Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 661–683.
DOI: 10.3934/math.2026029
Received: 04 September 2025
Revised: 12 December 2025
Accepted: 19 December 2025
Published: 09 January 2026

Research article

On generalized f-projection and generalized f-prox-regularity on Banach spaces

Ali Al-Tane[1,*], See Keong Lee[1] and Messaoud Bounkhel[2]

[1] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia

[2] Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: alialtane@student.usm.my.

Abstract: This paper investigates the connection between the generalized projection π_S and the generalized f-projection π_S^f. By introducing an f-proximal normal cone, defined via the generalized f-projection, we derive key properties of this set, generalizing established results on the classical V-proximal normal cone to Banach spaces. Additionally, we propose and examine a new regularity condition called uniform f-prox-regularity, which extends the standard notion of prox-regularity by leveraging the adaptability of f-projections.

Keywords: generalized f-projection; V-proximal subdifferential; generalized f-proximal normal cone; uniformly f-prox-regular sets; limiting V-proximal subdifferential; uniformly smooth Banach space; uniformly convex Banach space
Mathematics Subject Classification: 34A60, 49J53

1. Introduction

Projection operators in Banach spaces are of significant interest, owing to their applications in variational inequalities, optimization, and fixed-point theory. For related developments in Hilbert spaces, we refer the reader to [5, 11] and the references provided there.

In the setting of Banach spaces, a key advancement occurred with Alber [1], who introduced the generalized projection operator $\pi_S : X^* \to S$ for uniformly smooth and uniformly convex spaces. This operator became instrumental for addressing variational inequalities and optimization problems in this broader context. Extending Alber's work, Li [14] further refined the generalized projection for nonempty closed convex subsets of reflexive Banach spaces, investigating its implications for Banach space theory.

Further developments were achieved by Bounkhel [3, 6, 8], who generalized the operator π_S to nonconvex closed subsets in smooth Banach spaces. He derived essential properties of this operator and applied them to nonconvex variational problems.

Building on this, Wu and Huang [15] introduced the f-projection operator $\pi_S^f : X^* \to S$, defined by a proper, convex, lower semi-continuous function f. They established its fundamental properties and investigated its use for solving certain variational inequalities, primarily in reflexive Banach spaces.

Based on these advances, Bounkhel [9] introduced a further extension: the generalized (f, λ)-projection operator $\pi_S^{f,\lambda}$ and its corresponding V-Moreau envelope. He then studied the differentiability and regularity of this envelope, as well as the Hölder continuity of the projection itself.

The current paper extends this line of research on generalized projections for nonconvex sets, initiated by Bounkhel in [6, 8, 9]. We focus on the operators π_S and π_S^f. In Section 2, we investigate the relationship between the generalized f-projection and the standard generalized projection, providing illustrative examples and establishing key theoretical links.

In Section 3, we generalize the notion of the V-proximal normal cone, first introduced by Bounkhel and Al-Yusof in [2], employing the generalized f-projection within reflexive Banach spaces. We establish a range of properties for this extended normal cone, covering both convex and nonconvex settings.

Section 4 presents a new notion of prox-regularity for a set S relative to a given function f. This definition is expressed in terms of the generalized f-projection, and we prove several characteristic properties of this class of sets. Furthermore, we demonstrate the Lipschitz continuity of a specific subclass of the π_S^f operator.

2. Preliminaries

Throughout this work, let X be a uniformly smooth and uniformly convex Banach space (unless stated otherwise), and let X^* denote its topological dual. The closed unit balls in X^* and X are represented by $\mathbb{B}_*$ and $\mathbb{B}$, respectively. For definitions and foundational results on uniformly smooth and uniformly convex Banach spaces, we refer to the monographs [12, 13]. We recall that the normalized duality mapping $J : X \to X^*$ is defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|.\|x\| = \|x^*\|^2 = \|x\|^2\}.$$

Now, we recall several definitions and notations essential to our work. Let S be a closed nonempty subset of a Banach space X. The V-proximal normal cone of S at $x \in S$ (see [2]) is defined as

$$N^\pi(S, x) = \{x^* \in X^* \ : \ \exists\, \alpha > 0 \text{ such that } x \in \pi_S(J(x) + \alpha x^*)\},$$

where π_S denotes the generalized projection given by

$$\pi_S(x^*) = \{x \in S : V(x^*; x) = \inf_{s \in S} V(x^*; s)\},$$

and

$$V(x^*; x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2.$$

If S is convex, the V-proximal normal cone $N^{\pi}(S; x)$ coincides with the classical normal cone from convex analysis (see [10]):

$$N^{con}(S, x) = \{x^* \in X^* : \langle x^*, s - x\rangle \leq 0,\ \forall s \in S\}.$$

We also recall the well-known notions of the Fréchet subdifferential $\partial^F f(x)$ (see, for instance, [5]), the convex subdifferential $\partial^{\text{con}} f$ (see [5]), and the V-proximal subdifferential $\partial^{\pi} f$ (see [2]).

- For a lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom} f$, we say that $x^* \in \partial^F f(x)$ if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

  $$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \epsilon\|x' - x\|, \quad \forall x' \in x + \delta\mathbb{B}.$$

  By taking f to be the indicator function of a nonempty closed set, we get the definition of the Fréchet normal cone as follows:

  $$N^F(S, x) = \{x^* \in X^* : \forall\epsilon > 0,\ \exists\delta > 0 \text{ such that } \langle x^*, x' - x\rangle \leq \epsilon\|x' - x\|,\ \forall x' \in [x + \delta\mathbb{B}] \cap S\}.$$

- If we assume that f is a convex lower semi-continuous function, the convex subdifferential of f at the point x is given by the set:

  $$\partial^{\text{con}} f(x) = \{x^* \in X^* : \langle x^*, x' - x\rangle \leq f(x') - f(x),\ \forall x' \in X\}.$$

  The convex normal cone of a nonempty closed convex set S is obtained by taking f to be the indicator function of S as follows:

  $$N^{con}(S, x) = \{x^* \in X^* : \langle x^*, x' - x\rangle \leq 0,\ \forall x' \in S\}.$$

- For a lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom} f$, we say that $x^* \in \partial^{\pi} f(x)$, if and only if, there exist $\delta > 0$ and $\sigma > 0$ such that

  $$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \sigma V(J(x); x'), \quad \forall x' \in x + \delta\mathbb{B}.$$

We present the following definitions from [15, 16].

Definition 2.1. *For a given proper function $f : X \to \mathbb{R} \cup \{+\infty\}$, define the functional $V^f : X^* \times X \to \mathbb{R} \cup \{+\infty\}$ by*

$$V^f(x^*; x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2 + f(x), \qquad x^* \in X^*,\ x \in X.$$

Definition 2.2. *Let S be a nonempty closed subset of X. Denote*

$$d^V_{S,f}(x^*) := \inf_{v\in S} V^f(x^*; v),$$

and define the generalized f-projection operator $\pi^f_S : X^* \rightrightarrows X$ as

$$\pi^f_S(x^*) = \{x \in S : V^f(x^*; x) = d^V_{S,f}(x^*)\}, \qquad \forall x^* \in X^*.$$

It is important to note that when $f(x) = 0$ for every $x \in X$, the generalized f-projection $\pi_S^f(x^*)$ coincides with the generalized projection $\pi_S(x^*)$, which was introduced and analyzed by Alber [1] and Li [14], for closed convex sets and by Bounkhel [6] for nonempty closed (not necessarily convex) sets. Additionally, Bounkhel [6] provided an example involving nonconvex closed sets S in uniformly convex and uniformly smooth Banach spaces, demonstrating that $\pi_S(x^*) = \phi$. By employing a similar approach, we can show that for nonconvex closed sets S in a uniformly smooth and convex Banach space, the projection $\pi_S^f(x^*)$ may indeed be empty.

Example 2.3. *Let $X = l_p$ with $(p \geq 1)$, $0 = (0, 0, 0, \ldots) \in (l_p)^* = l_{p'}$ (with $\frac{1}{p} + \frac{1}{p'} = 1$), and let*

$$S = \{x_1, x_2, \ldots, x_n, \ldots\} \quad ; \quad x_n = (0, 0, \ldots, \frac{n+1}{n}, \ldots).$$

Let $f : X \to \mathbb{R}$ be $f(x) = \|x\|_{l_p}$. Then S is closed, not convex, with $\|x\|_{l_p} = 1 + \frac{1}{n} > 1, \quad \forall x \in S$. Hence,

$$V^f(0; x) = V(0; x) + f(x) = \|x\|_{l_p}(\|x\|_{l_p} + 1) > 2, \quad \forall x \in S. \tag{2.1}$$

Moreover, since $V^f(\cdot;\cdot)$ is continuous with respect to the second variable, we obtain

$$\begin{aligned} 2 \leq \inf_{x \in S} V^f(0; x) &\leq \lim_{n\to\infty} V^f(0; x_n) \\ &= \lim_{n\to\infty}(\|x_n\|_{l_p}^2 + \|x_n\|_{l_p}) \\ &= \lim_{n\to\infty}[(1 + \frac{1}{n})(2 + \frac{1}{n})] = 2. \end{aligned}$$

So,

$$\inf_{x \in S} V^f(0; x) = 2. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\pi_S^f(0) = \{x \in S : V^f(0; x) = \inf_{u \in S} V^f(0; u) = 2\} = \phi.$$

From this example, we see that even when f is a convex continuous function and X is a smooth reflexive Banach space, the generalized f-projection onto a nonconvex set may be empty. It is worth noting that the third author recently proved in [8] that the set of points $x^* \in X^*$ that admit a generalized f-projection is dense in X^* (see Theorem 2.1 in [8]), provided X is a reflexive Banach space whose dual norm is smooth.

Now, we move on to study the relationship between π_S^f and π_S. We start with the following result in which we need the definition of level sets $[f = \alpha] := \{x \in X : f(x) = \alpha\}$.

Proposition 2.4. *Let $\alpha \in \mathbb{R}$ and $x^* \in X^*$.*

(1) If f is bounded below by α on S, then $[f = \alpha] \cap \pi_S(x^*) \subseteq \pi_S^f(x^*)$.
(2) If f is bounded above by α on S, then $[f = \alpha] \cap \pi_S^f(x^*) \subseteq \pi_S(x^*)$.

Proof. (1) For any $x^* \in X^*$, we have

$$\begin{aligned} V^f(x^*; v) &= V(x^*; v) + f(v) \\ &\geq V(x^*; v) + \alpha, \quad \forall v \in S, \end{aligned}$$

and so $\inf_{v\in S} V^f(x^*; v) \geq \inf_{v\in S} V(x^*; v) + \alpha$. Let $x \in [f = \alpha] \cap \pi_S(x^*)$. Then $f(x) = \alpha$ and

$$V(x^*; x) = \inf_{v\in S} V(x^*; v).$$

Thus,

$$V^f(x^*; x) = V(x^*; x) + f(x) = \inf_{v\in S} V(x^*; v) + \alpha \leq \inf_{v\in S} V^f(x^*; v).$$

This ensures that $x \in \pi_S^f(x^*)$, and so we get the inclusion

$$[f = \alpha] \cap \pi_S(x^*) \subseteq \pi_S^f(x^*), \quad \forall x^* \in X^*.$$

(2) For any $x^* \in X^*$, we have

$$\begin{aligned} V^f(x^*; v) &= V(x^*; v) + f(v) \\ &\leq V(x^*; v) + \alpha; \quad \forall v \in S, \end{aligned}$$

and so $\inf_{v\in S} V^f(x^*; v) \leq \inf_{v\in S} V(x^*; v) + \alpha$. Let $x \in [f = \alpha] \cap \pi_S^f(x^*)$. Then, $f(x) = \alpha$ and

$$V^f(x^*; x) = \inf_{v\in S} V^f(x^*; v).$$

Thus,

$$V(x^*; x) = V^f(x^*; x) - f(x) = V^f(x^*; x) - \alpha = \inf_{v\in S} V^f(x^*; v) - \alpha \leq \inf_{v\in S} V(x^*; v).$$

Hence, $V(x^*; x) = \inf_{v\in S} V(x^*; v)$, and so $x \in \pi_S(x^*)$. This proves the inclusion

$$[f = \alpha] \cap \pi_S^f(x^*) \subseteq \pi_S(x^*),$$

and the proof of the proposition is complete. □

The first natural question that arises is whether the conclusions of the proposition remain valid if we remove the intersection with level sets. To address this question, either affirmatively or negatively, we present the following examples.

Example 2.5. *Let $X = l_1$, and let $x_1 = (1, 1, 0, 0, \cdots)$, $x_2 = (1, 0, 1, 0, \cdots)$, and $x_3 = (2, 0, 0, 1, 0, \cdots)$ be elements in l_1. Consider $S = co\{x_1, x_2, x_3\}$ and $f(x) = \|x\|_{l_1} - 2$. Then for $x^* = (1, 1, 1, \dots) \in X^* = l_\infty$, we have $\pi_S^f(x^*) = \pi_S(x^*)$.*

Proof. It has been proved in [14] that $\pi_S(x^*) = co\{x_1, x_2\}$. So, we have to show that $\pi_S^f(x^*) = co\{x_1, x_2\}$. For any $\lambda \in [0, 1]$, we set $v_\lambda := \lambda x_1 + (1 - \lambda)x_2$. First, we note that

$$v_\lambda = (1, \lambda, 1 - \lambda, 0, 0, 0, \cdots), \|v_\lambda\|_{l_1} = \|x_1\|_{l_1} = \|x_2\|_{l_1} = \|x_3\|_{l_1} = 2 \text{ and } \|x^*\|_{l_\infty} = 1.$$

Also, we have

$$\langle x^*; v_\lambda\rangle_{l_\infty,l_1} = 2.$$

Then we have, for any $\lambda \in [0, 1]$,

$$V^f(x^*, v_\lambda) = V(x^*, v_\lambda) + f(v_\lambda) = \|x^*\|^2_{l_\infty} - 2\langle x^*; v_\lambda\rangle + \|v_\lambda\|^2_{l_1} + \|v_\lambda\|_{l_1} - 2 = 1.$$

This ensures that $V^f(x^*; v) = 1, \forall v \in co\{x_1, x_2\}$. Now fix any $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and set $z := \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$. First, we notice that

$$z = (1 + \lambda_3, \lambda_1, \lambda_2, \lambda_3, 0, 0, 0, \cdots), \|z\|_{l_1} = 2 + \lambda_3, \text{ and } \langle x^*; z\rangle_{l_\infty,l_1} = 2 + \lambda_3.$$

Then we have

$$\begin{aligned} V^f(x^*, z) &= V(x^*, z) + f(z) = \|x^*\|^2_{l_\infty} - 2\langle x^*; z\rangle + \|z\|^2_{l_1} + \|z\|_{l_1} - 2 \\ &= 1 - 2(2 + \lambda_3) + (2 + \lambda_3)^2 + (2 + \lambda_3) - 2 \\ &= 1 + 3\lambda_3 + \lambda_3^2 \geq 1, \quad \forall z \in S. \end{aligned}$$

Thus, we obtain

$$1 \leq \inf_{z\in S} V^f(x^*, z) \leq \inf_{z\in co\{x_1,x_2\}} V^f(x^*, z) = 1,$$

that is,

$$\inf_{z\in S} V^f(x^*, z) = V^f(x^*; v) = 1, \quad \forall v \in co\{x_1, x_2\}.$$

This ensures that $co\{x_1, x_2\} \subset \pi^f_S(x^*)$. Now fix any element $w \in S \setminus co\{x_1, x_2\}$. Then there exist $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_3 \neq 0$ such that $w = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$. Finally $V^f(x^*, w) = 1 + 3\lambda_3 + \lambda_3^2 > 1 = \inf_{z\in S} V^f(x^*, z)$. This guarantees that $w \notin \pi^f_S(x^*)$, and so the unique elements of S belonging to $\pi^f_S(x^*)$ are $co\{x_1, x_2\}$, and so $\pi^f_S(x^*) = co\{x_1, x_2\}$. Therefore, $\pi_S(x^*) = \pi^f_S(x^*) = co\{x_1, x_2\}$. The proof of the example is finished. □

In this example, we have $f(x) \geq 0$ on S and $[f = 0] = \text{co}\{x_1, x_2\} = \pi_S(x^*) = \pi^f_S(x^*)$. Therefore, the equality case of the inclusion in Part (1) of Proposition 2.4 holds. The intersection with the level set $[f = 0]$ is implicitly satisfied, since $[f = 0] = \text{co}\{x_1, x_2\}$. This example also shows that the generalized projection $\pi_S(x^*)$ and the generalized f-projection $\pi^f_S(x^*)$ can coincide. To illustrate the importance of the intersection with the level set, we consider a different function f defined by $f(x) = \|x - x_1\|_{\ell_1}$, for all $x \in X$. Clearly, $f(x) \geq 0$ on S and $[f = 0] = \{x_1\}$. Moreover, we have $\pi^f_S(x^*) = \{x_1\}$. Indeed, for any element $z \in S = \text{co}\{x_1, x_2, x_3\}$, there exist $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ such that $z = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$. Then

$$z - x_1 = (\lambda_3, \lambda_1 - 1, \lambda_2, \lambda_3, 0, 0, 0, \cdots) \text{ and } \|z - x_1\|_{l_1} = 3\lambda_3 + 2\lambda_2.$$

Hence,

$$V^f(x^*, x_1) = V(x^*, x_1) + f(x_1) = \|x^*\|^2_{l_\infty} - 2\langle x^*; x_1\rangle + \|x_1\|^2_{l_1} = 1,$$

and

$$V^f(x^*, z) = V(x^*, z) + f(z) = \|x^*\|^2_{l_\infty} - 2\langle x^*; z\rangle + \|z\|^2_{l_1} + \|z - x_1\|_{l_1}$$

$$= 1 + 5\lambda_3 + \lambda_3^2 + 2\lambda_2 > 1 = V^f(x^*, x_1).$$

This ensures that z satisfies

$$V^f(x^*, z) = \inf_{s \in S} V^f(x^*, s) = 1$$

if and only if $\lambda_2 = \lambda_3 = 0$, which implies $\lambda_1 = 1$ and hence $z = x_1$. Consequently, $\pi_S^f(x^*) = \{x_1\}$.

Thus, in general, the inclusion $\pi_S(x^*) \subset \pi_S^f(x^*)$ does not hold. However, intersecting with the level set $[f = 0]$ yields

$$[f = 0] \cap \pi_S(x^*) \subset \pi_S^f(x^*).$$

This second example underscores the critical role of intersecting with the level set and demonstrates that attempting to prove the inclusion $\pi_S(x^*) \subset \pi_S^f(x^*)$ in general is not viable. A similar line of reasoning can be used to construct analogous examples for the second inclusion in Proposition 2.4.

Example 2.6. *For this example, we take the same space $X = \ell_1$ and the same set $S = \mathrm{co}\{x_1, x_2, x_3\}$ with x_1, x_2, x_3 as defined earlier. However, we now take $x^* = (0, 0, 0, \dots) \in \ell_\infty$ and consider the function $f(x) = \|x\|_{\ell_1}^2$. In this case, we obtain $\pi_S(x^*) = \pi_S^f(x^*)$.*

Proof. Since we have already established in the previous examples that $\pi_S(x^*) = \mathrm{co}\{x_1, x_2\}$, it remains to show that $\pi_S^f(x^*) = \mathrm{co}\{x_1, x_2\}$. First, observe that $V^f(0, x_1) = V^f(0, x_2) = 8$, $V^f(0, x_3) = 18$. Fix any $\mu \in [0, 1]$ and set $v_\mu := \mu x_1 + (1 - \mu)x_2$. Then we have $v_\lambda = (1, \mu, 1 - \mu, 0, 0, 0, \cdots)$, and so

$$V^f(0, v_\lambda) = V(0, v_\lambda) + f(v_\lambda) = 2\|v_\lambda\|_{l_1}^2 = 2[1 + \mu + (1 - \mu)]^2 = 8.$$

Now fix any $\mu_1, \mu_2, \mu_3 \in [0, 1]$ with $\mu_1 + \mu_2 + \mu_3 = 1$ and set $v := \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$. Hence, $v = (1 + \mu_3, \mu_1, \mu_2, \mu_3, 0, 0, 0, \cdots)$ and $\|v\|_{l_1} = 2 + \mu_3$. Then, we have

$$V^f(0; v) = \|v\|_{l_1}^2 + f(v) = 2\|v\|_{l_1}^2 = 2(2 + \mu_3)^2 \geq 8.$$

The equality in the last inequality holds if and only if $\mu_3 = 0$, which is equivalent to $v \in \mathrm{co}\{x_1, x_2\}$. Consequently, we obtain the desired equality

$$\pi_S^f(x^*) = \mathrm{co}\{x_1, x_2\} = \pi_S(x^*),$$

which completes the proof for this example. □

From this example, we conclude that while the inclusion $\pi_S(x^*) \subset \pi_S^f(x^*)$ generally cannot be guaranteed without intersecting with level sets, it remains possible to construct specific cases where the generalized projection $\pi_S(x^*)$ and the generalized f-projection $\pi_S^f(x^*)$ coincide, even without intersecting with any level set.

3. Generalized f-proximal normal cone

In this section, we extend the definition of the V-proximal cone introduced and studied in [2]. We define a new object, called the generalized f-proximal normal cone and denoted by $N_f^\pi(S,\bar{x})$, in terms of the generalized f-projection π_S^f as follows

Definition 3.1. *Given a nonempty closed subset S of a reflexive smooth Banach space X and a point $\bar{x} \in S$, the generalized f-proximal normal cone to S at $\bar{x}$ is defined as*

$$N_f^\pi(S,\bar{x}) = \{x^* \in X^* \ : \ \exists\, \alpha > 0 \ \text{ such that } \ \bar{x} \in \pi_S^f(J(\bar{x}) + \alpha x^*)\}.$$

Clearly, when $f \equiv 0$ on S, the set $N_f^\pi(S,\bar{x})$ reduces to the well-known V-proximal normal cone $N^\pi(S,\bar{x})$ introduced and studied in [2]. In the following proposition, we establish several important properties of this new set. For example, we will use it to:

- analyze special cases depending on the location of the point $\bar{x}$, as shown in Propositions 3.6 and 3.7.
- derive a relationship between the Fréchet normal cone and the generalized f-proximal normal cone, as demonstrated in Theorem 3.9.

Proposition 3.2. *Let S be a closed nonempty subset of a reflexive Banach space X, and let $\bar{x} \in S$. Then the following statements hold:*

(1) An element $x^* \in X^*$ belongs to $N_f^\pi(S,\bar{x})$ if and only if there exists a constant $\sigma > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \sigma\left[V^f(J(\bar{x}), x) - f(\bar{x})\right], \quad \forall x \in S.$$

(2) The set $N_f^\pi(S,\bar{x})$ is a convex cone in X^*, though not necessarily closed.

Proof. (1) Let $x^* \in N_f^\pi(S,\bar{x})$. Then there exists $\alpha > 0$ such that $\bar{x} \in \pi_S^f(J(\bar{x}) + \alpha x^*)$. By definition of the generalized f-projection, we have, for any $x \in S$,

$$V^f(J(\bar{x}) + \alpha x^*, \bar{x}) \le V^f(J(\bar{x}) + \alpha x^*, x).$$

We expand V^f using its definition

$$\|J(\bar{x}) + \alpha x^*\|^2 - 2\langle J(\bar{x}) + \alpha x^*, \bar{x}\rangle + \|\bar{x}\|^2 + f(\bar{x}) \le \|J(\bar{x}) + \alpha x^*\|^2 - 2\langle J(\bar{x}) + \alpha x^*, x\rangle + \|x\|^2 + f(x).$$

We cancel the common term $\|J(\bar{x}) + \alpha x^*\|^2$

$$-2\langle J(\bar{x}) + \alpha x^*, \bar{x}\rangle + \|\bar{x}\|^2 + f(\bar{x}) \le -2\langle J(\bar{x}) + \alpha x^*, x\rangle + \|x\|^2 + f(x).$$

We rearrange to bring terms involving $\bar{x}$ and x together, and grouping the pairing product terms gives

$$2\langle J(\bar{x}) + \alpha x^*, x - \bar{x}\rangle \le \|x\|^2 - \|\bar{x}\|^2 + f(x) - f(\bar{x}).$$

Equivalently, we obtain

$$2\alpha\langle x^*, x - \bar{x}\rangle \le \|x\|^2 - \|\bar{x}\|^2 + f(x) - f(\bar{x}) - 2\langle J(\bar{x}), x - \bar{x}\rangle.$$

We recognize and regroup terms to form $V(J(\bar{x}), x)$

$$2\alpha\langle x^*, x - \bar{x}\rangle \le V(J(\bar{x}), x) + f(x) - f(\bar{x}).$$

But $V(J(\bar{x}), x) + f(x) = V^f(J(\bar{x}), x)$. Hence

$$2\alpha\langle x^*, x - \bar{x}\rangle \le V^f(J(\bar{x}), x) - f(\bar{x}).$$

Since $\alpha > 0$, dividing this inequality by 2α yields

$$\langle x^*, x - \bar{x}\rangle \le \frac{1}{2\alpha}[V^f(J(\bar{x}), x) - f(\bar{x})], \quad \forall x \in S. \tag{8}$$

Thus, we obtain for $\sigma = (2\alpha)^{-1}$,

$$\langle x^*; x - \bar{x}\rangle \le \sigma[V^f(J(\bar{x}), x) - f(\bar{x})], \quad \forall x \in S.$$

Conversely, let $\sigma > 0$ so that

$$\langle x^*; x - \bar{x}\rangle \le \sigma[V^f(J(\bar{x}), x) - f(\bar{x})], \quad \forall x \in S.$$

Using the same reasoning as above, we can rewrite this inequality as

$$V^f(J(\bar{x}) + \frac{1}{2\sigma}x^*, \bar{x}) \le V^f(J(\bar{x}) + \frac{1}{2\sigma}x^*, x), \quad \forall x \in S,$$

which means that $\bar{x} \in \pi_S^f(J(\bar{x}) + \frac{1}{2\sigma}x^*)$, and so by Definition 3.1, we obtain $x^* \in N_f^\pi(S, \bar{x})$.

(2) First, observe that when $f \equiv 0$, the set $N_f^\pi(S, \bar{x})$ reduces to the V-proximal normal cone $N^\pi(S, \bar{x})$. It is known (see, e.g., [2]) that this cone is not necessarily closed, even in finite-dimensional spaces. Consequently, the newly defined set $N_f^\pi(S, \bar{x})$ is also not guaranteed to be closed in general.

Now, we prove that $N_f^\pi(S, \bar{x})$ is a cone. Let $\beta > 0$ and $x^* \in N_f^\pi(S, \bar{x})$. By Definition 3.1, there exists $\alpha > 0$ such that

$$\bar{x} \in \pi_S^f(J(\bar{x}) + \alpha x^*) = \pi_S^f\Big(J(\bar{x}) + \frac{\alpha}{\beta}(\beta x^*)\Big).$$

Hence, $\beta x^* \in N_f^\pi(S, \bar{x})$, establishing the positive homogeneity. To prove convexity, it suffices to show that $N_f^\pi(S, \bar{x})$ is closed under addition. Let $x_1^*, x_2^* \in N_f^\pi(S, \bar{x})$. By Part (1), there exist constants $\sigma_1, \sigma_2 > 0$ such that for all $x \in S$,

$$\langle x_1^*, x - \bar{x}\rangle \le \sigma_1[V^f(J(\bar{x}), x) - f(\bar{x})],$$

and

$$\langle x_2^*, x - \bar{x}\rangle \le \sigma_2[V^f(J(\bar{x}), x) - f(\bar{x})].$$

Adding these inequalities, we obtain for $x^* := x_1^* + x_2^*$,

$$\begin{aligned}\langle x^*, x - \bar{x}\rangle &= \langle x_1^*, x - \bar{x}\rangle + \langle x_2^*, x - \bar{x}\rangle \\ &\le (\sigma_1 + \sigma_2)[V^f(J(\bar{x}), x) - f(\bar{x})].\end{aligned}$$

Taking $\sigma := \sigma_1 + \sigma_2 > 0$, this inequality can be written as

$$\langle x^*, x - \bar{x}\rangle \le \sigma[V^f(J(\bar{x}), x) - f(\bar{x})].$$

Therefore, by Part (1), $x^* = x_1^* + x_2^* \in N_f^\pi(S, \bar{x})$. This completes the proof. □

The lemma below characterizes the generalized f-projection in terms of an associated inequality.

Lemma 3.3. *Let S be a nonempty closed subset of X, and let $x^* \in X^*$. Then $\bar{x} \in \pi_S^f(x^*)$ if and only if*

$$\langle x^* - J(\bar{x}); x - \bar{x}\rangle \leq \frac{1}{2}\left[f(x) - f(\bar{x}) + V(J(\bar{x}), x)\right], \quad \forall x \in S. \tag{3.1}$$

Proof. Let $x \in S$ be arbitrary. We develop the difference $V^f(x^*; \bar{x}) - V^f(x^*; x)$ by expanding each term using the definition of V^f. First, substituting the expression $V^f(x^*; y) = \|x^*\|^2 - 2\langle x^*, y\rangle + \|y\|^2 + f(y)$ yields

$$[\|x^*\|^2 - 2\langle x^*, \bar{x}\rangle + \|\bar{x}\|^2 + f(\bar{x})] - [\|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2 + f(x)].$$

Canceling the common term $\|x^*\|^2$ and regrouping gives

$$\|\bar{x}\|^2 - \|x\|^2 + 2\langle x^*, x - \bar{x}\rangle + f(\bar{x}) - f(x).$$

Inserting $J(\bar{x}) - J(\bar{x})$ into the pairing product leaves the value unchanged, and splitting the resulting sum leads to

$$\|\bar{x}\|^2 - \|x\|^2 + 2\langle J(\bar{x}), x - \bar{x}\rangle - 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(\bar{x}) - f(x).$$

Expanding $2\langle J(\bar{x}), x - \bar{x}\rangle$ as $2\langle J(\bar{x}), x\rangle - 2\langle J(\bar{x}), \bar{x}\rangle$ and using the property $\langle J(\bar{x}), \bar{x}\rangle = \|\bar{x}\|^2$ produces

$$\|\bar{x}\|^2 - \|x\|^2 + 2\langle J(\bar{x}), x\rangle - 2\|\bar{x}\|^2 - 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(\bar{x}) - f(x).$$

Simplifying the $\|\bar{x}\|^2$ terms yields

$$-\|\bar{x}\|^2 - \|x\|^2 + 2\langle J(\bar{x}), x\rangle - 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(\bar{x}) - f(x).$$

Recognizing the combination $-\|\bar{x}\|^2 + 2\langle J(\bar{x}), x\rangle - \|x\|^2$ as the negative of $V(J(\bar{x}), x)$ (since $\|J(\bar{x})\| = \|\bar{x}\|$), we obtain

$$-V(J(\bar{x}), x) - 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(\bar{x}) - f(x).$$

Finally, rewriting the last two terms as a single bracket gives the compact form

$$V^f(x^*; \bar{x}) - V^f(x^*; x) = -V(J(\bar{x}), x) - \left[\, 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(x) - f(\bar{x})\,\right].$$

Thus, if $\bar{x} \in \pi_S^f(x^*)$, we have $V^f(x^*, \bar{x}) = \inf_{s\in S} V^f(x^*, s) \leq V^f(x^*; x)$, and so

$$-V(J(\bar{x}), x) - \left[2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(x) - f(\bar{x})\right] = V^f(x^*, \bar{x}) - V^f(x^*; x) \leq 0,$$

which leads to

$$2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(x) - f(\bar{x}) + V(J(\bar{x}), x) \geq 0,$$

so the direct implication holds.

Conversely, assume that the inequality (3.1) holds. Then, using the identity previously derived, we obtain, for any $x \in S$,

$$V^f(x^*; \bar{x}) - V^f(x^*; x) = -V(J(\bar{x}), x) - \left[\, 2\langle J(\bar{x}) - x^*, x - \bar{x}\rangle + f(x) - f(\bar{x})\,\right] \leq 0,$$

where the inequality follows directly from (3.1). Consequently,

$$V^f(x^*; \bar{x}) \leq V^f(x^*; x) \quad \text{for all } x \in S,$$

which implies that $V^f(x^*; \bar{x}) = \inf_{s\in S} V^f(x^*; s)$. Hence, $\bar{x} \in \pi_S^f(x^*)$. □

Note that in the convex case, that is, when both S and f are convex, the inequality (3.1) is equivalent to the simpler inequality

$$2\langle x^* - J(\bar{x}), x - \bar{x}\rangle \leq f(x) - f(\bar{x}), \quad \forall x \in S. \tag{3.2}$$

Indeed, the direct implication is immediate since $V(J(\bar{x}), x) \geq 0$. To prove the reverse implication, assume (3.1) holds. Fix $x \in S$, and let $t \in (0, 1)$. Define $x_t := \bar{x} + t(x - \bar{x})$, which belongs to S by convexity. Applying (3.1) to x_t gives

$$2\langle x^* - J(\bar{x}), x_t - \bar{x}\rangle \leq f(x_t) - f(\bar{x}) + V(J(\bar{x}), x_t). \tag{3.3}$$

Since f is convex,

$$f(x_t) \leq tf(x) + (1 - t)f(\bar{x}),$$

so that $f(x_t) - f(\bar{x}) \leq t[f(x) - f(\bar{x})]$. Substituting this and the definition $x_t - \bar{x} = t(x - \bar{x})$ into (3.3) yields

$$2t\langle x^* - J(\bar{x}), x - \bar{x}\rangle \leq t[f(x) - f(\bar{x})] + V(J(\bar{x}), \bar{x} + t(x - \bar{x})).$$

Dividing by $t > 0$, we obtain

$$2\langle x^* - J(\bar{x}), x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \frac{V(J(\bar{x}), \bar{x} + t(x - \bar{x}))}{t}. \tag{3.4}$$

Now, we examine the limit of the last term as $t \to 0^+$. Observe that

$$\frac{V(J(\bar{x}), \bar{x} + t(x - \bar{x}))}{t} = \frac{V(J(\bar{x}), \bar{x} + t(x - \bar{x})) - V(J(\bar{x}), \bar{x})}{t},$$

which tends to the directional derivative of $V(J(\bar{x}), \cdot)$ at $\bar{x}$ in the direction $x - \bar{x}$. By direct computation, this derivative equals

$$\nabla_x V(J(\bar{x}), \cdot)(\bar{x}) = 2[J(\bar{x}) - J(\bar{x})] = 0.$$

Hence, taking the limit $t \to 0^+$ in (3.4) gives

$$2\langle x^* - J(\bar{x}), x - \bar{x}\rangle \leq f(x) - f(\bar{x}),$$

which is precisely (3.2). We summarize these observations in the following proposition.

Proposition 3.4. *Let S be a nonempty closed convex subset of a reflexive Banach space X, and let $x^* \in X^*$. Then the following statements are equivalent:*

(1) $\bar{x} \in \pi_S^f(x^*)$;
(2) $\langle x^* - J(\bar{x}); x - \bar{x}\rangle \leq \frac{1}{2}[f(x) - f(\bar{x}) + V(J(\bar{x}), x)], \quad \forall x \in S$;
(3) $\langle x^* - J(\bar{x}); x - \bar{x}\rangle \leq \frac{1}{2}[f(x) - f(\bar{x})], \quad \forall x \in S$.

We continue our study of the f-proximal normal cone by examining its properties in the convex setting in the following theorem.

Theorem 3.5. *Let S be a nonempty closed convex subset of X, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex continuous function. For any $x \in S$, we have*

$$N_f^\pi(S, x) = \mathbb{R}_+\partial^{con} f(x) + N^{con}(S, x).$$

Proof. Let $x^* \in N_f^\pi(S,x)$. Then by Definition 3.1, there exists a positive constant $\alpha > 0$ such that $x \in \pi_S^f(J(x) + \alpha x^*)$. By applying Proposition 3.4, we obtain

$$\langle 2\alpha x^*, x' - x\rangle \leq f(x') - f(x), \quad \forall x' \in S,$$

and so

$$\langle 2\alpha x^*, x' - x\rangle \leq (f + \psi_S)(x') - (f + \psi_S)(x), \quad \forall x' \in X,$$

where ψ_S is the indicator function associated with the set S. This inequality means, by the definition of the subdifferential of convex functions, that $2\alpha x^* \in \partial^{con}(f + \psi_S)(x)$. We use the exact sum rule for the subdifferential of convex continuous functions to write

$$2\alpha x^* \in \partial^{con}(f + \psi_S)(x) \subset \partial^{con} f(x) + \partial^{con}\psi_S(x) = \partial^{con} f(x) + N^{con}(S,x).$$

This implies that

$$x^* \in \frac{1}{2\alpha}\partial^{con} f(x) + N^{con}(S,x) \subset \mathbb{R}_+\partial^{con} f(x) + N^{con}(S,x),$$

and hence the direct inclusion $N_f^\pi(S,x) \subset \mathbb{R}_+\partial^{con} f(x) + N^{con}(S,x)$ holds. Conversely, we will prove that $\mathbb{R}_+\partial^{con} f(x) + N^{con}(S,x) \subset N_f^\pi(S,x)$. Let $v^* \in \mathbb{R}_+\partial^{con} f(x)$, i.e, there is $\alpha > 0$ so that $2\alpha v^* \in \partial^{con} f(x)$, and let $z^* \in N^{con}(S,x)$. We shall show that $v^* + z^* \in N_f^\pi(S,x)$. By the definition of the convex subdifferential and the convex normal cone, we have

$$\langle 2\alpha v^*, v - x\rangle \leq f(v) - f(x) \quad , \forall v \in X,$$

and

$$\langle z^*, v - x\rangle \leq 0, \quad \forall v \in S.$$

Hence,

$$2\langle [J(x) + \alpha(v^* + z^*)] - J(x), v - x\rangle = 2\alpha\langle v^* + z^*, v - x\rangle \leq f(v) - f(x), \quad \forall v \in S.$$

This ensures by Proposition 3.4 that $x \in \pi_S^f(J(x) + \alpha(v^* + z^*))$, and hence by Definition 3.1, we get $v^* + z^* \in N_f^\pi(S;x)$. The demonstration is complete. □

We now consider a specific class of sets defined via a convex function f. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function that is locally Lipschitz on X, and define

$$K := \{x \in X : 0 \notin \partial^{\text{con}} f(x)\},$$

where ∂^{con} denotes the convex subdifferential. Fix a point $\bar{x} \in K$, and define

$$S := \{x \in X : f(x) \leq f(\bar{x})\}.$$

For this set S and the point $\bar{x} \in S$, a consequence of Corollary 2.4.7 in [10] gives

$$\mathbb{R}_+ \partial^{\text{con}} f(\bar{x}) = N^{\text{con}}(S, \bar{x}),$$

where $N^{\text{con}}(S,\bar{x})$ is the normal cone in the sense of convex analysis. Combining this equality with our Theorem 3.5 yields

$$N^{\pi}_{f}(S,\bar{x}) = N^{\text{con}}(S,\bar{x}). \tag{3.5}$$

In the next two propositions, we examine special cases depending on the location of $\bar{x}$. The first deals with the situation in which $\bar{x}$ lies in the topological interior of S. It is well known that in this case, all classical normal cones (e.g., $N^{\text{con}}(S;\bar{x})$) reduce to the singleton $\{0\}$. It is therefore natural to investigate the behavior of our new function-dependent normal cone in this interior-point setting.

Proposition 3.6. *Let S be a nonempty closed subset of a reflexive Banach space X, and assume $\bar{x} \in \text{int}(S)$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any lower semicontinuous function. Then we have*

$$N^{\pi}_{f}(S;\bar{x}) \subset \mathbb{R}_{+}\partial^{\pi} f(\bar{x}). \tag{3.6}$$

Proof. Assume $\bar{x} \in \text{int}(S)$, and let $x^* \in N^{\pi}_{f}(S;\bar{x})$. By Proposition 3.2, there exists $\alpha > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \alpha\big[V^{f}(J(\bar{x}), x) - f(\bar{x})\big] = \alpha\big[V(J(\bar{x}), x) + f(x) - f(\bar{x})\big], \quad \forall x \in S.$$

Since $\bar{x} \in \text{int}(S)$, there exists $\delta > 0$ with $\bar{x} + \delta\mathbb{B} \subset S$. Restricting the inequality above to points $x \in \bar{x} + \delta\mathbb{B}$ yields

$$\langle \alpha^{-1}x^*, x - \bar{x}\rangle \le V(J(\bar{x}), x) + f(x) - f(\bar{x}), \quad \forall x \in \bar{x} + \delta\mathbb{B}.$$

By definition of the V-proximal subdifferential $\partial^{\pi} f$, this implies $\alpha^{-1}x^* \in \partial^{\pi} f(\bar{x})$. Hence, $x^* \in \mathbb{R}_{+}\,\partial^{\pi} f(\bar{x})$, completing the proof. □

From this proposition, we can deduce several important special cases depending on the choice of f.

(1) Case $f \equiv 0$: Here, $N^{\pi}_{f}(S,\bar{x}) = N^{\pi}(S,\bar{x}) = \{0\}$, which recovers the classical result for the V-proximal normal cone at an interior point.
(2) Constant case: If f is constant on a neighborhood of $\bar{x}$, then $\partial^{\pi} f(\bar{x}) = \{0\}$, and hence $N^{\pi}_{f}(S,\bar{x}) = \{0\}$.
(3) Smooth local minimum case: If f is of class C^2 around $\bar{x}$ and $\bar{x}$ is a local minimum of f, then again $\partial^{\pi} f(\bar{x}) = \{0\}$ (see [2]), and so $N^{\pi}_{f}(S,\bar{x}) = \{0\}$.

Thus, in each of these situations, the generalized f-proximal normal cone reduces to the trivial cone $\{0\}$ at interior points of S.

The second important case of the point $\bar{x}$ is:

$$\bar{x} \in \text{argmax}_S(f) := \{x \in S : f(x) = \sup_{s\in S} f(s)\}.$$

Proposition 3.7. *Let $\bar{x} \in \text{argmax}_S(f)$. Then we have*

$$N^{\pi}_{f}(S;\bar{x}) \subset N^{\pi}(S;\bar{x}). \tag{3.7}$$

Proof. Let $\bar{x} \in \text{argmax}_S(f)$, and let $x^* \in N^{\pi}_{f}(S;\bar{x})$. Then by Proposition 3.2, there exists $\alpha > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \alpha\left[V^{f}(J\bar{x}), x) - f(\bar{x})\right] = \alpha\left[V(J\bar{x}), x) + f(x) - f(\bar{x})\right], \quad \forall x \in S. \tag{3.8}$$

Since $\bar{x} \in \operatorname{argmax}_S(f)$, we have

$$f(x) \leq f(\bar{x}), \quad \forall x \in S.$$

Hence, the inequality (3.8) becomes

$$\langle x^*, x - \bar{x} \rangle \leq \alpha \left[V(J\bar{x}), x) + f(x) - f(\bar{x}) \right] \leq \alpha V(J\bar{x}), x), \quad \forall x \in S.$$

This implies by the definition of $N^\pi(S\,;\bar{x})$ that $x^* \in N^\pi(S\,;\bar{x})$, and so we finish the proof. □

Before concluding this section, we establish a connection between the Fréchet normal cone and the generalized f-proximal normal cone, as stated in the theorem below. The proof relies on the following auxiliary lemma.

Lemma 3.8. *If the space X is a q-uniformly smooth Banach space, then for any $M > 0$, there exists a constant $\beta_M > 0$ such that*

$$V(J(x), y) \leq \beta_M \|x - y\|^q, \quad \forall x, y \in M\mathbb{B}.$$

Theorem 3.9. *Let X be q-uniformly smooth, and let $\bar{x} \in X$. Assume that $\bar{x}$ is a local maximum of f. Then $N^\pi_f(S, \bar{x}) \subset N^F(S, \bar{x})$.*

Proof. Let $x^* \in N^\pi_f(S, \bar{x})$. Then by Proposition 3.2, there exists $\sigma > 0$ such that

$$\langle x^*; x - \bar{x} \rangle \leq \sigma[V^f(J(\bar{x}), x) - f(\bar{x})], \quad \forall x \in S. \tag{3.9}$$

First, we assume that $\bar{x}$ is a local maximum of f, and so there exists $\delta > 0$ such that

$$f(x) \leq f(\bar{x}), \quad \forall x \in \bar{x} + \delta\mathbb{B}. \tag{3.10}$$

Also, we use the fact that X is q-uniformly smooth to apply Lemma 3.8 with $M := \|\bar{x}\| + \delta > 0$ to find some constant $\beta_M > 0$ such that

$$V(J(\bar{x}), x) \leq \beta_M \|x - \bar{x}\|^q, \quad \forall x \in \bar{x} + \delta\mathbb{B}. \tag{3.11}$$

Let any $\epsilon > 0$. Put $\mu := \min\{\delta, \frac{1}{2}(\frac{\epsilon}{\sigma\beta_M})^{\frac{1}{q-1}}\}$. Then for $x \in S \cap (\bar{x} + \mu\mathbb{B})$, we get, by combining the above three inequalities, (3.9), (3.10), and (3.11),

$$\begin{aligned}
\langle x^*; x - \bar{x} \rangle &\leq \sigma[V^f(J(\bar{x}), x) - f(\bar{x})] \\
&\leq \sigma V(J(\bar{x}), x) + \sigma[f(x) - f(\bar{x})] \\
&\leq \sigma\beta_M \|x - \bar{x}\|^q \\
&= \sigma\beta_M \|x - \bar{x}\|^{q-1} \|x - \bar{x}\| \\
&\leq \sigma\beta_M \left[\frac{1}{2} (\frac{\epsilon}{\sigma\beta_M})^{\frac{1}{q-1}} \right]^{q-1} \|x - \bar{x}\| \\
&\leq \epsilon \|x - \bar{x}\|.
\end{aligned}$$

This ensures by the definition of Fréchet normal cone that $x^* \in N^F(S\,;\bar{x})$, and so the proof is complete. □

4. Uniformly f-prox-regular sets

In this section, we introduce and study the concept of uniform f-prox-regularity in reflexive Banach spaces with a smooth dual norm, which involves the use of π_S^f, the V-proximal subdifferential $\partial^\pi f(\cdot)$, and the V-proximal normal cone $N^\pi(S;\cdot)$.

Definition 4.1. *Let X be a reflexive smooth Banach space, and let S be a closed subset of X. The set S will be called uniformly f-prox-regular if and only if there is a couple of positive numbers (r_1, r_2), so that for every $x \in S$ and for all $x_1^* \in N^\pi(S;x)$ with $\|x_1^*\| < 1$ and for all $x_2^* \in \partial^\pi f(x)$, we have $x \in \pi_S^f(J(x) + r_1 x_1^* + r_2 x_2^*))$. We will say that S is (r_1, r_2)-uniformly f-prox-regular.*

Before starting the study of this new class of sets, we start by noting the following:

- We notice that when $f \equiv 0$ on S, the uniform f-prox-regularity coincides with the uniform generalized prox-regularity defined and studied in [3].
- All closed convex sets are uniformly f-prox-regular with respect to any convex continuous function f on S, as we will prove in Proposition 4.2.
- We will also prove later in Proposition 4.5, that the new class of sets contains nonconvex sets with respect to nonconvex functions.
- We will extend Theorem 3.5 to this new class of nonconvex sets.

We start with the following proposition.

Proposition 4.2. *Assume that S is a closed convex set and f is a convex continuous function on S. Then S is an (r_1, r_2)-uniformly f-prox-regular set with $r_2 = \frac{1}{2}$ and for any $r_1 > 0$.*

Proof. Let $x \in S$, $x_1^* \in N^\pi(S;x) = N^{con}(S;x)$ with $\|x_1^*\| < 1$, and let $x_2^* \in \partial^\pi f(x) = \partial^{con} f(x)$. Then,

$$\langle x_1^*, x' - x\rangle \leq 0, \quad \forall x' \in S, \tag{4.1}$$

and

$$\langle x_2^*, x' - x\rangle \leq f(x') - f(x), \quad \forall x' \in X. \tag{4.2}$$

Fix any $r_1 > 0$. By multiplying (4.1) by $2r_1$ and adding with (4.2), we obtain

$$\langle 2r_1 x_1^* + x_2^*, x' - x\rangle \leq f(x') - f(x), \quad \forall x' \in S. \tag{4.3}$$

This inequality can be rewritten as follows:

$$2\left\langle \left[r_1 x_1^* + \frac{1}{2} x_2^* + J(x) \right] - J(x), x' - x \right\rangle \leq f(x') - f(x), \quad \forall x' \in S. \tag{4.4}$$

Using Proposition 3.4, we obtain $x \in \pi_S^f(J(x) + r_1 x_1^* + \frac{1}{2} x_2^*)$, and so by Definition 4.1, the set S is an $(r_1, \frac{1}{2})$-uniformly f-prox-regular set. Hence, the proof is complete. □

Before proving the next result, we recall two definitions of regularity - one for sets and one for functions - which were introduced and studied in [3,4], respectively.

Definition 4.3. *A closed set S in a reflexive smooth Banach space X is said to be uniformly prox-regular if there exists a constant $r > 0$ such that, for every $x \in S$ and every $x^* \in N^\pi(S; x)$ with $\|x^*\| < 1$, one has*

$$x \in \pi_S\big(J(x) + rx^*\big).$$

In this case, we say that S is r-uniformly prox-regular.

Note that every closed convex set is r-uniformly prox-regular, for any $r > 0$. A nonconvex example of an r-uniformly prox-regular set is given in Example 4.10 of [3].

Definition 4.4. *Given a closed set S in a reflexive smooth Banach space X and a continuous function f defined on S, we say that f is uniformly prox-regular over S if there exists a constant $r > 0$ such that, for every $x \in S$ and every $x^* \in \partial^\pi f(x)$, we have*

$$\langle x^*, x' - x\rangle \le f(x') - f(x) + \frac{1}{2r} V(J(x), x'), \quad \forall x' \in S.$$

In this case, we say that f is r-uniformly prox-regular over S.

We observe that every convex continuous function is r-uniformly prox-regular over any closed convex subset S of its domain, for any $r > 0$. Furthermore, it has been established in [4] that the usual distance function d_S associated with an r-uniformly prox-regular set S is itself r-uniformly prox-regular over S with the same constant $r > 0$.

Proposition 4.5. *Assume that S is r_1-uniformly prox-regular and the continuous function f is r_2-uniformly prox-regular over S, with $r_2 > 1$. Then S is $(r_1, \frac{1}{2})$-uniformly f-prox-regular.*

Proof. Let $x \in S$, $x_1^* \in N^\pi(S; x)$ with $\|x_1^*\| < 1$, and $x_2^* \in \partial^\pi f(x)$. Then the following inequalities hold:

$$\langle x_1^*, x' - x\rangle \le \frac{1}{2r_1} V(J(x), x'), \qquad \forall x' \in S, \tag{4.5}$$

and

$$\langle x_2^*, x' - x\rangle \le f(x') - f(x) + \frac{1}{2r_2} V(J(x), x') \le f(x') - f(x) + \frac{1}{2} V(J(x), x'), \qquad \forall x' \in S. \tag{4.6}$$

Combining the inequalities (4.5) and (4.6) and using the assumption $r_2 > 1$ gives

$$2\Big\langle r_1 x_1^* + \frac{1}{2}x_2^*, x' - x\Big\rangle \le f(x') - f(x) + \Big(\frac{1}{2} + \frac{1}{2r_2}\Big) V(J(x), x') \le f(x') - f(x) + V(J(x), x'), \quad \forall x' \in S.$$

By Lemma 3.3, this implies

$$x \in \pi_S^f\Big(J(x) + r_1 x_1^* + \frac{1}{2}x_2^*\Big),$$

which completes the proof. □

We now prove the analogue of Theorem 3.5 for a subclass of uniformly f-prox-regular sets.

Theorem 4.6. *Let S be r_1-uniformly prox-regular, and let f be Lipschitz continuous and r_2-uniformly prox-regular over S. Then, for any $x \in S$,*

$$N_f^\pi(S, x) \subset \mathbb{R}_+ \partial^\pi f(x) + N^\pi(S, x).$$

If, in addition, we assume that both, $r_1, r_2 > 1$, then the inclusion becomes an equality.

Proof. Let $x^* \in N_f^\pi(S,x)$. By Proposition 3.2, there exists $\sigma > 0$ such that

$$\langle x^*, x' - x\rangle \le \sigma[V(J(x), x') + f(x') - f(x)], \qquad \forall x' \in S.$$

Since ψ_S is the indicator function of S, we have $\psi_S(x') = 0$ for $x' \in S$ and $+\infty$ otherwise. Hence, for any $x' \in X$,

$$\langle x^*, x' - x\rangle \le \sigma\Big[V(J(x), x') + (f + \psi_S)(x') - (f + \psi_S)(x)\Big].$$

By the definition of the V-proximal subdifferential ∂^π, this implies

$$\sigma^{-1} x^* \in \partial^\pi (f + \psi_S)(x) \subset \partial^{L\pi}(f + \psi_S)(x),$$

where $\partial^{L\pi} f$ denotes the limiting V-proximal subdifferential introduced in [7] and defined by

$$\partial^{L\pi} f(x) = \limsup_{x' \xrightarrow{f} x} \partial^\pi f(x') := \Big\{w\text{-}\lim x_n^* : x_n^* \in \partial^\pi f(x_n),\ x_n \xrightarrow{f} x\Big\}.$$

Applying the exact sum rule for the limiting V-proximal subdifferential (Theorem 13 in [7]), we obtain

$$\sigma^{-1} x^* \in \partial^{L\pi}(f + \psi_S)(x) \subset \partial^{L\pi} f(x) + \partial^{L\pi}\psi_S(x) = \partial^\pi f(x) + N^\pi(S,x).$$

This implies that

$$x^* \in \frac{1}{\sigma}\partial^\pi f(x) + N^\pi(S,x) \subset \mathbb{R}_+ \partial^\pi f(x) + N^\pi(S,x),$$

and hence, the direct inclusion $N_f^\pi(S,x) \subset \mathbb{R}_+\partial^\pi f(x) + N^\pi(S,x)$ is proved.

Conversely, since $N_f^\pi(S,x)$ is a cone, it suffices to prove the inclusion

$$\partial^\pi f(x) + N^\pi(S,x) \subset N_f^\pi(S,x).$$

Fix $v^* \in \partial^\pi f(x)$ and $z^* \in N^\pi(S,x)$. By the uniform prox-regularity of f over S and of the set S itself, we have

$$\langle v^*, v - x\rangle \le f(v) - f(x) + \frac{1}{2r_1} V(J(x), v), \qquad \forall v \in X,$$

and

$$\langle z^*, v - x\rangle \le \frac{1}{2r_2} V(J(x), v), \qquad \forall v \in S.$$

Adding these two inequalities yields, for all $x' \in S$,

$$\langle v^* + z^*, x' - x\rangle \le f(x') - f(x) + \left[\frac{1}{2r_1} + \frac{1}{2r_2}\right] V(J(x), x').$$

Because $r_1, r_2 > 1$, we have $\frac{1}{2r_1} + \frac{1}{2r_2} < 1$, and therefore

$$\langle v^* + z^*, x' - x\rangle \le f(x') - f(x) + V(J(x), x'), \qquad \forall x' \in S.$$

By Proposition 3.2, this implies $v^* + z^* \in N_f^\pi(S,x)$, which completes the proof. □

Following the same reasoning as in Proposition 4.2 of [3], we now prove a key property of uniformly f-prox-regular sets.

Proposition 4.7. *Let S be a nonempty closed subset of a reflexive Banach space X, and let f be a lower semi-continuous function defined on X. Assume that S is generalized (r_1, r_2)-uniformly f-prox-regular. Then, for any $x \in S$, any nonzero $x_1^* \in N^\pi(S; x)$, and any $x_2^* \in \partial^\pi f(x)$, we have*

$$x \in \pi_S^f\Big(J(x) + r_1 \frac{x_1^*}{\|x_1^*\|} + r_2 x_2^*\Big).$$

Proof. Assume that S is (r_1, r_2)-uniformly f-prox-regular. Let $x \in S$, $x_1^* \in N^\pi(S; x)$ with $x_1^* \neq 0$, and $x_2^* \in \partial^\pi f(x)$. Define

$$x_{1,n}^* := \frac{x_1^*}{\|x_1^*\| + \frac{1}{n}}, \qquad n \geq 1.$$

Then $x_{1,n}^* \in N^\pi(S, x)$ and $\|x_{1,n}^*\| < 1$, for all n. By the definition of generalized (r_1, r_2)-uniform f-prox-regularity, we have

$$x \in \pi_S^f(J(x) + r_1 x_{1,n}^* + r_2 x_2^*), \qquad \forall n \geq 1,$$

which means

$$V^f(J(x) + r_1 x_{1,n}^* + r_2 x_2^*; x) \leq V^f(J(x) + r_1 x_{1,n}^* + r_2 x_2^*; v), \qquad \forall v \in S,\ \forall n \geq 1.$$

Since V^f is continuous in its first argument and $x_{1,n}^* \to \frac{x_1^*}{\|x_1^*\|}$ as $n \to \infty$, passing to the limit yields

$$V^f\Big(J(x) + r_1 \frac{x_1^*}{\|x_1^*\|} + r_2 x_2^*; x\Big) \leq V^f\Big(J(x) + r_1 \frac{x_1^*}{\|x_1^*\|} + r_2 x_2^*; v\Big), \qquad \forall v \in S.$$

Hence, by the definition of π_S^f,

$$x \in \pi_S^f\Big(J(x) + r_1 \frac{x_1^*}{\|x_1^*\|} + r_2 x_2^*\Big),$$

which completes the proof. □

Before concluding this section, we introduce the following operator:

$$P_{f,S}^\pi : X^* \to S, \tag{4.7}$$

$$P_{f,S}^\pi(x^*) := \{x \in S : \quad 2[x^* - J(x)] \in N^\pi(S; x) \cap \mathrm{int}(\mathbb{B}) + \partial^\pi f(x)\}. \tag{4.8}$$

We now establish the Lipschitz continuity of this operator for uniformly prox-regular sets S and for convex differentiable functions f. First, observe that the new operator $P_{f,S}^\pi$ is contained in the generalized f-projection over S whenever the set S is uniformly generalized r_1-prox-regular and the function f is uniformly generalized r_2-prox-regular, provided $r_1, r_2 > 1$. More precisely, we have the following inclusion:

$$P_{f,S}^\pi(x^*) \subset \pi_S^f(x^*), \quad \forall x^* \in X^*. \tag{4.9}$$

Indeed, let $x^* \in X^*$, and let $x \in P^{\pi}_{f,S}(x^*)$. By definition of $P^{\pi}_{f,S}$, we have

$$2[x^* - J(x)] \in N^{\pi}(S; x) \cap \mathrm{int}(\mathbb{B}) + \partial^{\pi} f(x).$$

Choose $v^* \in \partial^{\pi} f(x)$ and $z^* \in N^{\pi}(S, x) \cap \mathrm{int}(\mathbb{B})$ such that

$$2[x^* - J(x)] = v^* + z^*.$$

By the uniform prox-regularity of f over S and of S itself, we obtain

$$\langle v^*, v - x\rangle \leq f(v) - f(x) + \frac{1}{2r_1} V(J(x), v), \qquad \forall v \in X,$$

and

$$\langle z^*, v - x\rangle \leq \frac{1}{2r_2} V(J(x), v), \qquad \forall v \in S.$$

Adding these inequalities yields, for every $x' \in S$,

$$\langle v^* + z^*, x' - x\rangle \leq f(x') - f(x) + \left[\frac{1}{2r_1} + \frac{1}{2r_2}\right] V(J(x), x').$$

Since $r_1, r_2 > 1$, we have $\frac{1}{2r_1} + \frac{1}{2r_2} < 1$, and therefore

$$\langle v^* + z^*, x' - x\rangle \leq f(x') - f(x) + V(J(x), x').$$

Recalling that $v^* + z^* = 2[x^* - J(x)]$, we obtain

$$\langle 2[x^* - J(x)], x' - x\rangle \leq f(x') - f(x) + V(J(x), x'), \qquad \forall x' \in S.$$

By Lemma 3.3, this implies $x \in \pi^f_S(x^*)$, completing the argument.

We recall (see for instance [3]) the following technical lemmas needed in our proof of the next theorem.

Lemma 4.8. *If the space X is q-uniformly convex, then for any $M > 0$, there exists some constant $R_M > 0$ such that*

$$\langle J(x) - J(v); x - v\rangle \geq R_M \|x - v\|^q, \quad \forall x, v \in M\mathbb{B}.$$

Consider now the following subset of X^*:

$$\mathbb{A}^*_{r,\alpha} = \{x^* \in X^*; \|x^*\| \leq \alpha \text{ and } d^V_{S,f}(x^*) < r\}. \tag{4.10}$$

Now, we are ready to prove the Lipschitz continuity of $P^{\pi}_{f,S}$ over $\mathbb{A}^*_{r,\alpha}$.

Theorem 4.9. *Let X be a 2-uniformly convex Banach space with a smooth norm. Assume that S is a generalized uniformly prox-regular set in X with constant $r > 0$, and let $\alpha > 0$. Suppose that f is convex and bounded below by some $\beta \in \mathbb{R}$ on S and f is of class $C^{1,1}$ on S, that is, f is continuously differentiable with an L-Lipschitz continuous gradient ∇f on S. Assume further that the parameters L, r, β, α satisfy*

$$r > 2 \quad \textit{and} \quad L < 2(r-2)R_T, \tag{4.11}$$

where $T = \alpha + \sqrt{r - \beta}$ and R_T is the constant given in Lemma 4.8. Then there exists $\lambda > 0$ such that

$$\|P^{\pi}_{f,S}(x^*_2) - P^{\pi}_{f,S}(x^*_1)\| \leq \lambda \|x^*_1 - x^*_2\|, \quad \forall x^*_1, x^*_2 \in \mathbb{A}^*_{r,\alpha} \cap \mathrm{dom}\, P^{\pi}_{f,S}.$$

Proof. Let $x_1^*, x_2^* \in \mathbb{A}_{r,\alpha}^* \cap \operatorname{dom} P_{f,S}^{\pi}$, and choose $x_i \in P_{f,S}^{\pi}(x_i^*)$ for $i = 1, 2$. By definition,

$$2[x_i^* - J(x_i)] \in [N^{\pi}(S, x_i) \cap \operatorname{int}(\mathbb{B})] + \partial^{\pi} f(x_i).$$

Thus, we can select $v_i^* \in \partial^{\pi} f(x_i)$ and $z_i^* \in N^{\pi}(S, x_i) \cap \operatorname{int}(\mathbb{B})$ such that

$$2[x_i^* - J(x_i)] = v_i^* + z_i^*.$$

Using the uniform prox-regularity of S (with constant $r > 0$) and the convexity of f on S, we obtain

$$x_i \in \pi_S^f(J(x_i) + rv_i^* + rz_i^*) = \pi_S^f(J(x_i) + r[x_i^* - J(x_i)]), \quad \forall i = 1, 2.$$

Then

$$V^f\,(J(x_i) + r[x_i^* - J(x_i)]; x_i) \le V^f(J(x_i) + r[x_i^* - J(x_i)]; s), \quad \forall s \in S.$$

Define

$$u_i^* := J(x_i) + r[x_i^* - J(x_i)], \qquad i = 1, 2.$$

The functional $V^f(u_i^*; \cdot)$ is convex and differentiable on S, with its Fréchet gradient given by

$$\nabla^F V^f(u_i^*; \cdot)(w) = 2[J(w) - u_i^*] + \nabla f(w) \in \partial^{\text{con}} V^f(u_i^*; \cdot)(w), \qquad \forall w \in S.$$

From the definition of the convex subdifferential, we have, for all $z, w \in S$,

$$2\Big\langle J(w) - u_1^* + \frac{1}{2}\nabla f(w),\ z - w\Big\rangle \le V^f(u_1^*; z) - V^f(u_1^*; w), \tag{4.12}$$

and

$$2\Big\langle J(w) - u_2^* + \frac{1}{2}\nabla f(w),\ z - w\Big\rangle \le V^f(u_2^*; z) - V^f(u_2^*; w). \tag{4.13}$$

Choosing $z = x_1$, $w = x_2$ in (4.12) and $z = x_2$, $w = x_1$ in (4.13) yields

$$2\Big\langle J(x_2) - u_1^* + \frac{1}{2}\nabla f(x_2),\ x_1 - x_2\Big\rangle \le V^f(u_1^*; x_1) - V^f(u_1^*; x_2) \le 0, \tag{4.14}$$

and

$$2\Big\langle J(x_1) - u_2^* + \frac{1}{2}\nabla f(x_1),\ x_2 - x_1\Big\rangle \le V^f(u_2^*; x_2) - V^f(u_2^*; x_1) \le 0. \tag{4.15}$$

Adding (4.14) and (4.15) gives

$$2\Big\langle J(x_2) - J(x_1) + u_2^* - u_1^* + \frac{1}{2}(\nabla f(x_2) - \nabla f(x_1)),\ x_1 - x_2\Big\rangle \le 0. \tag{4.16}$$

Rearranging (4.16) and recalling the definition of u_i^* leads to

$$\Big\langle (2 - r)[J(x_2) - J(x_1)] + r[x_2^* - x_1^*] + \frac{1}{2}(\nabla f(x_2) - \nabla f(x_1)),\ x_2 - x_1\Big\rangle \ge 0. \tag{4.17}$$

Equivalently,

$$\big\langle (r-2)[J(x_2)-J(x_1)],\ x_2-x_1\big\rangle \le r\big\langle x_2^*-x_1^*,\ x_2-x_1\big\rangle + \frac{1}{2}\langle \nabla f(x_2)-\nabla f(x_1),\ x_2-x_1\rangle. \tag{4.18}$$

Applying the Cauchy–Schwarz inequality to the right-hand side of (4.18) gives

$$\big\langle (r-2)[J(x_2)-J(x_1)],\ x_2-x_1\big\rangle \le r\|x_2^*-x_1^*\|\|x_2-x_1\| + \frac{1}{2}\|\nabla f(x_2)-\nabla f(x_1)\|\|x_2-x_1\|. \tag{4.19}$$

We now derive a bound for the points x_1 and x_2. Since f is bounded below on S, there exists a constant β such that $f(x) \ge \beta$, for all $x \in S$. Given that $x_i^* \in \mathbb{A}_{r,\alpha}^*$, we have $d_{S,f}^V(x_i^*) < r$ for $i = 1, 2$. Hence,

$$\|x_i\| \le \|x_i^*\| + (V^f(x_i^*, x_i) - f(x_i))^{\frac{1}{2}} = \|x_i^*\| + (d_{S,f}^V(x_i^*) - f(x_i))^{\frac{1}{2}} \le \alpha + (r-\beta)^{\frac{1}{2}} =: T.$$

Thus, by Lemma 4.8, there exists a constant $R_T > 0$ such that

$$R_T\|x_2-x_1\|^2 \le \langle J(x_2)-J(x_1),\ x_2-x_1\rangle. \tag{4.20}$$

Combining (4.20) with inequality (4.19) yields

$$\begin{aligned}(r-2)R_T\|x_2-x_1\|^2 &\le \big\langle (r-2)[J(x_2)-J(x_1)],\ x_2-x_1\big\rangle \\ &\le r\|x_2^*-x_1^*\|\|x_2-x_1\| + \frac{1}{2}\|\nabla f(x_2)-\nabla f(x_1)\|\|x_2-x_1\|.\end{aligned}$$

Dividing by $\|x_2-x_1\|$ (assuming $x_1 \ne x_2$) and using the Lipschitz continuity of ∇f gives

$$\begin{aligned}\|x_2-x_1\| &\le \frac{r}{(r-2)R_T}\|x_2^*-x_1^*\| + \frac{1}{2(r-2)R_T}\|\nabla f(x_2)-\nabla f(x_1)\| \\ &\le \frac{r}{(r-2)R_T}\|x_2^*-x_1^*\| + \frac{L}{2(r-2)R_T}\|x_2-x_1\|.\end{aligned}$$

Rearranging terms, we obtain

$$\Big[1-\frac{L}{2(r-2)R_T}\Big]\|x_2-x_1\| \le \frac{r}{(r-2)R_T}\|x_2^*-x_1^*\|.$$

By the assumption in (4.11), we have $\frac{L}{2(r-2)R_T} < 1$. Therefore,

$$\|x_2-x_1\| \le \frac{r}{(r-2)R_T-\frac{L}{2}}\|x_2^*-x_1^*\|.$$

Consequently,

$$\|P_{f,S}^{\pi}(x_2^*) - P_{f,S}^{\pi}(x_1^*)\| \le \lambda\|x_2^*-x_1^*\|,$$

with Lipschitz constant

$$\lambda := \frac{2r}{2(r-2)R_T-L}.$$

This completes the proof. □

5. Conclusions

This work has advanced the theory of generalized projection operators in Banach spaces by establishing a systematic bridge between the classical generalized projection π_S and the more flexible generalized f-projection π_S^f. By introducing the f-proximal normal cone $N_f^{\pi}(S, \bar{x})$ - defined through the generalized f-projection - we have unified and extended fundamental results previously known for the classical V-proximal normal cone to the broader setting of reflexive Banach spaces. Key structural properties of this cone, including its characterization via variational inequalities and its behavior in both convex and nonconvex settings, are rigorously established.

A central contribution of this paper is the introduction and analysis of the uniform f-prox-regularity of sets. This new regularity concept meaningfully incorporates the geometry of the set S together with the functional perturbation f, thereby extending the well-studied notion of prox-regularity. We demonstrate that this class of sets enjoys several desirable properties, particularly when S is uniformly prox-regular and f is sufficiently smooth. Under such conditions, we prove a precise decomposition formula for the f-proximal normal cone and establish the Lipschitz continuity of a natural subclass of the projection operator π_S^f. This stability result is significant for applications in variational analysis and optimization, where the continuity of solution mappings is essential.

Our investigation underscores the versatility of the f-projection framework. By choosing the function f appropriately, one can tailor the projection operator and the associated normal cone to specific problem structures, enabling finer analysis in nonconvex variational problems, sensitivity analysis, and the study of regularity properties in Banach spaces. The theoretical framework developed here thus provides a robust foundation for future research in areas such as nonsmooth optimization, equilibrium problems, and the analysis of geometric approximations in uniformly convex and uniformly smooth Banach spaces.

In summary, this paper deepens the understanding of projection-based constructions in variational analysis and opens several avenues for further exploration, including applications to more general classes of functions and connections with other normal cone constructions, as well as the use of these tools in the study of differential inclusions and evolution equations in Banach spaces.

Author contributions

Ali Al-Tane, See Keong Lee, and Messaoud Bounkhel: Writing original draft, Writing review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare they have no conflicts of interest.

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