AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 810–824.
DOI: 10.3934/math.2026035
Received: 16 October 2025
Revised: 25 November 2025
Accepted: 01 December 2025
Published: 12 January 2026

Research article

Recovering time-dependent coefficients in a parabolic equation from two nonlocal measurements

Shufang Qiu[1], Hang Deng[2], Zewen Wang[3,*] and Di Liu[3]

[1] School of Artificial Intelligence, Guangzhou Maritime University, Guangzhou 510725, China
[2] Army Infantry Academy of PLA, Nanchang 330103, China
[3] School of Arts and Sciences, Guangzhou Maritime University, Guangzhou 510725, China

* **Correspondence:** Email: wangzewen@gzmtu.edu.cn; zwwang6@163.com.

Abstract: This work investigates the inverse recovery of two time-dependent coefficients in a one-dimensional parabolic initial–boundary value problem using two nonlocal measurements. Under suitable regularity assumptions and a natural identifiability condition, we establish the existence and uniqueness of the reconstructed coefficients. We further clarify the intrinsic ill-posedness of the problem, noting that the reconstruction step inherently amplifies high-frequency noise, thereby necessitating regularization. Building on these insights, we develop a Crank–Nicolson finite-difference inversion method, which at each half time step reduces to solving a small linear system for the two coefficients. To stabilize the measured data, we employ mollification. Numerical experiments with synthetic data demonstrate accurate reconstructions under small noise levels and reveal that the recovery of the first coefficient is more sensitive to noise than that of the second. The results highlight the critical roles of identifiability and data smoothing in achieving stable performance.

Keywords: parabolic equation; inverse problem; finite difference; nonlocal measurement; mollification method
Mathematics Subject Classification: 65M32, 35R30

1. Introduction

Parabolic partial differential equations are fundamental tools for modeling diffusion and heat transfer in science and engineering. In many practical applications, certain coefficients in these models are not known a priori and must be inferred from measured data. This gives rise to inverse problems, where the objective is to recover model coefficients from indirect observations of the solution. A substantial body of research has focused on identifying a single unknown coefficient using boundary or interior measurements, and numerous analytical and numerical techniques have been developed for this purpose; see, for example, [1, 2]. Nonlocal measurements and overdetermination conditions have also been used to improve the information content of the data [3, 4]. In bioheat transfer, for example, time-varying perfusion and source rates are of particular interest and have been investigated using similar techniques [5–8]. Finite-difference schemes are widely adopted owing to their simplicity and flexibility [9–11]. Inverse problems are inherently ill-posed and highly sensitive to noise, necessitating regularization; discrete mollification provides a practical and effective remedy [12, 13]. Related approaches have also been explored in time-fractional diffusion models [14–18].

A growing body of literature addresses the simultaneous recovery of multiple unknowns in diffusion-type models. For classical parabolic equations, analytical and numerical strategies for recovering two time-dependent coefficients demonstrate that overposed data, combined with appropriate regularization, ensures identifiability and stable computation [19]. In transmission problems, the joint identification of the initial value and source strength underscores the importance of enriched interface measurements and yields stable reconstructions [20]. In higher-dimensional settings, simultaneous recovery of a right-hand side and time-dependent coefficients has been established, supported by convergent schemes robust to noise [21]. Parallel advances in fractional diffusion include joint inversion of the fractional order with a space-dependent source [22, 23], uniqueness results for multiple orders and parameters in multi-term models [24] and simultaneous identification of the fractional order together with the diffusion coefficient [25]. Furthermore, multi-term identification from nonlocal observations has been demonstrated for time-fractional diffusion with a symmetric potential [26].

Motivated by this context, we address a challenging inverse problem: the simultaneous recovery of two time-dependent coefficients in a one-dimensional parabolic problem from two nonlocal measurements. Such problems are inherently difficult, as they raise fundamental questions about whether the available data provide sufficient independent information and about the stability of the reconstruction process. In response, we first establish existence and uniqueness under suitable regularity assumptions and a natural identifiability (nondegeneracy) condition. We then design a Crank–Nicolson finite-difference inversion scheme which, at each half time step, reduces the recovery to solving a 2×2 linear system for the two coefficients. The inverse problem is formally described below.

Problem formulation. Consider the following initial-boundary value problem of parabolic equation:

$$\begin{cases} u_t = u_{xx} - p(t)u + q(t)f(x,t), & 0 < x < 1,\ 0 < t \le T, \\ u(x,0) = \varphi(x), & 0 < x < 1, \\ u(0,t) = a(t), \quad u(1,t) = b(t), & 0 < t \le T. \end{cases} \tag{1.1}$$

Given the functions $p(t)$, $q(t)$, $f(x,t)$, $\varphi(x)$, $a(t)$, and $b(t)$, determining the distribution $u(x,t)$ constitutes the direct problem. In contrast, our focus is on the inverse problem of reconstructing $p(t)$ and $q(t)$ from the following nonlocal conditions:

$$E_1(t) = \int_0^1 w_1(x)u(x,t)\,dx, \quad t \in [0,T], \tag{1.2}$$

and

$$E_2(t) = \int_0^1 w_2(x)u(x,t)\,dx, \quad t \in [0,T], \tag{1.3}$$

where $w_1(x)$ and $w_2(x)$ are prescribed weight functions. In practical applications, however, the data appearing in these nonlocal conditions—namely $E_1(t)$ and $E_2(t)$—are typically contaminated by random measurement noise errors. Consequently, one only has access to the noisy observations $E_1^\delta(t)$ and $E_2^\delta(t)$, which satisfy

$$\|E_1^\delta(t) - E_1(t)\| \le \delta, \quad \|E_2^\delta(t) - E_2(t)\| \le \delta, \tag{1.4}$$

where δ denotes the noise level.

The structure of the paper is organized as follows: Section 2 establishes existence and uniqueness results for recovering the time-dependent coefficients $p(t)$ and $q(t)$ from two nonlocal measurements. Section 3 develops a Crank–Nicolson–based numerical algorithm for the simultaneous inversion of these coefficients. Section 4 presents numerical experiments that demonstrate feasibility and robustness under noisy data. Section 5 offers a brief conclusion.

2. Existence and uniqueness for recovering $p(t)$ and $q(t)$

The main objective of this section is to establish the existence and uniqueness of recovering the pair (p, q) under suitable regularity and identifiability conditions. To this end, we begin by introducing a set of assumptions on the data and coefficients. These assumptions guarantee the well-posedness of the associated direct problem and provide the analytical framework required to prove the ensuing existence and uniqueness results for the inverse reconstruction.

Assumptions

(A1) Regularity: $\varphi \in L^2(0,1)$, $a, b \in H^{1/2}(0,T)$, $f \in C([0,T]; L^2(0,1))$, and $w_j \in H^2(0,1)$ with $w_j(0) = w_j(1) = 0$ for $j = 1, 2$. In particular, w_j' has well-defined endpoint traces.

(A2) Observation regularity and compatibility: $E_j(t) \in W^{1,1}(0,T)$ and $E_j(0) = \int_0^1 w_j(x)\,\varphi(x)\,dx$ for $j = 1, 2$.

(A3) Identifiability (nondegeneracy): with $F_j(t) := \int_0^1 w_j(x)\,f(x,t)\,dx$,

$$\Delta(t) := E_2(t)F_1(t) - E_1(t)F_2(t) \quad \text{satisfies} \quad |\Delta(t)| \ge \Theta > 0 \quad \text{for all } t \in [0,T].$$

Theorem 1 (Existence and Uniqueness)**.** *Under assumptions* (A1)–(A3)*, there exists* $\tau \in (0,T]$ *such that there is a unique pair*

$$(p, q) \in L^\infty(0,\tau) \times L^\infty(0,\tau)$$

for which the corresponding weak solution u of (1.1) *satisfies the given nonlocal conditions*

$$E_j(t) = \int_0^1 w_j(x)\,u(x,t)\,dx, \quad j = 1, 2, \text{ for a.e. } t \in [0,\tau].$$

Moreover, if $|\Delta(t)| \ge \Theta > 0$ *holds on the entire interval* $[0,T]$*, then the above construction can be continued in finitely many steps to the whole time range, and one obtains a unique recovery* $(p, q) \in L^\infty(0,T) \times L^\infty(0,T)$*. That is, given the exact data* $\{E_1(t), E_2(t)\}$ *on* $[0,T]$*, the coefficients p and q are uniquely determined.*

Proof. Under (A1), for any $p, q \in L^\infty(0,T)$ the forward problem admits a unique weak solution

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1(0,1)), \quad u_t \in L^2(0,T; H^{-1}(0,1)).$$

This is standard for one-dimensional parabolic equations with Dirichlet boundary conditions. From the definitions of $E_j(t)$ for $j = 1,2$, differentiating with respect to t and applying equation (1.1) yields

$$E_j'(t) = \int_0^1 w_j\, u_t\, dx = \int_0^1 w_j\, u_{xx}\, dx - p(t)\int_0^1 w_j\, u\, dx + q(t)\int_0^1 w_j\, f\, dx.$$

Integrating by parts twice and using $w_j(0) = w_j(1) = 0$ yields

$$\int_0^1 w_j u_{xx}\, dx = \int_0^1 w_j'' u\, dx - (w_j'(1)b(t) - w_j'(0)a(t)).$$

Hence, for $j = 1,2$,

$$-p(t)E_j(t) + q(t)F_j(t) = E_j'(t) + \left(w_j'(1)b(t) - w_j'(0)a(t)\right) - \int_0^1 w_j'' u\, dx. \tag{2.1}$$

Set

$$B_j(t) := [w_j'(1)\, b(t) - w_j'(0)\, a(t)], \qquad J_j[u](t) := \int_0^1 w_j''(x)\, u(x,t)\, dx,$$

and define the right-hand side

$$R_j[u](t) := E_j'(t) + B_j(t) - J_j[u](t), \qquad j = 1,2.$$

Hence, for the prescribed functions $E_j(t)$ and $F_j(t)$ for $j = 1,2$, the inverse problem under consideration reduces to solving the following nonlinear system of equations:

$$\begin{pmatrix} -E_1(t) & F_1(t) \\ -E_2(t) & F_2(t) \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} R_1[u](t) \\ R_2[u](t) \end{pmatrix}. \tag{2.2}$$

By $|\Delta(t)| \geq \Theta > 0$, (2.2) is invertible and yields

$$p(t) = \frac{R_1[u](t)\, F_2(t) - R_2[u](t)\, F_1(t)}{\Delta(t)}, \qquad q(t) = \frac{E_2(t)\, R_1[u](t) - E_1(t)\, R_2[u](t)}{\Delta(t)}. \tag{2.3}$$

Fix $\tau > 0$ (to be chosen) and consider $X := L^\infty(0,\tau) \times L^\infty(0,\tau)$ with norm $\|(p,q)\|_X = \|p\|_{L^\infty(0,\tau)} + \|q\|_{L^\infty(0,\tau)}$. Given $(p,q) \in X$, the forward problem admits a unique weak solution

$$u \in C([0,\tau]; L^2(0,1)) \cap L^2(0,\tau; H^1(0,1)), \quad u_t \in L^2(0,\tau; H^{-1}(0,1)).$$

Therefore, we can define a map $\Phi : X \to X$ by

$$\Phi(p,q) := (\tilde{p}, \tilde{q}), \quad \text{where } (\tilde{p}, \tilde{q}) \text{ are given by (2.3) with } u = u[p,q].$$

Let $(p_k, q_k) \in X$ with solutions $u_k = u[p_k, q_k]$ for $k = 1,2$, and set $v := u_1 - u_2$. Then

$$\begin{cases} v_t = v_{xx} - p_1\, v - (p_1 - p_2)u_2 + (q_1 - q_2)f, \\ v(\cdot,0) = 0, \quad v(0,t) = v(1,t) = 0. \end{cases}$$

Parabolic energy estimates and Grönwall's inequality yield a constant $C > 0$ such that

$$\|v\|_{C([0,\tau];L^2)} \leq C\,\tau\,\left(\|p_1 - p_2\|_\infty + \|q_1 - q_2\|_\infty\right), \tag{2.4}$$

hence

$$\left\|J_j[u_1] - J_j[u_2]\right\|_\infty \leq \|w_j''\|_{L^2}\,\|v\|_{C([0,\tau];L^2)} \leq C\,\tau\,\left(\|p_1 - p_2\|_\infty + \|q_1 - q_2\|_\infty\right).$$

Unless otherwise specified, throughout this paper, C denotes a generic constant whose value may change from line to line. Since (p_k, q_k) depend linearly on $R_j[u_k]$, we obtain

$$\|\Phi(p_1, q_1) - \Phi(p_2, q_2)\|_\infty \leq \frac{C}{\Theta}\,\|J_j[u_1] - J_j[u_2]\|_\infty \leq \frac{C}{\Theta}\,\tau\,\left(\|p_1 - p_2\|_\infty + \|q_1 - q_2\|_\infty\right).$$

Choose $\tau_0 > 0$ so that $\frac{C}{\Theta}\tau_0 < 1$. Then for any $\tau \leq \tau_0$, Φ is a contraction on X. By Banach's fixed point theorem, Φ has a unique fixed point $(p, q) \in X$, and the Picard iteration $(p^{(k+1)}, q^{(k+1)}) = \Phi(p^{(k)}, q^{(k)})$ converges to it.

Next, we verify that the unique solution obtained from Banach's fixed-point theorem indeed satisfies the prescribed nonlocal conditions. Let $(p^*, q^*) \in X$ denote the unique fixed point of the mapping Φ, i.e.,

$$\Phi(p^*, q^*) = (p^*, q^*).$$

Denote by u^* the corresponding unique solution of the forward problem associated with (p^*, q^*). Then, for each $j = 1, 2$, one has

$$-E_j p^* + F_j q^* = E_j' + B_j - J_j[u^*]. \tag{2.5}$$

On the other hand, multiplying the equation satisfied by u^* by w_j and integrating over $(0, 1)$ gives

$$\frac{d}{dt}\left(\int_0^1 w_j u^*\,dx\right) = B_j + J_j[u^*] - p^* \int_0^1 w_j u^*\,dx + q^* F_j. \tag{2.6}$$

Hence, define

$$G_j(t) := \int_0^1 w_j(x)\,u^*(x,t)\,dx.$$

Combining (2.5) and (2.6), we obtain for $j = 1, 2$ the initial-value problem

$$\begin{cases} (G_j - E_j)'(t) = -\,p^*(t)\,(G_j - E_j)(t), \\ (G_j - E_j)(0) = 0. \end{cases}$$

It follows that $G_j(t) \equiv E_j(t)$ for almost every $t \in [0, \tau]$, and thus the fixed-point solution (p^*, q^*) indeed satisfies the prescribed nonlocal conditions.

Finally, by partitioning $[0, T]$ into finitely many sub-intervals of length $\leq \tau$ and iterating the above construction, using $u(\cdot, t_m)$ as the initial condition on each sub-interval, this yields existence and uniqueness of (p, q) on $[0, T]$. □

Assumptions

(A4) Dirichlet eigenfunction weights: for $j = 1,2$, the weight $w_j(x)$ is a Dirichlet eigenfunction on $(0,1)$, i.e.,

$$w_j'' + \lambda_j w_j = 0, \quad w_j(0) = w_j(1) = 0, \quad \lambda_j > 0,$$

where λ_j is a Dirichlet eigenvalue of the one-dimensional Laplacian on $(0,1)$.

Corollary 1. *Under (A1)–(A4), there exists a unique pair* $(p,q) \in L^\infty(0,T) \times L^\infty(0,T)$ *and a unique solution u of* (1.1) *corresponding to the data* (1.2)–(1.3)*. Moreover, the coefficients admit the explicit representations*

$$p(t) = \frac{R_1(t)F_2(t) - R_2(t)F_1(t)}{\Delta(t)}, \qquad q(t) = \frac{E_2(t)R_1(t) - E_1(t)R_2(t)}{\Delta(t)}, \tag{2.7}$$

where

$$R_j(t) := E_j'(t) + B_j(t) + \lambda_j E_j(t), \quad B_j(t) := w_j'(1)\,b(t) - w_j'(0)\,a(t), \quad j = 1,2,$$

and $\Delta(t) = E_2(t)F_1(t) - E_1(t)F_2(t)$*.*

Remark 1. *In view of* $w_j'' = -\lambda_j w_j$ *on* $(0,1)$*, we have*

$$J_j[u](t) = \int_0^1 w_j''(x)\,u(x,t)\,dx = -\lambda_j \int_0^1 w_j(x)\,u(x,t)\,dx = -\lambda_j E_j(t),$$

hence $R_j(t) = E_j'(t) + B_j(t) - J_j[u](t) = E_j'(t) + B_j(t) + \lambda_j E_j(t)$*. Therefore, all quantities on the right-hand sides of* (2.7) *depend only on the prescribed data and known functions; in particular, the formulas for* $p(t)$ *and* $q(t)$ *are explicit and do not depend (recursively) on* (p,q)*. This is different from* (2.3)*, where* $R_j[u]$ *depends on u and thus on* (p,q)*.*

Although the reconstruction is well posed in the sense of existence and uniqueness, it is unstable with respect to perturbations in the data. The explicit formulas (2.3) involve time derivatives of the nonlocal observations through $R_j[u] = E_j'(t) + B_j(t) - J_j[u](t)$, and differentiation is an unbounded operator on the natural data spaces. Consequently, high-frequency noise in E_j is strongly amplified in E_j' [18]: if $E_j^\delta = E_j + \eta_j$ with $\|\eta_j\|_{L^\infty(0,T)} \le \delta$ and, say, $\eta_j(t) = \delta \sin(nt)$, then $\|E_j^{\delta\prime} - E_j'\|_{L^\infty(0,T)} \approx n\delta$. Even under the uniform nondegeneracy $|\Delta(t)| \ge \Theta > 0$, the reconstruction error is of order $n\delta/\Theta$ and can be arbitrarily large as $n \to \infty$. This is a classical Hadamard-type ill-posedness: small perturbations of the data may induce large errors in (p,q). In practice, regularization is indispensable,for instance, by smoothing or low-pass filtering of E_j prior to differentiation, by employing integral (Volterra) formulations that avoid explicit time derivatives, or by adding Tikhonov-type penalties on p and q.

As an illustration based on the explicit representations (2.7), we demonstrate that the recovery of $p(t)$ is typically more ill-conditioned (i.e., more sensitive to noise) than that of $q(t)$. Assume that $E_j(t)$ and $F_j(t)$ are exact and satisfy $|\Delta(t)| \ge \Theta > 0$ on $[0,T]$. Let the differentiated quantities be contaminated as

$$R_j^\delta(t) = R_j(t) + \eta_j(t), \qquad j = 1,2,$$

where η_j may be large due to the numerical differentiation contained in R_j. Define p^δ, q^δ by substituting R_j^δ into (2.7). A direct calculation yields

$$p^\delta(t) - p(t) = \frac{\eta_1(t)\,F_2(t) - \eta_2(t)\,F_1(t)}{\Delta(t)}, \quad q^\delta(t) - q(t) = \frac{E_2(t)\,\eta_1(t) - E_1(t)\,\eta_2(t)}{\Delta(t)}.$$

Hence, taking the supremum over t and using $|\Delta(t)| \geq \Theta$ gives the uniform bounds

$$\|p^{\delta} - p\|_{L^{\infty}(0,T)} \leq \frac{2\,\|F\|_{\infty}}{\Theta}\,\|\eta\|_{\infty}, \quad \|q^{\delta} - q\|_{L^{\infty}(0,T)} \leq \frac{2\,\|E\|_{\infty}}{\Theta}\,\|\eta\|_{\infty},$$

where $\|F\|_{\infty} = \max\{\|F_1\|_{L^{\infty}(0,T)}, \|F_2\|_{L^{\infty}(0,T)}\}$, $\|E\|_{\infty} = \max\{\|E_1\|_{L^{\infty}(0,T)}, \|E_2\|_{L^{\infty}(0,T)}\}$, $\|\eta\|_{\infty} = \max\{\|\eta_1\|_{L^{\infty}(0,T)}, \|\eta_2\|_{L^{\infty}(0,T)}\}$. In many practical settings one observes $\|F\|_{\infty} \gg \|E\|_{\infty}$ (due to parabolic smoothing of u, whereas F_j are direct projections of the source). Consequently, the reconstruction error for $p(t)$ is amplified by a larger factor than that for $q(t)$. Thus, even when E_j and F_j are exact, the differentiated-data noise in R_j typically affects p more strongly than q, a behavior that is corroborated by our numerical experiments.

In the next section, we introduce a finite-difference inversion scheme that reconstructs the coefficients p and q without explicitly computing numerical time derivatives. Nevertheless, despite circumventing numerical differentiation. This reconstruction remains computationally unstable; this instability is intrinsic to the inverse problem itself, rather than a consequence of the discretization.

3. Numerical algorithm for coefficients inversion

This section develops a numerical scheme for recovering the time-dependent coefficients $p(t)$ and $q(t)$ using a Crank–Nicolson finite-difference discretization, combined with composite numerical quadrature to enforce the nonlocal constraints. The space–time domain $[0, 1] \times [0, T]$ is uniformly partitioned with spatial step $h = 1/M$ and time step $\tau = T/N$. Grid points are (x_m, t_n) with $x_m = mh$ $(m = 0, 1, \ldots, M)$ and $t_n = n\tau$ $(n = 0, 1, \ldots, N)$. We set the mesh ratio $r = \tau/h^2$ and define

$$t_{n+\frac{1}{2}} = (n + \tfrac{1}{2})\tau,\ p^{n+\frac{1}{2}} = p(t_{n+\frac{1}{2}}),\ q^{n+\frac{1}{2}} = q(t_{n+\frac{1}{2}}),\ u_m^n = u(x_m, t_n),\ f_m^{n+\frac{1}{2}} = f(x_m, t_{n+\frac{1}{2}}).$$

The corresponding finite-difference approximations of $u(x_m, t_n)$, $p(t_{n+\frac{1}{2}})$ and $q(t_{n+\frac{1}{2}})$ are denoted by U_m^n, $P^{n+\frac{1}{2}}$, and $Q^{n+\frac{1}{2}}$, respectively.

By the Crank-Nicolson scheme, equation (1.1) is discretized for $1 \leq m \leq M - 1$ as

$$\frac{u_m^{n+1} - u_m^n}{\tau} = \frac{(u_{m+1}^n - 2u_m^n + u_{m-1}^n) + (u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1})}{2h^2} - p^{n+\frac{1}{2}}\,\frac{u_m^n + u_m^{n+1}}{2} + q^{n+\frac{1}{2}} f_m^{n+\frac{1}{2}} + R_{mn}, \tag{3.1}$$

where R_{mn} denotes the local truncation error. Under standard smoothness, there exists a constant $d_1 > 0$ such that

$$|R_{mn}| \leq d_1(\tau^2 + h^2).$$

The boundary and initial conditions are discretized by

$$u_0^n = a(t_n), \quad u_M^n = b(t_n), \quad u_m^0 = \varphi(x_m). \tag{3.2}$$

The nonlocal conditions (1.2)–(1.3) at t_{n+1} are approximated by the composite trapezoidal rule:

$$\frac{h}{2} w_{10}\, u_0^{n+1} + h \sum_{m=1}^{M-1} w_{1m}\, u_m^{n+1} + \frac{h}{2} w_{1M}\, u_M^{n+1} + Q_{1,n+1} = E_1^{n+1}, \tag{3.3}$$

$$\frac{h}{2} w_{20}\, u_0^{n+1} + h \sum_{m=1}^{M-1} w_{2m}\, u_m^{n+1} + \frac{h}{2} w_{2M}\, u_M^{n+1} + Q_{2,n+1} = E_2^{n+1}, \tag{3.4}$$

where $w_{jm} = w_j(x_m)$, and suppose that for each n there exists $d_2 > 0$ such that

$$|Q_{j,n+1}| \le d_2 h^2, \qquad j = 1, 2.$$

Neglecting the consistency remainders R_{mn} and $Q_{j,n+1}$ in (3.1), (3.3), and (3.4), and replacing $(u_m^n, p^{n+\frac{1}{2}}, q^{n+\frac{1}{2}})$ with the discrete unknowns $(U_m^n, P^{n+\frac{1}{2}}, Q^{n+\frac{1}{2}})$, we obtain

$$(1+r+\tfrac{\tau}{2}P^{n+\frac{1}{2}})U_m^{n+1} - \frac{r}{2}(U_{m+1}^{n+1}+U_{m-1}^{n+1}) = (1-r-\frac{\tau}{2}P^{n+\frac{1}{2}})U_m^n + \frac{r}{2}(U_{m+1}^n+U_{m-1}^n) + \tau Q^{n+\frac{1}{2}} f_m^{n+\frac{1}{2}}, \tag{3.5}$$

$$U_0^n = a(t_n), \quad U_M^n = b(t_n), \quad U_m^0 = \varphi(x_m), \tag{3.6}$$

$$\frac{h}{2} w_{10}\, U_0^{n+1} + h \sum_{m=1}^{M-1} w_{1m}\, U_m^{n+1} + \frac{h}{2} w_{1M}\, U_M^{n+1} = E_1^{n+1}, \tag{3.7}$$

$$\frac{h}{2} w_{20}\, U_0^{n+1} + h \sum_{m=1}^{M-1} w_{2m}\, U_m^{n+1} + \frac{h}{2} w_{2M}\, U_M^{n+1} = E_2^{n+1}. \tag{3.8}$$

Next, we rewrite (3.5)–(3.8) in matrix form. Let $U^n = [U_1^n, \dots, U_{M-1}^n]^T$, $W_j = [w_{j1}, \dots, w_{j,M-1}]^T$ for $j = 1, 2$. Define the tridiagonal matrices $A(P^{n+\frac{1}{2}}), B \in \mathbb{R}^{(M-1)\times(M-1)}$ by

$$A(P^{n+\frac{1}{2}}) = \begin{bmatrix} d & -\frac{r}{2} & & \\ -\frac{r}{2} & d & \ddots & \\ & \ddots & \ddots & -\frac{r}{2} \\ & & -\frac{r}{2} & d \end{bmatrix}, \quad B = \begin{bmatrix} 1-r & \frac{r}{2} & & \\ \frac{r}{2} & 1-r & \ddots & \\ & \ddots & \ddots & \frac{r}{2} \\ & & \frac{r}{2} & 1-r \end{bmatrix}, \quad d = 1 + r + \frac{\tau}{2} P^{n+\frac{1}{2}}.$$

Let $F^{n+\frac{1}{2}} = \left[f(x_1, t_{n+\frac{1}{2}}), \dots, f(x_{M-1}, t_{n+\frac{1}{2}}) \right]^T$ and define the boundary vector

$$C^n = \left[\frac{r}{2}(U_0^{n+1} + U_0^n),\, 0,\, \dots,\, 0,\, \frac{r}{2}(U_M^{n+1} + U_M^n) \right]^T,$$

where $U_0^n = a(t_n)$ and $U_M^n = b(t_n)$ are known. Then (3.5) is equivalent to

$$A(P^{n+\frac{1}{2}})U^{n+1} = BU^n - \frac{\tau}{2} P^{n+\frac{1}{2}} U^n + \tau Q^{n+\frac{1}{2}} F^{n+\frac{1}{2}} + C^n. \tag{3.9}$$

The nonlocal constraints (3.7)–(3.8) can be written as

$$h\, W_1^T U^{n+1} = E_1^{n+1} - \frac{h}{2} w_{10} U_0^{n+1} - \frac{h}{2} w_{1M} U_M^{n+1}, \tag{3.10}$$

$$h\, W_2^T U^{n+1} = E_2^{n+1} - \frac{h}{2} w_{20} U_0^{n+1} - \frac{h}{2} w_{2M} U_M^{n+1}. \tag{3.11}$$

From (3.9) we obtain

$$U^{n+1} = A^{-1}(P^{n+\frac{1}{2}})(BU^n + C^n) - \frac{\tau}{2}\, P^{n+\frac{1}{2}} A^{-1}(P^{n+\frac{1}{2}}) U^n + \tau\, Q^{n+\frac{1}{2}} A^{-1}(P^{n+\frac{1}{2}}) F^{n+\frac{1}{2}}. \tag{3.12}$$

Define

$$\alpha_j := E_j^{n+1} - \tfrac{h}{2} w_{j0} U_0^{n+1} - \tfrac{h}{2} w_{jM} U_M^{n+1} - h\, W_j^T A^{-1}(P^{n+\frac{1}{2}})(BU^n + C^n),$$
$$\beta_j := -\tfrac{1}{2}\, h\, W_j^T A^{-1}(P^{n+\frac{1}{2}}) U^n, \quad \gamma_j := h\, W_j^T A^{-1}(P^{n+\frac{1}{2}}) F^{n+\frac{1}{2}}, \quad j = 1, 2.$$

Substituting (3.12) into (3.10)–(3.11) yields a 2×2 linear system for $(P^{n+\frac{1}{2}}, Q^{n+\frac{1}{2}})$:

$$\begin{cases} \tau\beta_1\, P^{n+\frac{1}{2}} + \tau\gamma_1\, Q^{n+\frac{1}{2}} = \alpha_1, \\ \tau\beta_2\, P^{n+\frac{1}{2}} + \tau\gamma_2\, Q^{n+\frac{1}{2}} = \alpha_2, \end{cases} \tag{3.13}$$

and, provided $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$, then

$$P^{n+\frac{1}{2}} = \frac{1}{\tau} \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1}, \qquad Q^{n+\frac{1}{2}} = \frac{1}{\tau} \frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\beta_1\gamma_2 - \beta_2\gamma_1}. \tag{3.14}$$

Based on (3.9)–(3.14), the time-marching inversion proceeds as Algorithm 1.

Algorithm 1 : Finite-difference inversion scheme for reconstructing $p(t)$ and $q(t)$

for $n = 0, 1, \ldots, N-1$ **do**

Step 1: Given the tolerance $\varepsilon > 0$, choose an initial guess $P_0^{n+\frac{1}{2}}$ for $P^{n+\frac{1}{2}}$.

Step 2: Assemble the tridiagonal matrix $A\left(P_k^{n+\frac{1}{2}}\right)$ and compute

$$Y_U = A^{-1}\left(P_k^{n+\frac{1}{2}}\right) U^n, \; Y_F = A^{-1}\left(P_k^{n+\frac{1}{2}}\right) F^{n+\frac{1}{2}}, \; Y_{BF} = A^{-1}\left(P_k^{n+\frac{1}{2}}\right)(BU^n + C^n).$$

Step 3: Using the data E_j^{n+1}, define

$$\alpha_j = E_j^{n+1} - hW_j^T Y_{BF} - \tfrac{h}{2} w_{j0} U_0^{n+1} - \tfrac{h}{2} w_{jM} U_M^{n+1},$$

$$\beta_j = -\tfrac{1}{2} h W_j^T Y_U, \quad \gamma_j = h W_j^T Y_F, \quad j = 1, 2.$$

Step 4: If $|\beta_1\gamma_2 - \beta_2\gamma_1| < \mathrm{tol}_{\mathrm{det}}$, the system is ill-conditioned; stop. Otherwise, update

$$P_{k+1}^{n+\frac{1}{2}} = \frac{1}{\tau} \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1},$$

Step 5: If $\left|P_{k+1}^{n+\frac{1}{2}} - P_k^{n+\frac{1}{2}}\right| \leq \varepsilon$, set $P^{n+\frac{1}{2}} = P_{k+1}^{n+\frac{1}{2}}$ and proceed to Step 6. Otherwise, set $P_k^{n+\frac{1}{2}} \leftarrow P_{k+1}^{n+\frac{1}{2}}$ and return to Step 2.

Step 6: Update

$$Q^{n+\frac{1}{2}} = \frac{1}{\tau} \frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\beta_1\gamma_2 - \beta_2\gamma_1}$$

and the state variable:

$$U^{n+1} = Y_{BF} - \tfrac{\tau}{2} P^{n+\frac{1}{2}} Y_U + \tau Q^{n+\frac{1}{2}} Y_F.$$

Then move to the next time level $n \leftarrow n + 1$.

end for

Remark 2. *From* (3.9)–(3.12) *and the definitions of* $\alpha_j, \beta_j, \gamma_j$, *the formula* (3.14) *shows that* $P^{n+\frac{1}{2}}$ *appears on both sides through the tridiagonal matrix* $A(P^{n+\frac{1}{2}})$. *Thus, the update in Steps 1-5 of Algorithm 1 effectively solves a nonlinear equation for* $P^{n+\frac{1}{2}}$ *via an (approximate) fixed-point iteration*

$$P_{k+1}^{n+\frac{1}{2}} = G\Big(P_k^{n+\frac{1}{2}}\Big), \qquad G(P) := \frac{1}{\tau}\,\frac{\alpha_1(P)\,\gamma_2(P) - \alpha_2(P)\,\gamma_1(P)}{\beta_1(P)\,\gamma_2(P) - \beta_2(P)\,\gamma_1(P)},$$

where $\alpha_j(P), \beta_j(P), \gamma_j(P)$ *are obtained by evaluating* $A^{-1}(P)$ *in their definitions. For admissible* $P \geq 0$, *the matrix* $A(P)$ *is strictly diagonally dominant, so* $A^{-1}(P)$ *exists and depends smoothly on* P. *Consequently,* $\alpha_j(P), \beta_j(P), \gamma_j(P)$ *are locally Lipschitz in* P; *combined with the nondegeneracy* $|\beta_1\gamma_2 - \beta_2\gamma_1| \geq \Theta > 0$, *this implies that* G *is a contraction for sufficiently small spatial step* h, *ensuring convergence of the iteration. The detailed derivations are technical and omitted for brevity. In computations, we employ an absolute stopping rule with tolerance* $\varepsilon = 10^{-8}$.

In applications, only noisy measurements $E_j^\delta(t)$ are available. Accordingly, in (3.10)–(3.11) the exact samples E_j^{n+1} must be replaced by the discrete observations $E_j^{\delta,n+1} = E_j^\delta(t_{n+1})$. Although the inversion formulas do not explicitly differentiate the data, the recovered coefficients $P^{n+\frac{1}{2}}$ and $Q^{n+\frac{1}{2}}$ are obtained from ratios whose numerators contain the noisy data while the denominator is proportional to τ (cf. (3.14)). Hence, a perturbation of size δ in the numerators is effectively amplified by a factor of order $1/\tau$. This instability is therefore equivalent to the classical ill-posedness of numerical differentiation [18]: as $\tau \to 0$, the inversion deteriorates unless suitable regularization is employed.

A convenient stabilization is to mollify the noisy time series E_j^δ. Let $\omega > 0$, $p > 0$, and $A_p = \int_{-p}^{p} e^{-s^2}\,ds$. Define the compactly supported Gaussian mollifier

$$\rho_{\omega,p}(t) = \frac{1}{\omega A_p}\exp\Big(-\frac{t^2}{\omega^2}\Big)\,\mathbf{1}_{\{|t|\leq p\omega\}},$$

and set $(J_\omega E_j^\delta)(t) = (\rho_{\omega,p} * E_j^\delta)(t)$, where $*$ denotes convolution. In practice we use the discrete analogue on the samples $E_j^\delta(t_n)$:

$$(J_\omega E_j^\delta)(t_n) = \sum_{m=-\lfloor p\omega/\tau\rfloor}^{\lfloor p\omega/\tau\rfloor} \rho_{\omega,p}(m\tau)\,E_j^\delta(t_{n-m}),$$

with standard zero (or mirror) padding at the endpoints. Detailed error estimates for discrete mollification are available in [12, 13] and are therefore omitted here. Following [12, 13], we fix the truncation parameter at $p = 3$ and select the bandwidth ω using generalized cross-validation (GCV), which automatically balances bias and variance for each time series. This choice is consistent with the stability properties of discrete mollification and provides a robust, parameter-free denoising step for the inversion.

Remark 3 (Implementation with noisy data)**.** *To obtain a finite-difference inversion algorithm based on mollification regularization [12, 13, 27, 28], it suffices to modify* ***Step 3*** *of Algorithm 1 by replacing the raw inputs* E_j^{n+1} *with their mollified counterparts* $(J_\omega E_j^\delta)(t_{n+1})$, *computed via the MATLAB implementation of the discrete mollification method (with bandwidth* ω *selected by GCV; see [12, 13]). This simple modification suppresses high-frequency noise while preserving the essential dynamics of the measurements, thereby enhancing the stability and robustness of the recovered coefficients* $p(t)$ *and* $q(t)$.

4. Numerical examples

This section presents two numerical examples designed to evaluate the feasibility and robustness of the proposed reconstruction scheme. The nonlocal data are contaminated with relative uniform noise at levels $\delta^* \in \{0,\ 0.005,\ 0.01,\ 0.03\}$. Given the exact discrete data $E_j(t_n)$, we generate noisy observations according to

$$E_j^{\delta}(t_n) = E_j(t_n)(1 + \delta^* \, \xi_{j,n}),$$

where $\xi_{j,n} = 2\,\mathrm{rand}(t_n) - 1$ and `rand` returns independent and identically distributed samples from the standard uniform distribution on $(0, 1)$. We then apply mollification regularization to obtain stabilized inputs for the inversion. Throughout the experiments, we use the following weight functions

$$w_1(x) = \exp\!\left(-150\,(x - 0.3)^2\right), \quad w_2(x) = \exp\!\left(-150\,(x - 0.8)^2\right),$$

and set $T = 1.0$, $M = 200$, $N = 100$. Reconstructions of $p(t)$ and $q(t)$ are displayed in Figures 1–2, where the legends, δ^* denotes the prescribed relative noise level. The discrete (Euclidean) relative errors of the reconstructed coefficients for Example 1 and Example 2 are summarized in Table 1.

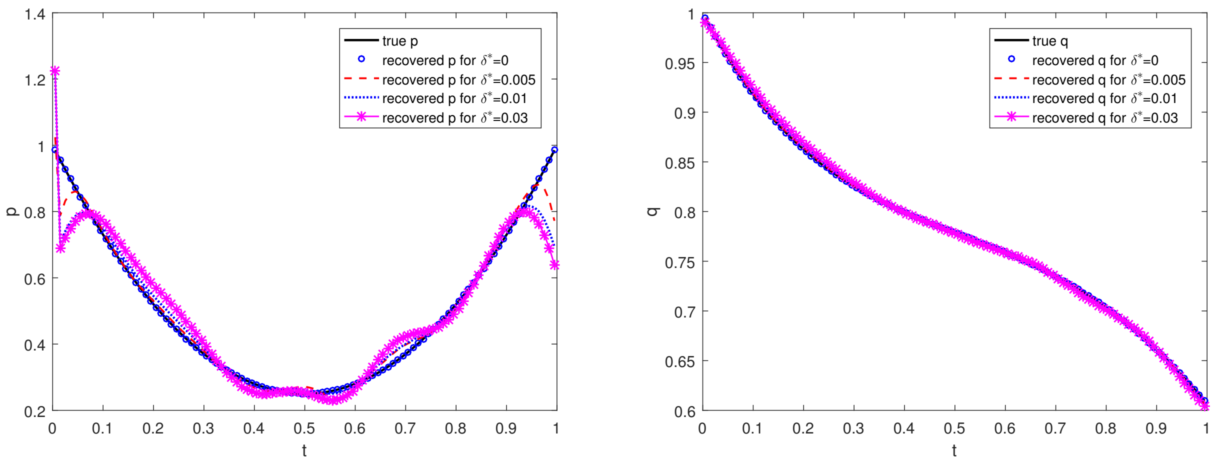
Example 1. Consider

$$\begin{cases} u_t = u_{xx} - p(t)\,u + q(t)\,(20x^3 - 2), & 0 < x < 1,\ 0 < t < T, \\ u(0, t) = u(1, t) = 0, & 0 < t \le T, \\ u(x, 0) = x^2(1 - x^3), & 0 < x \le 1, \end{cases}$$

with the exact coefficients and solution

$$p(t) = 1 - 3t + 3t^2, \quad q(t) = \exp\!\left(-(t - 1.5t^2 + t^3)\right), \quad u(x, t) = x^2(1 - x^3)\, e^{-(t + \frac{1}{2}t^2)}.$$

Figure 1 displays the reconstructions of p and q with respect to different noise levels.



(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 1. Reconstructed coefficients for noise levels $\delta^* \in \{0, 0.005, 0.01, 0.03\}$ in Example 1.

Table 1. Relative ℓ^2 errors of reconstructed coefficients under different noise levels δ^*.

	Example 1		Example 2	
δ^*	$\|\|p^{\delta^*} - p\|\|_2/\|\|p\|\|_2$	$\|\|q^{\delta^*} - q\|\|_2/\|\|q\|\|_2$	$\|\|p^{\delta^*} - p\|\|_2/\|\|p\|\|_2$	$\|\|q^{\delta^*} - q\|\|_2/\|\|q\|\|_2$
0	1.193e-08	4.046e-11	6.685e-10	3.191e-11
0.005	6.413e-02	1.462e-03	4.682e-02	1.981e-02
0.01	1.114e-01	2.938e-03	5.667e-02	2.867e-02
0.03	1.373e-01	4.746e-03	1.350e-01	8.877e-02

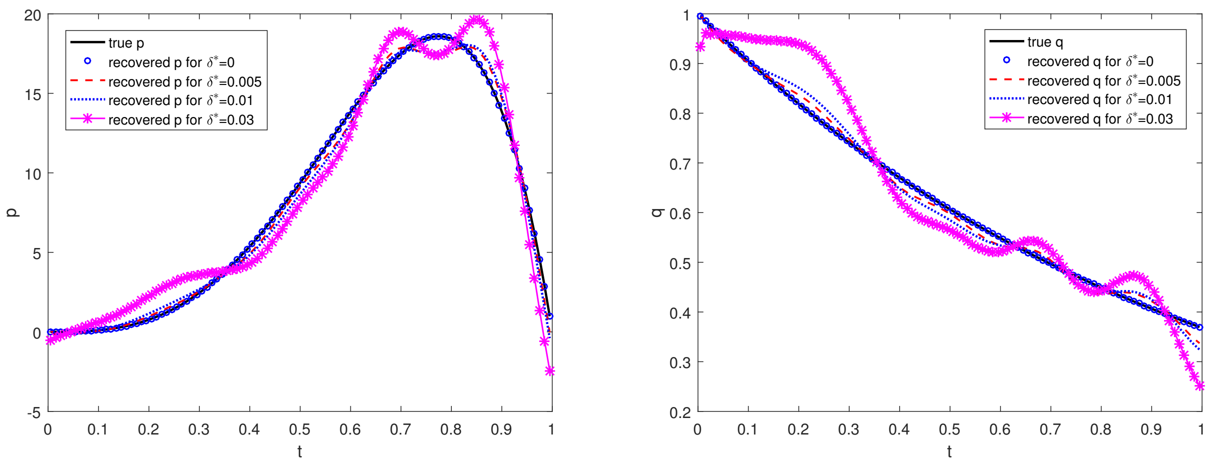
Example 2. Consider

$$\begin{cases} u_t = u_{xx} - p(t)\,u + q(t)(x^2 - x + 2 + 100\,t^3(1-t^2)(x-x^2)), & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t \le T, \\ u(x,0) = x - x^2, & 0 < x \le 1, \end{cases}$$

with the exact coefficients and solution

$$p(t) = 100\,t^3(1-t^2), \qquad q(t) = e^{-t}, \qquad u(x,t) = (x - x^2)\,e^{-t}.$$

Figure 2 reports the reconstructions of p and q with respect to different noise levels.



(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 2. Reconstructed coefficients for noise levels $\delta^* \in \{0, 0.005, 0.01, 0.03\}$ in Example 2.

Discussion. The numerical results demonstrate that the proposed finite-difference inversion, combined with mollification of the measured nonlocal data, yields stable and accurate reconstructions of the time-dependent coefficients when the noise level is small. For noise-free data ($\delta^* = 0$), the recovered p and q closely match the ground truth. As the relative noise level δ^* increases, however, the reconstruction quality degrades—an effect consistent with the intrinsic instability discussed earlier. In particular, the recovery of $p(t)$ is more sensitive to noise than that of $q(t)$, exhibiting noticeable deterioration at higher noise levels. While the smoothing step mitigates the amplification of measurement errors and improves robustness, it cannot fully compensate for the loss of accuracy at large δ^*.

5. Conclusions

We investigated the inverse identification of the time-dependent coefficients $p(t)$ and $q(t)$ in a parabolic IBVP from two nonlocal measurements. Under suitable regularity assumptions and the identifiability condition $|\Delta(t)| \geq \Theta > 0$, we established local existence and uniqueness and derived an explicit reconstruction formula that motivates a Crank–Nicolson finite-difference algorithm. The analysis highlights an intrinsic ill-posedness: although the scheme avoids explicit numerical differentiation, the algebraic step remains noise-sensitive, necessitating regularization. Numerical experiments with synthetic data confirm that the proposed scheme, combined with mollification of the measured data, yields accurate reconstructions for small noise levels. As the noise increases, the reconstruction quality degrades—particularly for $p(t)$, which is more sensitive than $q(t)$— consistent with the ill-posed nature of the problem.

Author contributions

Shufang Qiu: Conceptualization, Formal analysis, Validation, Writing – original draft, Writing – revised draft. Hang Deng: Investigation, Formal analysis, Validation, Writing – original draft. Zewen Wang: Conceptualization, Methodology, Writing – original draft, Writing – revised draft. Di Liu: Investigation, Validation. All authors have read and approved the final version of the paper.

Use of Generative-AI tools declaration

The authors would like to disclose that DeepSeek and OpenAI's ChatGPT were used to assist in improving English grammar and enhancing the clarity of writing in this manuscript.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (12261004), the Guangdong Basic and Applied Basic Research Foundation (2025A1515012248), and by the Innovation Team Project of Regular Universities in Guangdong Province (2025KCXTD037).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. V. Isakov, S. Kindermann, Identification of the diffusion coefficient in a one-dimensional parabolic equation, *Inverse Probl.,* **16** (2000), 665. https://doi.org/10.1093/oq/16.4.665
2. Z. Liu, B. Wang, Coefficient identification in parabolic equations, *Inverse Probl.,* **209** (2009), 379–390. https://doi.org/10.1016/j.amc.2008.12.062

3. N. B. Kerimov, M. I. Ismailov, An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions, *J. Math. Anal. Appl.,* **396** (2012), 546–554. https://doi.org/10.1016/j.jmaa.2012.06.046

4. M. I. Ismailov, F. Kanca, An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions, *Math. Meth. Appl. Sci.,* **34** (2011), 692–702. https://doi.org/10.1002/mma.1396

5. D. Trucu, D. B. Ingham, D. Lesnic, An inverse coefficient identification problem for the bio-heat equation, *Inverse Probl. Sci. En.,* **17** (2009), 65–83. https://doi.org/10.1080/17415970802082880

6. A. Hazanee, D. Lesnic, Determination of a time-dependent coefficient in the bioheat equation, *Int. J. Mech. Sci.,* **88** (2014), 259–266. https://doi.org/10.1016/j.ijmecsci.2014.05.017

7. M. I. Ismailov, F. S. V. Bazán, L. Bedin, Time-dependent perfusion coefficient estimation in a bioheat transfer problem, *Comput. Phys. Commun.,* **230** (2018), 50–58. https://doi.org/10.1016/j.cpc.2018.04.019

8. Q. Bi, Z. Lin, B. Chen, M. Lai, Y. Guo, Y. Lv, et al., Super-Resolution Reconstruction of Weak Targets on Water Surfaces: A Generative Adversarial Network Approach Based on Implicit Neural Representation, *Trait. Signal,* **40** (2023), 2701–2710. http://doi.org/10.18280/ts.400630

9. S. Wang, Y. Lin, A finite-difference solution to an inverse problem for determining a control function in a parabolic partial differential equation, *Inverse Probl.,* **5** (1989), 631. http://doi.org/10.1088/0266-5611/5/4/013

10. Z. Wang, Z. Ruan, H. Huang, S. Qiu , Determination of an unknown time-dependent heat source from a nonlocal measurement by finite difference method, *Acta Math. Appl. Sin,-E.,* **36** (2020), 151–165. http://dx.doi.org/10.1007/s10255-020-0918-3

11. Q. Cao, B. Hu, S. Wan, Z. Wang, Numerical inversion algorithm for the perfusion rate function in the bioheat transfer equation, *J. Jinggangshan. Univ. (Natural Sci.),* **43** (2022), 22–27. http://doi.org/10.3969/j.issn.1674-8085.2022.02.004

12. D. A. Murio, C. E. Mejía, S. Zhan, Discrete mollification and automatic numerical differentiation, *Comput. Math. Appl.,* **35** (1998), 1–16. https://doi.org/10.1016/S0898-1221(98)00001-7

13. D. A. Murio, On the stable numerical evaluation of caputo fractional derivatives, *Comput. Math. Appl.,* **51** (2006), 1539–1550. https://doi.org/10.1016/j.camwa.2005.11.037

14. J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation, *Inverse Probl.,* **25** (2009), 115002. http://doi.org/10.1088/0266-5611/25/11/115002

15. J. J. Liu, M. Yamamoto, A backward problem for the time-fractional diffusion equation, *Appl. Anal.,* **89** (2010), 1769–1788. https://doi.org/10.1080/00036810903479731

16. Y. Zhang, X. Xu, Inverse source problem for a fractional diffusion equation, *Inverse Probl.,* **27** (2011), 035010. http://doi.org/10.1088/0266-5611/27/3/035010

17. Q. Chen, Z. Wang, A double-parameter regularization scheme for the backward diffusion problem with a time-fractional derivative, *Fractal Fract.,* **9** (2025), 459. http://dx.doi.org/10.3390/fractalfract9070459

18. Z. Wang, S. Qiu, X. Rui, W. Zhang, Numerical differentiation of fractional order derivatives based on inverse source problems of hyperbolic equations, *Math. Model. Anal.,* **30** (2025), 74–96. https://doi.org/10.3846/mma.2025.19339
19. M. S. Hussein, D. Lesnic, M. I. Ivanchov, Simultaneous determination of time-dependent coefficients in the heat equation, *Comput. Math. Appl.,* **67** (2014), 1065–1091. https://doi.org/10.1016/j.camwa.2014.01.004
20. S. Chen, D. Jiang, H. Wang, Simultaneous identification of initial value and source strength in a transmission problem for a parabolic equation, *Adv. Comput. Math.,* **48** (2022), 77. http://dx.doi.org/10.1007/s10444-022-09983-x
21. Y. T. Mehraliyev, M. J. Huntul, E. I. Azizbayov, Simultaneous identification of the right-hand side and time-dependent coefficients in a two-dimensional parabolic equation, *Math. Model. Anal.,* **29** (2024), 90–108. https://doi.org/10.3846/mma.2024.17974
22. Z. Ruan, W. Zhang, Z. Wang, Simultaneous inversion of the fractional order and the space-dependent source term for the time-fractional diffusion equation, *Appl. Math. Comput.,* **328** (2018), 365–379. http://doi.org/10.1016/j.amc.2018.01.025
23. L. Sun, X. Yan, K. Liao, Simultaneous inversion of a fractional order and a space source term in an anomalous diffusion model, *J. Inverse Ill-pose. P.,* **30** (2022), 791–805. http://doi.org/10.1515/jiip-2021-0027
24. Y. Liu, M. Yamamoto, Uniqueness of orders and parameters in multi-term time-fractional diffusion equations by short-time behavior, *Inverse Probl.,* **39** (2022), 024003. http://doi.org/10.1088/1361-6420/acab7a
25. X. Jing, J. Jia, X. Song, Simultaneous uniqueness identification of the fractional order and diffusion coefficient in a time-fractional diffusion equation, *Appl. Math. Lett.,* **162** (2025), 109386. http://dx.doi.org/10.1016/j.amle.2024.406
26. Z. Wang, Z. Qiu, S. Qiu, Z. Ruan, Multiple terms identification of time fractional diffusion equation with symmetric potential from nonlocal observation, *Fractal Fract.,* **7** (2023), 778. http://doi.org/10.3390/fractalfract7110778
27. Z. Wang, Regularization methods for an inverse problem of a class of linear Burgers equations, *Jiangxi Sci.,* **2008** (2008), 175–178. http://doi.org/10.13990/j.issn1001-3679.2008.02.001
28. S. Qiu, Z. Wang, X. Zeng, B. Hu, Inversion method for the source term in a class of time-fractional diffusion equations, *J. Jiangxi Norm. Univ. (Natural Sci.),* **42** (2018), 610–615. http://doi.org/10.16357/j.cnki.issn1000-5862.2018.06.11