$$

It only makes sense if

$$f(x, y, 0) = 0.$$

Assume that there is a given l-dimensional manifold M_0, which is contained in the set

$$\{f(x, y, 0) = 0\}.$$

Definition 2.1. *The manifold M_0 is called normally hyperbolic provided that, for every point on M_0, the linearization of the layer system* (2.3) *possesses precisely l eigenvalues with zero real parts (i.e., located on the imaginary axis).*

Definition 2.2. *A set M is said to be locally invariant under the flow given in (2.1) if, for every $x \in M$, the following holds: Whenever the trajectory segment $x \cdot [0, t]$ remains within a certain neighborhood V of M, it must in fact remain entirely in M. The same condition applies in reverse time for $t < 0$ with the interval $[0, t]$ replaced by $[t, 0]$. This means there exists a neighborhood V such that no trajectory can exit M without first leaving V. Here, the notation $x \cdot t$ denotes the position at time t along the flow starting from the initial condition x.*

Assume that there is a C^∞ function $h^0(y)$, for $y \in K$, with K being a compact domain in R^l, such that

$$M_0 = \{(x, y) : x = h^0(y)\}.$$

Thus, we establish the following lemmas pertaining to the geometric theory of singular perturbations.

Lemma 2.1. *For $\varepsilon > 0$ is sufficiently small, there exists a manifold M_ε lying within $O(\varepsilon)$ of M_0. M_ε is diffeomorphic to M_0 and locally invariant under the flow of* (2.1)*, and is C^r in x, y and ε, for any $0 < r < +\infty$.*

Lemma 2.2. *For $\varepsilon > 0$ is sufficiently small, there exists a function*

$$x = h^\varepsilon(y)$$

defining on K such that the graph

$$M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\}$$

is locally invariant under (2.1)*. Additionally, $h^\varepsilon(y)$ is jointly C^r in y and ε for any $0 < r < +\infty$. Furthermore, M_ε admits locally invariant stable and unstable manifolds, denoted as $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$. These lie within an $O(\varepsilon)$ neighborhood and are C^r-diffeomorphic to their counterparts $W^s(M_0)$ and $W^u(M_0)$ associated with the critical manifold M_0.*

2.2. Near-Hamiltonian system

Consider a C^∞ system of the form

$$\begin{cases} x'(t) = H_y(x,y) + \varepsilon p(x,y,\varepsilon), \\ y'(t) = -H_x(x,y) + \varepsilon q(x,y,\varepsilon), \end{cases} \tag{2.5}$$

where $H(x,y)$, $p(x,y,\varepsilon)$ and $q(x,y,\varepsilon)$ are C^∞ functions and $0 < \varepsilon \ll 1$. For $\varepsilon = 0$, Eq (2.5) becomes

$$\begin{cases} x'(t) = H_y(x,y), \\ y'(t) = -H_x(x,y), \end{cases} \tag{2.6}$$

which is a Hamiltonian system. Thus, Eq (2.5) is called a near-Hamiltonian system. For Eq (2.6), we suppose there exists a family of periodic orbits given by

$$\Gamma_h : H(x,y) = h, \quad h \in (\alpha, \beta),$$

such that Γ_h tends to an elementary center Γ_α as $h \to \alpha$ and to an invariant curve Γ_β as $h \to \beta$. The boundary Γ_β typically forms either a homoclinic loop (comprising of a saddle and its connection) or a heteroclinic loop (involving at least two saddles and their connecting orbits).

If (2.5) has a limit cycle $\Gamma_{h,\varepsilon}$, then the limit of the cycle as $\varepsilon \to 0$ is either the center Γ_α, a periodic orbit Γ_h with $h \in (\alpha, \beta)$, or the boundary Γ_β. In this case, it is said that the limit cycle $\Gamma_{h,\varepsilon}$ is generated from Γ_h. In order to study the number of limit cycles, we usually consider the number of simple zero of the Melnikov function

$$M(h) = \oint_{H(x,y)=h} q dx - p dy, \tag{2.7}$$

which is also called an Abelian integral. The Melnikov function $M(h)$ can often be employed to determine the cyclicity associated with a periodic orbit. The search for the maximum number of isolated zeros of such integrals is called the weakened (or infinitesimal or tangential) Hilbert's 16th problem.

Lemma 2.3. *(Poincaré-Pontryagin-Andronov) Suppose that $M(h,\delta)$ has different zeros for $h \in J$. The following conclusions hold within the periodic annulus Γ_h:*

(i) *If there exists $h^* \in (\alpha, \beta)$ such that*

$$M_1(h^*, \delta) = 0 \quad \text{and} \quad M_1'(h^*, \delta) \neq 0,$$

then (2.5) *admits a unique $\Gamma_{h^*,\varepsilon}$ bifurcate from Γ_{h^*}. Further, $\Gamma_{h^*,\varepsilon} \to \Gamma_{h^*}$ as $\varepsilon \to 0$.*

(ii) *If*

$$M'(h^*, \delta) = M''(h^*, \delta) = \cdots = M^{(k-1)}(h^*, \delta) = 0, \quad M^{(k)}(h^*, \delta) \neq 0,$$

then (2.5) *admits at most k limit cycles bifurcating from Γ_{h^*} within the periodic annulus Γ_h.*

(iii) *The total number of limit cycles (counting multiplicities) of* (2.5) *that bifurcate from Γ_{h^*} within Γ_h is determined by the number of isolated zeros (counting multiplicities) of $M(h^*, \delta)$ for $h^* \in (\alpha, \beta)$.*

In the expressions above, the prime notation $'$ denotes the derivative with respect to the variable h.

2.3. The criteria

In this subsection, we introduce the criteria for the monotonicity of the ratio of two Abelian integrals proposed by Liu et al. [35].

Consider the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = -\Psi'(x), \tag{2.8}$$

which has a Hamiltonian function in the form

$$\mathcal{H}(x, y) = \frac{1}{2}y^2 + \Psi(x),$$

where $\Psi(x) \in C^2(\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R}$. Assume that there is a number $a \in (\alpha, \beta)$ such that $\Psi(a) = 0$, and the following hypothesis is satisfied:

$$(\tilde{H}_1): \quad \Psi'(x)(x - a) > 0, \quad \text{for all } x \in (\alpha, \beta) \setminus \{a\}.$$

It is obvious that $(a, 0)$ is a center of system (2.8) and $\Psi(x) > 0$ for $x \in (\alpha, \beta) \setminus \{a\}$. Let

$$h_s = \Psi(\alpha) = \Psi(\beta) > h_c = \mathcal{H}(a, 0) = \Psi(a) = 0$$

and denote by

$$\Gamma_h = \{(x, y) \mid \mathcal{H}(x, y) = h\}.$$

For any $h \in (h_c, h_s)$, Γ_h includes a closed orbit. It is obvious that there exists an involution σ defined in (α, β) such that

$$\Psi(x) = \Psi(\sigma(x)).$$

Here, a mapping σ: $I \to I$ is called an involution if

$$\sigma^2 = Id \quad \text{and} \quad \sigma \neq Id.$$

Note that

$$\sigma(a) = a$$

and

$$(x - a)(\sigma(x) - a) < 0$$

for $x \in (\alpha, \beta) \setminus \{a\}$.

Define two Abelian integrals

$$I_i(h) = \int_{\Gamma_h} f_i(x) y \mathrm{d}x, \tag{2.9}$$

where $f_i \in C^1(\alpha, \beta)$ and $i = 1, 2$.

Assume that the following hypothesis is also satisfied:

$$(\tilde{H}_2): \quad \frac{f_1(x)}{\Psi'(x)} - \frac{f_1(\sigma(x))}{\Psi'(\sigma(x))} > 0, \ \text{for all } x \in (a, \beta).$$

Define two functions $G_1(x)$ and $G_2(x)$ as follows:

$$G_1(x) = \int_{\sigma(x)}^{x} f_1(t)\mathrm{d}t, \quad G_2(x) = \int_{\sigma(x)}^{x} f_2(t)\mathrm{d}t. \tag{2.10}$$

It follows from

$$\Psi(x) = \Psi(\sigma(x))$$

and hypothesis $(\tilde{H}_1)$ that

$$\frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} = \frac{\Psi'(x)}{\Psi'(\sigma(x))} < 0$$

for $x \in (a, \beta)$. By direct calculation, we have

$$\begin{aligned} G_1'(x) &= f_1(x) - f_1(\sigma(x))\sigma'(x) \\ &= f_1(x) - f_1(\sigma(x))\frac{\Psi'(x)}{\Psi'(\sigma(x))} \\ &= \Psi'(x)\left(\frac{f_1(x)}{\Psi'(x)} - \frac{f_1(\sigma(x))}{\Psi'(\sigma(x))}\right). \end{aligned} \tag{2.11}$$

Thus, under the hypothesis $(\tilde{H}_1)$, the hypothesis $(\tilde{H}_2)$ can be replaced by the following form

$$(\tilde{H}_2') : \quad G_1'(x) > 0, \text{ for all } x \in (a, \beta).$$

Then, we have the following lemma, which is equivalent to [35, Theorem 2.1].

Lemma 2.4. *Suppose that the hypotheses $(\tilde{H}_1)$ and $(\tilde{H}_2')$ are satisfied. Let*

$$\xi(x) = \frac{G_2(x)}{G_1(x)} \quad \text{and} \quad P(h) = \frac{I_2(h)}{I_1(h)}. \tag{2.12}$$

Then $\xi'(x) < 0$ (resp. > 0) in (a, β) implies $P'(h) < 0$ (resp. > 0) in (h_c, h_s).

3. Proof of main results

According to the introduction on GSPT, a near-Hamiltonian system, we investigate the traveling wave solutions of Eq (1.2). The persistence of both periodic and kink (anti-kink) wave solutions is established by analyzing the corresponding traveling wave equation. Using GSPT, the singularly perturbed system is regularized. To handle the complexity arising from multiple parameters, a translation transformation is subsequently introduced. Finally, the uniqueness of the zero for the Melnikov function is confirmed via involution mapping criterion.

From the first equation of system (1.5), it yields

$$v = cu - a\eta u,$$

and then substituting v into the second equation results in

$$(1 - c^2)u + bcu^2 + ac\eta u - ab\eta u^2 + u'' = g_2. \tag{3.1}$$

To reformulate Eq (3.1), we introduce the variable

$$y = u'$$

and then

$$y' = u'' = g_2 - ((1-c^2)u + bcu^2 + ac\eta u - ab\eta u^2),$$

which leads to an equivalent three-dimensional system.

$$\begin{cases} u' = y, \\ y' = g_2 + (c^2-1)u - bcu^2 - ac\eta u + ab\eta u^2, \\ \tau\eta' = \dfrac{1}{c}(\eta - u). \end{cases} \tag{3.2}$$

The autonomy of system (3.2) is evident. It possesses the characteristics of a singularly perturbed system. Under the limit $\tau \to 0$, it can be observed that $\eta \to u$. To analyze the case of $\tau \neq 0$, we employ the time-scale transformation

$$\xi = \tau s,$$

which converts the slow system (3.2) into its corresponding fast-time scale formulation

$$\begin{cases} \dfrac{du}{ds} = \tau y, \\ \dfrac{dy}{ds} = \tau\left(g_2 + (c^2-1)u - bcu^2 - ac\eta u + ab\eta u^2\right), \\ \dfrac{d\eta}{ds} = \dfrac{1}{c}(\eta - u). \end{cases} \tag{3.3}$$

For $\tau > 0$, systems (3.2) and (3.3) are equivalent. In the limiting case $\tau = 0$, they respectively reduce to the following systems:

$$\begin{cases} u' = y, \\ y' = g_2 + (c^2-1)u - bcu^2 - ac\eta u + ab\eta u^2, \\ 0 = \dfrac{1}{c}(\eta - u), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \dfrac{du}{ds} = 0, \\ \dfrac{dy}{ds} = 0, \\ \dfrac{d\eta}{ds} = \dfrac{1}{c}(\eta - u). \end{cases} \tag{3.5}$$

We designate system (3.4) as the reduced system and system (3.5) as the layer system. Since the set

$$M_0 = \{(u, y, \eta) \in \mathbb{R}^3 : \eta = u\}$$

defines a slow invariant manifold, the linearized matrix of system $(3.5)|_{\tau=0}$ is given as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/c & 0 & 1/c \end{pmatrix}.$$

The eigenvalues of the system are readily found to be 0, 0, and $1/c$. Since the number of purely imaginary eigenvalues equals $\dim(M_0)$ and the remaining eigenvalue is hyperbolic, M_0 is normally hyperbolic. According to GSPT indicated in [36, 37], this guarantees, for sufficiently small $\tau > 0$, the existence of a locally invariant manifold M_τ for system (3.2) under the flow of (3.3). This manifold is diffeomorphic to M_0 and can be expressed as

$$M_\tau = \{(u, y, \eta) \in \mathbb{R}^3 : \eta = u + G(u, y, \tau)\},$$

with $G(u, y, \tau)$ being a smooth function and $G(u, y, 0) = 0$. Its Taylor expansion is

$$G(u, y, \tau) = \tau G_1(u, y) + O(\tau^2).$$

The manifold is

$$\eta = u + \tau G_1(u, y) + O(\tau^2)$$

and then

$$\eta' = y + O(\tau).$$

Substitution into the third equation of (3.2) obtains

$$\tau\eta = \tau y + O(\tau^2) = \tau G_1/c + O(\tau),$$

which gives

$$\tau(y + O(\tau)) = \tau G_1/c + O(\tau^2).$$

Matching the τ-coefficients determines

$$G_1(u, y) = cy,$$

leading to a regularly perturbed system on M_τ as the form

$$\begin{cases} u' = y, \\ y' = g_2 + (c^2 - 1)u - (bc + ac)u^2 + abu^3 + ac\tau(-cuy + bu^2y) + O(\tau^2), \end{cases} \tag{3.6}$$

which is a planar near-Hamiltonian system. Introducing

$$u = \phi + c(a + b)/(3ab)$$

and setting

$$g_2 = -c(a + b)/(3ab)(c^2 - 1 - 2c^2(a + b)^2/9ab),$$

Eq (3.6) is changed into

$$\begin{cases} \phi' = y, \\ y' = \left(c^2 - 1 - \dfrac{c^2(a + b)^2}{3ab}\right)\phi + ab\phi^3 + ac\tau R_1(\phi)y + O(\tau^2), \end{cases} \tag{3.7}$$

where

$$R_1(\phi) = -\frac{c^2(a+b)}{3ab}\left(1 - \frac{a+b}{3a}\right) - \left(c - \frac{2c(a+b)}{3a}\right)\phi + b\phi^2.$$

By introducing the new variables

$$\phi = m\mu, \quad \xi = nz, \quad \text{and} \quad \nu := d\mu/dz,$$

system (3.7) is converted to

$$\begin{cases} \frac{d\mu}{dz} = \nu, \\ \frac{d\nu}{dz} = -\mu + \mu^3 + \varepsilon R_2(\mu)\nu + O(\varepsilon^2), \end{cases} \tag{3.8}$$

where m, n are defined in Theorem 1.1 and $\varepsilon = acn\tau$, and

$$R_2(\mu) = -\frac{c^2(a+b)}{3ab}\left(1 - \frac{a+b}{3a}\right) - \left(c - \frac{2c(a+b)}{3a}\right)m\mu + bm^2\mu^2.$$

When $\varepsilon \to 0$, Eq (3.8) reduces to a Hamiltonian system

$$\begin{cases} \frac{d\mu}{dz} = \nu, \\ \frac{d\nu}{dz} = -\mu + \mu^3, \end{cases} \tag{3.9}$$

which has an energy function

$$H(\mu,\nu) = \frac{\nu^2}{2} + \frac{\mu^2}{2} - \frac{\mu^4}{4}. \tag{3.10}$$

The potential energy function and the configuration of its equilibrium points dictate the dynamics of system (3.9). The system possesses three equilibria: a center at $E_0(0,0)$ and two saddles at $E_1(1,0)$ and $E_2(-1,0)$. The corresponding energy levels are

$$H(0,0) = 0 \quad \text{and} \quad H(\pm 1, 0) = 1/4.$$

At the energy $h = 1/4$, there exist two heteroclinic orbits, denoted $\Gamma_{+1/4}$ and $\Gamma_{-1/4}$, which connect the saddles E_1 and E_2. For energies $h \in (0, 1/4)$, the level set

$$H(\mu,\nu) = h$$

defines a family of periodic orbits Γ_h, enclosed by the heteroclinic loop

$$\Gamma_{1/4} = \Gamma_{+1/4} \cup \Gamma_{-1/4} \cup E_{1,2},$$

as illustrated in Figure 1.

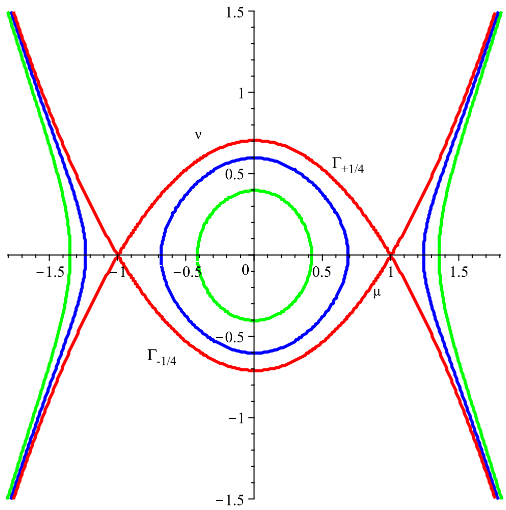


Figure 1. Phase portrait.

The period of Γ_h is denoted by $T(h)$. For $h \in (0, 1/4)$, it has

$$\nu = (\mu^2 - a_1^2)(\mu^2 - a_2^2)/2,$$

where

$$a_1 = (1 - \sqrt{1 - 4h})^{1/2}, \quad a_2 = (1 + \sqrt{1 - 4h})^{1/2}.$$

Combining (3.10) with an elliptic integral formula, we can get the exact explicit expression of Γ_h as the form

$$\mu_1^{peri}(z) = a_1 a_2 \left(\frac{\mathrm{sn}^2(\omega z, k) - 1}{a_2^2 \mathrm{sn}^2(\omega z, k) - a_1^2} \right)^{\frac{1}{2}}, \tag{3.11}$$

where $\mathrm{sn}(\cdot, \cdot)$ is the Jacobian elliptic function,

$$\omega := a_2/\sqrt{2}, \quad k := a_1/a_2,$$

and the exact expressions of $L_\pm$ are given as

$$\mu_\pm^{kink}(z) = \frac{e^{\pm\sqrt{2}z} - 1}{e^{\pm\sqrt{2}z} + 1}$$

via direct computations.

Bifurcation theory is employed to analyze the heteroclinic and periodic orbits in system (3.8). For an energy level $h \in (0, 1/4)$, the point $A(h)$ corresponds to the rightmost μ-axis intersection of the periodic orbit Γ_h at $z = 0$. When perturbed by a small ε and a slight energy shift h_ε, the orbit segment Γ_{h_ε} is defined as the trajectory starting at $A(h)$ and ending at the next positive μ-axis intersection $B(h_\varepsilon)$, reached at time

$$z = z(\varepsilon).$$

By continuity, we have

$$\lim_{\varepsilon \to 0} \Gamma_{h_\varepsilon} = \Gamma_h, \quad \lim_{\varepsilon \to 0} B(h_\varepsilon) = A(h), \quad \lim_{\varepsilon \to 0} z(\varepsilon) = T(h).$$

The displacement function between $B(h_\varepsilon)$ and $A(h)$ is

$$\begin{aligned}
d(h,\varepsilon) &= \int_{\widehat{AB}} dH = \int_{\widehat{AB}} H_\mu d\mu + H_\nu d\nu \\
&= \int_{\widehat{AB}} (-\mu+\mu^3)d\mu + \nu d\nu \\
&= \int_0^{z(\varepsilon)} \left\{(-\mu+\mu^3)\nu + \left(\mu-\mu^3+\varepsilon(R_2(\mu)+O(\varepsilon))\nu^2\right)\right\} dz \\
&= \varepsilon \int_0^{z(\varepsilon)} (R_2(\mu)\nu^2 + O(\varepsilon))dz \\
&\triangleq \varepsilon F(h,\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
F(h,\varepsilon) &= \int_0^T R_2(\mu)\nu^2 dz + O(\varepsilon) \\
&= \oint_{\Gamma_h} R_2(\mu)\nu d\mu + O(\varepsilon).
\end{aligned}$$

Given the symmetry of Γ_h (Figure 1), we have

$$\oint_{\Gamma_h} \mu\nu d\mu \equiv 0,$$

leading to

$$F(h,\varepsilon) = M(h) + O(\varepsilon),$$

where

$$\begin{aligned}
M(h) &= -\frac{c^2(a+b)}{3ab}\left(1-\frac{a+b}{3a}\right)J_0(h) + bm^2 J_2(h) \\
&= J_0(h)\left(-\frac{c^2(a+b)}{3ab}\left(1-\frac{a+b}{3a}\right) + bm^2\frac{J_2(h)}{J_0(h)}\right)
\end{aligned} \tag{3.12}$$

is called the first-order Melnikov function. Here,

$$J_0(h) := \oint_{\Gamma_h} \nu d\mu = \iint_{int\Gamma_h} d\mu d\nu > 0, \qquad J_2(h) := \oint_{\Gamma_h} \mu^2 \nu d\mu$$

for $h \in (0, 1/4)$. Define the ratio function as

$$P(h) := J_2(h)/J_0(h).$$

According to Poincaré bifurcation theory, the limit cycles of system (3.8) are in one-to-one correspondence with the isolated zeros of the displacement function $d(h,\varepsilon)$. Consequently, establishing the monotonicity of $P(h)$ plays a crucial role in uniqueness of the root for

$$M(h) = 0.$$

The properties of $P(h)$ are detailed in the following lemma.

Lemma 3.1. *Over the interval* $h \in (0, 1/4)$*,* $P(h)$ *increases monotonically from* 0 *to* $1/5$*.*

Proof. First, we consider $P(h)$ in a small neighborhood of the endpoints of $(0, 1/4)$. When $0 < h \ll 1$, setting

$$\mu = r\cos\theta, \nu = r\sin\theta,$$

it follows from

$$H(\mu, \nu) - h = 0$$

that

$$-\frac{1}{4}r^4\cos^4\theta + \frac{1}{2}r^2 - h = 0.$$

Define a function

$$\Phi(r, h, \theta) := \frac{r}{\sqrt{2}}\sqrt{1 - \frac{1}{2}r^2\cos^4\theta} - \sqrt{h}, \tag{3.13}$$

which satisfies that

$$\Phi(0, 0, \theta) = 0, \quad \left.\frac{\partial\widetilde{\Phi}(r, h, \theta)}{\partial r}\right|_{(r,h)=(0,0)} = \frac{1}{\sqrt{2}} \neq 0.$$

By implicit function theorem, for (r, h) near $(0, 0)$, there exists an analytic $r(h, \theta)$ such that

$$\Phi(r(h, \theta), h, \theta) = 0.$$

Thus, from (3.13), we obtain the expression of $r(h, \theta)$ as form

$$r(h, \theta) = \sqrt{2h} + \frac{\sqrt{2}h^{\frac{3}{2}}}{2}\cos^4\theta + o(h^{3/2}). \tag{3.14}$$

From Green's formula, it can be derived that

$$J_n(h) = \oint_{\Gamma_h} \mu^n \nu d\mu = \iint_{int\Gamma_h} \mu^n d\mu d\nu = \int_0^{2\pi} d\theta \int_0^{r(h,\theta)} (r\cos\theta)^n r dr. \tag{3.15}$$

Substituting (3.14) into (3.15), we obtain

$$J_0(h) = 2\pi h + \frac{3}{4}\pi h^2 + \frac{35}{256}\pi h^3 + O\left(h^4\right), \quad J_2(h) = \pi h^2 + \frac{5}{4}\pi h^3 + O\left(h^4\right),$$

for $0 < h \ll 1$. Thus,

$$\lim_{h\to 0} P(h) = \lim_{h\to 0}\frac{J_2(h)}{J_0(h)} = 0,$$

and then

$$\lim_{h\to 0} P(h) = 0$$

holds.

For

$$0 < 1/4 - h \ll 1,$$

Eq (3.10) gives

$$\nu = \pm(1 - \mu^2)/\sqrt{2}.$$

Performing a direct computation on this result shows

$$\lim_{h\to 1/4} J_0(h) = \oint_{\Gamma_{1/4}} \nu d\mu = \sqrt{2}\int_{-1}^{1}(1-\mu^2)d\mu = \frac{4\sqrt{2}}{3},$$
$$\lim_{h\to 1/4} J_2(h) = \oint_{\Gamma_{1/4}} \mu^2\nu d\mu = \sqrt{2}\int_{-1}^{1}\mu^2(1-\mu^2)d\mu = \frac{4\sqrt{2}}{15}.$$

Consequently, it has

$$\lim_{h\to 1/4} P(h) = 1/5.$$

We now prove the strict monotonicity of $P(h)$ on $h \in (0, 1/4)$ by involution mapping criterion. From the relation

$$H(\mu, \nu) = h,$$

we can express

$$\nu = \nu(\mu, h) = \sqrt{2h - \mu^2 + \mu^4/2}$$

for $h \in (0, 1/4)$. Differentiating this expression with respect to h to yields

$$\partial\nu/\partial h = 1/\nu.$$

Thus, we have

$$J_i'(h) = \oint_{\Gamma_h} \mu^i \frac{\partial \nu}{\partial h} d\mu = \oint_{\Gamma_h} \frac{\mu^i}{\nu} d\mu,$$

which implies

$$J_0'(h) = \oint_{\Gamma_h} \frac{1}{\nu} d\mu = \int_0^{T(h)} dz = T(h) > 0.$$

Thus, $J_0(h)$ is strictly increasing for $h \in (0, 1/4)$. As $h \to 0$, the orbit Γ_h approaches the point $(1, 0)$ implying that $\nu \to 0$. Consequently, we obtain

$$J_0(0) = \lim_{h\to 0} \oint_{\Gamma_h} \nu d\mu = \lim_{h\to 0} \int_0^{T(h)} \nu^2 dz = 0.$$

Combining with $J_0'(h) > 0$ to obtain $J_0(h) > 0$ for $h \in (0, 1/4)$, the potential energy function of (3.9) is denoted as

$$W(\mu) := (2\mu^2 - \mu^4)/4,$$

and a discriminant equation is defined as

$$C(\mu) := \mu^2/3.$$

Since

$$\mu W'(\mu) = \mu^2(1 - \mu^2) > 0, \quad \mu \in (-1, 0) \cup (0, 1)$$

always holds, it can be inferred from [35, Theorem 2.1] that the monotonicity of $P(h)$ is consistent with the monotonicity of $C(\mu)$. Obviously,

$$C'(\mu) = 2\mu/3 \neq 0, \quad \mu \in (-1, 0) \cup (0, 1),$$

which implies that $P'(h) \neq 0$ on $h \in (0, 1/4)$. The proof is finished.

□

Proof of Theorem 1.1. Equation (3.12) shows that a necessary and sufficient condition for

$$M(h) = 0$$

is

$$-\frac{c^2(a+b)}{3ab}\left(1 - \frac{a+b}{3a}\right) + bm^2P(h). \tag{3.16}$$

Notice that

$$m^2 = \frac{1}{ab}\left(1 - c^2 + \frac{c^2(a+b)^2}{3ab}\right),$$

when $a > 8b/7$. It follows from (3.16) that

$$c(h) = \sqrt{\frac{9abP(h)}{(a+b)(2a-b) - 3(a+b)^2P(h) + 9abP(h)}}.$$

Applying the conclusions of Lemma 3.1 yields

$$c'(h) = \frac{P'(h)[9ab(1-c^2) + 3c^2(a+b)^2]}{(a+b)(2a-b) - 3(a+b)^2P(h) + 9abP(h)} > 0,$$

which implies that $c(h)$ is strictly increasing for $h \in (0, 1/4)$. Here, we assume that $a > 0$ and $b > 0$. In order to find out the bound of $c(h)$, $a > 8b/7$ is restricted. Therefore, we obtain

$$\lim_{h\to 0} c(h) = 0, \quad c_1 := \lim_{h\to 1/4} c(h) = \sqrt{\frac{9ab}{7a^2 - 8b^2 + 8ab}} < 1.$$

Therefore, for any given $c^* \in (0, c_1)$ and $h \in (0, 1/4)$, there exists a unique h^* satisfying

$$M(h^*) = 0.$$

Moreover, since

$$\frac{\partial M(h,\delta)}{\partial h}|_{h=h^*} \neq 0,$$

this root is simple. An application of the implicit function theorem guarantees that, for sufficiently small $\tau > 0$, there corresponds a wave speed

$$c = c^* + O(\varepsilon),$$

such that the displacement function $d(h, \varepsilon)$ possesses one zero at

$$h = h^* + O(\varepsilon).$$

This implies that system (3.8) admits a unique limit cycle $\Gamma_{h^*,\varepsilon}$, which in turn establishes the existence of a unique periodic wave solution

$$u(x - ct) = m\mu((x - ct)/n) + c(a + b)/(3ab)$$

for Eq (1.2). Here, $\mu(z)$ with

$$z = (x - ct)/n$$

is the solution profile corresponding to $\Gamma_{h,\varepsilon}$. Furthermore, the limit cycle $\Gamma_{h^*,\varepsilon}$ exhibits the expected asymptotic behavior: It contracts to the center $(0, 0)$ as $(\varepsilon, c) \rightarrow (0, 0)$, and it approaches the heteroclinic loop $\Gamma_{\pm 1/4}$ as $(\varepsilon, c) \rightarrow (0, c_1)$. This completes the proof of Theorem 1.1 (i).

We remark that $a > 8b/7$ is limited to ensure the radicand in the expression for c_1 is positive. Since the radicands in the expressions for n and m contain c, they must likewise be positive. Therefore, we restrict to guarantee $a > 8b/7$ that all numerical calculations remain well-defined.

We now proceed to prove Theorem 1.1 (ii). As established in the analysis of system (3.9), the energy level

$$h = 1/4$$

corresponds to the heteroclinic loop $\Gamma_{\pm 1/4}$, whose profile satisfies

$$\nu = \pm(1 - \mu^2)/\sqrt{2}$$

according to (3.10). For a small perturbation $0 < \varepsilon \ll 1$, let $E_{1,\varepsilon}$ and $E_{2,\varepsilon}$ denote the saddles near E_1 and E_2, respectively. We then consider the unstable manifold of $E_{2,\varepsilon}$, denoted $\Gamma^u_{+,\varepsilon}$, and the stable manifold of $E_{1,\varepsilon}$, denoted $\Gamma^s_{+,\varepsilon}$. Suppose these manifolds intersect the ν-axis at ν_1 and ν_2, respectively. The distance between ν_1 and ν_2 is then expressed as

$$d_+(1/4, \varepsilon, c) = \varepsilon M_+(1/4, c) + O(\varepsilon^2),$$

where $M_+(1/4, c)$ is the Melnikov function for system (3.8). By [38], $M_+(1/4, c)$ has the form

$$\begin{aligned} M_+(1/4, c) &= \int_{-1}^{1} \left(2c^2 + 2\alpha\mu^2\right) \nu d\mu \\ &= \sqrt{2} \int_{-1}^{1} (c^2 + \alpha\mu^2)(1 - \mu^2) d\mu \\ &= 4\left(\frac{c^2}{3} + \frac{\alpha}{15}\right). \end{aligned}$$

Consequently, we have

$$M_+(1/4, c) = 0$$

at $c = c_1$. It can be further verified that

$$\frac{\partial M_+(1/4, c)}{\partial c} \neq 0$$

at this point. By the implicit function theorem, there exists a wave speed

$$c = c_1 + O(\varepsilon)$$

for which $\Gamma^u_{+,\varepsilon}$ and $\Gamma^s_{+,\varepsilon}$ intersect transversely, implying the existence of a unique heteroclinic orbit $\Gamma_{+,\varepsilon}$ for system (3.8) near $\Gamma_{+1/4}$, with $\Gamma_{+,\varepsilon} \rightarrow \Gamma_{+,\varepsilon}$ as $\varepsilon \rightarrow 0$. A parallel argument establishes the existence of another unique heteroclinic orbit $\Gamma_{-,\varepsilon}$ near $\Gamma_{-1/4}$ satisfying the same asymptotic condition. Given the correspondence between kink (anti-kink) wave solutions and heteroclinic orbits, we conclude that Eq (1.2) admits one kink and one anti-kink wave solution of the form

$$u(x-ct) = m\mu((x-ct)/n) + c(a+b)/(3ab)$$

with the wave speed

$$c = c_1 + O(\varepsilon),$$

where

$$\mu((x-ct)/n) = \mu(z)$$

is the solution of $\Gamma_{\pm 1/4}$. This completes the proof of Theorem 1.1 (ii). □

4. Conclusions and simulations

This paper establishes the existence of traveling wave solutions for the Broer-Kaup equation with distributed delay, a model for bi-directional long wave propagation. The analysis proceeds by first employing GSPT to construct a locally invariant manifold near the critical manifold, thereby reducing the singularly perturbed system to a regularly perturbed one. To simplify the resulting near-Hamiltonian system, a translation transformation is applied to remove the quadratic term, and a variables transformation is further introduced to normalize the coefficients of the unperturbed terms. The crux of the proof lies in demonstrating the existence and uniqueness of a zero of the Melnikov function, which is achieved by proving the monotonicity of a ratio of Abelian integrals via involution mapping criterion. Under specific parametric conditions, this ensures the existence and uniqueness of periodic wave solutions, for which the admissible wave speed range is determined. For kink (anti-kink) solutions, the analysis leverages the explicit form of the Melnikov function to identify the condition for a unique zero, leading to the persistence criteria for these kink (anti-kink) solutions. These mathematical results have direct hydrodynamic interpretations: Periodic waves represent oscillatory surface patterns, while kink (anti-kink) waves describe transition fronts between distinct steady states. The distributed delay models the retarded waveform evolution due to viscosity, memory effects, or multi-path scattering, aligning the model more closely with physical wave propagation.

A numerical study of system (3.8) is conducted to verify the theoretical findings. In particular, the unique periodic orbit identified in the analysis serves as the counterpart to the unique periodic wave solution of Eq (1.2). To analyze the behavior of the function $P(h)$, we plot its profile over the interval $h \in (0, 1/4)$ by using Maple-2021. The resulting graph is presented in Figure 2a. This graph demonstrates the strict monotonicity of the ratio of Abelian integrals. We take $a = 2$ and $b = 1$, then

$$c_1 = 1/\sqrt{2} \approx 0.0707106781, \quad m = 0.075, \quad n \approx 0.9428090414.$$

Thus, we take $c = 0.5$ and then obtain the coefficients of $J_0(h)$ and $J_2(h)$ to derive

$$M(h) = -0.0625 J_0(h) + 0.5625 J_2(h).$$

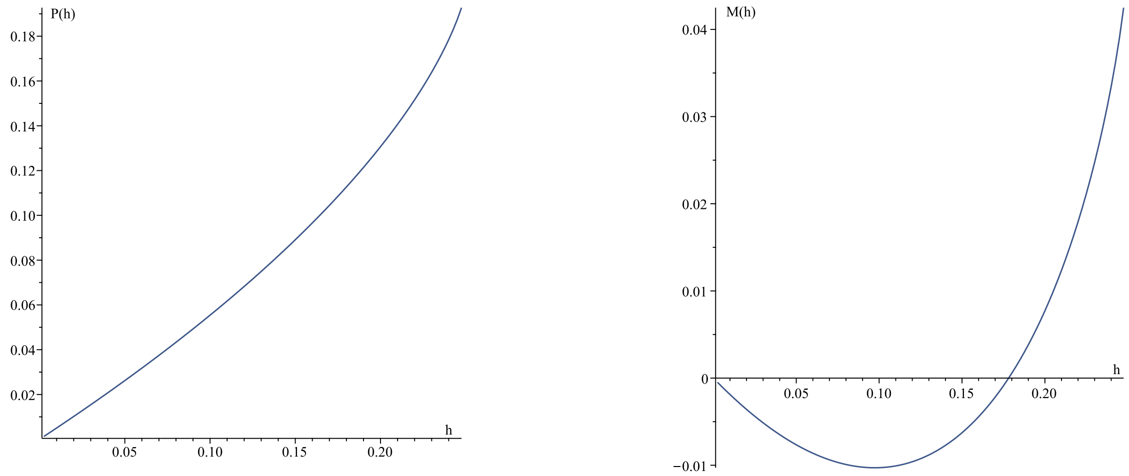
We plot $M(h)$ in Figure 2b, where a unique zero is observable at $h^* \approx 0.1781$. From

$$\mu^4/2 - \mu^2 + 2h^* = 0,$$

we obtain two roots

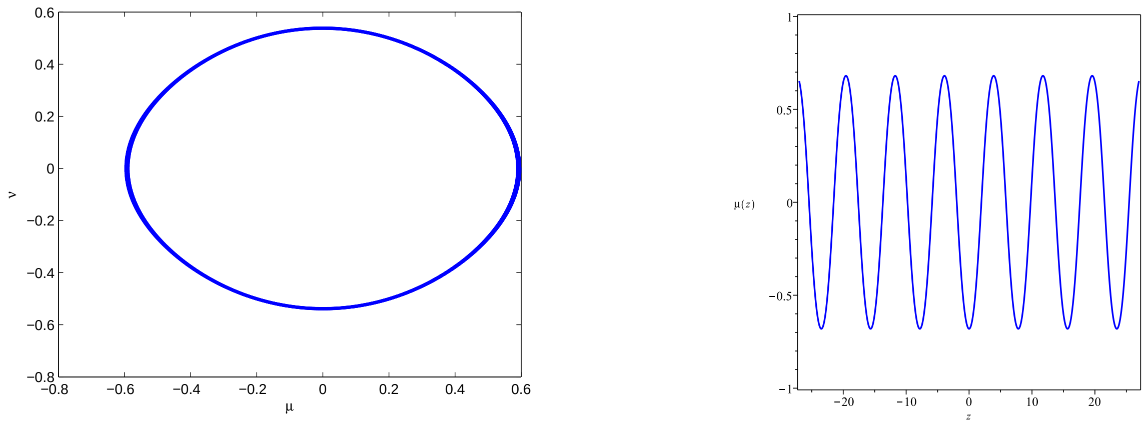
$$\mu_1 \approx -0.6809673215 \quad \text{and} \quad \mu_2 \approx 0.6809673215$$

in $(-1, 1)$. With the parameter fixed at $\tau = 0.01$, the time interval $z \in (0, 300)$, and the initial value set to $(0.6, 0)$, the resulting trajectory is observed to converge to a unique limit cycle, as depicted in Figure 3a. The profile of the corresponding periodic wave solution, which oscillates between μ_1 and μ_2, is shown in Figure 3b. This numerical result is consistent with the statement of Theorem 1.1 (i).



(a) **(b)**

Figure 2. Taking $a = 2$, $b = 1$, $c = 0.5$, and $\tau = 0.01$ in system (3.8): (a) the graph of $P(h)$ for $h \in (0, 1/4)$, which is strictly increasing; (b) the graph of $M(h)$ with a unique zero for $h \in (0, 1/4)$.



(a) **(b)**

Figure 3. Taking $a = 2$, $b = 1$, $c = 0.5$, and $\tau = 0.01$ in system (3.8): (a) a trajectory approaching to a unique limit cycle; (b) periodic wave corresponding to the unique limit cycle.

To numerically confirm the existence of the kink (anti-kink) wave solutions stated in Theorem 1.1

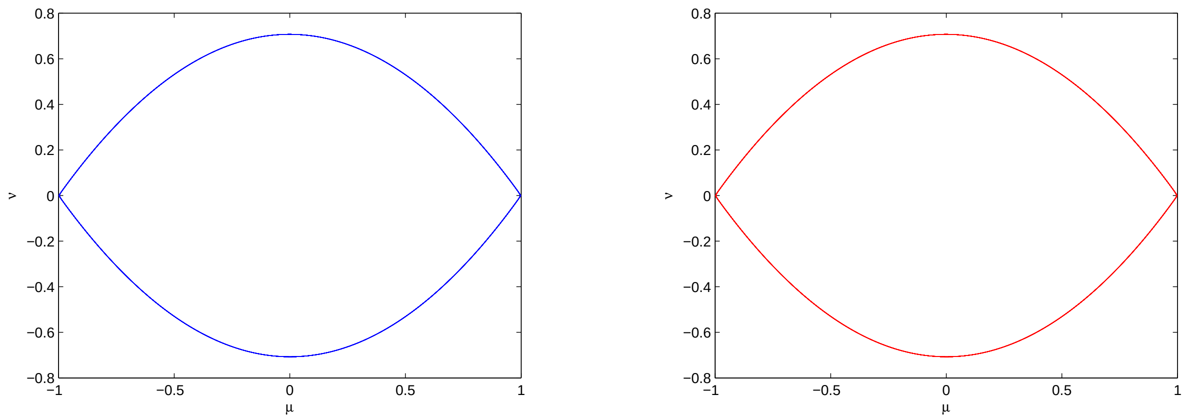
(ii), a series of simulations were conducted with the parameters fixed at $a = 2$, $b = 1$, and $\tau = 0.01$. For this purpose, the wave speed c was first set to a value slightly above the critical speed c_1, specifically

$$c = c_1 + 10^{-4}.$$

The system was then integrated with initial conditions chosen near the expected saddle points, namely $(0, \pm\sqrt{2} - 10^{-4})$. The phase trajectories obtained from these simulations are presented in Figure 4a, which clearly depicts orbits connecting the unstable and stable manifolds in the vicinity of $(0, 1/\sqrt{2})$ and $(0, -1/\sqrt{2})$. Conversely, when the wave speed is set just less than c_1, that is,

$$c = c_1 - 10^{-4},$$

and the initial points are taken as $(0, \pm\sqrt{2} + 10^{-4})$, the resulting trajectories are shown in Figure 4b. Collectively, these numerical findings demonstrate that for wave speeds within the narrow interval $c \in (c_1 - 10^{-4}, c_1 + 10^{-4})$, there exists a unique heteroclinic orbit passing near $(0, 1/\sqrt{2})$ and another unique heteroclinic orbit near $(0, -1/\sqrt{2})$. Consequently, the simulations offer direct numerical confirmation of the analytical results, showing perfect alignment with Theorem 1.1 (ii).



(a) **(b)**

Figure 4. Orbits of system (3.8) with $a = 2$, $b = 1$, and $\tau = 0.01$ connecting the saddles near $(0, 1/\sqrt{2})$ and $(0, -1/\sqrt{2})$, which correspond to the kink (anti-kink) wave solutions stated in Theorem 1.1 (ii): (a) $c = c_1 + 10^{-4}$, $(\mu_0, \nu_0) = (\sqrt{2} - 10^{-4}, 0)$; (b) $c = c_1 - 10^{-4}$, $(\mu_0, \nu_0) = (\sqrt{2} + 10^{-4}, 0)$.

Author contributions

Minzhi Wei: conceptualization, formal analysis, investigation, methodology, writing-original draft; Feiting Fan: formal analysis, funding acquisition, investigation, methodology, resources, software, supervision, validation, writing-review & editing; Xinxin Liu: formal analysis, supervision, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank the support of Natural Science Foundation of Guangxi Province of China (No. 2024GXNSFBA010384), Zhejiang Provincial Natural Science Foundation of China (No. LQ24A010001), the Natural Science Foundation of Sichuan Province of China (No. 2024NSFSC1403), National Natural Science Foundation of China (Nos. 12461031, 12301637, 12501233).

Conflict of interest

This work does not have any conflicts of interest.

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