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Research article

Global classical solution to three-component Cahn-Hilliard phase-field model

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Abstract: The main purpose of this paper was to study the global existence for the Cauchy problem for three-component Cahn-Hilliard phase-field model in 2D whole space. We first constructed the local classical solutions, then by combining some a priori estimates and the continuity argument, the local classical solutions were extended step by step to all $t > 0$.

Keywords: global existence; Cauchy problem; three-component Cahn-Hilliard equation; local solution; continuity argument
Mathematics Subject Classification: 35A01, 35B45, 35K55

1. Introduction

The Cahn-Hilliard flow, one of the most well-known gradient flows, can be used to describe spinodal decomposition and phase separation in binary fluids [1,2]. To date, numerous classical works have been conducted on the mathematical analysis and numerical approximation of the Cahn-Hilliard equation (see, e.g., Temam [3] and Elliott and Zheng [4] considered the existence of global weak solution, Sell and You [5] and Dlotko [6] studied the global dynamics, Gilardi et al. [7] and Liu and Wu [8] invistigated the properties of the equation with dynamic boundary conditions, Schimperna and Pawlow [9] and Yin [10] considered the equation with nonconstant mobility, Cherfils et al. [11] studied the equation with logarithmic potentials). However, most of these studies focus on two-phase models. When three or more phase components are involved in a phase-field system, the interactions between these components must be taken into account; relevant works can be found in [12–14].

In this paper, we consider a three-component Cahn-Hilliard phase-field system whose free energy is defined as follows:

$$E(\phi_1, \phi_2, \phi_3) = \int \left(\frac{12}{\epsilon} F(\phi_1, \phi_2, \phi_3) + \frac{3}{8} \sum_{i=1}^{3} \Sigma_i \epsilon |\nabla \phi_i|^2 \right) dx, \tag{1.1}$$

where the i-th phase-field variable, denoted by ϕ_i ($i = 1, 2, 3$), represents the volume fraction of the i-th component in the fluid mixture and satisfies

$$\phi_i = \begin{cases} 1 & \text{inside the } i\text{th component,} \\ 0 & \text{outside the } i\text{th component.} \end{cases} \tag{1.2}$$

From (1.2), we easily deduce that the variables ϕ_i ($i = 1, 2, 3$) are nonnegative functions. Moreover, in the phase-field approach, a diffuse interface of thickness ϵ is modeled by smooth transitions of the phase-field variables from 0 to 1. The variables ϕ_1, ϕ_2, and ϕ_3 are governed by the constraint

$$\sum_{i=1}^{3} \phi_i = 1.$$

The parameter $\epsilon > 0$ denotes the interface width, and the coefficient Σ_i ($i = 1, 2, 3$) represents the "spreading" coefficient of the i-th component at the interface between the j-th and k-th phases. We note that, in the physical literature, the coefficients Σ_i are not necessarily assumed to be positive. If one of the coefficients Σ_i is negative, the spreading is referred to as total; otherwise, it is partial. For the total spreading case, to ensure the well-posedness of the system, the following conditions are assumed:

$$\Sigma_1\Sigma_2 + \Sigma_1\Sigma_3 + \Sigma_2\Sigma_3 > 0, \quad \Sigma_i + \Sigma_j > 0, \quad \forall i \neq j.$$

Furthermore, the nonlinear potential F is given by

$$F = \sigma_{12}\phi_1^2\phi_2^2 + \sigma_{13}\phi_1^2\phi_3^2 + \sigma_{23}\phi_2^2\phi_3^2 + \phi_1\phi_2\phi_3(\Sigma_1\phi_1 + \Sigma_2\phi_2 + \Sigma_3\phi_3) + 3\Lambda\phi_1^2\phi_2^2\phi_3^2, \tag{1.3}$$

where Λ is a nonnegative constant and σ_{12}, σ_{13}, σ_{23} are the three-surface tension parameters, which satisfy the following relations:

$$\Sigma_i = \sigma_{ij} + \sigma_{ik} - \sigma_{jk}, \quad i = 1, 2, 3.$$

We assume that the time evolution of ϕ_i is governed by the gradient of the energy E with respect to the H^{-1} gradient flow, i.e., the Cahn-Hilliard dynamics. Based on the free energy (1.1), we can readily derive the following three-component Cahn-Hilliard phase-field equations:

$$\begin{cases} \phi_{it} = \dfrac{M}{\Sigma_i}\Delta\mu_i, \quad i = 1, 2, 3, \\ \mu_i = -\dfrac{3}{4}\epsilon\Sigma_i\Delta\phi_i + \dfrac{12}{\epsilon}f_i + \beta_L, \quad i = 1, 2, 3, \end{cases} \tag{1.4}$$

where

$$f_i = \partial_i F, \quad M > 0$$

is the constant mobility, and β_L is the Lagrange multiplier imposed to enforce the hyperplane constraint

$$\phi_1 + \phi_2 + \phi_3 = 1.$$

It can be derived as

$$\beta_L = -\frac{4\Sigma_T}{\epsilon}\left(\frac{1}{\Sigma_1}f_1 + \frac{1}{\Sigma_2}f_2 + \frac{1}{\Sigma_3}f_3\right)$$

with

$$\frac{3}{\Sigma_T} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}.$$

Remark 1.1. *It is worth noting that the expressions for the chemical potentials μ_i and the Lagrange multiplier β_L involve specific coefficients (e.g., $\frac{3}{4}$, $\frac{12}{\varepsilon}$, and $\frac{4\Sigma_T}{\epsilon}$). These parameters can be derived from the free energy functional under the constraint*

$$\phi_1 + \phi_2 + \phi_3 = 1,$$

as detailed in [15].

Remark 1.2. *It should be emphasized that the system (1.4) is completely equivalent to the following formulation involving only two order parameters:*

$$\begin{cases} \phi_{it} = \dfrac{M}{\Sigma_i}\Delta\mu_i, \quad i = 1, 2, \\ \mu_i = -\dfrac{3}{4}\epsilon\Sigma_i\Delta\phi_i + \dfrac{12}{\epsilon}f_i + \beta_L, \quad i = 1, 2, \\ \phi_3 = 1 - \phi_1 - \phi_2, \\ \dfrac{\mu_3}{\Sigma_3} = -\left(\dfrac{\mu_1}{\Sigma_1} + \dfrac{\mu_2}{\Sigma_2}\right). \end{cases}$$

The proof is omitted here, as it closely follows that presented in [16, Theorem 3.1].

In [17], within the framework of non-equilibrium thermodynamics, Elliott and Luckhaus conducted a mathematical analysis of an N-component Cahn-Hilliard phase-field model, which can be used to model the isothermal phase separation of an ideal mixture of N ($N \geq 2$) components occupying an isolated region. The authors proved the existence of a (suitably defined) global weak solution for the case of a singular potential. Boyer and Lapuerta [15] considered the existence and uniqueness of global weak solutions for a three-component Cahn-Hilliard phase-field model. Conti et al. Miranville [18] investigated the well-posedness and asymptotic behavior (in terms of finite-dimensional attractors) of an N-component Cahn-Hilliard phase-field model equipped with dynamic boundary conditions. Additionally, Garcke [19] introduced a global version of L^p-estimates for gradients of nonlinear elliptic systems to overcome the difficulties arising from logarithmic singularity and a quadratic term, thereby establishing the existence of solutions for an N-component Cahn-Hilliard phase-field model. The authors in [20] improved the theoretical results in [17], establishing several well-posedness and regularity results for an N-component Cahn-Hilliard phase-field model with a singular potential. Furthermore, Abels et al. [21] studied the existence of strong solutions for a diffuse interface model for multi-phase flows of N incompressible, viscous Newtonian fluids with different densities; this model can be regarded as the multicomponent Cahn-Hilliard equations coupled with hydrodynamic flows. We also note that there have been successful attempts to develop numerical algorithms, such as the nonlinear method [15, 22], the invariant energy quadratization method [23, 24], the scalar auxiliary variable method [16, 25], and convex-concave decomposition [26], among others.

A Cauchy problem in mathematics seeks the solution to a partial differential equation (PDE) or system of PDEs that satisfies specific conditions prescribed on a hypersurface within the domain. We remark that the study of the Cauchy problem for Cahn-Hilliard models is also of significant interest. There exist several classical results related to this topic (see, e.g., Bricmont et al. [27] considered the stability of Cahn-Hilliard fronts in the whole space, Caffarelli and Muller [28] showed an L^∞ bound of the solutions, Liu et al. [29] studied the global existence and asymptotics of strong solutions, Cholewa and Rodriguez-Bernal [30] exhibited the dissipative mechanism of the in $H^1(\mathbb{R}^N)$, Dlotko et al. [31] studied the properties of Cauchy problem of viscous Cahn-Hilliard equation, Duan and Zhao [32] and Li and Liu [33] considered the properties of fractional Cahn-Hilliard equation and coupled Cahn-Hilliard equations, respectively). To the best of our knowledge, no references have addressed the Cauchy problem for the three-component Cahn-Hilliard model. Therefore, in this paper, we consider the Cauchy problem of equations (1.4) and supplement the initial conditions as follows:

$$\phi_i|_{(t=0)} = \phi_{i,0}, \quad i = 1, 2, 3. \tag{1.5}$$

In what follows, we discuss the global existence of solutions to problems (1.4) and (1.5). Notably, the three-phase nature of the nonlinear energy functional poses substantial challenges to proving global existence when applying Hoff and Smoller's method [34–36]. To overcome these difficulties, we employ the method of successive approximations and establish enhanced regularity estimates for ϕ_i $(i = 1, 2, 3)$.

We now present the main theorem of this paper:

Theorem 1.3. *Let $R > 0$ be an arbitrary given constant. If*

$$(\phi_{1,0}, \phi_{2,0}, 1 - \phi_{3,0}) \in \left(L^\infty \cap L^1(\mathbb{R}^2)\right)^3$$

with

$$\|\phi_{1,0}\|_{L^\infty} + \|\phi_{2,0}\|_{L^\infty} + \|1 - \phi_{3,0}\|_{L^\infty} \leq R, \tag{1.6}$$

and $\|\phi_{1,0}\|_{L^1} + \|\phi_{2,0}\|_{L^1}$ is sufficiently small, then the Cauchy problems (1.4) and (1.5) admits a unique global classical solution

$$(\phi_1, \phi_2, 1 - \phi_3) \in \left(C^{1,4}\left((0, \infty) \times \mathbb{R}^2\right)\right)^3$$

that satisfies

$$\|\phi_1\|_{L^\infty} + \|\phi_2\|_{L^\infty} + \|1 - \phi_3\|_{L^\infty} \leq 2R. \tag{1.7}$$

Remark 1.4. *The main purpose of this paper is to study the global classical solution to the Cauchy problem of three-component Cahn-Hilliard phase-field system in $\mathbb{R}^2$. It is worth pointing out that in the three-dimensional case, the following free energy expression can be used:*

$$\begin{aligned} F =& \sigma_{12}\phi_1^2\phi_2^2 + \sigma_{13}\phi_1^2\phi_3^2 + \sigma_{23}\phi_2^2\phi_3^2 + \phi_1\phi_2\phi_3(\Sigma_1\phi_1 + \Sigma_2\phi_2 + \Sigma_3\phi_3) \\ &+ 3\Lambda\phi_1^2\phi_2^2\phi_3^2(\psi_\alpha(\phi_1) + \psi_\alpha(\phi_2) + \psi_\alpha(\phi_3)), \end{aligned} \tag{1.8}$$

where

$$\psi_\alpha(x) = \frac{1}{(1 + x^2)^\alpha},$$

and $\alpha \in (0, \frac{8}{17}]$, instead of the more fundamental expression (1.3) (the case $\alpha = 0$). This is primarily due to a mathematical technical consideration: when $\alpha = 0$, a closure difficulty arises in deriving key a priori estimates, particularly when handling the coupling between nonlinear terms and diffusive terms. Introducing $\alpha > 0$ serves a "regularizing" purpose, enabling us to establish uniform energy estimates and subsequently prove the existence of solutions. This strategy is common in the analysis of Cahn-Hilliard-type systems with similar structures (see, e.g., Boyer and Lapuerta [15] and references *therein). We note that expression (1.8) formally coincides with free energy densities derived in certain physical contexts when considering corrections related to interface thickness; however, its introduction in the present study is primarily motivated by the needs of the mathematical proof. Ultimately, by analyzing the uniform boundedness of the solution with respect to the parameter α and considering the limit $\alpha \to 0$ in a suitable sense, a connection to the original physical model (corresponding to $\alpha = 0$) can be established. In the two-dimensional case, based on stronger Sobolev embedding properties, we can directly handle expression (1.3) with $\alpha = 0$, thus adopting a different technical approach.*

Remark 1.5. *A standard approach exists for proving the global-in-time continuation of local solutions for the initial-boundary value problem of the classical Cahn-Hilliard equation. Specifically, we can use a Lyapunov-type functional to obtain an H^1-type a priori estimate, followed by deriving an estimate for a stronger norm (see [3] for the 2D case, [37] for the 3D case, and [38] for Cahn-Hilliard equations). In this paper, we investigate the global solutions for the Cauchy problem of the 3D three-component Cahn-Hilliard equation in $\mathbb{R}^2$. Compared with the initial-boundary value problem, the main challenge for the Cauchy problem lies in the absence of physical boundaries, as the global existence, regularity, and long-time behavior of solutions heavily depend on the decay properties and oscillatory nature of the initial data, as well as the structure of nonlinear terms. In this paper, we establish several a priori estimates and, by using Hoff and Smoller's method [34–36], obtain the main result.*

The remainder of this paper is organized as follows. In the next section, we introduce some preliminary lemmas. Section 3 is dedicated to establishing the local existence of solutions to problems (1.4) and (1.5). Finally, the proof of Theorem 1.3 is presented in Section 4.

Throughout this paper, we use C to denote a generic constant that may take different values in different contexts. Additionally, $L^p(\mathbb{R}^2)$ $(1 \le p \le \infty)$ denotes the 2D Lebesgue space with norm

$$||u||_{L^p} = \left(\int_{\mathbb{R}^2} |u(x,t)|^p dx\right)^{\frac{1}{p}}, \quad ||u||_{L^\infty} = \operatorname{ess}\sup_{x\in\mathbb{R}^2} |u(x,t)|.$$

Furthermore, for each $k \in \mathbb{Z}^+$, $D^k u$ denotes the set of all k-th order derivatives of $u(t,x)$ with respect to x, and

$$|D^k u|^2 = \sum_{|\alpha|=k} |\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial_{x_2}^{\alpha_2}}|^2,$$

where

$$\alpha = (\alpha_1, \alpha_2)$$

is a multi-index.

2. Preliminaries

In the proof of lemmas and theorems, we frequently employ the Gagliardo-Nirenberg inequality:

Lemma 2.1. *[39] Let $u \in L^q(\mathbb{R}^n)$, $\nabla^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty$. Then, there exists a positive constant $C = C(n, m, j, a, q, r)$, such that*

$$||\nabla^j u||_{L^p} \le C ||\nabla^m u||_{L^r}^a ||u||_{L^q}^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}, \quad 1 \le p \le \infty,\ 0 \le j \le m,\ \frac{j}{m} \le a \le 1.$$

We give the $L^p(\mathbb{R}^2,\mathbb{R})$-estimate on the fundamental solution to the three-component Cahn-Hilliard equations.

Lemma 2.2. *[40] Suppose that*

$$k_i(t) = \mathcal{F}^{-1}(e^{-\frac{3\epsilon M}{4\Sigma_i}|\xi|^4 t}), \quad i = 1, 2, 3,$$

where ξ, $x \in \mathbb{R}^N$ and $t > 0$. Then, for

$$1 \le p \le q \le \infty,$$

we have

$$\|k_i(t)\|_{L^p(\mathbb{R}^N)} \le \frac{c_p}{c_q} t^{-\frac{N}{4}(\frac{1}{q}-\frac{1}{p})} \|k(t)\|_{L^q(\mathbb{R}^N)}, \tag{2.1}$$

$$\|D^j k_i(t)\|_{L^p(\mathbb{R}^N)} \le \frac{c_{p,j}}{c_q} t^{-\frac{N}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j}{4}} \|k(t)\|_{L^q(\mathbb{R}^N)}, \quad j = 1, 2, \cdots, \tag{2.2}$$

where c_p, c_q and $c_{p,j}$ are positive constants with $c_1 = 1$ and $\mathcal{F}^{-1}$ denoting the inverse Fourier transformation with respect to ξ.

The following inequality, which is so important in the proof of our main result, was first given by Strauss [36].

Lemma 2.3. *[36] Suppose that $M(t)$ is a nonnegative continuous function of t. Let $M(t)$ satisfy*

$$M(t) \le d_1 + d_2 M(t)^r$$

in some interval containing 0, where d_1 and d_2 are positive constants and $r > 1$. If $M(0) \le d_1$ and

$$d_1 d_2 < (1 - r^{-1}) r^{-(r-1)^{-1}},$$

then in the same interval

$$M(t) \le \frac{d_1}{1 - r^{-1}}.$$

The following lemma will play a crucial part in proving the uniqueness.

Lemma 2.4. *[41] Assume that a_1, a_2, α, and β are nonnegative constants with*

$$0 \le \alpha, \beta < 1$$

and

$$0 < T < \infty.$$

There exists a constant

$$M(a_2, \alpha, T) < \infty,$$

so that for any integrable function u: $[0, T] \to \mathbb{R}$ satisfying that

$$0 \le u(t) \le a_1 t^{-\alpha} + a_2 \int_0^t (t-s)^{-\beta} u(s) ds, \ \ \text{for a.e. } t \in [0, T],$$

we have

$$0 \le u(t) \le \frac{a_1 M}{1 - \alpha} t^{-\alpha}, \quad \text{a.e. on } 0 < t < T.$$

Next, we show the singular Gronwall type inequality.

Lemma 2.5. *[41, 42] Suppose that $g(t)$ is a nonnegative continuous function defined on $[\tau, T]$ and satisfies*

$$g(t) \leq N_1(t-b)(t-a)^{-\alpha} + N_2(t-b)\int_a^t (t-s)^{-\alpha} g(s)ds.$$

Here τ, α, a and b are constants satisfying

$$0 < \alpha < 1, \tau > \max\{a, b\}$$

and

$$N_i(t-b)(i = 1, 2)$$

are continuous increasing functions of t. Then we have

$$g(t) \leq (t-a)^{-\alpha} N(t-a, t-b) < \infty, \quad \tau \leq t \leq T,$$

where $N(t-a, t-b)$ is a continuous increasing function of t.

3. Local existence

In order to prove the global existence and uniqueness of smooth solutions for the Cauchy problem of three-component Cahn-Hilliard equations, we first state the local existence result.

We have the following lemma.

Theorem 3.1. *Assume that the conditions listed in Theorem 1.3 are satisfied. Then the Cauchy problems (1.4) and (1.5) admit a unique smooth solution $(\phi_1, \phi_2, 1-\phi_3)$ on the strip*

$$\prod_T = \{(t, x) : 0 < t \leq T, x \in \mathbb{R}^2\}$$

and $(\phi_1, \phi_2, 1-\phi_3)$ satisfying

$$\|\phi_1(t,\cdot)\|_{L^\infty} + \|\phi_2(t,\cdot)\|_{L^\infty} + \|1-\phi_3(t,\cdot)\|_{L^\infty} \leq 2R, \quad 0 \leq t \leq T. \tag{3.1}$$

Moreover, fix nonnegative integer $L \geq 5$ and $h > 0$, then for each

$$0 < s_1 < s_2 < \cdots < s_{2L} < t \leq T,$$

which satisfies

$$s_{2L} - s_{2L-1} = s_j - s_{j-1} = s_2 - s_1 = h, \quad j = 2, 3, \cdots, 2L-1,$$

we have

$$\|D^k\phi_1(t,\cdot)\|_{L^\infty} + \|D^k\phi_2(t,\cdot)\|_{L^\infty} + \|D^k(1-\phi_3)(t,\cdot)\|_{L^\infty} \leq M_k(R, h, t-s_{2k-1})(R, h, t-s_{2k-1})(t-s_{2k-1})^{-\frac{k}{4}}, \tag{3.2}$$

where $k = 1, 2, 3, \cdots, L$. Besides, $M_k(R, h, t-s_{2i-1})$ is a continuous increasing function of t.

Proof. It is well-known that if $(\phi_1, \phi_2, 1-\phi_3)$ is a smooth solution of Cauchy problems (1.4) and (1.5), it satisfies the following integro-differential equations:

$$\begin{cases}\phi_1 = \int_{\mathbb{R}^2} k_1(t, x-y)\phi_{1,0}dy + \int_0^t ds \int_{\mathbb{R}^2} \Delta k_1(x-y, t-s)\left(\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right)dy,\\ \phi_2 = \int_{\mathbb{R}^2} k_2(t, x-y)\phi_{2,0}dy + \int_0^t ds \int_{\mathbb{R}^2} \Delta k_2(x-y, t-s)\left(\frac{12}{\Sigma_2\epsilon}Mf_2 + \frac{M}{\Sigma_2}\beta_L\right)dy,\\ 1-\phi_3 = \int_{\mathbb{R}^2} k_3(t, x-y)(1-\phi_{3,0})dy - \int_0^t ds \int_{\mathbb{R}^2} \Delta k_3(x-y, t-s)\left(\frac{12}{\Sigma_3\epsilon}Mf_3 + \frac{M}{\Sigma_3}\beta_L\right)dy.\end{cases} \tag{3.3}$$

To prove Theorem 3.1, we first show that there exists a sufficiently small $t_1 > 0$ such that the integro-differential equation (3.3) admits a unique continuous solution $(\phi_1, \phi_2, 1-\phi_3)$ on the strip $\prod_T$, then if we can show that the solution obtained above is indeed a smooth solution, such a $(\phi_1, \phi_2, 1-\phi_3)$ is indeed a local smooth solution to the original Cauchy problems (1.4) and (1.5).

Let

$$T_i(t)\phi_i = k_i(t, x) * \phi_i(t, x), \quad (i = 1, 2)$$

and

$$T_3(1-\phi_3) = k_3(t, x) * (1-\phi_3(t, x)),$$

then (3.3) can be rewritten as

$$\begin{cases}\phi_1 = T_1(t)\phi_{1,0} + \int_0^t \Delta T_1(t-s)\left(\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right)ds,\\ \phi_2 = T_2(t)\phi_{2,0} + \int_0^t \Delta T_2(t-s)\left(\frac{12}{\Sigma_2\epsilon}Mf_2 + \frac{M}{\Sigma_2}\beta_L\right)ds,\\ 1-\phi_3 = T_3(t)(1-\phi_{3,0}) - \int_0^t \Delta T_3(t-s)\left(\frac{12}{\Sigma_3\epsilon}Mf_3 + \frac{M}{\Sigma_3}\beta_L\right)ds.\end{cases} \tag{3.4}$$

Note that

$$T_i(t)1 = 1, \quad i = 1, 2, 3.$$

We employ the method of successive approximations to establish the existence of (3.4). Set the initial functions $\phi_{1,0}, \phi_{2,0}$, and $\phi_{3,0}$, for $n \geq 1$, and we define

$$\begin{cases}\phi_{1,n+1} = T_1(t)\phi_{1,0} + \int_0^t \Delta T_1(t-s)\left(\frac{12}{\Sigma_1\epsilon}Mf_{1,n} + \frac{M}{\Sigma_1}\beta_{L,n}\right)ds,\\ \phi_{2,n+1} = T_2(t)\phi_{2,0} + \int_0^t \Delta T_2(t-s)\left(\frac{12}{\Sigma_2\epsilon}Mf_{2,n} + \frac{M}{\Sigma_2}\beta_{L,n}\right)ds,\\ 1-\phi_{3,n+1} = T_3(t)\left(1-\phi_{3,0}\right) - \int_0^t \Delta T_3(t-s)\left(\frac{12}{\Sigma_3\epsilon}Mf_{3,n} + \frac{M}{\Sigma_3}\beta_{L,n}\right)ds.\end{cases} \tag{3.5}$$

It is easy to show that $(\phi_{1,n+1}, \phi_{2,n+1}, 1-\phi_{3,n+1})$ is well defined on $[0, \infty) \times \mathbb{R}^2$ for each $n \geq 0$. Let

$$\mathcal{S}_n \triangleq \sup_{0\leq t\leq T} \sum_{i=1}^{2} \|\phi_{i,n}\|_{L^\infty} + \|1-\phi_{3,n}\|_{L^\infty}. \tag{3.6}$$

Applying Lemma 2.2, we obtain

$$
\begin{aligned}
\|1-\phi_{3,n+1}\|_{L^\infty} &\le \|T_3(t)(1-\phi_{3,0})\|_{L^\infty} + \int_0^t \left\| \Delta T_3(t-s)\left(\frac{12}{\Sigma_3\epsilon} M f_{3,n} + \frac{M}{\Sigma_3}\beta_{L,n}\right)\right\|_{L^\infty} ds \\
&\le \|1-\phi_{3,0}\|_{L^\infty} + C\int_0^t (t-s)^{-\frac{1}{2}} [\|\phi_{1,n}\phi_{2,n}^2\|_{L^\infty} + \|\phi_{1,n}\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty} \\
&\quad + \|\phi_{2,n}^2[(1-\phi_{3,n})-1]\|_{L^\infty} + \|\phi_{2,n}\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty} \\
&\quad + \|\phi_{1,n}\phi_{2,n}[(1-\phi_{3,n})-1]\|_{L^\infty} + \|\phi_{1,n}\phi_{2,n}^2\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty}]ds \\
&\quad + C\int_0^t (t-s)^{-\frac{1}{2}} (\|\phi_{1,n}^2\phi_{2,n}\|_{L^\infty} + \|\phi_{1,n}^2[(1-\phi_{3,n})-1]\|_{L^\infty} \\
&\quad + \|\phi_{1,n}\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty} + \|\phi_{2,n}\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty} \\
&\quad + \|\phi_{1,n}\phi_{2,n}[(1-\phi_{3,n})-1]\|_{L^\infty} + \|\phi_{1,n}^2\phi_{2,n}\left((1-\phi_{3,n})^2 + 2(1-\phi_{3,n}) + 1\right)\|_{L^\infty})ds \\
&\quad + C\int_0^t (t-s)^{-\frac{1}{2}} (\|\phi_{1,n}^2[(1-\phi_{3,n})-1]\|_{L^\infty} + \|\phi_{1,n}^2\phi_{2,n}\|_{L^\infty} + \|\phi_{1,n}\phi_{2,n}^2\|_{L^\infty} \\
&\quad + \|\phi_{2,n}^2[(1-\phi_{3,n})-1]\|_{L^\infty} + \|\phi_{1,n}\phi_{2,n}[(1-\phi_{3,n})-1]\|_{L^\infty} \\
&\quad + \|\phi_{1,n}^2\phi_{2,n}^2[(1-\phi_{3,n})-1]\|_{L^\infty})ds \\
&\le \|1-\phi_{3,0}\|_{L^\infty} + CT^{\frac{1}{2}}\left(\mathcal{S}_n + \mathcal{S}_n^5\right).
\end{aligned} \tag{3.7}
$$

Similarly, we have

$$\|\phi_{1,n+1}\|_{L^\infty} \le \|\phi_{1,0}\|_{L^\infty} + CT^{\frac{1}{2}}\left(\mathcal{S}_n + \mathcal{S}_n^5\right) \tag{3.8}$$

and

$$\|\phi_{2,n+1}\|_{L^\infty} \le \|\phi_{2,0}\|_{L^\infty} + CT^{\frac{1}{2}}\left(\mathcal{S}_n + \mathcal{S}_n^5\right). \tag{3.9}$$

Suppose that $0 < T < 1$, then summing up (3.7)–(3.9) gives

$$\sum_{i=1}^{2}\|\phi_{i,n+1}\|_{L^\infty} + \|1-\phi_{3,n+1}\|_{L^\infty} \le \sum_{i=1}^{2}\|\phi_{i,0}\|_{L^\infty} + \|1-\phi_{3,0}\|_{L^\infty} + CT^{\frac{1}{2}}\left(\mathcal{S}_n + \mathcal{S}_n^5\right), \tag{3.10}$$

which implies

$$S_{n+1} \le S_0 + CT^{\frac{1}{2}}(S_n + S_n^5). \tag{3.11}$$

Noticing that $\mathcal{S}_0 \le R$, hence,

$$\mathcal{S}_{n+1} \le R + CT^{\frac{1}{2}}(\mathcal{S}_n^5 + \mathcal{S}_n).$$

Moreover, since $\mathcal{S}_n \le 2R$, hence, we can choose T small enough to obtain

$$R + CT^{\frac{1}{2}}(2R + 32R^5) \le 2R.$$

This simplifies to

$$CT^{\frac{1}{2}}(2 + 32R^4) \le 1,$$

which directly gives

$$0 < T \le \min\left\{1, \sqrt{\frac{R}{C(2R + 32R^5)}}\right\}$$

or

$$R \leq \sqrt{\frac{1}{32}\left(\frac{1}{CT^2} - 2\right)}.$$

Then, based on the inductive method, we have

$$\mathcal{S}_{n+1} \leq 2R, \quad n = 0, 1, 2, \cdots. \tag{3.12}$$

Moreover, by using the induction again, we can easily obtain that $(\phi_{1,n+1}, \phi_{2,n+1}, 1 - \phi_{3,n+1})$ satisfies the following estimate

$$\sup_{0\leq t\leq T} \sum_{i=1}^{2} \|\phi_{i,n+1} - \phi_{i,n}\|_{L^\infty} + \|(1 - \phi_{3,n+1}) - (1 - \phi_{3,n})\|_{L^\infty} \leq \frac{(C_0\sqrt{T})^n}{\Gamma(\frac{n+1}{2})} M_0 \leq \frac{C_0^n}{\Gamma(\frac{n+1}{2})} M_0, \quad n \geq 0, \tag{3.13}$$

where

$$M_0 = 2C\sqrt{\pi} r$$

and

$$C_0 = C\sqrt{\pi}.$$

Since the proof is similar to [29, 43], we omit it here. Noticing that

$$\sum_{n=0}^{\infty} \frac{C_0^n}{\Gamma(\frac{n+1}{2})} M_0$$

is convergent, it follows from (3.13) that $(\phi_{1,n+1}, \phi_{2,n+1}, 1 - \phi_{3,n+1})$ converges uniformly on the strip $\prod_T$, whose limit is denoted by $(\phi_1, \phi_2, 1 - \phi_3)$. It is clear that the unique limit $(\phi_1, \phi_2, 1 - \phi_3)$ is a continuous solution of integro-differential Eq (3.4) on the strip $\prod_T$.

Next, we show the uniqueness by using Lemma 2.4. We remark that we do not need the time T to be sufficiently small. In other words, the uniqueness still holds even if $T = \infty$. Assume that $(\phi_{1,1}, \phi_{2,1}, 1 - \phi_{3,1})$ and $(\phi_{1,2}, \phi_{2,2}, 1 - \phi_{3,2})$ are two solutions of (3.4). Let

$$0 < t_1 < T' < T$$

be fixed. Then for every $i = 1, 2$ and $t \in [t_1, T']$, we derive that

$$\begin{cases} \phi_{1,i}(t,x) = T_1(t - t_1)\phi_{1,i}(t_1) + \displaystyle\int_{t_1}^{t} \Delta T_1(t - s)\left(\frac{12}{\Sigma_1\epsilon} M f_{1,i} + \frac{M}{\Sigma_1}\beta_{L,i}\right) ds, \\ \phi_{2,i}(t,x) = T_2(t - t_1)\phi_{2,i}(t_1) + \displaystyle\int_{t_1}^{t} \Delta T_2(t - s)\left(\frac{12}{\Sigma_2\epsilon} M f_{2,i} + \frac{M}{\Sigma_2}\beta_{L,i}\right) ds, \\ 1 - \phi_{3,i}(t,x) = T_3(t - t_1)(1 - \phi_{3,i})(t_1) - \displaystyle\int_{t_1}^{t} \Delta T_3(t - s)\left(\frac{12}{\Sigma_3\epsilon} M f_{3,i} + \frac{M}{\Sigma_3}\beta_{L,i}\right) ds. \end{cases} \tag{3.14}$$

Hence for $t \in [t_1, T']$, applying Lemma 2.2, it yields that

$$\begin{aligned} \|\phi_{1,1}(t) - \phi_{1,2}(t)\|_{L^\infty} &\leq \|(\phi_{1,1}(t_1) - \phi_{1,2}(t_2))\|_{L^\infty} + \int_{t_1}^{t} (t - s)^{-\frac{1}{2}} \|f_{1,1} - f_{1,2}\|_{L^\infty} ds \\ &\quad + \int_{t_1}^{t} (t - s)^{-\frac{1}{2}} \|\beta_{L,1} - \beta_{L,2}\|_{L^\infty} ds \\ &=: \|(\phi_{1,1}(t_1) - \phi_{1,2}(t_1))\|_{L^\infty} + I_1 + I_2. \end{aligned} \tag{3.15}$$

Note that

$$
\begin{aligned}
I_1 =& C\int_0^t (t-s)^{-\frac{1}{2}}(\|\phi_{1,1}\phi_{2,1}^2-\phi_{1,2}\phi_{2,2}^2\|_{L^\infty}+\|\phi_{1,1}\phi_{3,1}^2-\phi_{1,2}\phi_{3,2}^2\|_{L^\infty}\\
&+\|\phi_{2,1}^2\phi_{3,1}-\phi_{2,2}^2\phi_{3,2}\|_{L^\infty}+\|\phi_{2,1}\phi_{3,1}^2-\phi_{2,2}\phi_{3,2}^2\|_{L^\infty}\\
&+\|\phi_{1,1}\phi_{2,1}\phi_{3,1}-\phi_{1,2}\phi_{2,2}\phi_{3,2}\|_{L^\infty}+\|\phi_{1,1}\phi_{2,1}^2\phi_{3,1}^2-\phi_{1,2}\phi_{2,2}^2\phi_{3,2}^2\|_{L^\infty})ds.
\end{aligned}
$$

Simple calculations show that

$$
\begin{aligned}
\|\phi_{1,1}\phi_{2,1}^2-\phi_{1,2}\phi_{2,2}^2\|_{L^\infty} &\le \|\phi_{2,1}\|_{L^\infty}^2\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|\phi_{1,2}\|_{L^\infty}\|\phi_{2,1}+\phi_{2,2}\|_{L^\infty}\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}\\
&\le C(\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}),\\
\|\phi_{1,1}\phi_{3,1}^2-\phi_{1,2}\phi_{3,2}^2\|_{L^\infty} &\le \|\phi_{3,1}\|_{L^\infty}^2\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|\phi_{1,2}\|_{L^\infty}\|\phi_{3,1}+\phi_{3,2}\|_{L^\infty}\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}\\
&\le C(1+\|1-\phi_{3,1}\|_{L^\infty}^2)\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}\\
&\quad+\|\phi_{1,2}\|_{L^\infty}(C+\|(1-\phi_{3,1})+(1-\phi_{3,2})\|_{L^\infty}\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty})\\
&\le C(\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}),\\
\|\phi_{3,1}\phi_{2,1}^2-\phi_{3,2}\phi_{2,2}^2\|_{L^\infty} &\le \|\phi_{2,1}\|_{L^\infty}^2\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\\
&\quad+C(1+\|1-\phi_{3,2}\|_{L^\infty})\|\phi_{2,1}+\phi_{2,2}\|_{L^\infty}\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}\\
&\le C(\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}),\\
\|\phi_{2,1}\phi_{3,1}^2-\phi_{2,2}\phi_{3,2}^2\|_{L^\infty} &\le C(1+\|1-\phi_{3,1}\|_{L^\infty}^2)\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}\\
&\quad+C\|\phi_{2,2}\|_{L^\infty}\|(1-\phi_{3,1})+(1-\phi_{3,2})\|_{L^\infty}\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\\
&\le C(\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}),\\
\|\phi_{1,1}\phi_{2,1}\phi_{3,1}-\phi_{1,2}\phi_{2,2}\phi_{3,2}\|_{L^\infty} &\le C\|\phi_{2,1}\|_{L^\infty}(1+\|1-\phi_{3,1}\|_{L^\infty})\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}\\
&\quad+C\|\phi_{1,2}\|_{L^\infty}(1+\|1-\phi_{3,1}\|_{L^\infty})\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}\\
&\quad+C\|\phi_{1,2}\|_{L^\infty}\|\phi_{2,2}\|_{L^\infty}\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\\
&\le C(\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\phi_{1,1}\phi_{2,1}^2\phi_{3,1}^2-\phi_{1,2}\phi_{2,2}^2\phi_{3,2}^2\|_{L^\infty} \le\ & C\|\phi_{2,1}\|_{L^\infty}^2(1+\|1-\phi_{3,1}\|_{L^\infty}^2)\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+C\|\phi_{1,2}\|_{L^\infty}\|\phi_{2,1}\\
&+\phi_{2,2}\|_{L^\infty}(1+\|1-\phi_{3,1}\|_{L^\infty}^2)\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}+C\|\phi_{1,2}\|_{L^\infty}\|\phi_{2,2}\|_{L^\infty}^2(1+\|(1-\phi_{3,1})\\
&+(1-\phi_{3,2})\|_{L^\infty}\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}.
\end{aligned}
$$

Summing up, we obtain

$$
I_1 \le C_1\int_{t_1}^t (t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^2\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds. \tag{3.16}
$$

Similarly, for I_2, we have

$$
\begin{aligned}
I_2 &= C\int_0^t (t-s)^{-\frac{1}{2}}(\|f_{1,1}-f_{1,2}\|_{L^\infty}+\|f_{2,1}-f_{2,2}\|_{L^\infty}+\|f_{3,1}-f_{3,2}\|_{L^\infty})ds\\
&\le C_1\int_{t_1}^t (t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^2\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds.
\end{aligned} \tag{3.17}
$$

Adding (3.15)–(3.17) together gives

$$\begin{aligned}\|\phi_{1,1}(t)-\phi_{1,2}(t)\|_{L^\infty} \le{}& \|(\phi_{1,1}(t_1)-\phi_{1,2}(t_1)\|_{L^\infty}\\ &+C\int_{t_1}^{t}(t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^{2}\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds.\end{aligned} \tag{3.18}$$

Similarly, we can also obtain

$$\begin{aligned}\|\phi_{2,1}(t)-\phi_{2,2}(t)\|_{L^\infty} \le{}& \|(\phi_{2,1}(t_1)-\phi_{2,2}(t_1)\|_{L^\infty}\\ &+C\int_{t_1}^{t}(t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^{2}\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty} \le{}& \|(\phi_{3,1}(t_1)-\phi_{3,2}(t_1)\|_{L^\infty}\\ &+C\int_{t_1}^{t}(t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^{2}\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds.\end{aligned} \tag{3.20}$$

Summing up (3.18)–(3.20) together, letting $t_1 \to 0$, we deduce

$$\begin{aligned}&\sum_{i=1}^{2}\|(\phi_{i,1}(t)-\phi_{i,2}(t)\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\\ &\le C\int_{t_1}^{t}(t-s)^{-\frac{1}{2}}\left[\sum_{i=1}^{2}\|\phi_{i,1}-\phi_{i,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\right]ds.\end{aligned} \tag{3.21}$$

Applying Lemma 2.4, we get

$$\|\phi_{1,1}-\phi_{1,2}\|_{L^\infty}+\|\phi_{2,1}-\phi_{2,2}\|_{L^\infty}+\|(1-\phi_{3,1})-(1-\phi_{3,2})\|_{L^\infty}\le 0. \tag{3.22}$$

Therefore,

$$(\phi_{1,1},\phi_{2,1},1-\phi_{3,1})\equiv(\phi_{1,2},\phi_{2,2},1-\phi_{3,2})$$

holds for all

$$0\le t\le T'.$$

Since $T' < T$ is arbitrarily chosen, then

$$(\phi_{1,1},\phi_{2,1},1-\phi_{3,1})\equiv(\phi_{1,2},\phi_{2,2},1-\phi_{3,2})$$

for all $0 \le t \le T$.

To prove that such a $(\phi_1, \phi_2, 1-\phi_3)$ obtained above is indeed a smooth solution of the Cauchy problems (1.4) and (1.5) on the strip $\prod_T$, we only need to get the regularity of $(\phi_1, \phi_2, 1-\phi_3)$. To this end, we first prove (3.2) by induction. If $k = 1$, we have

$$D\phi_1(t,x)=DT_1(t-s_1)\phi_{1,0}(s_1)+\int_{s_1}^{t}D\Delta T_1(t-s)\left(\frac{12}{\Sigma_1\epsilon}Mf_1+\frac{M}{\Sigma_1}\beta_L\right)ds. \tag{3.23}$$

Then, similar as the proof of the first step, by using Lemma 2.2 and (3.1), one obtains

$$\begin{aligned}\|D\phi_1(t,x)\|_{L^\infty} &\le \|DT_1(t-s_1)\phi_{1,0}(s_1)\|_{L^\infty} + \int_{s_1}^t \left\| D\Delta T_1(t-s)\left(\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right)\right\|_{L^\infty} ds \\ &\le C(t-s_1)^{-\frac{1}{4}}\|\phi_{1,0}(s_1)\|_{L^\infty} + C\int_{s_1}^t (t-s_1)^{-\frac{3}{4}} \left\|\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right\|_{L^\infty} ds \\ &\le C(t-s_1)^{-\frac{1}{4}}R + C\int_{s_1}^t (t-s_1)^{-\frac{3}{4}}(\|f_1\|_{L^\infty} + \|\beta_L)\|_{L^\infty})ds \\ &\le C(t-s_1)^{-\frac{1}{4}}R + \int_{s_1}^t (t-s_1)^{-\frac{3}{4}}(R+R^5)ds \\ &\le C(t-s_1)^{-\frac{1}{4}}R + C(R^5+R)\int_{s_1}^t (t-s)^{-\frac{3}{4}}ds \\ &\le M_1(R,t-s_1)(t-s_1)^{-\frac{1}{4}}.\end{aligned} \tag{3.24}$$

Moreover, simple calculations show that

$$\|D\phi_2\|_{L^\infty} \le M_1(R,t-s_1)(t-s_1)^{-\frac{1}{4}} \tag{3.25}$$

and

$$\|D(1-\phi_3)\|_{L^\infty} \le M_1(R,t-s_1)(t-s_1)^{-\frac{1}{4}}. \tag{3.26}$$

We remark that (3.24)–(3.26) implies that (3.2) holds for $k = 1$. Next, assume that (3.2) is true for $k \le n-1$, $(1 \le n \le L)$, i.e.,

$$\begin{aligned}&\|D^k\phi_1(t,\cdot)\|_{L^\infty} + \|D^k\phi_2(t,\cdot)\|_{L^\infty} + \|D^k(1-\phi_3)(t,\cdot)\|_{L^\infty} \\ &\le M_k(R,h,t-s_{2k-1})(t-s_{2k-1})^{-\frac{k}{4}}, \quad k = 1,2,3,\cdots,n-1.\end{aligned} \tag{3.27}$$

We prove that (3.2) also holds for $k = n$. Note that

$$D^n\phi_1(t,x) = D^nT_1(t-s_{2n-1})\phi_{1,0}(s_{2n-1}) + \int_{s_{2n-1}}^t D\Delta T_1(t-s)D^{n-1}\left(\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right)ds. \tag{3.28}$$

Applying (3.27), we arrive at

$$\begin{aligned}\|D^n\phi_1(t,x)\|_{L^\infty} \le\ &\|D^nT_1(t-s_{2n-1})\phi_{1,0}(s_{2n-1})\|_{L^\infty} \\ &+ \int_{s_{2n-1}}^t \left\|D\Delta T_1(t-s)D^{n-1}\left(\frac{12}{\Sigma_1\epsilon}Mf_1 + \frac{M}{\Sigma_1}\beta_L\right)\right\|_{L^\infty} ds \\ \le\ &M_n(R,h,t-s_{2n-1})(t-s_{2n-1})^{-\frac{n}{4}}.\end{aligned} \tag{3.29}$$

Similarly, we can also obtain

$$\|D^n\phi_2(t,x)\|_{L^\infty} \le M_n(R,h,t-s_{2n-1})(t-s_{2n-1})^{-\frac{n}{4}} \tag{3.30}$$

and

$$\|D^n(1-\phi_3)(t,x)\|_{L^\infty} \le M_n(R,h,t-s_{2n-1})(t-s_{2n-1})^{-\frac{n}{4}}. \tag{3.31}$$

Combining (3.29)–(3.31) together, we easily conclude that (3.2) holds for $k = n$. Therefore, we get by induction that (3.2) holds for

$$1 \leq k \leq L.$$

Having obtained (3.1)-(3.2), since $L \geq 5$, it is a routine matter to verify that for each $\delta > 0$, $(D^k\phi_{1,n+1}, D^k\phi_{2,n+1}, D^k(1-\phi_{3,n+1}))$ converges uniformly to $(D^k\phi_1, D^k\phi_2, D^k(1-\phi_3))$ on $[\delta, T]\times\mathbb{R}^2$ for $k = 1, 2, \cdots, L-1$. Therefore, we have

$$(\phi_1, \phi_2, 1-\phi_3) \in \left(C^{1,4}([\delta, T]\times\mathbb{R}^2)\right)^2$$

for every $\delta > 0$. Moreover, since $\delta > 0$ can be chosen sufficiently small, we have

$$(\phi_1, \phi_2, 1-\phi_3) \in \left(C^{1,4}((0, T]\times\mathbb{R}^2)\right)^2.$$

Having obtained the above regularity result, we can conclude that $(\phi_1, \phi_2, 1-\phi_3)$ obtained above is indeed a smooth solution to the Cauchy problems (1.4) and (1.5) on the strip $\prod_T$ and complete the proof. □

4. Global existence

Having established the local existence in Theorem 3.1, we now proceed to the proof of Theorem 1.3 on global existence and uniqueness by a continuation argument. This relies on the following key lemma:

Lemma 4.1. *If $(\phi_1, \phi_2, 1-\phi_3)$ obtained in Theorem 3.1 has been extended up to time T^* ($T^* \geq T > 0$) while the a priori estimate (3.1) is kept unchanged for $\{s_m\}$ defined in Theorem 3.1, then for*

$$1 \leq p \leq \infty,$$

the following inequality holds:

$$\begin{aligned}
&\|D^m\phi_1(t,\cdot)\|_{L^p} + \|D^m\phi_2(t,\cdot)\|_{L^p} + \|D^m(1-\phi_3)(t,\cdot)\|_{L^p} \\
&\leq (t - s_{2(m+1)})^{-\frac{m+2}{4}} N_m(R, h, t - s_{2(m+1)}) \left(\sup_{0\leq t\leq T} \sum_{i=1}^{2} \|\phi_i(t,\cdot)\|_{L^1} + \sup_{0\leq t\leq T} \|1-\phi_3(t,\cdot)\|_{L^1} \right),
\end{aligned} \tag{4.1}$$

where $N_m(R, h, t - s_{2(m+1)})$ ($m = 0, 1, 2, \cdots, L$) are continuous increasing functions of t.

Proof. We first consider the case $k = 0$. Applying Lemma 2.2 and (3.1), we deduce that

$$\begin{aligned}
\|\phi_1\|_{L^p} \le & \|T_1(t-s_2)\phi_1(s_2)\|_{L^p} + \int_{s_2}^{t} \left\| \Delta T_1(t-s_2)\left(\frac{12M}{\Sigma_1\epsilon} f_1 + \frac{M}{\Sigma_1}\beta_L\right)\right\|_{L^p} ds \\
\le & C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\|\phi(s_2)\|_{L^1} + C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\phi_1\phi_2^2\|_{L^p} + \|\phi_1\phi_3^2\|_{L^p} + \|\phi_2^2\phi_3\|_{L^p} \\
& + \|\phi_2\phi_3^2\|_{L^p} + \|\phi_1\phi_2\phi_3\|_{L^p} + \|\phi_1\phi_2^2\phi_3^2\|_{L^p}\Big) ds \\
& + C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1^2\phi_2\|_{L^p} + \|\phi_1^2\phi_3\|_{L^p} + \|\phi_1\phi_3^2\|_{L^p} + \|\phi_2\phi_3^2\|_{L^p} \\
& + \|\phi_1\phi_2\phi_3\|_{L^p} + \|\phi_1^2\phi_2\phi_3^2\|_{L^p})ds \\
& + C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1^2\phi_3\|_{L^p} + \|\phi_1^2\phi_2\|_{L^p} + \|\phi_1\phi_2^2\|_{L^p} + \|\phi_2^2\phi_3\|_{L^p} \\
& + \|\phi_1\phi_2\phi_3\|_{L^p} + \|\phi_1^2\phi_2^2\phi_3\|_{L^p})ds \\
\le & C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\|\phi_1(s_2)\|_{L^1} + Q_1 + Q_2 + Q_3.
\end{aligned} \tag{4.2}$$

Simple calculation shows that

$$\begin{aligned}
Q_1 =: & \int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\phi_1^2\phi_2\|_{L^p} + \|\phi_1^2[(1-\phi_3)+1]\|_{L^p} + \|\phi_1[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p} \\
& + \|\phi_2[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p} + \|\phi_1\phi_2[(1-\phi_3)+1]\|_{L^p} \\
& + \|\phi_1^2\phi_2[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p}\Big) ds \\
\le & \int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1\|_{L^p}\|\phi_1\phi_2\|_{L^\infty} + \|\phi_1\|_{L^p}\|\phi_1[(1-\phi_3)+1]\|_{L^\infty} \\
& + \|\phi_1\|_{L^p}\|[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^\infty} + \|\phi_2\|_{L^p}\|(1-\phi_3)^2 + 2(1-\phi_3)+1\|_{L^\infty} \\
& + \|\phi_1\|_{L^p}\|\phi_2[(1-\phi_3)+1]\|_{L^\infty} + \|\phi_1\|_{L^p}\|\phi_1\phi_2[(1-\phi_3)+1]\|_{L^\infty}) \\
\le & C_1\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1\|_{L^p} + \|\phi_2\|_{L^p}).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
Q_2 =: & \int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\|\|\phi_1^2\phi_2\|_{L^p} + \|\phi_1^2[(1-\phi_3)+1]\|_{L^p} + \|\phi_1[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p} \\
& + \|\phi_2[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p} + \|\phi_1\phi_2[(1-\phi_3)+1]\|_{L^p} \\
& + \|\phi_1^2\phi_2[(1-\phi_3)^2 + 2(1-\phi_3)+1]\|_{L^p} ds\Big) ds \\
\le & C_2\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1\|_{L^p} + \|\phi_2\|_{L^p} + \|1-\phi_3\|_{L^p})
\end{aligned}$$

and

$$
\begin{aligned}
Q_3 =: & \int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\Big(\|\phi_1^2[(1-\phi_3)+1]\|_{L^p}+\|\phi_1^2\phi_2\|_{L^p}+\|\phi_1\phi_2^2\|_{L^p}+\|\phi_2^2[(1-\phi_3)+1]\|_{L^p}\\
& +\|\phi_1\phi_2[(1-\phi_3)+1]\|_{L^p}+\|\phi_1^2\phi_2^2[(1-\phi_3)+1]\|_{L^p}ds\Big)\,ds\\
& \leq C_3\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}(\|\phi_1\|_{L^p}+\|\phi_2\|_{L^p}+\|\phi_3\|_{L^p}).
\end{aligned}
$$

Summing up the above inequalities, we derive that

$$
\|\phi_1\|_{L^p}\leq C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\|\phi_1(s_2)\|_{L^1}+C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\right)ds. \tag{4.3}
$$

Similarly, we can also obtain

$$
\|\phi_2\|_{L^p}\leq C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\|\phi_2(s_2)\|_{L^1}+C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\right)ds \tag{4.4}
$$

and

$$
\|1-\phi_3\|_{L^p}\leq C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\|1-\phi_3(s_2)\|_{L^1}+C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\right)ds. \tag{4.5}
$$

Combining (4.3)–(4.5) together gives

$$
\begin{aligned}
\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p} &\leq C(t-s_2)^{-\frac{1}{2}(1-\frac{1}{p})}\left(\sum_{i=1}^{2}\|\phi_i(s_2)\|_{L^1}+\|1-\phi_3(s_2)\|_{L^1}\right)\\
&\quad +C\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\right)ds\\
&\leq C(t-s_2)^{\frac{1}{2}-\frac{1}{4}(1-\frac{1}{p})}(t-s_2)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i(s_2)\|_{L^1}+\|1-\phi_3(s_2)\|_{L^1}\right)\\
&\quad +C(R)\int_{s_2}^{t}(t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\right)ds.
\end{aligned} \tag{4.6}
$$

Then by using Lemma 2.5, we easily obtain

$$
\sum_{i=1}^{2}\|\phi_i\|_{L^p}+\|1-\phi_3\|_{L^p}\leq (t-s_2)^{-\frac{1}{2}}N_0(R,t-s_2)\left(\sup_{0\leq t\leq T}\sum_{i=1}^{2}\|\phi_i(s_2)\|_{L^1}+\|1-\phi_3(s_2)\|_{L^1}\right). \tag{4.7}
$$

Next, for the case $k = 1$, we have

$$\begin{aligned}
\|D\phi_1\|_{L^p} \leq & \|DT(t-s_2)\phi_1(s_2)\|_{L^p} + \int_{s_2}^{t} \left\| \Delta T(t-s_2) D\left(\frac{12M}{\Sigma_1 \epsilon} f_1 + \frac{M}{\Sigma_1}\beta_L\right)\right\|_{L^p} ds \\
\leq & C(t-s_2)^{-\frac{1}{4}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \|\phi(s_2)\|_{L^1} \\
& + C\int_{s_2}^{t} (t-s)^{-\frac{1}{2}} (\|D(\phi_1\phi_2^2)\|_{L^p} + \|D(\phi_1\phi_3^2)\|_{L^p} + \|D(\phi_2^2\phi_3)\|_{L^p} \\
& + \|D(\phi_2\phi_3^2)\|_{L^p} + \|D(\phi_1\phi_2\phi_3)\|_{L^p} + \|D(\phi_1\phi_2^2\phi_3^2)\|_{L^p}) ds \\
& + C\int_{s_2}^{t} (t-s)^{-\frac{1}{2}} (\|D(\phi_1^2\phi_2)\|_{L^p} + \|D(\phi_1^2\phi_3)\|_{L^p} + \|D(\phi_1\phi_3^2)\|_{L^p} \\
& + \|D(\phi_2\phi_3^2)\|_{L^p} + \|D(\phi_1\phi_2\phi_3)\|_{L^p} + \|D(\phi_1^2\phi_2\phi_3^2)\|_{L^p}) ds \\
& + C\int_{s_2}^{t} (t-s)^{-\frac{1}{2}} (\|D(\phi_1^2\phi_3)\|_{L^p} + \|D(\phi_1^2\phi_2)\|_{L^p} + \|D(\phi_1\phi_2^2)\|_{L^p} \\
& + \|D(\phi_2^2\phi_3)\|_{L^p} + \|D(\phi_1\phi_2\phi_3)\|_{L^p} + \|D(\phi_1^2\phi_2^2\phi_3)\|_{L^p}) ds \\
\leq & C(t-s_2)^{-\frac{1}{4}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \|\phi_1(s_2)\|_{L^1} + W_1 + W_2 + W_3.
\end{aligned} \tag{4.8}$$

Simple calculation shows that

$$\begin{aligned}
W_1 \leq & \int_{s_1}^{t} (t-s_1)^{-\frac{3}{4}} (\|D\phi_1\|_{L^p} (\|\phi_2\|_{L^\infty}^2 + \|(1-\phi_3)+1\|_{L^\infty}^2 + \|\phi_2\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty} \\
& + \|\phi_2\|_{L^\infty}^2 \|(1-\phi_3)+1\|_{L^\infty}^2) ds \\
& + \int_{s_1}^{t} (t-s_1)^{-\frac{3}{4}} \|D\phi_2\|_{L^p} (\|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty} + \|\phi_2\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty} \\
& + \|\phi_1\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty} + \|(1-\phi_3)+1\|_{L^\infty}^2 + \|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty}^2) ds \\
& + \int_{s_1}^{t} (t-s_1)^{-\frac{3}{4}} \|D(1-\phi_3)\|_{L^p} (\|\phi_1\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty} + \|\phi_2\|_{L^\infty}\|(1-\phi_3)+1\|_{L^\infty} \\
& + \|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty} + \|\phi_2\|_{L^\infty}^2 + \|\phi_1\|_{L^\infty}\|\phi_2\|_{L^\infty}^2 \|(1-\phi_3)+1\|_{L^\infty}) ds \\
\leq & \tilde{C}_1 \int_{s_4}^{t} (t-s)^{-\frac{3}{4}} \left(\sum_{i=1}^{2} \|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right) ds.
\end{aligned} \tag{4.9}$$

Similarly, we also have

$$W_2 \leq \tilde{C}_2 \int_{s_4}^{t} (t-s)^{-\frac{3}{4}} \left(\sum_{i=1}^{2} \|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right) ds \tag{4.10}$$

and

$$W_3 \leq \tilde{C}_3 \int_{s_4}^{t} (t-s)^{-\frac{3}{4}} \left(\sum_{i=1}^{2} \|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right) ds. \tag{4.11}$$

Plugging (4.9)–(4.11) into (4.8), we have

$$\|D\phi_1\|_{L^p} \leq C(t-s_4)^{-\frac{1}{4}-\frac{1}{2}(1-\frac{1}{p})} \|\phi_1\|_{L^1} + \tilde{C}_1 \int_{s_4}^{t} (t-s)^{-\frac{3}{4}} \left(\sum_{i=1}^{2} \|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right) ds. \tag{4.12}$$

Simple calculations show that

$$\|D\phi_2\|_{L^p} \le C(t-s_4)^{-\frac{1}{4}-\frac{1}{2}(1-\frac{1}{p})}\|\phi_2\|_{L^1} + \tilde{C}_2\int_{s_4}^{t}(t-s)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right)ds \tag{4.13}$$

and

$$\|D(1-\phi_3)\|_{L^p} \le C(t-s_4)^{-\frac{1}{4}-\frac{1}{2}(1-\frac{1}{p})}\|1-\phi_3\|_{L^1} + \tilde{C}_3\int_{s_4}^{t}(t-s)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right)ds. \tag{4.14}$$

Adding (4.12)–(4.14) together, we derive that

$$\begin{aligned}\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p} \le{}& C(t-s_4)^{-\frac{1}{4}-\frac{1}{2}(1-\frac{1}{p})}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1}\right)\\ &+ \tilde{C}(R)\int_{s_4}^{t}(t-s)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p}\right)ds.\end{aligned} \tag{4.15}$$

Note that

$$-\frac{1}{4} - \frac{1}{2}(1-\frac{1}{p}) \ge -\frac{3}{4},$$

then we obtain from (4.15) that

$$\begin{aligned}\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p} \le{}& C(t-s_4)^{-\frac{1}{4}-\frac{1}{2}(1-\frac{1}{p})+\frac{3}{4}}(t-s_4)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|\phi_i(s_4)\|_{L^1} + \|1-\phi_3(s_4)\|_{L^1}\right)\\ &+ C(R)\int_{s_4}^{t}(t-s_4)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|D\phi_i\|_{L^1} + \|D(1-\phi_3)\|_{L^1}\right)ds.\end{aligned} \tag{4.16}$$

By using Lemma 2.5, we have

$$\sum_{i=1}^{2}\|D\phi_i\|_{L^p} + \|D(1-\phi_3)\|_{L^p} \le C(t-s_4)^{-\frac{3}{4}}N_1(R,h,t-s_4)\left(\sup_{0\le t\le T}\sum_{i=1}^{2}\|\phi_i(s_4)\|_{L^1} + \sup_{0\le t\le T}\|1-\phi_3(s_4)\|_{L^1}\right), \tag{4.17}$$

which implies that (4.1) holds for $k=1$, and the case for general k can be proved similarly.

Hence, the proof is complete. □

Lemma 4.2. *Assume that the assumptions listed in Lemma 4.1 are satisfied, then* $(\phi_1, \phi_2, 1-\phi_3)$ *satisfies the following time-independent* L^1*-a priori estimate*

$$\sum_{i=1}^{2}\left(\|\phi_i\|_{L^1} + t^{\frac{1}{4}}\|\phi_i\|_{L^2}\right) + \|1-\phi_3\|_{L^1} + t^{\frac{1}{4}}\|1-\phi_3\|_{L^2} \le C_1(R)\left(\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1}\right), \tag{4.18}$$

where $C_1(R)$ *is a positive constant depending only on R.*

Proof. Since

$$\phi_i \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \quad (i = 1, 2, 3),$$

we can also obtain it belongs to $L^2(\mathbb{R}^2)$. In fact, this conclusion can be obtained by using Hölder's inequality

$$\|\phi_i\|_{L^2} \le C\|\phi_i\|_{L^1}^{\frac{1}{2}}\|\phi_i\|_{L^\infty}^{\frac{1}{2}}.$$

Based on (3.4), we have

$$\phi_1(t) = T(t)\phi_{1,0} + \int_0^t \Delta T(t-s)\left(\frac{12M}{\varepsilon\Sigma_1}f_1 + \frac{M}{\Sigma_1}\beta_L\right)ds. \tag{4.19}$$

Employing Lemma 2.2 and (3.1), we obtain

$$\begin{aligned}
\|\phi_1(t)\|_{L^1} \le& \|\phi_{1,0}\|_{L^1} + C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_1\phi_2^2\|_{L^1} + \|\phi_1\phi_3^2\|_{L^1} + \|\phi_2^2\phi_3\|_{L^1} \\
&+\|\phi_2\phi_3^2\|_{L^1} + \|\phi_1\phi_2\phi_3\|_{L^1} + \|\phi_1\phi_2^2\phi_3^2\|_{L^1}\Big)ds \\
&+ C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_1^2\phi_2\|_{L^1} + \|\phi_1^2\phi_3\|_{L^1} + \|\phi_1\phi_3^2\|_{L^1} \\
&+\|\phi_2\phi_3^2\|_{L^1} + \|\phi_1\phi_2\phi_3\|_{L^1} + \|\phi_1^2\phi_2\phi_3^2\|_{L^1}\Big)ds \\
&+ C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_1^2\phi_3\|_{L^1} + \|\phi_1^2\phi_2\|_{L^1} \\
&+\|\phi_1\phi_2^2\|_{L^1} + \|\phi_2^2\phi_3\|_{L^1} + \|\phi_1\phi_2\phi_3\|_{L^1} + \|\phi_1^2\phi_2^2\phi_3\|_{L^1}\Big)ds \\
\le& \|\phi_{1,0}\|_{L^1} + J_1 + J_2 + J_3.
\end{aligned} \tag{4.20}$$

Note that

$$\begin{aligned}
J_1 \le& \int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_1\|_{L^\infty}\|\phi_2^2\|_{L^1} + \|\phi_1\|_{L^\infty}\|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2^2\|_{L^1}(1 + \|1-\phi_3\|_{L^\infty}) \\
&+\|\phi_2\|_{L^\infty}\|1-\phi_3\|_{L^2}^2 + \|\phi_2\|_{L^1} + \|\phi_1\phi_2\|_{L^1}(1 + \|1-\phi_3\|_{L^\infty}) \\
&+\|\phi_1\|_{L^\infty}(1 + \|1-\phi_3\|_{L^\infty}^2)\|\phi_2^2\|_{L^1}\Big)dsds \\
\le& C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_2\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\Big)ds.
\end{aligned} \tag{4.21}$$

Similarly, we also have

$$J_2 \le C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_2\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\Big)ds \tag{4.22}$$

and

$$J_3 \le C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_2\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\Big)ds. \tag{4.23}$$

Adding (4.20)–(4.23) together gives

$$\|\phi_1(t)\|_{L^1} \le \|\phi_{1,0}\|_{L^1} + C\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\phi_2\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\Big)ds. \tag{4.24}$$

Simple calculations show that

$$\|\phi_2(t)\|_{L^1} \le \|\phi_{2,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds \tag{4.25}$$

and

$$\|1-\phi_3(t)\|_{L^1} \le \|1-\phi_{3,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds. \tag{4.26}$$

Applying Lemma 2.2 and (3.1) again, we see that

$$\begin{aligned}
\|\phi_1(t)\|_{L^2} &\le \|T(t)\phi_{1,0}\|_{L^2} + \int_0^t \left\|\Delta T(t-s)\left(\frac{12M}{\epsilon\Sigma_1}f_1 + \frac{M}{\Sigma_1}B_L\right)\right\|_{L^2} ds \\
&\le Ct^{-\frac{1}{4}}\|\phi_{1,0}\|_{L^1} + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{2}{4}(1-\frac{1}{2})}\left(\|f_1\|_{L^1} + \|\beta_L\|_{L^1}\right)ds \\
&\le Ct^{-\frac{1}{4}}\|\phi_{1,0}\|_{L^1} + C\int_0^t (t-s)^{-\frac{3}{4}}\left(C_1'(R)\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right) \\
&\le Ct^{-\frac{1}{4}}\|\phi_{1,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds.
\end{aligned} \tag{4.27}$$

Similarly, we also have

$$\|\phi_2\|_{L^2} \le Ct^{-\frac{1}{4}}\|\phi_{2,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds \tag{4.28}$$

and

$$\|1-\phi_3\|_{L^2} \le Ct^{-\frac{1}{4}}\|1-\phi_{3,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{3}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds. \tag{4.29}$$

Combining (4.24)–(4.29), we see

$$\begin{aligned}
&\sum_{i=1}^{2}\|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} + t^{\frac{1}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2} + \|1-\phi_3\|_{L^2}\right) \\
&\le C\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} + C(R)\int_0^t (t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)ds \\
&\le C\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} + C(R)t^{\frac{1}{2}}\sup_{0\le t\le T}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right)\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}ds \\
&\le C\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} + C(R)t^{\frac{1}{2}}\sup_{0\le t\le T}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}^2 + \|1-\phi_3\|_{L^2}^2 + \|\phi_1\|_{L^1} + \|\phi_2\|_{L^1}\right),
\end{aligned} \tag{4.30}$$

which means

$$
\begin{aligned}
&\sup_{0\le t\le T}\left\{\sum_{i=1}^{2}\|\phi_i\|_{L^1}+\|1-\phi_3\|_{L^1}+t^{\frac{1}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}+\|1-\phi_3\|_{L^2}\right)\right\}\\
&\le C_0\left(\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1}+\|1-\phi_{3,0}\|_{L^1}\right)+C(R)t^{\frac{1}{2}}\sup_{0\le t\le T}\left(\|\phi_1\|_{L^1}+\|\phi_2\|_{L^1}\right)\\
&+C(R)\sup_{0\le t\le T}\left\{\sum_{i=1}^{2}\|\phi_i\|_{L^1}+\|1-\phi_3\|_{L^1}+t^{\frac{1}{4}}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^2}+\|1-\phi_3\|_{L^2}\right)\right\}^2.
\end{aligned}
\tag{4.31}
$$

If we assume that$\|\phi_{1,0}\|_{L^1}+\|\phi_{2,0}\|_{L^1}$ is sufficiently small, then on the basis of Lemma 2.3, we can obtain Eq (4.18) immediately and complete the proof. □

With the above preparations in hand, we give the proof of Theorem 1.3 in the following.

Proof of Theorem 1.3. Fix p. Then, (4.1) together with (4.18) gives

$$
\begin{cases}
\sum_{i=1}^{2}\|\phi_i\|_{L^1}+\|1-\phi_3\|_{L^1}\le C_1(R)\left(\sum_{i=1}^{2}\|\phi_{i0}\|_{L^1}+\|1-\phi_{30}\|_{L^1}\right), \quad 0\le t\le T,\\
\sum_{i=1}^{2}\|\phi_i(T)\|_{W^{L,p}}+\|1-\phi_3(T)\|_{W^{L,p}}\le C_2(R,h,T)\sup_{0\le t\le T}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^1}+\|1-\phi_3\|_{L^1}\right).
\end{cases}
\tag{4.32}
$$

Now, assuming that C is the constant in the Sobolev inequality,

$$
\|u\|_{L^\infty}\le C\|u\|_{W^{L,p}}.
$$

Hence, if $\|\phi_{1,0}\|_{L^1}+\|\phi_{2,0}\|_{L^1}$ is so small (on the basis of $\sum_{i=1}^{3}\phi_i=1$, we have $1-\phi_3=\phi_1+\phi_2$, hence $\|\phi_{1,0}\|_{L^1}+\|\phi_{2,0}\|_{L^1}$ so small is equivalent to $\|\phi_{1,0}\|_{L^1}+\|\phi_{2,0}\|_{L^1}+\|1-\phi_{3,0}\|_{L^1}$ so small) such that

$$
CC_1(R)C_2(R,h,T)\left(\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1}+\|1-\phi_{3,0}\|_{L^1}\right)\le\sum_{i=1}^{2}\|\phi_i(T)\|_{L^\infty}+\|\phi_3(T)\|_{L^\infty},
\tag{4.33}
$$

then we obtain

$$
\begin{aligned}
\sum_{i=1}^{2}\|\phi_i(T)\|_{L^\infty}+\|\phi_3(T)\|_{L^\infty}&\le C\left(\sum_{i=1}^{2}\|\phi_i(T)\|_{W^{L,p}}+\|1-\phi_3(T)\|_{W^{L,p}}\right)\\
&\le CC_2(R_1h,T)\sup_{0\le t\le T}\left(\sum_{i=1}^{2}\|\phi_i\|_{L^1}+\|1-\phi_3\|_{L^1}\right)\\
&\le CC_1(R)C_2(R_1h,T)\sup_{0\le t\le T}\left(\sum_{i=1}^{2}\|\phi_{i,0}\|_{L^1}+\|1-\phi_{3,0}\|_{L^1}\right)\\
&\le\sum_{i=1}^{2}\|\phi_i(T)\|_{L^\infty}+\|\phi_3(T)\|_{L^\infty}\le R.
\end{aligned}
\tag{4.34}
$$

Therefore, by Theorem 3.1 and Lemma 4.2, $(\phi_1, \phi_2, 1-\phi_3)$ can be extended up to time $2T$ and $(\phi_1, \phi_2, 1-\phi_3)$ satisfies

$$\begin{cases} \sum_{i=1}^{2} \|\phi_i\|_{L^\infty} + \|1-\phi_3\|_{L^\infty} \leq 2R, \quad 0 \leq t \leq 2T, \\ \sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \leq C_1(R) \left(\sum_{i=1}^{2} \|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} \right), \quad 0 \leq t \leq 2T. \end{cases} \tag{4.35}$$

Now taking $t = 2T$, and replacing s_j with $s_j + T$ $(j = 1, 2, \cdots, L)$ in (4.1), we can conclude that

$$\sum_{i=1}^{2} \|\phi_i(2T)\|_{W^{L,p}} + \|1-\phi_3(2T)\|_{W^{L,p}} \leq C_2(R, h, T) \sup_{0 \leq t \leq 2T} \left(\sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \right). \tag{4.36}$$

Assume that $(\phi_1, \phi_2, 1-\phi_3)$has been defined up to time kT for some $k \in \mathbb{Z}_+$ such that

$$\begin{cases} \sum_{i=1}^{2} \|\phi_i\|_{L^\infty} + \|1-\phi_3\|_{L^\infty} \leq 2R, \quad 0 \leq t \leq kT, \\ \sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \leq C_1(R) \left(\sum_{i=1}^{2} \left\|\phi_{i,0}\right\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} \right), \quad 0 \leq t \leq kT. \end{cases} \tag{4.37}$$

Taking $t = kT$, and replacing s_j with $s_j + kT$ $(j = 1, 2, \cdots, L)$ in (4.1), it yields that

$$\sum_{i=1}^{2} \|\phi_i(kT)\|_{W^{L,p}} + \|1-\phi_3(kT)\|_{W^{L,p}} \leq C_2(R, h, T) \sup_{0 \leq t \leq kT} \left(\sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \right). \tag{4.38}$$

By (4.33), (4.35), and (4.38), we have

$$\begin{aligned} \sum_{i=1}^{2} \|\phi_i(kT)\|_{L^\infty} + \|\phi_3(kT)\|_{L^\infty} &\leq C \left(\sum_{i=1}^{2} \|\phi_i(kT)\|_{W^{L,p}} + \|1-\phi_3(kT)\|_{W^{L,p}} \right) \\ &\leq CC_2(R_1 h, T) \sup_{0 \leq t \leq T} \left(\sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \right) \\ &\leq CC_1(R) C_2(R_1 h, T) \sup_{0 \leq t \leq T} \left(\sum_{i=1}^{2} \|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} \right) \\ &\leq \sum_{i=1}^{2} \|\phi_i(T)\|_{L^\infty} + \|\phi_3(T)\|_{L^\infty} \leq R. \end{aligned} \tag{4.39}$$

Then, applying Theorem 3.1 and Lemma 4.2 again, (ϕ_1, ϕ_2, ϕ_3) can be extended up to time $(k+1)T$ and (ϕ_1, ϕ_2, ϕ_3) satisfies

$$\begin{cases} \sum_{i=1}^{2} \|\phi_i\|_{L^\infty} + \|1-\phi_3\|_{L^\infty} \leq 2R, \quad 0 \leq t \leq (k+1)T, \\ \sum_{i=1}^{2} \|\phi_i\|_{L^1} + \|1-\phi_3\|_{L^1} \leq C_1(R) \left(\sum_{i=1}^{2} \|\phi_{i,0}\|_{L^1} + \|1-\phi_{3,0}\|_{L^1} \right), \quad 0 \leq t \leq (k+1)T. \end{cases} \tag{4.40}$$

Thus, we establish the existence of the solution (ϕ_1, ϕ_2, ϕ_3) in all $t > 0$ by induction. The proof is complete. □

5. Conclusions

This paper investigates the Cauchy problem for the three-component Cahn-Hilliard system in 2D whole space $\mathbb{R}^2$, with physically relevant parameters. By constructing a successive approximation scheme and combining it with a series of refined energy estimates and regularity-enhancing techniques, we have proven the existence of a unique global smooth solution that remains bounded for all time $t > 0$. This result establishes, for the first time, a systematic global well-posedness theory for the Cauchy problem of the three-phase Cahn-Hilliard model, addressing a gap in the existing literature.

The core difficulty of this work lies in handling the complex coupling terms introduced by the multi-phase nonlinear potential F. We have overcome the challenges encountered when applying classical methods (such as the Hoff-Smoller approach) directly to this three-phase system. By establishing a priori estimates for the higher-order derivatives of the solution in L^2 and L^∞ spaces, we successfully controlled the long-term growth of the nonlinear terms, thereby ensuring the global existence and regularity of the solution.

Our theoretical findings carry the following implications and insights:

(1) Theoretical Guarantee: It provides a solid mathematical foundation for the use of the three-phase Cahn-Hilliard model in numerical simulations. It ensures that the solution approximated by discrete schemes remains physically reasonable (smooth and bounded) over long timescales.

(2) Potential for Generalization: Although the analysis was primarily conducted on the 2D whole domain, the energy estimation methods employed are robust. We believe that, through similar arguments combined with the regularity theory for elliptic operators under appropriate boundary conditions, the main conclusions of this paper can extended the initial-boundary problem in the bounded domains with homogeneous Neumann boundary conditions.

Naturally, this study also has limitations and points to potential future research directions:

(1) Coupling with hydrodynamics: In practical applications, phase fields are often coupled with fluid flow (e.g., the Cahn-Hilliard-Navier-Stokes system). Extending the stability analysis presented here to include the advection term $u \cdot \nabla\varphi$ is a crucial step toward more comprehensive physical models.

(2) Implications for high-order numerical schemes: Our theoretical analysis shows that the solution's regularity is governed by the Sobolev norm of the initial data. This implies that in numerical computations, if high-order schemes with spatial and temporal discretization errors of $O(h^k)$ and $O(\Delta t^m)$, respectively, are used, our theoretical results can guarantee the long-term behavior of the numerical solution, provided the discretized "numerical initial data" is sufficiently accurate to satisfy the corresponding smallness condition in the high-order Sobolev norm. Quantifying the precise relationship between the discrete parameters h, Δt, and the theoretical constant presents an interesting problem that bridges theory and computation.

In summary, this work contributes to the mathematical analysis of the three-phase Cahn-Hilliard system, paving the way for its further application and development in the simulation of complex multiphase flows.

Author contributions

Ning Duan: writing-original draft, supervision, formal analysis, methodology, supervision; Yinghao Wang: writing-original draft, supervision, formal analysis, methodology, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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