Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 717–733.
DOI: 10.3934/math.2026031
Received: 01 November 2025
Revised: 22 December 2025
Accepted: 04 January 2026
Published: 09 January 2026

Research article

A characterization of infra-Lindelöf spaces

Ohud F. Alghamdi[1,*] **and Ahmad Al-Omari**[2]

[1] Department of Mathematics, Faculty of Science, Al-Baha University, Al-Baha, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan; omarimutah1@yahoo.com

* **Correspondence:** Email: ofalghamdi@bu.edu.sa.

Abstract: In this paper, we provided a new topological structure called co-infra topological space. We provided new results regarding co-infra open and closed sets. Moreover, we investigated a new characterization of Infra-Lindelöf spaces.

Keywords: infra-topology; infra-open; Lindelöf; infra-Lindelöf; co-infra open
Mathematics Subject Classification: 54A05, 54A10

1. Introduction

General topology represents the foundational branch of topology concerned with the study of fundamental set-theoretic principles and methodologies that underpin the discipline. It serves as the theoretical basis for most other subfields of topology, including algebraic, geometric, and differential topology. The field attained maturity through the continual development of various classes of topological spaces, their properties, examples, and interrelationships. As a result, mathematicians began to explore possible extensions and generalizations of the classical notion of a topological space. The extension of the standard concept of a topological space is not a recent innovation. Pre-topologies were first introduced by Choquet [1] in the 1940s. Subsequently, in the 1960s, Levine identified several families of sets with weaker properties than regular open sets, including α-, semi-, pre-, b-, and β-open sets; see [2]. During the 1980s, Masshour [3] proposed the concept of supra-topological spaces. In the decade that followed, Császár [2] was among the first to conduct a systematic analysis of families closed under arbitrary unions, marking a significant step in the formal development of topological generalizations. This line of inquiry attracted considerable attention from researchers worldwide and has experienced substantial growth over the past two decades. Contemporary studies have introduced a variety of generalized structures, including weak structures, peri-topologies, minimal structures, reduced topologies, and generalized weak structures. As reported in [4], the latter can be characterized simply as an arbitrary family of subsets. Foundational topological concepts such as continuity, convergence, filtering, density, compactness, connectedness, and even topological groups have been reformulated within these generalized frameworks, thereby extending the applicability and scope of classical topology. The notion of infra-topological spaces was first introduced and formalized by Al-Odhari [5]. Subsequently, Al-Shami et al. [6] developed the concepts of continuity and separation axioms within this framework. Furthermore, Al-Shami et al. [7] employed fixed-point theorems to establish new notions of connectedness and covering properties in the context of infra-topologies. Reference [8] addresses the concept of fixed soft points within the framework of infra soft topological spaces and presents the fundamental properties of this concept. It also examines the transfer of fixed soft points between infra soft topology and classical infra topology in both directions. This work aims to pave the way for further studies in the field of fixed point theory.

In a later contribution, Al-Shami et al. [9] utilized infra-topological structures to define novel approximation operators and address specific problems arising in medical applications. Collectively, these studies highlight the significance of topological generalizations in modeling and interpreting various real-world phenomena. They have also paved the way for continued research in this evolving field. Subsequent contributions by other authors have further expanded this area of study (see, for example, [10–13]). Notably, authors in [14] also adopted the term "infra-topology" in their investigations, underscoring its growing relevance in contemporary topological research.

Husain [15] introduced the concept of infra-topological spaces with respect to a subset U of a universe $\mathbb{X}$. Building upon this foundational idea, the present study investigates the relationships among various types of near-infra open sets within infra-topological spaces. By exploring these interconnections, we aim to deepen the understanding of the structural behavior of such sets and their implications for the broader theory of generalized topologies.

In this paper, we introduce a new structural concept in topology, namely co-infra-open sets and co-infra-closed sets. The proposed framework employs the class of co-infra-open sets within an infra-topological space to define an associated topological structure. This construction provides a new approach to studying generalized topologies through the lens of infra-topological systems.

Infra-topology itself represents one of the weak structural frameworks that can be further extended and interpreted in various mathematical contexts. It has meaningful connections with simplicial complexes [16] and matroids associated with rough set theory [17]. Moreover, a new form of betweenness relation can be derived within this framework [18], and its concepts can also be effectively applied in nano-topology [19–22]. Over the past two decades, research on generalizations of topology has significantly expanded, with potential applications in computer science and related fields. These generalized structures incorporate concepts analogous to continuity, connectedness, Lindelöfness and compactness. This manuscript is prepared to contribute to this ongoing line of research.

2. Preliminaries

This section collects the definitions, properties, and results that will be used throughout the paper.

Definition 1. [15] The structure $(\mathbb{X}, \mathbb{F})$ constitutes an infra-topological space (abbreviated as $\mathbb{ITS}$) if:

(1) $\mathbb{X}, \emptyset \in \mathbb{F}$.

(2) If $\mathcal{H}_1$ and $\mathcal{H}_2$ are members of $\mathbb{F}$, then $\mathcal{H}_1 \cap \mathcal{H}_2 \in \mathbb{F}$.

If $\mathcal{H} \in \mathbb{F}$, we say that a set $\mathcal{H}$ is infra-open, whereas its complement $\mathcal{H}^c$ is referred to as infra-closed.

From the definition, every topological space is an $\mathbb{ITS}$. The converse implication does not generally apply, as evidenced by the following example:

Example 2. Let $X = \mathbb{R}$; where $\mathbb{R}$ denotes the collection of all real numbers. Define

$$\mathbb{F} = \{\mathbb{R}, \emptyset\} \cup \{\{x\} \mid x \in \mathbb{R}\}.$$

Then, $(\mathbb{R}, \mathbb{F})$ is an $\mathbb{ITS}$ but it doesn't form a topological space on $\mathbb{R}$.

Definition 3. [6] Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{S} \subseteq \mathbb{X}$. We define the infra-interior of $\mathcal{S}$, which is denoted by $iInt(\mathcal{S})$, and infra-closure of $\mathcal{S}$, denoted by $iCl(\mathcal{S})$, as follows:

(1) $iInt(\mathcal{S}) = \bigcup\{\mathcal{H} : \mathcal{H} \in \mathbb{F}, \ \mathcal{H} \subseteq \mathcal{S}\}$.

(2) $iCl(\mathcal{S}) = \bigcap\{F : F \ \text{ is an infra-closed set}, \ \mathcal{S} \subseteq F\}$.

Proposition 4. [6] Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{L}, \mathcal{S}$ be any subsets of $\mathbb{X}$. Hence, the following properties are valid:

(1) $iInt(\mathbb{X}) = \mathbb{X}$;

(2) If $\mathcal{L} \in \mathbb{F}$, then $iInt(\mathcal{L}) = \mathcal{L}$;

(3) If $\mathcal{S}$ is an infra-closed set, then $iCl(\mathcal{S}) = \mathcal{S}$;

(4) $l \in iInt(\mathcal{L})$ iff there exists $\mathcal{H} \in \mathbb{F}$ such that $l \in \mathcal{H} \subseteq \mathcal{L}$;

(5) $s \in iCl(\mathcal{S})$ iff for each $\mathcal{H} \in \mathbb{F}$ such that $s \in \mathcal{H}$, $\mathcal{H} \cap \mathcal{S} \neq \emptyset$;

(6) $iInt(\mathbb{X} - \mathcal{H}) = \mathbb{X} - iCl(\mathcal{H})$;

(7) $iCl(\mathbb{X} - \mathcal{S}) = \mathbb{X} - iInt(\mathcal{S})$;

(8) $iInt(\mathcal{H}) \subseteq \mathcal{H} \subseteq iCl(\mathcal{H})$;

(9) If $\mathcal{H} \subseteq \mathcal{S}$, then $iInt(\mathcal{H}) \subseteq iInt(\mathcal{S})$;

(10) $iInt(iInt(\mathcal{H})) = iInt(\mathcal{H})$;

(11) $iInt(\mathcal{H} \cap \mathcal{S}) = iInt(\mathcal{H}) \cap iInt(\mathcal{S})$;

(12) $iCl(\emptyset) = \emptyset$;

(13) If $\mathcal{H} \subseteq \mathcal{S}$, then $iCl(\mathcal{H}) \subseteq iCl(\mathcal{S})$;

(14) $iCl(iCl(\mathcal{H})) = iCl(\mathcal{H})$;

(15) $iCl(\mathcal{H} \cup \mathcal{S}) = iCl(\mathcal{H}) \cup iCl(\mathcal{S})$.

Definition 5. [6] Let $(\mathbb{X}, \mathbb{F})$ and $(\mathbb{Y}, \mathbb{G})$ be two $\mathbb{ITS}$s and let $g \colon (\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ be a function. Then, g is called:

(1) Infra-continuous function if for every $\mathcal{H} \in \mathbb{G}$ we get $g^{-1}(\mathcal{H}) \in \mathbb{F}$.

(2) Infra-open function if for every $\mathcal{H} \in \mathbb{F}$ we get $g(\mathcal{H}) \in \mathbb{G}$.

(3) Infra-closed function if for every infra-closed set $\mathcal{S} \subseteq \mathbb{X}$ we get that $g(\mathcal{S}) \subseteq \mathbb{Y}$ is an infra-closed set in $(\mathbb{Y}, \mathbb{G})$.

Definition 6. [7] Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Then,

(1) A collection $\{V_\beta \subseteq \mathbb{X} : \beta \in \Delta\}$ is an infra-open cover of $\mathbb{X}$ if

$$\mathbb{X} = \bigcup_{\beta\in\Delta} V_\beta,$$

and $V_\beta \in \mathbb{F}$ for each $\beta \in \Delta$.

(2) $\mathbb{X}$ is called an infra-Lindelöf if every infra-open cover of $\mathbb{X}$ has a countable subcover.

(3) A set $\mathcal{H} \subseteq \mathbb{X}$ is said to be an infra-Lindelöf subspace of $\mathbb{F}$ if every infra-open cover of $\mathcal{H}$ has a countable subcover.

3. Co-infra-topological spaces

In this section, we present a new topological structure derived from co-infra-open sets in an infra-topological space ($\mathbb{ITS}$).

Definition 7. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. A subset $\mathcal{Y} \subseteq \mathbb{X}$ is called co-infra-open if for each $y \in \mathcal{Y}$, there is an infra-open set $\mathcal{H}_y$ where $y \in \mathcal{H}_y$ and $\mathcal{H}_y - \mathcal{Y}$ is a countable set. Moreover, $\mathbb{X} - \mathcal{Y}$ is called co-infra-closed set. We will denote the collection of all co-infra-open sets in $(\mathbb{X}, \mathbb{F})$ by $\mathbb{F}^*$.

Remark 8. Observe that every infra-open set is a co-infra-open set. To show that, let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{H} \in \mathbb{F}$. Then, for every $x \in \mathcal{H}$, pick

$$\mathcal{H}_x = \mathcal{H}.$$

The converse of Remark 8 is not, in general, valid, as illustrated by the following example:

Example 9. Consider the usual (Euclidean) topological space $\mathcal{U}$ on the set of the real numbers $\mathbb{R}$. Then, $(\mathbb{R}, \mathcal{U})$ is an $\mathbb{ITS}$ on $\mathbb{R}$. The set of all irrational numbers $\mathbb{I}$ is a co-infra-open set but it is not an infra-open set.

We conclude that in an $\mathbb{ITS}$, any set that has a countable complement is a co-infra-open set.

Lemma 10. Let $(\mathbb{X}, \mathbb{F})$ be $\mathbb{ITS}$. A subset $\mathcal{Y} \subseteq \mathbb{X}$ is a co-infra-open set if and only if for each $x \in \mathcal{Y}$, there is a countable subset $C_x \subseteq \mathbb{X}$ and an infra-open set U_x where $x \in U_x$ such that

$$U_x - C_x \subseteq \mathcal{Y}.$$

Proof. Let $\mathcal{Y}$ be a co-infra-open set and let $x \in \mathcal{Y}$. For each point x, there exists an infra-open set U_x containing x where the difference $U_x - \mathcal{Y}$ is at most countable. Suppose

$$C_x = U_x - \mathcal{Y} = U_x \cap (\mathbb{X} - \mathcal{Y}).$$

Then,

$$U_x - C_x \subseteq \mathcal{Y}.$$

Conversely, let $x \in \mathcal{Y}$. Then, there is a countable subset C_x and an infra-open set U_x for which $x \in U_x$ and

$$U_x - C_x \subseteq \mathcal{Y}.$$

Therefore,

$$U_x - \mathcal{Y} \subseteq C_x,$$

which implies that $U_x - \mathcal{Y}$ is a countable set; hence, $\mathcal{Y}$ is a co-infra-open set. □

Lemma 11. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $A \subseteq \mathbb{X}$. If A is a co-infra-closed set, then

$$A \subseteq H \cup V,$$

such that $V \subseteq \mathbb{X}$ is a countable set and $H \subseteq \mathbb{X}$ is an infra-closed set.

Proof. Let A be a co-infra-closed set. Then, $\mathbb{X} - A$ is a co-infra-open set which yields that for each $x \in \mathbb{X} - A$, there is a countable set V and an infra-open set U such that $x \in U$ and

$$U - V \subseteq \mathbb{X} - A.$$

Thus,

$$A \subseteq \mathbb{X} - (U - V) = \mathbb{X} - (U \cap (\mathbb{X} - V)) = (\mathbb{X} - U) \cup V.$$

Let

$$H = \mathbb{X} - U;$$

hence, H is an infra-closed set and

$$A \subseteq H \cup V.$$

This completes the proof. □

Lemma 12. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Then,

$$[\mathbb{F}^*]^* = \mathbb{F}^*.$$

Proof. Since every infra-open set is a co-infra-open set, then we have $\mathbb{F}^* \subseteq [\mathbb{F}^*]^*$. Let $A \in [\mathbb{F}^*]^*$. By Lemma 10, for each $x \in A$, there is a countable set C_x and $U_x \in \mathbb{F}^*$ for which $x \in U_x$ and

$$U_x - C_x \subseteq A.$$

Furthermore, by Lemma 10, there is a countable set D_x and $V_x \in \mathbb{F}$ for which $x \in V_x$ and

$$V_x - D_x \subseteq U_x.$$

Hence, we have

$$V_x - (D_x \cup C_x) = (V_x - D_x) - C_x \subseteq U_x - C_x \subseteq A.$$

Since $D_x \cup C_x$ is a countable set, we obtain that $A \in \mathbb{F}^*$; hence,

$$[\mathbb{F}^*]^* = \mathbb{F}^*.$$

This completes the proof. □

The example below illustrates that a co-infra topological space is not equivalent to its corresponding $\mathbb{ITS}$.

Example 13. Consider

$$\mathbb{X} = \mathbb{R},$$

where $\mathbb{R}$ the collection of real numbers. Define $\mathbb{F}$ as follows:

$$\mathbb{F} = \{\emptyset, \mathcal{H} : \mathcal{H} \subseteq \mathbb{R} \text{ and } \sqrt{2} \in \mathcal{H}\}.$$

Then, $(\mathbb{R}, \mathbb{F})$ is an $\mathbb{ITS}$. We can find the corresponding co-infra topological space $\mathbb{F}^*$ on $\mathbb{R}$ as follows: Let $S \subseteq \mathbb{R}$ be a nonempty set. Let $s \in S$. We have two cases:

- Case 1. $s = \sqrt{2}$. Then, pick $\{\sqrt{2}\} \in \mathbb{F}$ and $\{\sqrt{2}\} - S = \emptyset$ is countable.
- Case 2. $s \neq \sqrt{2}$. Then, pick $\{\sqrt{2}, s\} \in \mathbb{F}$ and $\{\sqrt{2}, s\} - S$ is countable.

Hence, for all $S \subseteq \mathbb{R}$, we have $S \in \mathbb{F}^*$ and thus, $\mathbb{F}^*$ is the discrete topology.

Proposition 14. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. If $\mathcal{K}, \mathcal{H} \subseteq \mathbb{X}$ are both co-infra-open sets, then $\mathcal{K} \cap \mathcal{H}$ is a co-infra-open set.

Proof. Let $\mathcal{K}$ and $\mathcal{H}$ be co-infra-open sets in $\mathbb{X}$ and $x \in \mathcal{K} \cap \mathcal{H}$. Since $\mathcal{K}$ is a co-infra-open set, there is an infra-open set $V_{\mathcal{K}}$ for which $x \in V_{\mathcal{K}}$ and $V_{\mathcal{K}} - \mathcal{K}$ is a countable set. Moreover, since $\mathcal{H}$ is a co-infra-open set, then there exists an infra-open set $V_{\mathcal{H}}$ where $x \in V_{\mathcal{H}}$ and $V_{\mathcal{H}} - \mathcal{H}$ is a countable set. Now, we have $V_{\mathcal{K}} \cap V_{\mathcal{H}}$ is an infra-open set where $x \in V_{\mathcal{K}} \cap U_{\mathcal{H}}$,

$$\begin{aligned}(V_{\mathcal{K}} \cap V_{\mathcal{H}}) - (\mathcal{K} \cap \mathcal{H}) &= (V_{\mathcal{K}} \cap V_{\mathcal{H}}) \cap [(\mathbb{X} - \mathcal{K}) \cup (\mathbb{X} - \mathcal{H})] \\ &= [V_{\mathcal{K}} \cap V_{\mathcal{H}} \cap (\mathbb{X} - \mathcal{K})] \cup [V_{\mathcal{K}} \cap V_{\mathcal{H}} \cap (\mathbb{X} - \mathcal{H})] \\ &\subseteq (V_{\mathcal{K}} \cap (\mathbb{X} - \mathcal{K})) \cup (V_{\mathcal{H}} \cap (\mathbb{X} - \mathcal{H})).\end{aligned}$$

Since

$$(V_{\mathcal{K}} \cap (\mathbb{X} - \mathcal{K})) \cup (V_{\mathcal{H}} \cap (\mathbb{X} - \mathcal{H})) = [V_{\mathcal{K}} - \mathcal{K}] \cup [V_{\mathcal{H}} - \mathcal{H}]$$

is a countable set, then $(V_{\mathcal{K}} \cap V_{\mathcal{H}}) - (\mathcal{K} \cap \mathcal{H})$ is a countable set. This demonstrates that $\mathcal{K} \cap \mathcal{H}$ is a co-infra-open set. □

The previous result can be generalized as follows: The finite intersection of co-infra-open sets is a co-infra-open set.

Proposition 15. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. The union of co-infra-open sets is a co-infra-open set.

Proof. Let $\{\mathcal{H}_\alpha : \alpha \in \Delta\}$ be any family of co-infra-open sets in $(\mathbb{X}, \mathbb{F})$ and let

$$x \in \bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha.$$

Then, there exists $\beta \in \Delta$ such that $x \in \mathcal{H}_\beta$ which implies that there is an infra-open set V such that $x \in V$ and $V - \mathcal{H}_\beta$ is a countable set. Since

$$V - \left(\bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha\right) \subseteq V - \mathcal{H}_\beta;$$

thus, $V - \left(\bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha\right)$ is a countable set. Therefore, $\bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha$ is a co-infra-open set. □

Corollary 16. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Then, $\mathbb{F}^*$ forms a topological space on $\mathbb{X}$.

Remark 17. If $\mathbb{X}$ is a countable set, then $\mathbb{F}^*$ is equivalent to the discrete topological space on X.

Example 18. Let $\mathbb{X} = \mathbb{R}$, where $\mathbb{R}$ represents the collection of all real numbers. Define $\mathbb{F}$ as follows:

$$\mathbb{F} = \{\emptyset, \mathcal{H} : \mathcal{H} \subseteq \mathbb{R} \text{ and } \mathbb{R} - \mathcal{H} \text{ is finite}\}.$$

Then, $(\mathbb{R}, \mathbb{F})$ is an $\mathbb{ITS}$. We can define $\mathbb{F}^*$ on $\mathbb{R}$ as follows:

$$\mathbb{F}^* = \{\emptyset\} \cup \{\mathcal{H} : \mathcal{H} \subseteq \mathbb{R} \text{ and } \mathbb{R} - \mathcal{H} \text{ is countable}\}.$$

Example 19. Let

$$\mathbb{X} = \mathbb{R},$$

where $\mathbb{R}$ denotes the collection of all real numbers. Define $\mathbb{F}$ as follows:

$$\mathbb{F} = \{\emptyset,\ \mathbb{R}\} \cup \{\mathcal{H} \ :\ \sqrt{2} \in \mathcal{H},\ \sqrt{3} \notin \mathcal{H} \text{ and } \mathcal{H} \subseteq \mathbb{R}\}.$$

Then, $(\mathbb{R}, \mathbb{F})$ is an $\mathbb{ITS}$. Let $\mathcal{S} \subseteq \mathbb{R}$ be any nonempty set. Then, we have two cases:

Case 1. $\sqrt{3} \in \mathcal{S}$. Then, the only infra-open set that contains $\sqrt{3}$ is $\mathbb{R}$. Then, $\mathcal{S}$ is a co-infra-open set if and only if it has a countable complement.

Case 2. $\sqrt{3} \notin \mathcal{S}$. Then, $\mathcal{S}$ is a co-infra-open set. Indeed, let $s \in \mathcal{S}$. Then, we have two cases:

(a) $s = \sqrt{2}$. Let $\mathcal{H} = \{\sqrt{2}\} \in \mathbb{F}$. Then, $\mathcal{H} - \mathcal{S}$ is a countable set.

(b) $s \neq \sqrt{2}$. Let $\mathcal{H} = \{\sqrt{2}, s\} \in \mathbb{F}$. Then, $\mathcal{H} - \mathcal{S}$ is a countable set.

Hence,

$$\mathbb{F}^* = \{\mathcal{S} : \text{such that either } \mathbb{R} - \mathcal{S} \text{ is countable or } \sqrt{3} \notin \mathcal{S}\}.$$

Now, we are going to show that $\mathbb{F}^*$ is a topology on $\mathbb{R}$.

(1)
- $\mathbb{R} - \mathbb{R} = \emptyset$ is a countable set which implies that $\mathbb{R} \in \mathbb{F}^*$.
- $\sqrt{3} \notin \emptyset$ which implies that $\emptyset \in \mathbb{F}^*$.

(2) Let $U, V \in \mathbb{F}^*$ we have two cases:
- Case 1. Suppose that $\mathbb{R} - U$ and $\mathbb{R} - V$ are both countable. Then,

$$\mathbb{R} - (U \cap V) = (\mathbb{R} - U) \cup (\mathbb{R} - V),$$

  is a countable set which yields that $U \cap V \in \mathbb{F}^*$.
- Case 2. Suppose that $\sqrt{3} \notin U$ or $\sqrt{3} \notin V$. Then, $\sqrt{3} \notin U \cap V$. Hence, $U \cap V \in \mathbb{F}^*$.

(3) Let $U_\alpha \in \mathbb{F}^*$ for all $\alpha \in \Lambda$. Then, we have two cases:
- Case 1. There exists $\beta \in \Lambda$ such that $\mathbb{R} - U_\beta$ is countable. Hence,

$$\mathbb{R} - \left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) = \bigcap_{\alpha \in \Lambda} (\mathbb{R} - U_\alpha) \subseteq \mathbb{R} - U_\beta,$$

  which is countable; thus, $\mathbb{R} - (\cup_{\alpha \in \Lambda} U_\alpha)$ is countable. Then,

$$\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathbb{F}^*.$$

- Case 2. Suppose that $\sqrt{3} \notin U_\alpha$ for all $\alpha \in \Lambda$. Then,

$$\sqrt{3} \notin \bigcup_{\alpha\in\Lambda} U_\alpha,$$

which implies that

$$\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathbb{F}^*.$$

Hence, $\mathbb{F}^*$ is form a topology on $\mathbb{R}$.

Example 20. Let $(\mathbb{R}, \mathbb{F})$ be defined as in Example 2. Let $\mathcal{H} \subseteq \mathbb{R}$ be a nonempty set and let $h \in \mathcal{H}$. Hence, $\{h\} \in \tau$ and

$$\{h\} - \mathcal{H} = \emptyset.$$

Then, $\mathcal{H} \in \mathbb{F}^*$ if and only if $\mathcal{H} \subseteq \mathbb{R}$ which yields that $\mathbb{F}^*$ is equivalent to the discrete topological space defined on $\mathbb{R}$.

Definition 21. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{S} \subseteq \mathbb{X}$. We define the co-infra-interior of $\mathcal{S}$, denoted by $i_cInt(\mathcal{S})$, and denote the co-infra closure of $\mathcal{S}$ by $i_cCl(\mathcal{S})$, as follows:

(1) $i_cInt(\mathcal{S}) = \bigcup\{\mathcal{H} : \mathcal{H} \in \mathbb{F}^c,\ \mathcal{H} \subseteq \mathcal{S}\}$.
(2) $i_cCl(\mathcal{S}) = \bigcap\{F : F$ is a co-infra-closed set , $\mathcal{S} \subseteq F\}$.

Proposition 22. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Let $\mathcal{H}$ be any subset of $\mathbb{X}$. Then,

(1) $\mathcal{H} \in \mathbb{F}^*$ if and only if $i_cInt(\mathcal{H}) = \mathcal{H}$;
(2) $\mathcal{H}$ is a co-infra-closed set if and only if $i_cCl(\mathcal{H}) = \mathcal{H}$;
(3) $h \in i_cInt(\mathcal{H})$ if and only if there exists $\mathcal{Y} \in \mathbb{F}^*$ such that $h \in \mathcal{Y} \subseteq \mathcal{H}$;
(4) $h \in i_cCl(\mathcal{H})$ if and only if $\mathcal{H} \cap \mathcal{Y} \neq \emptyset$ for each $\mathcal{Y} \in \mathbb{F}^*$ with $h \in \mathcal{Y}$;
(5) $i_cInt(\mathbb{X} - \mathcal{H}) = \mathbb{X} - i_cCl(\mathcal{H})$;
(6) $i_cCl(\mathbb{X} - \mathcal{H}) = \mathbb{X} - i_cInt(\mathcal{H})$.

Proposition 23. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Let $\mathcal{H}, \mathcal{S}$ be any subsets of $\mathbb{X}$. Then,

(1) $i_cInt(\mathbb{X}) = \mathbb{X}$;
(2) $iInt(\mathcal{H}) \subseteq i_cInt(\mathcal{H}) \subseteq \mathcal{H} \subseteq i_cCl(\mathcal{H}) \subseteq iCl(\mathcal{H})$;
(3) If $\mathcal{H} \subseteq \mathcal{S}$, then $i_cInt(\mathcal{H}) \subseteq i_cInt(\mathcal{S})$;
(4) $i_cInt(i_cInt(\mathcal{H})) = i_cInt(\mathcal{H})$;
(5) $i_cInt(\mathcal{H} \cap \mathcal{S}) = i_cInt(\mathcal{H}) \cap i_cInt(\mathcal{S})$;
(6) $i_cInt(\emptyset) = \emptyset$;
(7) If $\mathcal{H} \subseteq \mathcal{S}$, then $i_cCl(\mathcal{H}) \subseteq i_cCl(\mathcal{S})$;
(8) $i_cCl(i_cCl(\mathcal{H})) = i_cCl(\mathcal{H})$;
(9) $i_cCl(\mathcal{H} \cup \mathcal{S}) = i_cCl(\mathcal{H}) \cup i_cCl(\mathcal{S})$.

Proof. It follows immediately from Definition 21. □

Theorem 24. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ such that all non-empty infra-open sets are uncountable. Then,

$$iCl(\mathcal{H}) = i_cCl(\mathcal{H}),$$

for all $\mathcal{H} \in \mathbb{F}$.

Proof. We know that

$$i_cCl(\mathcal{H}) \subseteq iCl(\mathcal{H})$$

from Proposition 23. Conversely, let $h \in iCl(\mathcal{H})$ and let $\mathcal{K}$ be a co-infra-open set where $h \in \mathcal{K}$. By Lemma 10, there is a countable set C and an infra-open set A such that $h \in A$ and

$$A - C \subseteq \mathcal{K}.$$

Thus,

$$(A - C) \cap \mathcal{H} \subseteq \mathcal{K} \cap \mathcal{H},$$

which implies that

$$(A \cap \mathcal{H}) - C \subseteq \mathcal{K} \cap \mathcal{H}.$$

Since $h \in A$ and $h \in iCl(\mathcal{H})$, then

$$A \cap \mathcal{H} \neq \emptyset,$$

and $A \cap \mathcal{H}$ is an infra-open set since A and $\mathcal{H}$ are both infra-open sets. According to the hypothesis that all non-empty infra-open sets are uncountable; hence, $(A \cap \mathcal{H}) - C$ is an uncountable set which implies that

$$\mathcal{K} \cap \mathcal{H} \neq \emptyset;$$

hence, $h \in i_cCl(\mathcal{H})$. Therefore,

$$iCl(\mathcal{H}) = i_cCl(\mathcal{H}).$$

This completes the proof. □

Corollary 25. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ such that every non-empty infra-open set is uncountable. Then,

$$iInt(\mathcal{H}) = i_cInt(\mathcal{H}),$$

for each $\mathcal{H} \in \mathbb{F}$.

Definition 26. Let $(\mathbb{X}, \mathbb{F})$ and $(\mathbb{Y}, \mathbb{G})$ be two $\mathbb{ITS}$s. A function g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ is called a co-infra-continuous function if for every $k \in \mathbb{X}$ and for every infra-open set $\mathcal{H} \subseteq \mathbb{Y}$ for which $g(k) \in \mathcal{H}$, there is a co-infra-open set $\mathcal{S} \subseteq \mathbb{X}$ where $k \in \mathcal{S}$ and $g(\mathcal{S}) \subseteq \mathcal{H}$.

Lemma 27. Let $(\mathbb{X}, \mathbb{F})$ and $(\mathbb{Y}, \mathbb{G})$ be two $\mathbb{ITS}$s and let g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ be a function. It follows that the following statements are equal in meaning:

(1) g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ is a co-infra-continuous function.

(2) g: $(\mathbb{X}, \mathbb{F}^*) \to (\mathbb{Y}, \mathbb{G})$ is an infra-continuous function.

Proof. Let g: $(\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G})$ be a co-infra-continuous function and let

$$\mathcal{S} \in \mathbb{G} \subseteq \mathbb{G}^*.$$

Then, $g^{-1}(\mathcal{S}) \in \mathbb{F}^*$ since every infra-open set is a co-infra-open set. Conversely, let g: $(\mathbb{X}, \mathbb{F}^*) \rightarrow (\mathbb{Y}, \mathbb{G})$ be an infra-continuous function. Assume that $s \in \mathbb{X}$ and $\mathcal{S} \in \mathbb{G}$ for which $g(s) \in \mathcal{S}$. Hence,

$$s \in g^{-1}(\mathcal{S}) \in \mathbb{F}^*,$$

as g is an infra-continuous and

$$g(g^{-1}(\mathcal{S})) \subseteq \mathcal{S}.$$

This completes the proof. □

Definition 28. Let $(\mathbb{X}, \mathbb{F})$ and $(\mathbb{Y}, \mathbb{G})$ be two $\mathbb{ITS}$s. A function g: $(\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G})$ is called a co-infra-closed function if for every co-infra-closed set $\mathcal{S} \subseteq \mathbb{X}$, $g(\mathcal{S})$ is a co-infra-closed set.

Lemma 29. Let $(\mathbb{X}, \mathbb{F})$ and $(\mathbb{Y}, \mathbb{G})$ be two $\mathbb{ITS}$s and let g: $(\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G})$ be a function. We conclude that the following properties are equivalent:

(1) g is a co-infra-closed function.
(2) $g : (\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G}^c)$ is an infra-closed function.

Theorem 30. If g: $(\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G})$ is an infra-open function, then the image of a co-infra-open set of $\mathbb{X}$ is a co-infra-open set in $\mathbb{Y}$.

Proof. Let $\mathcal{H} \in \mathbb{F}^c$ be a nonempty set and let $h \in \mathcal{H}$. Then, there exists an infra-open set $K \subseteq \mathbb{X}$ where $h \in K$ and

$$K - \mathcal{H} = C$$

is a countable set. Hence,

$$g(K) - g(\mathcal{H}) \subseteq g(K - \mathcal{H}) = g(C)$$

is a countable set and $g(K)$ is an infra-open set since g is an infra-open function. Thus, $g(\mathcal{H})$ is a co-infra-open set. □

Theorem 31. If g: $(\mathbb{X}, \mathbb{F}) \rightarrow (\mathbb{Y}, \mathbb{G})$ is a one to one infra-continuous function, then the inverse image of a co-infra open set in $\mathbb{Y}$ is a co-infra open set in $\mathbb{X}$.

Proof. Let $\mathcal{S} \in \mathbb{G}^c$ be a co-infra-open set for which $g^{-1}(S) \neq \emptyset$. Let $s \in g^{-1}(\mathcal{S})$. Then, $g(s) \in \mathcal{S}$ which implies that there is an infra-open set $\mathcal{H}$ where $s \in \mathcal{H}$ and

$$\mathcal{H} - \mathcal{S} = C$$

is a countable set. Thus,

$$g^{-1}(\mathcal{H}) - g^{-1}(\mathcal{S}) \subseteq g^{-1}(\mathcal{H} - \mathcal{S}) = g^{-1}(C)$$

is a countable set because g is one to one and $g^{-1}(\mathcal{H})$ is an infra-open set since g is an infra-continuous function. Then, $g^{-1}(\mathcal{S})$ is a co-infra open set. □

4. Infra-Lindelöf spaces

This section is dedicated to presenting the concept of infra-Lindelöf spaces, providing a new characterization, and examining some of their fundamental properties.

Theorem 32. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{H} \subseteq \mathbb{X}$. Then, $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{F}$ if and only if $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{F}^*$.

Proof. Let us assume that

$$\emptyset \neq \mathcal{H} \subseteq \mathbb{X}$$

is an infra-Lindelöf subspace of $\mathbb{F}$. Let $\{\mathcal{A}_\beta : \beta \in \Lambda\}$ be a cover of $\mathcal{H}$ where $\mathcal{A}_\beta \in \mathbb{F}^*$ for each $\beta \in \Lambda$. Let $x \in \mathcal{H}$ which implies that there exists $\beta(x) \in \Lambda$ for which $x \in \mathcal{A}_{\beta(x)}$. Since $\mathcal{A}_{\beta(x)}$ is a co-infra-open set, there exists an infra-open set $B_{\beta(x)} \in \mathbb{F}$ where $x \in B_{\beta(x)}$ and $B_{\beta(x)} - \mathcal{A}_{\beta(x)}$ is a countable set. Now, the collection $\{B_{\beta(x)} \mid x \in H\}$ is a cover of $\mathcal{H}$ by infra-open sets in $\mathbb{X}$; hence, there is a countable subset, say $\beta(x_1), \beta(x_2), \cdots \beta(x_n), \cdots$, such that

$$\mathcal{H} \subseteq \bigcup_{i \in \mathbb{N}} B_{\beta(x_i)}.$$

Thus, we have

$$\mathcal{H} \subseteq \bigcup_{i \in \mathbb{N}} \left[(B_{\beta(x_i)} - \mathcal{A}_{\beta(x_i)}) \bigcup \mathcal{A}_{\beta(x_i)} \right] = \left[\bigcup_{i \in \mathbb{N}} (B_{\beta(x_i)} - \mathcal{A}_{\beta(x_i)}) \right] \bigcup \left[\bigcup_{i \in \mathbb{N}} \mathcal{A}_{\beta(x_i)} \right].$$

Hence,

$$\mathcal{H} \subseteq \left[\bigcup_{i \in \mathbb{N}} (B_{\beta(x_i)} - \mathcal{A}_{\beta(x_i)}) \bigcap \mathcal{H} \right] \bigcup \left[\bigcup_{i \in \mathbb{N}} \mathcal{A}_{\beta(x_i)} \right].$$

For each $\beta(x_i)$,

$$(B_{\beta(x_i)} - \mathcal{A}_{\beta(x_i)}) \cap \mathcal{H}$$

is a countable set; then, there exists a countable subset

$$\Lambda_{\beta(x_i)} \subseteq \Lambda,$$

for which

$$[B_{\beta(x_i)} - \mathcal{A}_{\beta(x_i)}] \cap \mathcal{H} \subseteq \bigcup_{\beta \in \Lambda_{\beta(x_i)}} \mathcal{A}_\beta.$$

Therefore, we have

$$\mathcal{H} \subseteq \left[\bigcup_{i \in \mathbb{N}} \left(\bigcup_{\beta \in \Lambda_{\beta(x_i)}} \mathcal{A}_\beta \right) \right] \bigcup \left[\bigcup_{i \in \mathbb{N}} A_{\beta(x_i)} \right].$$

Hence, $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{F}^*$.

Let $\mathcal{H}$ be an infra-Lindelöf relative to $\mathbb{F}^c$. Since every infra-open set is a co-infra open set, then $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{F}$. □

Corollary 33. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Then, $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space if and only if $(\mathbb{X}, \mathbb{F}^*)$ is an infra-Lindelöf space.

Theorem 34. Let g: $(\mathbb{X},\mathbb{F}) \to (\mathbb{Y},\mathbb{G})$ be a co-infra-continuous mapping. If $(\mathbb{X},\mathbb{F})$ is an infra-Lindelöf space, then the image space $(\mathbb{Y},\mathbb{G})$ inherits the infra-Lindelöf property; that is, $\mathbb{Y}$ is infra-Lindelöf.

Proof. Let $\{\mathcal{A}_\alpha : \alpha \in \Lambda\}$ be an infra-open cover of $\mathbb{Y}$. Then, $\{g^{-1}(\mathcal{A}_\alpha) : \alpha \in \Lambda\}$ is a co-infra-open cover of $\mathbb{X}$. Since $(\mathbb{X},\mathbb{F})$ is an infra-Lindelöf space, then $(\mathbb{X},\mathbb{F}^*)$ is an infra-Lindelöf space by Corollary 33. Therefore,

$$\mathbb{X} \subseteq \bigcup_{i=1}^{\infty} g^{-1}(A_{\alpha i}),$$

which implies that

$$\mathbb{Y} = g(\mathbb{X}) \subseteq g\left(\bigcup_{i=1}^{\infty} g^{-1}(A_{\alpha i})\right) \subseteq \bigcup_{i=1}^{\infty} A_{\alpha i}.$$

Then, $\{A_{\alpha i}\}_{i=1}^{\infty}$ is a countable subcover of $\mathbb{Y}$. Hence, $(\mathbb{Y},\mathbb{G})$ is an infra-Lindelöf space.

This completes the proof. □

Theorem 35. Let $(\mathbb{X},\mathbb{F})$ be an $\mathbb{ITS}$ and let $\mathcal{H},\mathcal{K} \subseteq \mathbb{X}$. If $\mathcal{H}$ is an infra-Lindelöf and $\mathcal{K}$ is a co-infra-closed set, then $\mathcal{H} \cap \mathcal{K}$ is an infra-Lindelöf subspace of $\mathbb{X}$.

Proof. Let $\{\mathcal{A}_\alpha : \alpha \in \Lambda\}$ be an infra-open cover of $\mathcal{H} \cap \mathcal{K}$. Let $x \in \mathcal{H} - \mathcal{K}$. Then, $x \in \mathbb{X} - \mathcal{K}$ which implies that there exists an infra-open set B_x for which $x \in B_x$ and $B_x - (\mathbb{X} - \mathcal{K})$ is a countable set since $\mathcal{K}$ is a co-infra-closed set. Hence,

$$\mathcal{W} = \{\mathcal{A}_\alpha : \alpha \in \Lambda\} \bigcup \{B_x : x \in \mathcal{H} - \mathcal{K}\}$$

is an infra-open cover of $\mathcal{H}$. Thus,

$$\mathcal{H} \subseteq \left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_{\alpha_i}\right) \bigcup \left(\bigcup_{i\in\mathbb{N}} B_{x_i}\right),$$

which is a countable subcover of $\mathcal{W}$ since $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{X}$. Therefore,

$$\mathcal{H} \cap \mathcal{K} \subseteq \left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_{\alpha_i}\right) \bigcup \left(\bigcup_{i\in\mathbb{N}} (B_{x_i} \cap \mathcal{K})\right).$$

Since

$$\bigcup_{i\in\mathbb{N}} (B_{x_i} \cap \mathcal{K})$$

is a countable set, then for each

$$x_j \in \cup(B_{x_j} \cap K),$$

pick $\mathcal{A}_{\alpha(x_j)} \in \mathcal{W}$ such that $x_j \in \mathcal{A}_{\alpha(x_j)}$ and $j \in \mathbb{N}$. Then,

$$\mathcal{H} \cap \mathcal{K} \subseteq \left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_{\alpha_i}\right) \bigcup \left(\bigcup_{j\in\mathbb{N}} \mathcal{A}_{\alpha(x_j)}\right).$$

Hence, $\mathcal{H} \cap \mathcal{K}$ is an infra-Lindelöf subspace of $\mathbb{X}$. □

Corollary 36. If $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space and $\mathcal{H} \subseteq \mathbb{X}$ is a co-infra-closed set, then $\mathcal{H}$ is an infra-Lindelöf subspace.

Corollary 37. Let $(\mathbb{X}, \mathbb{F})$ be an infra-Lindelöf space and $\mathcal{H} \subseteq \mathbb{X}$ is an infra-closed set, then $\mathcal{H}$ is an infra-Lindelöf subspace.

Corollary 38. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. From this, we see that the following properties are equivalent:

(1) $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space.

(2) Every proper co-infra closed subset of $\mathbb{X}$ is an infra-Lindelöf subspace.

(3) Every proper infra closed subset of $\mathbb{X}$ is an infra-Lindelöf subspace.

Proof. (1) ⇒ (2): By Corollary 36, we know that every co-infra-closed subset of an infra-Lindelöf space is an infra-Lindelöf subspace.

(2) ⇒ (3): Since $\mathbb{F} \subseteq \mathbb{F}^*$, the proof is obvious.

(3) ⇒ (1): Let $\{\mathcal{A}_\beta \subseteq \mathbb{X} : \beta \in \Lambda\}$ be a family of infra-open sets such that

$$\mathbb{X} = \bigcup_{\beta \in \Lambda} \mathcal{A}_\beta.$$

Let $\beta^* \in \Lambda$ where $\mathbb{X} - \mathcal{A}_{\beta^*}$ is a proper subset of $\mathbb{X}$. Hence, $\{\mathcal{A}_\beta : \beta \in \Lambda - \{\beta^*\}\}$ is an infra-open cover of $\mathbb{X} - \mathcal{A}_{\beta^*}$. Since $\mathbb{X} - \mathcal{A}_{\beta^*}$ is an infra-closed set, then there is a countable subset $\Lambda_0 \subseteq \Lambda$ for which

$$\mathbb{X} - \mathcal{A}_{\beta^*} \subseteq \bigcup_{\beta \in \Lambda_0} \mathcal{A}_\beta.$$

Hence,

$$\mathbb{X} = \bigcup_{\beta \in \Lambda_0 \cup \{\beta^*\}} \mathcal{A}_\beta,$$

which implies that $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space. □

The following example illustrates that an infra-closed subspace of a non infra-Lindelöf space may not be an infra-Lindelöf subspace.

Example 39. Let $\mathbb{F}$ be defined on $\mathbb{R}$ as in Example 2. Then, $\{\{x\} \mid x \in \mathbb{R}\}$ is an infra-open cover that has no countable subcover. Hence, $(\mathbb{R}, \mathbb{F})$ is not infra-Lindelöf space. Let $\mathcal{S}$ be a nonempty proper infra-closed subspace. Then,

$$\mathcal{S} = \mathbb{R} - \{s\},$$

for some $s \in \mathbb{R}$. Hence, $\{\{x\} \mid x \in \mathbb{R} - \{s\}\}$ is an infra-open cover of $\mathcal{S}$ that has no countable subcover which implies that $\mathcal{S}$ is not an infra-Lindelöf subspace.

Theorem 40. If g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ is a co-infra-continuous function and $\mathcal{H} \subseteq \mathbb{X}$ is an infra-Lindelöf subspace of $\mathbb{X}$, then $g(\mathcal{H}) \subseteq \mathbb{G}$ is an infra-Lindelöf subspace of $\mathbb{G}$.

Proof. Let $\{\mathcal{A}_\beta : \beta \in \Lambda\}$ be an infra-open cover of $g(\mathcal{H})$. Then, $\{g^{-1}(\mathcal{A}_\beta) : \beta \in \Lambda\}$ is a co-infra-open cover of $\mathcal{H}$. Since $\mathcal{H}$ is an infra-Lindelöf subspace of $(\mathbb{X}, \mathbb{F}^c)$ by Theorem 32, then there is a countable subset $\Lambda_0 \subseteq \Lambda$ for which

$$\mathcal{H} \subseteq \bigcup_{\beta \in \Lambda_0} g^{-1}(\mathcal{A}_\beta).$$

Therefore,

$$g(\mathcal{H}) \subseteq \bigcup_{\beta \in \Lambda_0} \mathcal{A}_\beta.$$

Hence, $g(\mathcal{H})$ is an infra-Lindelöf subspace of $\mathbb{G}$. □

Corollary 41. If g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ is a surjective co-infra-continuous function and $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space, then $(\mathbb{Y}, \mathbb{G})$ is an infra-Lindelöf space.

Corollary 42. If g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ is a surjective infra-continuous function and $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space, then $(\mathbb{Y}, \mathbb{G})$ is an infra-Lindelöf space.

Definition 43. Let $(\mathbb{X}, \mathbb{F})$ be an $\mathbb{ITS}$. Then, the family $\mathbb{F}$ has property **B** if, for every countable collection of elements in $\mathbb{F}$, their union remains an element of $\mathbb{F}$.

Theorem 44. Let g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ be a co-infra-closed function such that $g^{-1}(b)$ is an infra-Lindelöf subspace of $\mathbb{F}$ for each $b \in \mathbb{Y}$. Suppose that $\mathbb{F}$ satisfies the property **B**. If $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{G}$, then $g^{-1}(\mathcal{H})$ is an infra-Lindelöf subspace of $\mathbb{F}$.

Proof. Let $\{\mathcal{A}_\beta : \beta \in \Lambda\}$ be an infra-open cover of $g^{-1}(\mathcal{H})$ and let $b \in \mathcal{H}$. Since $g^{-1}(b)$ is an infra-Lindelöf subspace of $\mathbb{F}$, there is a countable set $\Lambda_1(b) \subseteq \Lambda$ for which

$$g^{-1}(b) \subseteq \bigcup_{\beta \in \Lambda_1(b)} \mathcal{A}_\beta.$$

Let

$$A(b) = \bigcup_{\beta \in \Lambda_1(b)} \mathcal{A}_\beta.$$

Then, since g is a co-infra closed function, there is a co-infra open set $B(b) \subseteq \mathbb{Y}$ and $b \in B(b)$, where

$$g^{-1}(B(b)) \subseteq A(b).$$

Since $B(b)$ is a co-infra-open set, there is an infra-open-set $C(b)$ such that $b \in C(b)$ and $C(b) - B(b)$ is a countable set. Thus,

$$C(b) \subseteq (C(b) - B(b)) \cup B(b);$$

hence,

$$g^{-1}(C(b)) \subseteq g^{-1}(C(b) - B(b)) \cup g^{-1}(B(b)) \subseteq g^{-1}(C(b) - B(b)) \cup A(b).$$

Since $g^{-1}(b)$ is an infra-Lindelöf subspace of $\mathbb{F}$ for every $b \in \mathbb{Y}$ and $C(b) - B(b)$ is a countable set, then there is a countable set $\Lambda_2(b) \subseteq \Lambda$ such that:

$$g^{-1}[(C(b) - B(b)) \cap \mathcal{H}] \subseteq \bigcup_{\beta \in \Lambda_2(b)} \mathcal{A}_\beta.$$

Hence,

$$g^{-1}(C(b) \cap \mathcal{H}) \subseteq \bigcup_{\beta \in \Lambda_2(b)} \mathcal{A}_\beta \cup (\mathcal{A}(b)).$$

Since $\{C(b) : b \in \mathcal{H}\}$ is an infra-open cover $\mathcal{H}$ and $\mathcal{H}$ is an infra-Lindelöf subspace of $\mathbb{G}$, then

$$\mathcal{H} \subseteq \bigcup_{i \in \mathbb{N}} C(b_i).$$

Therefore, we get

$$\begin{aligned} g^{-1}(\mathcal{H}) &\subseteq \bigcup_{i \in \mathbb{N}} g^{-1}(C(b_i) \cap \mathcal{H}) \\ &\subseteq \bigcup_{i \in \mathbb{N}} \left[\left(\bigcup_{\beta \in \Lambda_2(b_i)} \mathcal{A}_\beta \right) \bigcup \left(\bigcup_{\beta \in \Lambda_1(b_i)} \mathcal{A}_\beta \right) \right] \\ &= \bigcup_{i \in \mathbb{N}} \left(\bigcup_{\beta \in \Lambda_1(b_i) \cup \Lambda_2(b_i)} \mathcal{A}_\beta \right). \end{aligned}$$

Then, $g^{-1}(\mathcal{H})$ is an infra-Lindelöf subspace of $\mathbb{F}$. □

Corollary 45. Let g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ be a co-infra-closed function and let $g^{-1}(b)$ be an infra-Lindelöf subspace of $\mathbb{F}$ for every $b \in \mathbb{Y}$. Suppose that $\mathbb{F}$ satisfies the property **B**. If $(\mathbb{Y}, \mathbb{G})$ is an infra-Lindelöf space, then $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space .

Corollary 46. Let g: $(\mathbb{X}, \mathbb{F}) \to (\mathbb{Y}, \mathbb{G})$ be a surjective co-infra-continuous and co-infra-closed function such that $g^{-1}(b)$ is an infra-Lindelöf subspace of $\mathbb{F}$ for every $b \in \mathbb{Y}$. Suppose that $\mathbb{F}$ satisfies the property **B**. If $(\mathbb{X}, \mathbb{F})$ is an infra-Lindelöf space, then $(\mathbb{Y}, \mathbb{G})$ is an infra-Lindelöf space.

5. Conclusions

In this work, we have explored the concept of co-infra topological spaces, providing a systematic treatment of co-infra open and co-infra-closed sets and showing that their collection forms a topological space. Through this approach, we established a bridge between infra-topological and co-infra-topological spaces and demonstrated several classical properties. We proved that an $\mathbb{ITS}$ is an infra-Lindelöf if and only if its corresponding co-infra-topology is infra-Lindelöf. Furthermore, we examined the behavior of co-infra-continuous and co-infra-closed functions and showed that they maintain the infra-Lindelöf property under suitable assumptions. The study thus provides a deeper insight into the structural harmony between infra and co-infra systems, enriching the theory of generalized topologies and paving the way for potential applications in mathematical modeling and abstract analysis. In forthcoming papers, we intend to employ infra-open sets to formally define the notion of infra-paracompact spaces and to investigate the properties of functional separation axioms, including infra-second countability.

Author contributions

Ohud F. Alghamdi: conceptualization, formal analysis, investigation, visualization, writing–original draft, writing–review and editing; Ahmad Al-Omari: conceptualization, formal analysis, methodology, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors wish to thank the referees for useful comments and suggestions.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. G. Choquet, Convergences, *Ann. Univ. Grenoble. Sect. Sci. Math. Phys.*, **23** (1948), 57–112.
2. Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hung.*, **96** (2002), 351–357. https://doi.org/10.1023/A:1019713018007
3. A. S. Mashhour, A. A. Allam, F. S. Mahmoud, F. H. Khedr, Supratopological spaces, *Indian J. Pure Appl. Math.*, **14** (1990), 502–510.
4. J. Avila, F. Molina, Generalized weak structures, *Int. Math. Forum*, **7** (2012), 2589–2595.
5. A. M. Al-Odhari, On infra-topological spaces, *Int. J. Math. Arch.*, **6** (2015), 179–184.
6. T. M. Al-shami, Z. A. Ameen, R. Abu-Gdairid, A. Mhemdi, Continuity and separation axioms via infra topological spaces, *J. Math. Comput. Sci.*, **30** (2023), 213–225. https://doi.org/10.22436/jmcs.030.03.03
7. T. M. Al-shami, A. Rawshdeh, H. H. Al-jarrah, A. Mhemdi, Connectedness and covering properties via infra topologies with application to fixed point theorem, *AIMS Math.*, **8** (2023), 8928–8948. https://doi.org/10.3934/math.2023447
8. T. M. Al-shami, Infra soft compact spaces and application to fixed point theorem, *J. Funct. Spaces*, **2021** (2021), 3417096. https://doi.org/10.1155/2021/3417096
9. T. M. Al-shami, A. Mhemdi, Approximation operators and accuracy measures of rough sets from an infra-topology view, *Soft Comput.*, **27** (2023), 1317–1330. https://doi.org/10.1007/s00500-022-07627-2
10. S. Dhanalakshmi, R. M. Devi, On generalized regular infra-closed sets, *Int. J. Math. Arch.*, **7** (2016), 33–36.
11. K. Vaiyomathi, F. N. Irudayam, Infra generalized *b*-closed sets in infratopological space, *Int. J. Math. Trends Technol.*, **47** (2017), 56–65.
12. I. Alshammari, A. Al-Omari, Infra cluster topology via infra soft topology and ideal, *J. Math.*, **2025** (2025), 8904772. https://doi.org/10.1155/jom/8904772
13. A. Al-Omari, O. Alghamdi, Regularity and normality on primal spaces, *AIMS Math.*, **9** (2024), 7662–7672. https://doi.org/10.3934/math.2024372

14. S. Chakrabarti, H. Dasgupta, Infra-topological space and its applications, *Rev. Bull. Calcutta Math. Soc.*, **17** (2009), 13–22.

15. T. Husain, *Topology and maps*, Academic Press, 1977.

16. A. A. El-Atik, A. A. Azzam, Topological representations of simplicial complexes and their applications, *Afr. Mat.*, **34** (2023), 22. https://doi.org/10.1007/s13370-023-01060-w

17. A. A. El-Atik, Approximation of simplicial complexes using matroids and rough sets, *Soft Comput.*, **27** (2023), 2217–2229. https://doi.org/10.1007/s00500-022-07774-6

18. A. A. El-Atik, A. Wahba, Some betweenness relation topologies induced by simplicial complexes, *Hacettepe J. Math. Stat.*, **51** (2022), 981–994. https://doi.org/10.15672/hujms.787479

19. A. A. El-Atik, A topological approach of a human heart via Nano pre-ideality, *Thai J. Math.*, **20** (2022), 1273–1285.

20. A. A. Nasef, A. A. El-Atik, G. K. Revathi, A model of hepatitis C and blood pressure via nano topological spaces, *Thai J. Math.*, **20** (2022), 669–677.

21. O. Alghamdi, A. Al-Omari, M. H. Alqahtani, Novel operators in the frame of primal topological spaces, *AIMS Math.*, **9** (2024), 25792–25808. https://doi.org/10.3934/math.20241260

22. T. Witczak, Infra-topologies revisited: logic and clarification of basic notions, *Commun. Korean Math. Soc.*, **37** (2022), 279–292. https://doi.org/10.4134/CKMS.c200455