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Research article

Neutrosophic primal structure with closure and proximity operators

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Abstract: This paper aimed to introduce novel neutrosophic primal and neutrosophic proximity operators, derived from a new abstract framework called "neutrosophic primal topology". We began by examining the core properties of neutrosophic primal operators. Next, we defined a neutrosophic primal closure operator derived from the neutrosophic primal operator and explored the relationships between them. Based on this neutrosophic primal closure operator, we constructed a neutrosophic topology and identified the conditions under which the image of a neutrosophic primal remained a neutrosophic primal. In the next stage, we defined the neutrosophic point-primal proximity operator and explored a range of fundamental properties characterizing neutrosophic primal proximity topological spaces derived from this operator. We also introduced the concept of neutrosophic proximal closed sets and demonstrated that the collection of complements of neutrosophic primal closed sets constituted a neutrosophic topology. Finally, we defined a neutrosophic operator on a neutrosophic primal proximity topological space that satisfies the neutrosophic Kuratowski closure axioms and used it to construct a neutrosophic topology. All results established in this study were thoroughly supported and clarified through illustrative examples.

Keywords: neutrosophic primal space; neutrosophic primal proximity space; neutrosophic primal operator; neutrosophic primal proximity topology
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1. Introduction

In 1995, Zadeh [1] revolutionized classical set theory by introducing fuzzy set theory, a profound generalization that has since transformed the mathematical landscape, sparking significant interest among researchers. This led to further developments such as intuitionistic fuzzy sets [2] and interval-valued intuitionistic fuzzy sets [3]. Sarkar [4] introduced the notion of fuzzy ideals within the framework of ideal fuzzy topology. Koam [5] subsequently investigated several properties associated with ideal fuzzy topological spaces, while Gähler [6] utilized fuzzy filters to construct particular classes of fuzzy topologies. Later, Azad [7] defined the structure of fuzzy grills to facilitate the study of fuzzy approximation spaces. Building on this, Mukherjee and Das [8] developed a form of fuzzy topology that can be referred to as grill fuzzy topology.

Recently, Acharjee et al. [9] introduced a novel structure termed a primal, which is regarded as the dual of the concept of grills. This structure led to the development of a new topology known as the primal topology. Furthermore, several new operators have been defined within primal topological spaces, giving rise to various derived topologies. Al-Omari et al. [10] explored this concept from multiple perspectives. Subsequently, in [11, 12], primal structure was utilized to construct additional operators in primal topological spaces. Al-Shami et al. [13] contributed by constructing a primal soft topological space, which integrates a soft topological space with a soft primal. Then, some operators within soft primal topological spaces were proposed in [14]. Various types of topologies have been generated within different structural frameworks using soft topological operators. Notable examples include grill soft topologies [15], cluster soft topologies [16], and density soft topologies [17, 18].

Proximity, a fundamental notion in topology [19] has also received considerable attention. Various generalizations, including multiset proximity [20], I-proximity [21], μ-proximity [22, 23], and quasi-proximity [24] have been explored by numerous researchers. More recently, Al-Omari et al. [25] introduced and analyzed primal proximity spaces along with their properties.

Later, in 1998, Smarandache [26] pioneered the concept of neutrosophic sets, significantly broadening the scope of intuitionistic fuzzy set theory. Subsequently, Salama and Alblowi [27] have provided a neutrosophic topology on an extension of the concept of fuzzy topology and intuitionistic topology. This topology's generalization has proven to be an interesting area of research. Jafari et al. [28] pioneered the idea of neutrosophic singletons along with the corresponding induced neutrosophic topology. In the neutrosophic context, numerous researchers have investigated neutrosophic ideals and neutrosophic filters across diverse frameworks and mathematical structures [29–31]. Then, Pal et al. [32] later defined the neutrosophic grill topological space. Building on this, Selvaraj [33] developed various types of neutrosophic grill structures, including neutrosophic grill α-open, neutrosophic grill pre-open, among others.

The motivations for writing this paper are as follows: First, we aim to introduce new neutrosophic primal and neutrosophic proximity operators grounded in neutrosophic primal topology. This is achieved by developing distinctive frameworks that facilitate the construction of new neutrosophic topologies, thereby offering a novel approach to generating topological structures. Subsequently, we examine several fundamental topological properties derived from these constructions. Ultimately, this work lays a foundational basis for further research across various branches of mathematics.

The subsequent sections of this paper are structured as follows. Some fundamentals of neutrosophic sets and neutrosophic topology concepts required in our work are briefly recalled in Section 2. In Section 3, we establish the foundational framework of the neutrosophic primal to obtain a new structure called neutrosophic primal topological spaces. The neutrosophic primal operator is introduced through the concept of neutrosophic topology and neutrosophic primal in detail. We provide some comparisons between neutrosophic primal and neutrosophic grill. In Section 4, we present a new neutrosophic primal closure operator based on the neutrosophic primal operator, and study some connections between them. In addition, we present a neutrosophic topology via a neutrosophic primal closure operator. Finally, we successfully identify the conditions under which the image of a neutrosophic primal is a neutrosophic primal. In Section 5, we define the neutrosophic point primal proximity operator and introduce many features of a neutrosophic primal proximity topological space through this operator. In Section 6, we define a neutrosophic proximal closed set. Then, different results between a neutrosophic primal proximity space and a neutrosophic primal topological space are introduced via neutrosophic proximal closed sets and related concepts.

2. Fundamentals of neutrosophic sets and neutrosophic topology

We now obtain the following notions and findings, which are necessary for the following section:

Definition 2.1. *[26] Let $S \neq \emptyset$ be a fixed set. A neutrosophic set Q is an object having the form $Q = \{< s, \xi_Q(s), \zeta_Q(s), \lambda_Q(s) >: s \in S\}$, where $\xi_Q : S \to]^-0, 1^+[$, $\zeta_Q : S \to]^-0, 1^+[$, and $\lambda_Q : S \to]^-0, 1^+[$, respectively, denote the degree membership, indeterminacy, and non-membership of an object $s \in S$ to the set $]^-0, 1^+[$ such that $^-0 \leq \xi_Q(s) + \zeta_Q(s) + \lambda_Q(s) \leq 3^+$ for each $s \in S$.*

$\mathcal{Q}(S)$ refers to the collection of all neutrosophic sets over S. For short, we will use the symbol $Q =< s, \xi_Q(s), \zeta_Q(s), \lambda_Q(s) >$ instead of $Q = \{< s, \xi_Q(s), \zeta_Q(s), \lambda_Q(s) >: s \in S\}$.

We must introduce the empty and universal neutrosophic sets $\widetilde{\emptyset}$ and $\widetilde{S}$ over S, respectively, as follows:

Definition 2.2. *[27] $\widetilde{\emptyset} = \{< s, 0, 0, 1 >: s \in S\}$ and $\widetilde{S} = \{< s, 1, 1, 0 >: s \in S\}$.*

Proposition 2.1. *[27] For any neutrosophic set $Q \in \mathcal{Q}(S)$, the following hold:*

(1) $\widetilde{\emptyset} \subseteq Q$, $\widetilde{\emptyset} \subseteq \widetilde{\emptyset}$;
(2) $Q \subseteq \widetilde{S}$, $\widetilde{S} \subseteq \widetilde{S}$.

Definition 2.3. *[27] Let $Q_1 =< s, \xi_{Q_1}(s), \zeta_{Q_1}(s), \lambda_{Q_1}(s) >$ and $Q_2 =< s, \xi_{Q_2}(s), \zeta_{Q_2}(s), \lambda_{Q_2}(s) >$ be two neutrosophic sets over S. Then,*

(1) $Q_1 \cap Q_2 =< s, \xi_{Q_1}(s) \wedge \xi_{Q_2}(s), \zeta_{Q_1}(s) \wedge \zeta_{Q_2}(s), \lambda_{Q_1}(s) \vee \lambda_{Q_2}(s) >$;
(2) $Q_1 \cup Q_2 =< s, \xi_{Q_1}(s) \vee \xi_{Q_2}(s), \zeta_{Q_1}(s) \vee \zeta_{Q_2}(s), \lambda_{Q_1}(s) \wedge \lambda_{Q_2}(s) >$;
(3) $Q_1^c =< s, \lambda_{Q_1}(s), 1 - \zeta_{Q_1}(s), \xi_{Q_1}(s) >$ [Complement of Q_1].

Definition 2.4. *[27] A neutrosophic topology on a set $S \neq \emptyset$ is a collection $\mathcal{J}$ of neutrosophic sets over S complying with the ensuing requirements:*

(1) $\widetilde{\emptyset}, \widetilde{S} \in \mathcal{J}$.
(2) $Q_1 \cap Q_2 \in \mathcal{J}$ for every $Q_1, Q_2 \in \mathcal{J}$.
(3) $\bigcup Q_\alpha \in \mathcal{J}$ for every $\{Q_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{J}$.

Remark 2.1. *(1) The pair $(S, \mathcal{J})$ is called a neutrosophic topology space over S, where the members of $\mathcal{J}$ are called neutrosophic open sets;*
(2) A neutrosophic closed set in $(S, \mathcal{J})$ is the neutrosophic complement of a neutrosophic open set. The collection of all neutrosophic closed sets is denoted by $\mathcal{CN}(S)$.

Each neutrosophic set in a neutrosophic topology is called a neutrosophic open set. Its complements are called neutrosophic closed sets.

Definition 2.5. *[27] Let $(S, \mathcal{J})$ be a neutrosophic topological space and $Q =< s, \xi_Q(s), \zeta_Q(s), \lambda_Q(s) >$ be a neutrosophic set in S. Then,*

(1) neutrosophic closure of $Q = \bigcap\{Q_1 : Q_1$ is a neutrosophic closed and $Q \subseteq Q_1\}$ and it is termed by n-cl(Q);
(2) neutrosophic interior of $Q = \bigcup\{Q_1 : Q_1$ is a neutrosophic open set and $Q_1 \subseteq Q\}$ and it is termed by n-int(Q).

Definition 2.6. *[32] Let $S \neq \emptyset$ be a set. A collection $\mathcal{G}$ of neutrosophic sets over S is termed a grill on S if it satisfies the criteria outlined below.*

(1) $\widetilde{\emptyset} \notin \mathcal{G}$.
(2) If $Q_1 \in \mathcal{G}$ and $Q_1 \subseteq Q_2$, then $Q_2 \in \mathcal{G}$.
(3) If $Q_1, Q_2 \subseteq \widetilde{S}$ and $Q_1 \cup Q_2 \in \mathcal{G}$, then $Q_1 \in \mathcal{G}$ or $Q_2 \in \mathcal{G}$.

Definition 2.7. *[34] Let S be a non-empty set. If $p, q, r \in]^-0, 1^+[$, then the neutrosophic set $e_{p,q,r}$ is called a neutrosophic point or singleton over S given by*

$$e_{p,q,r}(s_e) = \begin{cases} (p, q, r), & if\ e = s_e, \\ \widetilde{\emptyset}, & if\ e \neq s_e. \end{cases}$$

We call $s_e \in S$ is the support of $e_{p,q,r}$, where p denotes the degree of membership value, q denotes the degree of indeterminacy and r is the degree of non-membership value of $e_{p,q,r}$.

We denote a neutrosophic point by $e =< \xi_e, \zeta_e, \lambda_e >$ and the collection of all neutrosophic singleton points over S by $\mathcal{E}(S)$. Let $\mathcal{J}(e)$ stand for the collection of all neutrosophic open neighbourhoods of (p, q, r).

It is obvious that $\mathcal{E}(S) \subseteq \mathcal{Q}(S)$. Throughout this paper any neutrosophic point $e \in \mathcal{E}(S)$, we have $e \neq \widetilde{\emptyset}$.

Definition 2.8. *[34] Let $S \neq \phi$ be a set. A neutrosophic singleton e defined on S is said to belong to a neutrosophic set $Q =< s, \xi_Q(s), \zeta_Q(s), \lambda_Q(s) >$ $(e \in Q)$ if $\xi_e \leq \xi_Q, \zeta_e \leq \zeta_Q, \lambda_e \geq \lambda_Q$.*

Definition 2.9. *[25] A binary relation $\hookrightarrow$ on $\mathcal{P}(S)$ ($\mathcal{P}(S)$ the power set of S) with a primal $\mathcal{L}$ on a nonempty set S is termed a primal proximity on S if $\hookrightarrow$ satisfies the criteria outlined below for any $J_1, J_2 \subseteq S$.*

(1) If $J_1 \hookrightarrow J_2$, then $J_2 \hookrightarrow J_1$.
(2) If $J_1 \hookrightarrow (J_2 \cup J_3) \Leftrightarrow J_1 \hookrightarrow J_2$ or $J_1 \hookrightarrow J_3$.
(3) If $J_1^c \notin \mathcal{L}$, then $J_1 \not\hookrightarrow J_2$, for each $J_2 \subseteq S$.
(4) If $(J_1 \cap J_2)^c \in \mathcal{L}$, then $J_1 \hookrightarrow J_2$.
(5) If $J_1 \not\hookrightarrow J_2$, then there exist $H_1, H_2 \subseteq S$ such that $J_1 \not\hookrightarrow H_1^c$, $H_2^c \not\hookrightarrow J_2$, and $(H_1 \cap H_2)^c \notin \mathcal{L}$.

Definition 2.10. *[25] A primal-proximity space is a pair $(S, \hookrightarrow)$ comprising of a nonempty set S and primal-proximity relation on S. Then, $J_1 \hookrightarrow J_2$, if the sets $J_1, J_2 \subseteq S$ are $\hookrightarrow$-related, otherwise $J_1 \not\hookrightarrow J_2$.*

Definition 2.11. *[35] Let S be a nonempty set. Then, the operator $\mathfrak{h} : 2^S \to 2^S$ is a Kuratowski closure operator if the following statements hold:*

(1) $\mathfrak{h}(\emptyset) = \emptyset$.
(2) For each $J \in 2^S$, $J \subseteq \mathfrak{h}(J)$.
(3) For each $J, H \in 2^S$, $\mathfrak{h}(J \cup H) = \mathfrak{h}(J) \cup \mathfrak{h}(H)$.
(4) For each $J \in 2^S$, $\mathfrak{h}(\mathfrak{h}(J)) = \mathfrak{h}(J)$.

3. Neutrosophic topological structure and neutrosophic operators

Here, we introduce a novel class of operators associated with neutrosophic primal topological spaces and examine their essential properties.

Definition 3.1. *Let $S \neq \emptyset$. A family $\mathcal{L}$ of neutrosophic sets over S ($\mathcal{L} \subseteq Q(S)$) is a neutrosophic primal over S if, and only if, it adheres to the following stipulations:*

(1) $\widetilde{S} \notin \mathcal{L}$.
(2) If $Q_1 \in \mathcal{L}$ and $Q_2 \subseteq Q_1$, then $Q_2 \in \mathcal{L}$.
(3) If $Q_1 \cap Q_2 \in \mathcal{L}$, then $Q_1 \in \mathcal{L}$ or $Q_2 \in \mathcal{L}$.

Corollary 3.1. *Let $S \neq \emptyset$. A collection $\mathcal{L}$ of neutrosophic sets over S ($\mathcal{L} \subseteq Q(S)$) is referred to as a neutrosophic primal on S if it adheres to the following stipulations:*

(1) $\widetilde{S} \notin \mathcal{L}$.
(2) If $Q_2 \notin \mathcal{L}$ and $Q_2 \subseteq Q_1$, then $Q_1 \notin \mathcal{L}$.
(3) If $Q_1 \notin \mathcal{L}$ and $Q_2 \notin \mathcal{L}$, then $Q_1 \cap Q_2 \notin \mathcal{L}$.

Proof. (1) Obvious;
(2) Suppose that $Q_1 \in \mathcal{L}$. Since $Q_2 \subseteq Q_1$, then by part (2) of Definition 3.1, $Q_2 \in \mathcal{L}$. Thus, contradicts the assumption that $Q_2 \notin \mathcal{L}$. Hence, $Q_1 \notin \mathcal{L}$.
(3) Suppose that $Q_1 \cap Q_2 \in \mathcal{L}$, then by part (3) of Definition 3.1, $Q_1 \in \mathcal{L}$ or $Q_2 \in \mathcal{L}$. Thus, contradicts the assumption that $Q_1 \notin \mathcal{L}$ and $Q_2 \notin \mathcal{L}$. Hence, $Q_1 \cap Q_2 \notin \mathcal{L}$.

Example 3.1. Suppose $S \neq \phi$ is a set, then $Q(S) \setminus \{\widetilde{S}\}$ is a neutrosophic primal on S, where $Q(S)$ denote to all neutrosophic sets over S.

Example 3.2. For $S = \{s\}$, the family $\mathcal{L} = \{\{< s, a, b, 1, >\} : a, b \in [0, 1]\}$ is a neutrosophic primal on S.

The following theorem explains the relationship between primal and grill of neutrosophic sets over S.

Theorem 3.1. *Let $\mathcal{G} \subseteq Q(S)$ be a grill on S. Then, $\{Q | Q^c \in \mathcal{G}\}$ is a primal on S.*

Proof. Suppose that $\mathcal{G} \subseteq Q(S)$ is a grill on S and $\mathcal{L} = \{Q | Q^c \in \mathcal{G}\}$. Since $\widetilde{\emptyset} \notin \mathcal{G}$ and $(\widetilde{S})^c = \widetilde{\emptyset}$, then $\widetilde{S} \notin \mathcal{L}$. Let $Q_1 \in \mathcal{L}$ and $Q_2 \subseteq Q_1$. Then, $Q_1^c \subseteq Q_2^c$, hence, $Q_2^c \in \mathcal{G}$. Thus, $Q_2 \in \mathcal{L}$. Now, let $Q_1 \cap Q_2 \in \mathcal{L}$. Then, $Q_1^c \cup Q_2^c = (Q_1 \cap Q_2)^c \in \mathcal{G}$. Therefore, we get $Q_1^c \in \mathcal{G}$ or $Q_2^c \in \mathcal{G}$. Hence, $Q_1 \in \mathcal{L}$ or $Q_2 \in \mathcal{L}$. Therefore, $\mathcal{L}$ is a primal on S.

Theorem 3.2. *The union of two neutrosophic primals on S is a neutrosophic primal on S.*

Proof. Let $\mathcal{K},\mathcal{H}$ be two neutrosophic primals on S. Then, $\widetilde{S} \notin \mathcal{K}$ and $\widetilde{S} \notin \mathcal{H}$. Hence, $\widetilde{S} \notin \mathcal{K} \cup \mathcal{H}$. Now, let $Q_1 \in \mathcal{K} \cup \mathcal{H}$ and $Q_2 \subseteq Q_1$. Then, $Q_1 \in \mathcal{K}$ or $Q_1 \in \mathcal{H}$. Hence, $Q_2 \in \mathcal{K}$ or $Q_2 \in \mathcal{H}$. Therefore, $Q_2 \in \mathcal{K} \cup \mathcal{H}$. Finally, let $Q_1 \cap Q_2 \in \mathcal{K} \cup \mathcal{H}$. Then, $Q_1 \cap Q_2 \in \mathcal{K}$ or $Q_1 \cap Q_2 \in \mathcal{H}$. If $Q_1 \cap Q_2 \in \mathcal{K}$, then either $Q_1 \in \mathcal{K}$ or $Q_2 \in \mathcal{K}$. Also, if $Q_1 \cap Q_2 \in \mathcal{H}$, then either $Q_1 \in \mathcal{H}$ or $Q_2 \in \mathcal{H}$. Therefore, $Q_1 \in \mathcal{K} \cup \mathcal{H}$ or $Q_2 \in \mathcal{K} \cup \mathcal{H}$.

Definition 3.2. *A neutrosophic topological space $(S,\mathcal{J})$ with a neutrosophic primal $\mathcal{L} \subseteq Q(S)$ is called a neutrosophic primal topological space $(S,\mathcal{J},\mathcal{L})$ and indicated by $\mathcal{NPTS}$.*

Example 3.3. Suppose that $(S,\mathcal{J})$ be an $\mathcal{NTS}$, where $S = \{s\}$, $\mathcal{J} = \{\widetilde{\emptyset}, \widetilde{S}, \{< s, 0.5, 0.5, 0.4 >: s \in S\}, \{< s, 0.4, 0.6, 0.8 >: s \in S\}, \{< s, 0.5, 0.6, 0.4 >: s \in S\}, \{< s, 0.4, 0.5, 0.8 >: s \in S\}\}$ and $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$. Then, $(S,\mathcal{J},\mathcal{L})$ is an $\mathcal{NPTS}$.

Remark 3.1. *Any neutrosophic set can be written as a union of neutrosophic points.*

Based on what was presented earlier, we now define a new type of a neutrosophic primal operator based on primal. The new structure is presented below.

Definition 3.3. *Let $(S,\mathcal{J},\mathcal{L})$ be an $\mathcal{NPTS}$. For $J \in Q(S)$, we define a map $\zeta : Q(S) \to Q(S)$ as $\zeta(J)(S,\mathcal{J},\mathcal{L}) = \bigcup\{e \in \mathcal{E}(S) : J^c \cup G^c \in \mathcal{L}, \ for\ each\ G \in \mathcal{J}(e)\}$. To be clear, $\zeta(J)(S,\mathcal{J},\mathcal{L})$ is denoted as $\zeta(J)$ for brevity and is called the neutrosophic primal operator of J with respect to $\mathcal{J}$ and $\mathcal{L}$.*

Theorem 3.3. *Let $(S,\mathcal{J},\mathcal{L})$ be an $\mathcal{NPTS}$. Then, the following statements hold for each $J \in Q(S)$:*

(1) $\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) $n\text{-}cl(\zeta(J)) = \zeta(J)$.
(3) $\zeta(J) \subseteq cl(J)$.
(4) $\zeta(\zeta(J)) \subseteq \zeta(J)$ *for each* $\zeta(J) \subseteq J$.
(5) If $J^c \in \mathcal{J}$, *then* $\zeta(J) \subseteq J$.

Proof. (1) From Definition 3.3, for any neutrosophic point $e \in \mathcal{E}(S)$ and $G \in \mathcal{J}(e)$, we have $(\widetilde{\emptyset})^c \cup G^c = (\widetilde{S} \cup G^c) = \widetilde{S} \notin \mathcal{L}$, hence, $\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) We have always $\zeta(J) \subseteq n\text{-}cl(\zeta(J))$. Conversely, let $e \in n\text{-}cl(\zeta(J))$, where $e \in \mathcal{E}(S)$ is a neutrosophic point and $G \in \mathcal{J}(e)$. Hence, $G \cap \zeta(J) \neq \widetilde{\emptyset}$, which implies there exists a neutrosophic point $e_1 \in G \cap \zeta(J)$. Thus, $J^c \cup G^c \in \mathcal{L}$, because $e_1 \in \zeta(J)$. Hence, for any $G \in \mathcal{J}(e)$, we have $J^c \cup G^c \in \mathcal{L}$, then $e \in \zeta(J)$. Therefore, $n\text{-}cl(\zeta(J)) = \zeta(J)$.
(3) Suppose that $e \notin cl(J)$, then there exists $G \in \mathcal{J}(e)$ such that $G \cap J = \widetilde{\emptyset}$, hence, $G^c \cup J^c = \widetilde{S} \notin \mathcal{L}$. Thus, $e \notin \zeta(J)$. Therefore, $\zeta(J) \subseteq cl(J)$.
(4) Let $e \in \zeta(\zeta(J))$, where $e \in \mathcal{E}(S)$ is a neutrosophic point and $G \in \mathcal{J}(e)$. From Definition 3.3, $(\zeta(J))^c \cup G^c \in \mathcal{L}$. Since $\zeta(J) \subseteq J$, then $(J^c \cup G^c) \subseteq ((\zeta(J))^c \cup G^c)$. By the primality, we have $(J^c \cup G^c) \in \mathcal{L}$, hence, $e \in \zeta(J)$. Therefore, $\zeta(\zeta(J)) \subseteq \zeta(J)$.
(5) Let $e \in \zeta(J)$. Suppose that $e \notin J$, which implies $J^c \in \mathcal{J}(e)$. Hence, $(J^c)^c \cup J^c = \widetilde{S} \in \mathcal{L}$, which is a contradiction since $\widetilde{S} \notin \mathcal{L}$. Hence, $e \in J$, and so, $\zeta(J) \subseteq J$.

Theorem 3.4. *Let $(S,\mathcal{J},\mathcal{L})$ be an $\mathcal{NPTS}$. Then, the following statements hold for each $J_1, J_2 \in Q(S)$.*

(1) If $J_1 \subseteq J_2$, *then* $\zeta(J_1) \subseteq \zeta(J_2)$.
(2) $\zeta(J_1) \cup \zeta(J_2) = \zeta(J_1 \cup J_2)$.

(3) $\zeta(J_1 \cap J_2) \subseteq \zeta(J_1) \cap \zeta(J_2)$.

Proof. (1) Suppose that $J_1 \subseteq J_2$ and $e \in \zeta(J_1)$, then for all $G \in \mathcal{J}(e)$, we have $J_1^c \cup G^c \in \mathcal{L}$. Since $J_1 \subseteq J_2$, then $(J_2^c \cup G^c) \subseteq (J_1^c \cup G^c) \in \mathcal{L}$. By primality, $(J_2^c \cup G^c) \in \mathcal{L}$. Hence, $e \in \zeta(J_2)$. Therefore, $\zeta(J_1) \subseteq \zeta(J_2)$.
(2) Obvious from (1), $\zeta(J_1) \subseteq \zeta(J_1 \cup J_2)$ and $\zeta(J_2) \subseteq \zeta(J_1 \cup J_2)$. Hence, $\zeta(J_1) \cup \zeta(J_2) \subseteq \zeta(J_1 \cup J_2)$. On the other hand, suppose that $e \notin \zeta(J_1) \cup \zeta(J_2)$, then $e \notin \zeta(J_1)$ and $e \notin \zeta(J_2)$. Consequently, there are neutrosophic open sets G_1 and G_2 including e, for which $J_1^c \cup G_1^c \notin \mathcal{L}$ and $J_2^c \cup G_2^c \notin \mathcal{L}$. Now, put $G = (G_1 \cap G_2)$, hence, $G \in \mathcal{J}(e)$. Since $G \subseteq G_1$ and $G \subseteq G_2$, then $(J_1^c \cup G_1^c) \subseteq (J_1^c \cup G^c) \notin \mathcal{L}$ and $(J_2^c \cup G_2^c) \subseteq (J_2^c \cup G^c) \notin \mathcal{L}$. Thus, $(J_1^c \cup G^c) \notin \mathcal{L}$ and $(J_2^c \cup G^c) \notin \mathcal{L}$. Then, we have $((J_1 \cup J_2)^c \cup G^c) = (J_1^c \cap J_2^c) \cup G^c = ((J_1^c \cup G^c) \cap (J_2^c \cup G^c)) \notin \mathcal{L}$. Hence, $e \notin \zeta(J_1 \cup J_2)$. Therefore, $\zeta(J_1 \cup J_2) \subseteq \zeta(J_1) \cup \zeta(J_2)$.
(3) From (1), we have the desired result.

Here is an example to illustrate that the inclusion of part (3) of Theorem 3.4 need not be reversible.

Example 3.4. Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$, where $S = \{s\}$, $\mathcal{J} = \{\widetilde{\emptyset}, \widetilde{S}\}$ and $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$. Put $J_1 = \{< s, 0.5, 0, 1 >: s \in S\}$ and $J_2 = \{< s, 0, 0.5, 1 >: s \in S\}$, then $\zeta(J_1) = \widetilde{S}$ and $\zeta(J_2) = \widetilde{S}$. Hence, $\zeta(J_1) \cap \zeta(J_2) = \widetilde{S} \neq \widetilde{\emptyset} = \zeta(\widetilde{\emptyset}) = \zeta(J_1 \cap J_2)$.

Lemma 3.1. *Let* $(S, \mathcal{J}, \mathcal{L})$ *be an* $\mathcal{NPTS}$ *and* $J_1, J_2 \in Q(S)$. *If* $J_1 \in \mathcal{J}$. *Then,* $J_1 \cap \zeta(J_2) \subseteq \zeta(J_1 \cap J_2)$.

Proof. Let $e \in J_1 \cap \zeta(J_2)$. Then, $e \in J_1$ and $e \in \zeta(J_2)$. Since $J_1 \in \mathcal{J}$ and $e \in J_1$, then for all $G \in \mathcal{J}(e)$, we have $G \cap J_1 \in \mathcal{J}(e)$. From Definition 3.3, we have $(J_2^c \cup (G \cap J_1)^c) \in \mathcal{L}$. Hence, $(J_1 \cap J_2)^c \cup G^c = J_2^c \cup J_1^c \cup G^c = (J_2^c \cup (G \cap J_1)^c) \in \mathcal{L}$. Thus, $e \in \zeta(J_1 \cap J_2)$. Therefore, $J_1 \cap \zeta(J_2) \subseteq J_1 \cap \zeta(J_2)$.

Lemma 3.2. *Let* $(S, \mathcal{J}, \mathcal{L})$ *be an* $\mathcal{NPTS}$. *If* $J^c \notin \mathcal{L}$, *then* $\zeta(J) = \widetilde{\emptyset}$.

Proof. Suppose that $\zeta(J) \neq \widetilde{\emptyset}$, then there exists a neutrosophic point $e \neq \widetilde{\emptyset}$ and $e \in \zeta(J)$. Hence, for all $G \in \mathcal{L}(e)$, we have $G^c \cup J^c \in \mathcal{L}$. Since $J^c \notin \mathcal{L}$, then $G^c \cup J^c \notin \mathcal{L}$ for some neutrosophic open set $G \in \mathcal{J}(e)$. This is a contradiction. Therefore, $\zeta(J) = \widetilde{\emptyset}$.

Theorem 3.5. *Let* $(S, \mathcal{J}, \mathcal{L})$ *be an* $\mathcal{NPTS}$. *If* $\mathcal{CN}(S) \setminus \{\widetilde{S}\} \subseteq \mathcal{L}$, *then* $J \subseteq \zeta(J)$ *for all* $J \in \mathcal{J}$.

Proof. If $J = \widetilde{\emptyset}$, we are done, because $\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$. Suppose that $J \neq \widetilde{\emptyset}$. Now, let $J = \widetilde{S}$ and $e \notin \zeta(\widetilde{S})$. Then, for all $G \in \mathcal{J}(e)$, we have $G^c \cup (\widetilde{S})^c = G^c \cup \widetilde{\emptyset} = G^c \notin \mathcal{L}$, but $G^c \in \mathcal{CN}(S)$. This contradicts the fact that $\mathcal{CN}(S) \setminus \{\widetilde{S}\} \subseteq \mathcal{L}$, hence, $\widetilde{S} \subseteq \zeta(\widetilde{S})$. Since we always have $\zeta(\widetilde{S}) \subseteq \widetilde{S}$, then $\zeta(\widetilde{S}) = \widetilde{S}$. Assume that $J \neq \widetilde{S}$, then by Lemma 3.1, for any $J \in \mathcal{J}$ we have $J = J \cap \widetilde{S} = J \cap \zeta(\widetilde{S}) \subseteq \zeta(J \cap \widetilde{S}) = \zeta(J)$. Therefore, $J \subseteq \zeta(J)$.

4. Neutrosophic primal closure operators and its main characteristics

In this section, we introduce a neutrosophic primal closure operator derived from the neutrosophic primal operator and investigate the interrelations between them. Utilizing this neutrosophic primal closure operator, we construct a neutrosophic topology and determine the conditions under which the image of a neutrosophic primal remains a neutrosophic primal.

Definition 4.1. *Let* $(S, \mathcal{J}, \mathcal{L})$ *be an* $\mathcal{NPTS}$. *We define a map* $cl_\zeta : Q(S) \to Q(S)$ *by* $cl_\zeta(J) = J \cup \zeta(J)$ *for any* $J \in Q(S)$.

Theorem 4.1. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, the following statements hold for each $J \in Q(S)$:*

(1) $cl_\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) $cl_\zeta(\widetilde{S}) = \widetilde{S}$.
(3) $J \subseteq cl_\zeta(J)$.

Proof. (1) From part (1) of Theorem 3.3, $\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$. Hence, $\widetilde{\emptyset} \cup \zeta(\widetilde{\emptyset}) = \widetilde{\emptyset} \cup \widetilde{\emptyset} = \widetilde{\emptyset}$, which implies $cl_\zeta(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) Since $\widetilde{S} \cup \zeta(\widetilde{S}) = \widetilde{S}$, then $cl_\zeta(\widetilde{S}) = \widetilde{S}$.
(3) Since $J \subseteq (J \cup \zeta(J))$, then $J \subseteq cl_\zeta(J)$.

Theorem 4.2. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, the following statements hold for each $J_1, J_2 \in Q(S)$:*

(1) If $J_1 \subseteq J_2$, *then* $cl_\zeta(J_1) \subseteq cl_\zeta(J_2)$.
(2) $cl_\zeta(J_1) \cup cl_\zeta(J_2) = cl_\zeta(J_1 \cup J_2)$.
(3) $cl_\zeta(cl_\zeta(J_1)) = cl_\zeta(J_1)$ *for each* $\zeta(J_1) \subseteq J_1$.

Proof. (1) Let $J_1 \subseteq J_2$. Then, by part (1) of Theorem 3.4, $\zeta(J_1) \subseteq \zeta(J_2)$, hence, $J_1 \cup \zeta(J_1) \subseteq J_2 \cup \zeta(J_2)$. Thus, $cl_\zeta(J_1) \subseteq cl_\zeta(J_2)$.
(2) Since $cl_\zeta(J_1 \cup J_2) = (J_1 \cup J_2) \cup \zeta(J_1 \cup J_2)$, then by part (2) of Theorem 3.4, $cl_\zeta(J_1 \cup J_2) = (J_1 \cup J_2) \cup (\zeta(J_1) \cup \zeta(J_2)) = (J_1 \cup \zeta(J_1)) \cup (J_2 \cup \zeta(J_2)) = cl_\zeta(J_1) \cup cl_\zeta(J_2)$.
(3) It is clear by part (3) of Theorem 4.1, $cl_\zeta(J_1) \subseteq cl_\zeta(cl_\zeta(J_1))$. Conversely, $cl_\zeta(cl_\zeta(J_1)) = cl_\zeta(J_1) \cup \zeta(cl_\zeta(J_1)) = cl_\zeta(J_1) \cup \zeta(J_1 \cup \zeta(J_1))$. From part (2) of Theorem 3.4, $cl_\zeta(cl_\zeta(J_1)) = cl_\zeta(J_1) \cup \zeta(J_1) \cup \zeta(\zeta(J_1))$ Since $\zeta(J_1) \subseteq J_1$ and by part (4) of Theorem 3.3, $cl_\zeta(cl_\zeta(J_1)) \subseteq cl_\zeta(J_1) \cup \zeta(J_1) \cup \zeta(J_1) = cl_\zeta(J_1) \cup \zeta(J_1) \subseteq cl_\zeta(J_1) \cup J_1 = cl_\zeta(J_1)$. Therefore, $cl_\zeta(cl_\zeta(J_1)) = cl_\zeta(J_1)$ for each $\zeta(J_1) \subseteq J_1$.

Theorem 4.3. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, the collection $\mathcal{J}^\zeta_\mathcal{L} = \{J \in Q(S) : cl_\zeta(J^c) = J^c\}$ is a neutrosophic topology in S induced by neutrosophic topology $\mathcal{J}$ and neutrosophic primal $\mathcal{L}$.*

Proof. For condition (1): From part (1) of Theorem 3.3, $cl_\zeta((\widetilde{S})^c) = (\widetilde{S})^c \cup \zeta((\widetilde{S})^c) = \widetilde{\emptyset} \cup \zeta(\widetilde{\emptyset}) = \widetilde{\emptyset} \cup \widetilde{\emptyset} = \widetilde{\emptyset} = (\widetilde{S})^c$. Also, $cl_\zeta((\widetilde{\emptyset})^c) = (\widetilde{\emptyset})^c \cup \zeta((\widetilde{\emptyset})^c) = \widetilde{S} \cup \zeta(\widetilde{S}) = \widetilde{S} = (\widetilde{\emptyset})^c$. Therefore, $\widetilde{\emptyset}, \widetilde{S} \in \mathcal{J}^\zeta_\mathcal{L}$. For condition (2): Let $J_1, J_2 \in \mathcal{J}^\zeta_\mathcal{L}$. Then, $cl_\zeta(J_1^c) = J_1^c$ and $cl_\zeta(J_2^c) = J_2^c$. Now, $cl_\zeta((J_1 \cap J_2)^c) = (J_1 \cap J_2)^c \cup \zeta[(J_1 \cap J_2)^c] = J_1^c \cup J_2^c \cup \zeta[J_1^c \cup J_2^c]$. From part (2) of Theorem 3.4, $cl_\zeta((J_1 \cap J_2)^c) = J_1^c \cup J_2^c \cup \zeta(J_1^c) \cup \zeta(J_2^c) = cl_\zeta(J_1^c) \cup cl_\zeta(J_1^c) = J_1^c \cup J_2^c = (J_1 \cap J_2)^c$. Thus, $J_1 \cap J_2 \in \mathcal{J}^\zeta_\mathcal{L}$. Now, let $\{J_\xi : \xi \in \Lambda\} \subseteq \mathcal{J}^\zeta_\mathcal{L}$. Then, we have always $(\bigcup_{\xi\in\Lambda} J_\xi)^c \subseteq cl_\zeta((\bigcup_{\xi\in\Lambda} J_\xi)^c)$. Conversely, $cl_\zeta(J_\xi^c) = J_\xi^c$ for each $\xi \in \Lambda$, hence, $cl_\zeta[(\bigcup_{\xi\in\Lambda} J_\xi)^c] = (\bigcup_{\xi\in\Lambda} J_\xi)^c \cup \zeta[(\bigcup_{\xi\in\Lambda} J_\xi)^c] = (\bigcap_{\xi\in\Lambda} J_\xi^c) \cup \zeta(\bigcap_{\xi\in\Lambda} J_\xi^c) \subseteq (\bigcap_{\xi\in\Lambda} J_\xi^c) \cup (\bigcap_{\xi\in\Lambda} \zeta(J_\xi^c)) = \bigcap_{\xi\in\Lambda}(J_\xi^c \cup \zeta(J_\xi^c)) = \bigcap_{\xi\in\Lambda} cl_\zeta(J_\xi^c) = \bigcap_{\xi\in\Lambda} J_\xi^c = (\bigcup_{\xi\in\Lambda} J_\xi)^c$. Hence, $cl_\zeta((\bigcup_{\xi\in\Lambda} J_\xi)^c) = (\bigcup_{\xi\in\Lambda} J_\xi)^c$. Thus, $\bigcup_{\xi\in\Lambda} J_\xi \in \mathcal{J}^\zeta_\mathcal{L}$. Therefore, $\mathcal{J}^\zeta_\mathcal{L}$ is a neutrosophic topology in S induced by neutrosophic topology $\mathcal{J}$ and neutrosophic primal $\mathcal{L}$.

For simplicity, we will use the symbol $\mathcal{J}^\zeta$ instead of $\mathcal{J}^\zeta_\mathcal{L}$ if there is no confusion.

Theorem 4.4. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, the floating topology $\mathcal{J}^\zeta$ is finer than $\mathfrak{J}$.*

Proof. Let $J \in \mathcal{J}$. Then, J^c is $\mathcal{J}$-closed in S, and by part (5) of Theorem 3.3, $\zeta(J^c) \subseteq J^c$. Thus, $cl_\zeta(J^c) = J^c \cup \zeta(J^c) = J^c$, which implies $J \in \mathcal{J}^\zeta$. Therefore, $\mathcal{J} \subseteq \mathcal{J}^\zeta$.

Theorem 4.5. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, the following statements hold:*

(1) If $\mathcal{L} = \{\widetilde{\emptyset}\}$, then $\mathcal{J}^{\zeta} = Q(S)$.
(2) If $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$, then $\mathcal{J}^{\zeta} = \mathcal{J}$.

Proof. (1) We have always $\mathcal{J}^{\zeta} \subseteq Q(S)$. Conversely, we have two cases:

Case 1. If $J \in Q(S) \setminus \{\widetilde{S}\}$, then $\zeta(J) = \widetilde{\emptyset}$. Suppose that $\zeta(J) \neq \widetilde{\emptyset}$. Consequently, a neutrosophic point $e \neq \widetilde{\emptyset}$ can be found where $e \in \zeta(J)$, hence, for any $G \in \mathcal{J}(e)$, we have $G^c \cup J^c \in \mathcal{L}$. Thus, $G = J = \widetilde{S}$, because $\mathcal{L} = \{\widetilde{\emptyset}\}$ and thus contradicts the fact $J \in Q(S) \setminus \{\widetilde{S}\}$. Hence, $\zeta(J) = \widetilde{\emptyset}$, which implies $cl_{\zeta}(J^c) = J^c$. Therefore, $J \in \mathcal{J}^{\zeta}$.

Case 2. If $J = \widetilde{S}$, then $cl_{\zeta}(\widetilde{S}^c) = \widetilde{S}^c \cup \zeta((\widetilde{S})^c) = (\widetilde{S})^c \cup \zeta(\widetilde{\emptyset}) = (\widetilde{S})^c$. Hence, $J \in \mathcal{J}^{\zeta}$. Therefore, $\mathcal{J}^{\zeta} = Q(S)$.

(2) By Theorem 4.4, $\mathcal{J} \subseteq \mathcal{J}^{\zeta}$. Now, let $J \in \mathcal{J}^{\zeta}$. If $J = \widetilde{\emptyset}$, then $J \in \mathcal{J}$. Suppose that $J \neq \widetilde{\emptyset}$. Since $J \in \mathcal{J}^{\zeta}$, then $J^c \cup \zeta(J^c) = J^c$, which implies $\zeta(J^c) \subseteq J^c$. Assume that $e \notin \zeta(J^c)$, then there exists $G \in \mathcal{J}(e)$ such that $(J^c)^c \cup G^c \notin \mathcal{L}$. Since $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$, then $(J^c)^c \cup G^c = \widetilde{S}$ and so, $J^c \cap G = \widetilde{\emptyset}$. Hence, $e \notin cl(J^c)$. Thus, we have $cl(J^c) \subseteq \zeta(J^c) \subseteq J^c$. Therefore, $cl(J^c) = J^c$, which implies J^c is $\mathcal{J}$-closed in S and so, $J \in \mathcal{J}$. Hence, $\mathcal{J}^{\zeta} \subseteq \mathcal{J}$. Therefore, $\mathcal{J}^{\zeta} = \mathcal{J}$.

Theorem 4.6. *Let $(S, \mathcal{J}, \mathcal{L})$ and $(S, \mathcal{J}, \mathcal{F})$ be two $\mathcal{NPTS}$s. If $\mathcal{L} \subseteq \mathcal{F}$, then $\mathcal{J}^{\zeta}_{\mathcal{F}} \subseteq \mathcal{J}^{\zeta}_{\mathcal{L}}$.*

Proof. Let $J \in \mathcal{J}^{\zeta}_{\mathcal{F}}$. Then, $cl_{\zeta_{\mathcal{F}}}(J^c) = J^c \cup \zeta_{\mathcal{F}}(J^c) = J^c$, hence, $\zeta_{\mathcal{F}}(J^c) \subseteq J^c$. Let $e \notin J^c$. Then, $e \notin \zeta_{\mathcal{F}}(J^c)$, hence, there exists $G \in \mathcal{J}(e)$ such that $G^c \cup J \notin \mathcal{F}$. Since $\mathcal{J}^{\zeta}_{\mathcal{F}} \subseteq \mathcal{J}^{\zeta}_{\mathcal{J}}$, then we obtain $G \in \mathcal{J}(e)$ such that $G^c \cup J \notin \mathcal{L}$. Thus, we have $e \notin \zeta_{\mathcal{L}}(J^c)$, and so, $\zeta_{\mathcal{L}}(J^c) \subseteq J^c$. Hence, $cl_{\zeta_{\mathcal{L}}}(J^c) = J^c \cup \zeta_{\mathcal{L}}(J^c) = J^c$. Thus, $J \in \mathcal{J}^{\zeta}_{\mathcal{L}}$. Therefore, $\mathcal{J}^{\zeta}_{\mathcal{F}} \subseteq \mathcal{J}^{\zeta}_{\mathcal{L}}$.

Lemma 4.1. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$ and $J \in Q(S)$. Then, the followings hold:*

(1) $J \in \mathcal{J}^{\zeta}$ if, and only if, for all neutrosophic point $e \in J$, there exists a neutrosophic open set G containing e such that $G^c \cup J \notin \mathcal{L}$.
(2) If $J \notin \mathcal{L}$, then $J \in \mathcal{J}^{\zeta}$.

Proof. (1) Let $J \in \mathcal{J}^{\zeta}$. Then,

$$\begin{aligned}
J \in \mathcal{J}^{\zeta} &\Leftrightarrow cl_{\zeta}(J^c) = J^c \\
&\Leftrightarrow J^c \cup \zeta(J^c)) = J^c \\
&\Leftrightarrow \zeta(J^c) \subseteq J^c \\
&\Leftrightarrow J \subseteq (\zeta(J^c))^c \\
&\Leftrightarrow (\forall e \in J)(e \notin \zeta(J^c)) \\
&\Leftrightarrow (\forall e \in J)(\exists G \in \mathcal{J}(e))(G^c \cup (J^c)^c = G^c \cup J \notin \mathcal{L}).
\end{aligned}$$

(2) Let $J \notin \mathcal{L}$ and $e \in J$. Put $G = \widetilde{S}$, then G is a neutrosophic $\mathcal{J}$-open set containing e. Since $J \notin \mathcal{L}$ and $G^c \cup J = J$, then we have $G^c \cup J \notin \mathcal{L}$. By part (1), we have $J \in \mathcal{J}^{\zeta}$.

Theorem 4.7. *Let $(S, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPTS}$. Then, $\mathcal{H}^{\mathcal{J}}_{\mathcal{L}} = \{G \cap J : G \in \mathcal{J}$ and $J \notin \mathcal{L}\}$ is a neutrosophic open base for the neutrosophic topology $\mathcal{J}^{\zeta}$ on S.*

Proof. Let $H \in \mathcal{H}^{\mathcal{J}}_{\mathcal{L}}$. Then, there exist $G \in \mathcal{J}$ and $J \notin \mathcal{L}$, where $H = G \cap J$. From Theorem 4.4, we have $\mathcal{J} \subseteq \mathcal{J}^{\zeta}$, hence, $G \in \mathcal{J}^{\zeta}$. Also, from part (2) of Lemma 4.1, $J \in \mathcal{J}^{\zeta}$. Thus, $H = G \cap J \in \mathcal{J}^{\zeta}$. Therefore,

$\mathcal{H}_{\mathcal{L}}^{\mathcal{J}} \subseteq \mathcal{J}^{\zeta}$. Now, let $J \in \mathcal{J}^{\zeta}$ and a neutrosophic point e in J. Then, from part (1) of Lemma 4.1, there exists a neutrosophic open set G containing e such that $G^c \cup J \notin \mathcal{L}$. Put $H = G \cap (G^c \cup J)$. Hence, we get $H \in \mathcal{H}_{\mathcal{L}}^{\mathcal{J}}$, where $e \in H \subseteq J$.

Theorem 4.8. *Let $\Pi : Q(S) \to Q(F)$ be a one-to-one map and $\mathcal{L} \subseteq Q(S)$. If $\mathcal{L}$ is a primal over S and $\Pi(J) \neq \widetilde{F}$ for all $J \in \mathcal{L}$, then $\mathcal{F} = \{\Pi(J) : J \in \mathcal{L}\}$ is a primal over F.*

Proof. (1) Since $\Pi(J) \neq \widetilde{F}$ for all $J \in \mathcal{L}$, then $\widetilde{F} \notin \mathcal{F}$.
(2) Let $J_1 \in \mathcal{F}$ and $J_2 \subseteq J_1$. Then, there exists $J \in \mathcal{L}$ such that $\Pi(J) = J_1$. Put $D = \Pi^{-1}(J_2)$. Since Π is one-to-one, then $D \subseteq J$, which implies $D \in \mathcal{L}$. Hence, $J_2 = \Pi(D)$. Therefore, $J_2 \in \mathcal{F}$.
(3) Let $J_1 \cap J_2 \in \mathcal{F}$. Then, there exists $J, D \in \mathcal{L}$ such that $\Pi(J) = J_1$ and $\Pi(D) = J_2$. Since $J \cap D \subseteq J$, then by primality, $J \cap D \in \mathcal{L}$, hence, $J \in \mathcal{L}$ or $D \in \mathcal{L}$. Thus, $J_1 \in \mathcal{F}$ or $J_2 \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a primal over F.

In general, the following example shows that condition $\Pi(J) \neq \widetilde{S}$ for all $J \in \mathcal{L}$ in the previous theorem is necessary.

Example 4.1. Let $\Pi : Q(S) \to Q(S)$ be a one-to-one map defined by $\Pi(J) = J^c$ for all $J \in Q(S)$. Hence, $\Pi(\widetilde{\emptyset}) = (\widetilde{\emptyset})^c = \widetilde{S}$. Thus, $\widetilde{S} \in \mathcal{F}$, hence, $\mathcal{F}$ is not primal over F.

5. Neutrosophic primal proximity spaces

Here, we present the concept of neutrosophic primal proximity and study several of its master characteristics.

Definition 5.1. *A binary relation $\vdash$ on $Q(S)$ with a neutrosophic primal $\mathcal{L}$ over a nonempty set S is designated as a neutrosophic primal proximity on S if $\vdash$ meets the ensuing criteria:*

(1) If $J_1 \vdash J_2$, then $J_2 \vdash J_1$.
(2) If $J_1 \vdash (J_2 \cup J_3) \Leftrightarrow J_1 \vdash J_2$ or $J_1 \vdash J_3$.
(3) If $J_1 \vdash J_2$, then $J_1^c \in \mathcal{L}$ and $J_2^c \in \mathcal{L}$.
(4) If $(J_1 \cap J_2)^c \in \mathcal{L}$, then $J_1 \vdash J_2$.
(5) If $J_1 \nvdash J_2$, then there exist $F_1, F_2 \in Q(S)$ such that $J_1 \nvdash F_1^c$, $F_2^c \nvdash J_2$, and $(F_1 \cap F_2) = \widetilde{\emptyset}$.

Definition 5.2. *A neutrosophic primal proximity space (briefly NPPS) is a pair $(S, \vdash)$ consisting of a set S and neutrosophic primal proximity relation (briefly NPPR) on a nonempty set S. We write $J_1 \vdash J_2$, if the sets $J_1, J_2 \in Q(S)$ are $\vdash$-related, otherwise we denote $J_1 \nvdash J_2$.*

Corollary 5.1. *Let $\vdash$ be an NPPR on a nonempty set S. Then,*

(1) if $J_2 \nvdash J_1$, then $J_1 \nvdash J_2$;
(2) if $J_1 \nvdash (J_2 \cup J_3) \Leftrightarrow J_1 \nvdash J_2$ and $J_1 \nvdash J_3$;
(3) if there exist $J_1^c \notin \mathcal{L}$ or $J_2^c \notin \mathcal{L}$, then $J_1 \nvdash J_2$;
(4) if $J_1 \nvdash J_2$, then $(J_1 \cap J_2)^c \notin \mathcal{L}$;
(5) if $J_1 \nvdash J_2$, then there exist $F_1, F_2 \in Q(S)$ such that $J_1 \nvdash F_1^c$, $F_2^c \nvdash J_2$, and $(F_1 \cap F_2) = \widetilde{\emptyset}$.

Example 5.1. Let $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$ be a neutrosophic primal over a nonempty set S and $J_1, J_2 \in Q(S)$. We define a binary relation $\vdash$ on $Q(S)$ as follows:

$$J_1 \vdash J_2 \Leftrightarrow J_1^c, J_2^c \in \mathcal{L}.$$

Then, $\vdash$ is an $NPPR$ on S.

Proof. (1) Let $J_1 \vdash J_2$. Then, $J_1^c, J_2^c \in \mathcal{L}$. Hence, $J_2^c, J_1^c \in \mathcal{L}$, thus $J_2 \vdash J_1$.
(2) Let $J_1 \vdash (J_2 \cup J_3)$. Then, $J_1^c, (J_2 \cup J_3)^c \in \mathcal{L}$. Hence, $J_2^c \cap J_3^c \in \mathcal{L}$. By primality, $J_2^c \in \mathcal{L}$ or $J_3^c \in \mathcal{L}$. Hence, $J_1^c, J_2^c \in \mathcal{L}$ or $J_1^c, J_3^c \in \mathcal{L}$. Therefore, $J_1 \vdash J_2$ *or* $J_1 \vdash J_3$. Conversely, let $J_1 \vdash J_2$ or $J_1 \vdash J_3$. Then, $J_1^c, J_2^c \in \mathcal{L}$ or $J_1^c, J_3^c \in \mathcal{L}$. If $J_1^c, J_2^c \in \mathcal{L}$ or $J_1^c, J_3^c \in \mathcal{L}$, then $J_2^c \cap J_3^c \subseteq J_2^c$ or $J_2^c \cap J_3^c \subseteq J_3^c$. Then, by primality, $(J_2 \cup J_3)^c = (J_2^c \cap J_3^c) \in \mathcal{L}$. Thus, $J_1^c, (J_2 \cup J_3)^c \in \mathcal{L}$. Therefore, $J_1 \vdash (J_2 \cup J_3)$.
(3) Obvious by definition.
(4) Let $(J_1 \cap J_2)^c \in \mathcal{L}$. Then, $J_1^c \cup J_2^c \in \mathcal{L}$. Since $J_1^c \subseteq J_1^c \cup J_2^c$ and $J_2^c \subseteq J_1^c \cup J_2^c$, then by primality, $J_1^c, J_2^c \in \mathcal{L}$. Therefore, $J_1 \vdash J_2$.
(5) Let $J_1 \nvdash J_2$. Then, $J_1^c \notin \mathcal{L}$ or $J_2^c \notin \mathcal{L}$. If $J_1^c \notin \mathcal{L}$, then put $F_1 = J_1^c$ and $F_2 = J_1$. Hence, $J_1 \nvdash F_1^c$, $F_2^c \nvdash J_2$. Since $J_1^c \notin \mathcal{L}$ and $J_1^c \subseteq (F_1^c \cup F_2^c) = (F_1 \cap F_2)^c$, then by Corollary 3.1, $(F_1 \cap F_2)^c \notin \mathcal{L}$. However, $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$, hence, $F_1 = \widetilde{\emptyset}$ or $F_2 = \widetilde{\emptyset}$. Hence, $(F_1 \cap F_2) = \widetilde{\emptyset}$. If $J_2^c \notin \mathcal{L}$, then in the same way, we get the required.

Example 5.2. Let $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$ be a neutrosophic primal over a nonempty set S and $J_1, J_2 \in Q(S)$. We define a binary relation $\vdash$ on $Q(S)$ as follows:

$$J_1 \vdash J_2 \Leftrightarrow (J_1 \cap J_2)^c \in \mathcal{L}.$$

Then, $\vdash$ is an $NPPR$ on S.

Proof. (1), (2), (3) and (4) all follow directly by the definition.

(5) Let $J_1 \nvdash J_2$, then $(J_1 \cap J_2)^c \notin \mathcal{L}$. Since $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$, then $J_1 = \widetilde{\emptyset}$ or $J_2 = \widetilde{\emptyset}$. If $J_1 = \widetilde{\emptyset}$, then put $F_1 = J_1^c$, $F_2 = J_1$, and use the same process of part (5) of Example 5.1, hence, we get the required. Also, if $J_2 = \widetilde{\emptyset}$ then put $F_1 = J_2^c$, $F_2 = J_2$, and use the same process of part (5) of Example 5.1, hence, we get the required.

In the following, we define the neutrosophic point-primal proximity operator and study its main properties.

Definition 5.3. *Let $(S, \vdash)$ be an $\mathcal{NPPS}$. Then, we define a map $(\cdot)^{\vdash} : Q(S) \to Q(S)$ as $J^{\vdash}(S, \vdash, \mathcal{J}, \mathcal{L}) = \bigcup\{e \in \mathcal{E}(S) : e \vdash J\}$. To be clear, $J^{\vdash}(S, \vdash, \mathcal{J}, \mathcal{L})$ is denoted as $J^{\vdash}$ for brevity and is designated as the neutrosophic point-primal proximity operator of J with respect to $\mathcal{J}$, $\vdash$ and $\mathcal{L}$. Moreover, $J^{\vdash}$ It is termed the neutrosophic point-primal proximity of J.*

Theorem 5.1. *Let S be a nonempty set, $e \in \mathcal{E}(S)$ and $J_1, J_2 \in Q(S)$, where $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$. Then, the followings hold:*

(1) If $J_1^c \notin \mathcal{L}$, then $J_1 \cap J_2^{\vdash} = \widetilde{\emptyset}$.
(2) $e \vdash J_1$ for each $e \in J_1$.
(3) If $J_1 \nvdash J_2$, then $J_1 \cap J_2 = \widetilde{\emptyset}$.

Proof. (1) Let $J_1^c \notin \mathcal{L}$. Since $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$, then $J_1 = \widetilde{\emptyset}$. Therefore, $J_1 \cap J_2^{\vdash} = \widetilde{\emptyset}$.
(2) Let $e \in J_1$. Then, $(e \cap J_1)^c = e^c \in \mathcal{L}$, because $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$. By part (4) of Definition 5.1, $e \vdash J_1$ for each $e \in J_1$.
(3) Let $J_1 \nvdash J_2$. Assume that $J_1 \cap J_2 \neq \widetilde{\emptyset}$, then there exists a neutrosophic point $e \in J_1 \cap J_2$. Hence, $(J_1 \cap J_2)^c \neq \widetilde{S}$. Thus, $(J_1 \cap J_2)^c \in \mathcal{L}$, because $\mathcal{L} = Q(S) \setminus \{\widetilde{S}\}$. Hence, $J_1 \vdash J_2$, and this is a contradiction the fact $J_1 \nvdash J_2$. Therefore, $J_1 \cap J_2 = \widetilde{\emptyset}$.

Corollary 5.2. *Let S be a nonempty set, $e \in \mathcal{E}(S)$, where $\mathcal{L} = \mathcal{Q}(S) \setminus \{\widetilde{S}\}$. Then,*

(1) $e \vdash \widetilde{S}$ for each $e \in \widetilde{S}$.
(2) $(\widetilde{S})^{\vdash} = \widetilde{S}$.

Lemma 5.1. *Let $\mathcal{L}$ be a neutrosophic primal on $S \neq \emptyset$. If $J_1 \vdash J_2$, $J_1 \subseteq D_1$ and $J_2 \subseteq D_2$, then $D_1 \vdash D_2$.*

Proof. Let $J_1 \vdash J_2$, $J_1 \subseteq D_1$ and $J_2 \subseteq D_2$. Suppose that $D_1 \nvdash D_2$, then by part (5) of Definition 5.1, there exist $F_1, F_2 \in \mathcal{Q}(S)$ such that $D_1 \nvdash F_1^c$, $F_2^c \nvdash D_2$, and $(F_1 \cap F_2) = \widetilde{\emptyset}$. Hence, $F_1 = \widetilde{\emptyset}$ or $F_2 = \widetilde{\emptyset}$. If $F_1 = \widetilde{\emptyset}$, then by $D_1 \nvdash F_1^c$, and by part (4) of Corollary 5.1, $(D_1 \cap F_1^c)^c \notin \mathcal{L}$. Thus, $D_1^c = D_1^c \cup F_1 = (D_1 \cap F_1^c)^c \notin \mathcal{L}$. Since $J_1 \subseteq D_1$, then $D_1^c \subseteq J_1^c$, and by primality, $J_1^c \notin \mathcal{L}$. Hence, by part (3) of Corollary 5.1, $J_1 \nvdash J_2$, and this is a contradiction the fact $J_1 \vdash J_2$. Therefore, $D_1 \vdash D_2$. If $F_2 = \widetilde{\emptyset}$, then we use the same previous way to get the required.

Lemma 5.2. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$ and $J_1, J_2 \in \mathcal{Q}(S)$. If $J_2 \nvdash J_1$, then $J_1^{\vdash} \subseteq J_2^c$.*

Proof. Suppose that $J_1^{\vdash} \cap J_2 \neq \widetilde{\emptyset}$. Then, there exists a neutrosophic point e such that $e \in J_1^{\vdash} \cap J_2$. Hence, $e \vdash J_1$ and $e \subseteq J_2$, then by Lemma 5.1, $J_2 \vdash J_1$, which is a contradiction. Therefore, $J_1^{\vdash} \subseteq J_2^c$.

Theorem 5.2. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$ and $J_1, J_2 \in \mathcal{Q}(S)$. If $J_2 \nvdash J_1$, then $J_2 \nvdash J_1^{\vdash}$.*

Proof. Let $J_2 \nvdash J_1$. Then, by part (5) of Definition 5.1, there exist $F_1, F_2 \in \mathcal{Q}(S)$ such that $J_2 \nvdash F_1^c$, $F_2^c \nvdash J_1$, and $(F_1 \cap F_2) = \widetilde{\emptyset}$. Hence, $F_1 = \widetilde{\emptyset}$ or $F_2 = \widetilde{\emptyset}$. If $F_1 = \widetilde{\emptyset}$, then by $J_2 \nvdash F_1^c$, and by part (4) of Corollary 5.1, $(J_2 \cap F_1^c)^c \notin \mathcal{L}$. Thus, $J_2^c = J_2^c \cup F_1 = (J_2 \cap F_1^c)^c \notin \mathcal{L}$. Hence, by part (3) of Corollary 5.1, $J_2 \nvdash J_1^{\vdash}$. If $F_2 = \widetilde{\emptyset}$, then by $F_2^c \nvdash J_1$, and Lemma 5.2, $J_1^{\vdash} \subseteq F_2$. Hence, $(J_1^{\vdash})^c = \widetilde{S} \notin \mathcal{L}$, and by part (3) of Corollary 5.1, $J_2 \nvdash J_1^{\vdash}$.

Theorem 5.3. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$ and $J_1, J_2 \in \mathcal{Q}(S)$. Then, the following statements hold:*

(1) $(\widetilde{\emptyset})^{\vdash} = \widetilde{\emptyset}$.
(2) If $J_1 \subseteq J_2$, then $J_1^{\vdash} \subseteq J_2^{\vdash}$.
(3) $(J_1 \cap J_2)^{\vdash} \subseteq J_1^{\vdash} \cap J_2^{\vdash}$.
(4) $(J_1 \cup J_2)^{\vdash} = J_1^{\vdash} \cup J_2^{\vdash}$.
(5) If $J_1^c \notin \mathcal{L}$, then $J_1^{\vdash} = \widetilde{\emptyset}$.
(6) $(J_1^{\vdash})^{\vdash} \subseteq J_1^{\vdash}$.

Proof. (1) let $e \in \mathcal{E}(S)$. Then, by part (3) of Definition 5.1, $e \nvdash \widetilde{\emptyset}$, because $(\widetilde{\emptyset})^c = \widetilde{S} \notin \mathcal{L}$. Hence, $(\widetilde{\emptyset})^{\vdash} = \widetilde{\emptyset}$.
(2) Let $J_1 \subseteq J_2$ and $e \in J_1^{\vdash}$. Then, $e \vdash J_1$. Since $J_1 \subseteq J_2$, and by Lemma 5.1, $e \vdash J_2$. Hence, $e \in J_2^{\vdash}$. Therefore, $J_1^{\vdash} \subseteq J_2^{\vdash}$.
(3) Let $J_1, J_2 \in \mathcal{Q}(S)$. Since $J_1 \cap J_2 \subseteq J_1$ and $J_1 \cap J_2 \subseteq J_2$, then by part (2), $(J_1 \cap J_2)^{\vdash} \subseteq J_1^{\vdash}$ and $(J_1 \cap J_2)^{\vdash} \subseteq J_2^{\vdash}$. Thus, $(J_1 \cap J_2)^{\vdash} \subseteq J_1^{\vdash} \cap J_2^{\vdash}$.
(4) Let $J_1, J_2 \in \mathcal{Q}(S)$. Since $J_1, J_2 \subseteq (J_1 \cup J_2)$ and by part (2), we have $J_1^{\vdash} \subseteq (J_1 \cup J_2)^{\vdash}$ and $J_2^{\vdash} \subseteq (J_1 \cup J_2)^{\vdash}$. Hence, $J_1^{\vdash} \cup J_2^{\vdash} \subseteq (J_1 \cup J_2)^{\vdash}$. Conversely, let $e \in (J_1 \cup J_2)^{\vdash}$. Then, $e \vdash J_1 \cup J_2$, and by part (2) of Definition 5.1, $e \vdash J_1$ or $e \vdash J_2$. Hence, $e \in J_1^{\vdash}$ or $e \in J_2^{\vdash}$, which implies $e \in J_1^{\vdash} \cup J_2^{\vdash}$. Thus, $(J_1 \cup J_2)^{\vdash} \subseteq J_1^{\vdash} \cup J_2^{\vdash}$. Therefore, $(J_1 \cup J_2)^{\vdash} = J_1^{\vdash} \cup J_2^{\vdash}$.
(5) Let $J_1^c \notin \mathcal{L}$. Then, by part (3) of Corollary 5.1, $e \nvdash J_1$. Hence, $e \nvdash J_1$ for each $e \in \mathcal{E}(S)$. Therefore, $J_1^{\vdash} = \widetilde{\emptyset}$.

(6) Let $J_1 \in \mathcal{Q}(S)$ and $e \notin J_1^{\vdash}$. Then, $e \nvdash J_1$, and by Theorem 5.2, $e \nvdash J_1^{\vdash}$. Hence, $e \nvdash (J_1^{\vdash})^{\vdash}$. Thus, $(J_1^{\vdash})^{\vdash} \subseteq J_1^{\vdash}$.

6. Neutrosophic proximal closed sets and neutrosophic topologies

This section introduces neutrosophic proximal closed sets. Additionally, various results relating $\mathcal{NPPS}$ and $\mathcal{NPTS}$ are established using these neutrosophic proximal closed sets.

Definition 6.1. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$. Then, a subset J of $\mathcal{Q}(S)$ is termed neutrosophic proximity closed set (briefly $\mathcal{NPCS}$) if, and only if, $e \vdash J$ means $e \in J$.*

Lemma 6.1. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$. Then, the following statements hold:*

(1) Arbitrary intersections of neutrosophic proximity closed sets are neutrosophic proximity closed.
(2) Finite unions of neutrosophic proximity closed sets are neutrosophic proximity closed.

Proof. (1) Let $\{J_j : j \in \Lambda\}$ be a family of $\mathcal{NPCS}$. Let $e \vdash \bigcap\{J_j : j \in \Lambda\}$. Since $\bigcap J_j \subseteq J_j$ for each $j \in \Lambda$, then by Lemma 5.1, $e \vdash J_j$ for each $j \in \Lambda$. Since J_j is neutrosophic proximity closed for each $e \in J_j$, then $e \in J_j$ for each $j \in \Lambda$. Hence, $e \in \bigcap\{J_j : j \in \Lambda\}$, which implies $\bigcap\{J_j : j \in \Lambda\}$, is neutrosophic proximity closed.
(2) Let J_j be a neutrosophic proximity closed set for each $j \in \{1, 2, 3, \ldots, n\}$. Let $e \vdash \bigcup_{j=1}^{n} J_j$. Then, by part (2) of Definition 5.1 and $e \vdash [J_1 \cup (\bigcup_{j=2}^{n} J_j)]$, we have $e \vdash J_1$ or $e \vdash \bigcup_{j=2}^{n} J_j$. If $e \vdash J_1$, then $e \vdash \bigcup_{j=1}^{n} J_j$ (because J_1 is neutrosophic proximity closed, hence, $e \in J_1$, which implies $e \in \bigcup_{j=1}^{n} J_j$. Assume that $e \nvdash J_1$, then $e \vdash \bigcup_{j=2}^{n} J_j$. Then, we use the same process until $e \vdash (J_{n-1} \cup J_n)$. Hence, $e \vdash J_{n-1}$ or $e \vdash J_n$, and in both cases we have $e \vdash \bigcup_{j=1}^{n} J_j$. Therefore, $\bigcup_{j=1}^{n} J_j$ is neutrosophic proximity closed.

Theorem 6.1. *The family of complements of all $\mathcal{NPCS}$s of $(S, \vdash, \mathcal{J}, \mathcal{L})$ forms a neutrosophic topology on S. This neutrosophic topology is denoted by $\mathcal{J}^{\vdash}$.*

Proof. For condition (1): From part (1) of Corollary 5.2 and $e \vdash \widetilde{S}$ for each $e \in \widetilde{S}$, then $\widetilde{S}$ is a neutrosophic proximity closed set. Thus, $\widetilde{\emptyset} \in \mathcal{J}^{\vdash}$. Also, $e \vdash \widetilde{\emptyset}$ for each $e \in \widetilde{\emptyset}$, hence, $\widetilde{\emptyset}$ is a neutrosophic proximity closed set. Thus, $\widetilde{S} \in \mathcal{J}^{\vdash}$.

For condition (2): Let $J_1, J_2 \in \mathcal{J}^{\vdash}$. Then, J_1^c and J_2^c are neutrosophic proximity closed sets. By part (2) of Lemma 6.1, $J_1^c \cup J_2^c$ is neutrosophic proximity closed. Since $(J_1 \cap J_2)^c = J_1^c \cup J_2^c$, then $J_1 \cap J_2 \in \mathcal{J}^{\vdash}$.

For condition (3): Let $\{J_j : j \in \Lambda\} \subseteq \mathcal{J}^{\vdash}$. Then, J_j^c for each $j \in \Lambda$ is a neutrosophic proximity closed set. By part (1) of Lemma 6.1, $\bigcap\{J_j^c : j \in \Lambda\}$ is neutrosophic proximity closed. Since $[\bigcup\{J_j : j \in \Lambda\}]^c = [\bigcap\{J_j^c : j \in \Lambda\}]$, then $\bigcup\{J_j : j \in \Lambda\} \in \mathcal{J}^{\vdash}$.

Theorem 6.2. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$ and $J \subseteq J^{\vdash}$. The set $J^{\vdash}$ is the neutrosophic closure of the neutrosophic set J in which the neutrosophic closure is taken with respect to the neutrosophic topology $\mathcal{J}^{\vdash}$ and symbolized by $cl_{\mathcal{J}^{\vdash}}(J)$.*

Proof. Let e be a neutrosophic point such that $e \in J^{\vdash}$. Then, $e \vdash J$. Since $J \subseteq cl_{\mathcal{J}^{\vdash}}(J)$, and by Lemma 5.1, we have $e \vdash cl_{\mathcal{J}^{\vdash}}(J)$. Since $cl_{\mathcal{J}^{\vdash}}(J)$ is neutrosophic proximity closed, then $e \in cl_{\mathcal{J}^{\vdash}}(J)$. Hence, $J^{\vdash} \subseteq cl_{\mathcal{J}^{\vdash}}(J)$. Conversely, Let e be a neutrosophic point such that $e \notin J^{\vdash}$. Then, $e \nvdash J$. Assume that $e \vdash cl_{\mathcal{J}^{\vdash}}(J)$. Since $J \subseteq cl_{\mathcal{J}^{\vdash}}(J)$, and by Lemma 5.1, we have $e \vdash J$. Since $J \subseteq J^{\vdash}$. Hence, by Lemma 5.1, $e \vdash J^{\vdash}$. However, this is a contradictions the fact $e \nvdash J^{\vdash}$. Hence, $e \nvdash cl_{\mathcal{J}^{\vdash}}(J)$. Thus, $cl_{\mathcal{J}^{\vdash}}(J) \subseteq J^{\vdash}$. Therefore, $cl_{\mathcal{J}^{\vdash}}(J) = J^{\vdash}$.

Definition 6.2. *The operator $\Xi : \mathcal{Q}(S) \to \mathcal{Q}(S)$ is a neutrosophic Kuratowski closure operator if it satisfies the following conditions:*

(1) $\Xi(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) For each $J_1 \in \mathcal{Q}(S)$, $J_1 \subseteq \Xi(J_1)$.
(3) For each $J_1, J_2 \in \mathcal{Q}(S)$, $\Xi(J_1 \cup J_2) = \Xi(J_1) \cup \Xi(J_2)$.
(4) For each $J_1 \in \mathcal{Q}(S)$, $\Xi(\Xi(J_1)) = \Xi(J_1)$.

Theorem 6.3. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$, where $\mathcal{L} = \mathcal{Q}(S) \setminus \{\widetilde{S}\}$. Then, the neutrosophic operator $J^{\vdash} = \bigcup\{e \in \mathcal{E}(S) : e \vdash J\}$ on a neutrosophic primal proximity space $(S, \vdash, \mathcal{J}, \mathcal{L})$ is a neutrosophic Kuratowski closure operator.*

Proof. (1) By part (1) of Theorem 5.3, $(\widetilde{\emptyset})^{\vdash} = \widetilde{\emptyset}$.
(2) If $e \in J_1$, then by part (2) of Theorem 5.1, $e \vdash J_1$. Hence, $e \in J_1^{\vdash}$. Therefore, $J_1 \subseteq J_1^{\vdash}$.
(3) By part (4) of Theorem 5.3, $(J_1 \cup J_2)^{\vdash} = J_1^{\vdash} \cup J_2^{\vdash}$ for each $J_1, J_2 \in \mathcal{Q}(S)$.
(4) By part (6) of Theorem 5.3, $(J_1^{\vdash})^{\vdash} \subseteq J_1^{\vdash}$. On the other side, let $e \notin (J_1^{\vdash})^{\vdash}$. Then, $e \nvdash J_1^{\vdash}$. By part (4) of Corollary 5.1, $(e \cap J_1^{\vdash})^c \notin \mathcal{L}$. Since $\mathcal{L} = \mathcal{Q}(S) \setminus \{\widetilde{S}\}$, then $(e \cap J_1^{\vdash})^c = \widetilde{S}$, which implies $e \cap J_1^{\vdash} = \widetilde{\emptyset}$. Hence, $e = \widetilde{\emptyset}$ or $J_1^{\vdash} = \widetilde{\emptyset}$, and by part (3) of Corollary 5.1, $e \notin J_1^{\vdash}$. Thus, $J_1^{\vdash} \subseteq (J_1^{\vdash})^{\vdash}$. Therefore, $(J_1^{\vdash})^{\vdash} = J_1^{\vdash}$ for each $J_1 \in \mathcal{Q}(S)$.

Theorem 6.4. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$. For any $J \in \mathcal{Q}(S)$, we define a map $cl_{\Upsilon} : \mathcal{Q}(S) \to \mathcal{Q}(S)$ by $cl_{\Upsilon}(J) = J \cup J^{\vdash}$ satisfies neutrosophic Kuratowski closure axioms.*

Proof. (1) By part (1) of Theorem 5.3, $cl_{\Upsilon}(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(2) Let $J \in \mathcal{Q}(S)$. Since $cl_{\Upsilon}(J) = J \cup J^{\vdash}$, then $J \subseteq cl_{\Upsilon}(J)$.
(3) Let $J_1, J_2 \in \mathcal{Q}(S)$. Then, by part (4) of Theorem 5.3, we get

$$\begin{aligned} cl_{\Upsilon}(J_1 \cup J_2) &= (J_1 \cup J_2) \cup (J_1 \cup J_2)^{\vdash} \\ &= (J_1 \cup J_2) \cup (J_1^{\vdash} \cup J_2^{\vdash}) \\ &= (J_1 \cup J_1^{\vdash}) \cup (J_2 \cup J_2^{\vdash}) \\ &= cl_{\Upsilon}(J_1) \cup cl_{\Upsilon}(J_2). \end{aligned}$$

(4) Let $J \in \mathcal{Q}(S)$. Then, by parts (4) and (6) of Theorem 5.3, we get

$$\begin{aligned} cl_{\Upsilon}(cl_{\Upsilon}(J)) &= cl_{\Upsilon}(J) \cup [cl_{\Upsilon}(J)]^{\vdash} \\ &= (J \cup J^{\vdash}) \cup [J \cup J^{\vdash}]^{\vdash} \\ &= (J \cup J^{\vdash}) \cup [J^{\vdash} \cup (J^{\vdash})^{\vdash}] \\ &= (J \cup J^{\vdash}) \cup J^{\vdash} \\ &= J \cup J^{\vdash} \\ &= cl_{\Upsilon}(J). \end{aligned}$$

Theorem 6.5. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$. Then, the collection $\mathcal{J}^{\Upsilon}_{\mathcal{L}} = \{J \in \mathcal{Q}(S) : cl_{\Upsilon}(J^{c}) = J^{c}\}$ is a neutrosophic topology in S induced by neutrosophic topology $\mathcal{J}$, neutrosophic primal $\mathcal{L}$ and neutrosophic primal proximity $\vdash$ on a nonempty set S.*

Proof. From part (1) of Theorem 5.3, $cl_{\Upsilon}((\widetilde{S})^{c}) = (\widetilde{S})^{c} \cup ((\widetilde{S})^{c})^{\vdash} = \widetilde{\emptyset} \cup (\widetilde{\emptyset})^{\vdash} = \widetilde{\emptyset} \cup \widetilde{\emptyset} = \widetilde{\emptyset} = (\widetilde{S})^{c}$. Also, $cl_{\Upsilon}((\widetilde{\emptyset})^{c}) = (\widetilde{\emptyset})^{c} \cup ((\widetilde{\emptyset})^{c})^{\vdash} = \widetilde{S} \cup (\widetilde{S})^{\vdash} = \widetilde{S} = (\widetilde{\emptyset})^{c}$. Therefore, $\widetilde{\emptyset}, \widetilde{S} \in \mathcal{J}^{\vdash}_{\mathcal{L}}$.

Now, let $J_1, J_2 \in \mathcal{J}^{\vdash}_{\mathcal{L}}$. Then, $cl_{\Upsilon}(J_1^{c}) = J_1^{c}$ and $cl_{\Upsilon}(J_2^{c}) = J_2^{c}$. Now, $cl_{\Upsilon}((J_1 \cap J_2)^{c}) = (J_1 \cap J_2)^{c} \cup [(J_1 \cap J_2)^{c}]^{\vdash} = J_1^{c} \cup J_2^{c} \cup [J_1^{c} \cup J_2^{c}]^{\vdash}$. From part (4) of Theorem 5.3, $cl_{\Upsilon}((J_1 \cap J_2)^{c}) = J_1^{c} \cup J_2^{c} \cup (J_1^{c})^{\vdash} \cup (J_2^{c})^{\vdash} = cl_{\Upsilon}(J_1^{c}) \cup cl_{\Upsilon}(J_1^{c}) = J_1^{c} \cup J_2^{c} = (J_1 \cap J_2)^{c}$. Thus, $J_1 \cap J_2 \in \mathcal{J}^{\vdash}_{\mathcal{L}}$. Now, let $\{J_{\xi} : \xi \in \Lambda\} \subseteq \mathcal{J}^{\vdash}_{\mathcal{L}}$. Then, we have always $(\bigcup_{\xi \in \Lambda} J_{\xi})^{c} \subseteq cl_{\Upsilon}((\bigcup_{\xi \in \Lambda} J_{\xi})^{c})$. Conversely, $cl_{\Upsilon}(J_{\xi}^{c}) = J_{\xi}^{c}$ for each $\xi \in \Lambda$, hence, $cl_{\Upsilon}[(\bigcup_{\xi \in \Lambda} J_{\xi})^{c}] = (\bigcup_{\xi \in \Lambda} J_{\xi})^{c} \cup [(\bigcup_{\xi \in \Lambda} J_{\xi})^{c}]^{\vdash} = (\bigcap_{\xi \in \Lambda} J_{\xi}^{c}) \cup (\bigcap_{\xi \in \Lambda} J_{\xi}^{c})^{\vdash} \subseteq (\bigcap_{\xi \in \Lambda} J_{\xi}^{c}) \cup (\bigcap_{\xi \in \Lambda} (J_{\xi}^{c})^{\vdash}) = \bigcap_{\xi \in \Lambda} (J_{\xi}^{c} \cup (J_{\xi}^{c})^{\vdash}) = \bigcap_{\xi \in \Lambda} cl_{\Upsilon}(J_{\xi}^{c}) = \bigcap_{\xi \in \Lambda} J_{\xi}^{c} = (\bigcup_{\xi \in \Lambda} J_{\xi})^{c}$. Hence, $cl_{\Upsilon}((\bigcup_{\xi \in \Lambda} J_{\xi})^{c}) = (\bigcup_{\xi \in \Lambda} J_{\xi})^{c}$. Thus, $\bigcup_{\xi \in \Lambda} J_{\xi} \in \mathcal{J}^{\vdash}_{\mathcal{L}}$. Therefore, $\mathcal{J}^{\vdash}_{\mathcal{L}}$ is a neutrosophic topology in S induced by neutrosophic topology $\mathcal{J}$ and neutrosophic primal $\mathcal{L}$.

For simplicity, we will use the symbol $\mathcal{J}^{\Upsilon}$ instead of $\mathcal{J}^{\Upsilon}_{\mathcal{L}}$ if there is no confusion.

Theorem 6.6. *Let $(S, \vdash, \mathcal{J}, \mathcal{L})$ be an $\mathcal{NPPTS}$ and $J, F \in \mathcal{Q}(S)$. Then, the following statements hold:*

(1) $J \nvdash F \Leftrightarrow J \nvdash cl_{\Upsilon}(F)$.
(2) $cl_{\Upsilon}(J^{\vdash}) = J^{\vdash}$.
(3) $cl_{\Upsilon}(J^{\vdash}) = [cl_{\Upsilon}(J)]^{\vdash}$.

Proof. (1) Let $J \nvdash F$. Then, by Theorem 5.2, we get $J \nvdash F^{\vdash}$. By part (2) of Definition 5.1, $J \nvdash (F \cup F^{\vdash}) = cl_{\Upsilon}(F) \Leftrightarrow J \nvdash F$ *and* $J \nvdash F^{\vdash}$.
(2) By part (6) of Theorem 5.3, we have $cl_{\Upsilon}(J^{\vdash}) = J^{\vdash} \cup (J^{\vdash})^{\vdash} = J^{\vdash}$.
(3) By part (4) of Theorem 5.3, we have $cl_{\Upsilon}(J^{\vdash}) = J^{\vdash} \cup (J^{\vdash})^{\vdash} = (J \cup J^{\vdash})^{\vdash} = [cl_{\Upsilon}(J)]^{\vdash}$.

7. Conclusions and further work

Our work concentrated on creating a new abstract structure called the "neutrosophic primal topology". Then, we have shown some comparisons between neutrosophic primal and neutrosophic grill. After that, we have introduced new operators via neutrosophic primal and neutrosophic primal proximity spaces, respectively, and studied their basic properties. In addition, we have presented a new neutrosophic topology induced by the neutrosophic primal closure operator.We also have defined the neutrosophic point primal proximity operator and introduced many features of a neutrosophic primal proximity space. Furthermore, we have defined a neutrosophic proximal closed set. Finally, we have introduced different facts between a neutrosophic primal and a neutrosophic primal proximity spaces through neutrosophic proximal closed sets and related concepts.

The results we have obtained in this work are early. Further research into the features of the neutrosophic primal topology may provide more insights. This work intends to contribute to the direction of merging neutrosophic primal structures with other fields of sciences.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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