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Research article

Global robust stability of switched nonlinear positive time-varying delay systems with interval uncertainties and all unstable subsystems

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Abstract: This paper reports on results concerning the global robust stability and global asymptotic stability of switched nonlinear positive time-varying delay systems when all subsystems are unstable. Based on the assumptions of admissible sector nonlinearity and interval uncertainty, a new nonlinear multiple discretized copositive Lyapunov-Krasovskii functional was constructed, and a mode-dependent dwell time switching technique was utilized to derive sufficient conditions for global robust stability of the system under consideration. Furthermore, when the considered systems did not have without interval uncertainty, adequacy criteria were also proposed to ensure global asymptotic stability. By comparing with previous results, it was verified that our results were less conservative and more general than those of older studies. The theoretical results obtained were presented in illustrative examples applicable to delayed neural networks.

Keywords: nonlinear dynamical systems; global robust stability; global asymptotic stability; switched positive systems; all unstable subsystems; time-varying delay systems; interval uncertainty analysis
Mathematics Subject Classification: 34D05, 34D23, 93D05, 93D20

1. Introduction

The mathematical modeling of complex physical processes often leads to nonlinear dynamical systems (NDSs) [1], which are found in various areas such as mechanical systems, biological systems, control systems [2, 3], shallow water wave models [4, 5], and multi-agent systems [6]. Among these, a vital class of hybrid systems is made up of switched systems, which include several subsystems and a governing law for switching between them [7]. For decades, the stability problem of switched systems, which refers to their ability to return to a normal state when an external disturbance ceases, has garnered considerable attention [8, 9]. In particular, numerous results on the stability of switched nonlinear systems (SNSs) have been widely investigated [10–12]. Furthermore, there exists a considerable body of literature on stability of stochastic (switched) systems which are in the form of linear systems [13, 14] and nonlinear systems [15, 16].

There exists a special class of positive systems whose initial states and long-term state trajectories always remain nonnegative; these are called switched positive systems (SPSs). Examples of SPSs applications can be found extensively, such as in nonnegative dynamics of biochemical reaction networks [17], wireless power control [18], water-quality model [19], consensus of multi-agent systems [20], and delayed neural networks (DNNs) model [21]. It is worth noting that the aforementioned works analyze both the switching techniques and the types of subsystems. A common strategy used to study switching signals involves time-dependent switching techniques, such as dwell time (DT) and mode-dependent dwell time (MDDT) methods. There are three major conceptual frameworks for analyzing the stability of SPSs based on the specific characteristics of their subsystems. First is the stability analysis of SPSs when all subsystems are stable; most of the research in this field is aimed at solving this problem [22–24]. Second is when partial subsystems are unstable [25–27]. Third, a challenging problem that arises in this domain is how all unstable subsystems (AUSs) interact and whether the entire switched system can control its trajectory behavior to remain stable over a long period. This problem has received substantial interest and serves as the initial inspiration for this research. As mentioned in [28], a practical example of a switching system consisting entirely of unstable subsystems that work together to achieve overall stability is a highly maneuverable aircraft technology flight control system. The aircraft's natural aerodynamic subsystems are all unstable due to it being designed for high agility. The stability of the aircraft is achieved through a fast-acting, switched flight control system by employing techniques like state-dependent or DT switching. By carefully designing the switching strategy, the overall aircraft system remains stable and controllable, allowing the pilot to safely fly. However, without active control, they would quickly become uncontrollable and crash. Moreover, in real-world systems, unavoidable time delays often occur, which can cause poor system performance and instability [29, 30]. In practice, the existence of uncertainties that may arise from modeling errors, external perturbations, and parameter fluctuations can also directly affect the system and cause instability [31–33]. Thus, it is necessary and meaningful to further investigate the robust stability of SPSs with delays and uncertainties, which further motivates our research. With these inspirations, we have therefore investigated the global robust stability (GRS) and global asymptotic stability (GAS) for SPSs, including delays, uncertainties, and AUSs. Related research articles will be briefly reviewed below.

For linear switched systems with AUSs, Feng et al. [34] utilized the MDDT switching approach to examine both GRS and GAS of linear SPSs with interval uncertainties, but the existence of time-delay was not considered. Meanwhile, Liu et al. [35] showed that the DT switching law was applied to stabilize linear SPSs with time-varying delay. The problem of GRS for linear SPSs with time-varying delay and interval uncertainties was studied by employing the MDDT switching technique in the context of continuous-time systems [36] and discrete-time systems [37]. Recently, Zhang et al. [38] derived stability conditions of linear SPSs with delays using the DT switching scheme. However, the linear SPSs studied in [35, 38] did not take into account interval uncertainties or the MDDT switching rule. More recently, Zhang et al. [39] established sufficient conditions for stability and L_1-gain analysis of time-varying linear impulsive SPSs by employing interval DT and mode-dependent interval DT switching laws. It is essential to highlight that NDSs are often considered more applicable and general than linear systems. Therefore, the issue of stability analysis for SNSs has received much attention, and its literature review is summarized as follows. In [40], the stabilization for SNSs with AUSs using DT switching was addressed, but this result did not study time-delay or interval uncertainties. Adequacy criteria to guarantee exponential stability of nonlinear SPSs with time-delays and partially unstable subsystems for continuous-time systems [21] and discrete-time systems [27] were obtained. As far as we know, no previous research has analyzed the GRS and GAS for nonlinear SPSs containing time-varying delays and interval uncertainties where all subsystems are allowed to be unstable.

For this study, it was of interest to investigate the stability analysis of SNSs with time-varying delay as below

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)} g(x(t)) + B_{\sigma(t)} g(x(t-\beta(t))) + C_{\sigma(t)} x(t), & t \geq 0, \\ x(t) = \psi(t), & t \in [-\beta, 0], \end{cases} \tag{1.1}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), ..., g_n(x_n(t)))^T \in \mathbb{R}^n$ are the system state, and nonlinear function, respectively. $\sigma(t)$ is a piecewise constant function defining the switching signal, $\sigma(t) : [0, +\infty) \to \underline{N} = \{1, 2, ..., N\}$, where N stands for the number of subsystems or modes of the switched system and $N > 1$. Also, it is assumed to be continuous from the right everywhere for a switching sequence $0 = t_0 < t_1 < t_2 < \cdots < t_{l-1} < t_l < \cdots < +\infty$, where t_l represents the lth switching instant, and l belongs to the set of all nonnegative integers ($l \in \mathbb{N}_0$). For the switching mechanism, at instant t_l, the $\sigma(t_l)$th (or ith) subsystem is activated for $t \in [t_l, t_{l+1})$. Consequently, the system (1.1) switches from the $\sigma(t_{l-1})$th (or jth) subsystem to the ith subsystem, where $i, j \in \underline{N}$ and $i \neq j$. As in [34, 36], we assume that the state of system (1.1) dose not jump at each switching instant. A_i, B_i, and C_i are constant matrices with appropriate dimensions for each $i \in \underline{N}$. The time-varying delay function $\beta(t)$ satisfies $0 \leq \beta(t) \leq \beta$ and $\dot{\beta}(t) \leq d < 1$, where β and d are known constants. In addition, $\psi(t) : [-\beta, 0] \to \mathbb{R}^n$ is a vector-valued initial function with $\|\psi\|_\beta = \sup_{-\beta \leq t \leq 0} \|\psi(t)\|_2$. The main contributions of this work are as follows:

1. Both the positivity and novel nonlinear multiple discretized copositive Lyapunov-Krasovskii functionals for system (1.1), including interval uncertainties and AUSs, are proved and established, respectively. The MDDT switching approach is applied to formulate adequacy criteria for the GRS of system (1.1) with interval uncertainties and for the GAS of system (1.1) without interval uncertainties. Moreover, it is worth noting that system (1.1) incorporating interval uncertainties and AUSs is positive. Thus, in this paper, its "global" stability refers to stability in the nonnegative quadrant $\mathbb{R}^n_+$ ($\mathbb{R}^n_+ = \{\nu \in \mathbb{R}^n : \nu$ is the nonnegative vector and $\nu \neq 0\}$) rather than the entire $\mathbb{R}^n$ space.

2. Unlike [21,41], where the studied subsystems are all stable and partially unstable, respectively, we assume that all subsystems are unstable. It should be pointed out that the structure of system zhijie (1.1) represents a large class of SNSs and is much more general than the switched linear systems studied in [35,38,39].
3. The derived theoretical results can be applied to the DNNs modeling in [42] and compared with [36] to show that less conservative results can be obtained by our method.

The rest of this paper is organized as follows. Section 2 presents the system descriptions and preliminaries. Main theoretical achievements are shown in Section 3. Numerical examples are demonstrated in Section 4. Finally, Section 5 concludes this paper.

Notations: The following notations are used throughout this article. $\mathbb{N}$ denotes the set of positive integers. For any $K \in \mathbb{N}$, $\underline{K}_0 = \{0, 1, 2, ..., K\}$. $\mathbf{0}_n$ is the n-dimensional column vector with all entries being 0. Let $v \in \mathbb{R}^n$. A nonnegative vector and a positive vector are denoted by $v \succeq 0$ and $v \succ 0$, respectively. For a given $v \in \mathbb{R}^n$, its 1−norm and Euclidean norm are defined by $\|v\|_1 = \sum_{i=1}^n |v_i|$ and $\|v\|_2 = \left(\sum_{i=1}^n v_i^2\right)^{\frac{1}{2}}$, respectively. $\underline{\omega}(v)$ stands for the minimal elements of $v \in \mathbb{R}^n$. I_n is the $n \times n$ dimensional identity matrix, and A^T is the transpose of matrix A. The notation $A \succeq 0$ indicates that all entries of matrix $A \in \mathbb{R}^{n\times n}$ are nonnegative. A matrix is said to be a Metzler matrix if its off-diagonal elements are all nonnegative. The notation diag{...} is a block-diagonal matrix. For a function $\varphi(t)$ defined on $[0, +\infty)$, $\varphi(t^-) = \lim_{t\to t^-} \varphi(t)$ and $\varphi(t^+) = \lim_{t\to t^+} \varphi(t)$. Furthermore, $D^+ f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$, where D^+ represents the upper right Dini derivative of the function $f(x)$.

2. System descriptions and preliminaries

First, we introduce some definitions and assumptions, which play a significant role in obtaining our theoretical results in this paper.

Definition 2.1. *[35] System (1.1) is said to be positive if for any initial condition $\psi(t) \succeq 0$, $t \in [-\beta, 0]$ and any switching signal $\sigma(t)$, the corresponding trajectory $x(t) \succeq 0$ holds for all $t \geq 0$.*

Assumption 2.2. *The nonlinear function $g(x)$ lies in sector fields satisfying*

$$\varepsilon_1 x_p^2 \leq g_p(x_p)x_p \leq \varepsilon_2 x_p^2,$$

for $x_p \in \mathbb{R}$ and $p = 1, 2, ..., n$, where $0 < \varepsilon_1 \leq \varepsilon_2$, and $g_p(0) = 0$.

Remark 2.3. *According to the proof in [41] and the remark in [21], system (1.1) with $\psi(t) \succeq 0$, $t \in [-\beta, 0]$ and any switching signal is positive under Assumption 2.2 if and only if A_i and C_i are Metzler matrices, and $B_i \succeq 0$ for all $i \in \underline{N}$. Moreover, without loss of generality, it can be referred to Assumption 2.2 and $x_p \geq 0$ as $\varepsilon_1 x_p \leq g_p(x_p) \leq \varepsilon_2 x_p$ for $p = 1, 2, ..., n$, where $0 < \varepsilon_1 \leq \varepsilon_2$, and $g_p(0) = 0$.*

Definition 2.4. *[35] For any admissible sector nonlinearities satisfying Assumption 2.2 and arbitrary switching signal, system (1.1) under $\sigma(t)$ is said to be*

1. *uniformly stable (US) with respect to $\sigma(t)$ if $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\|x(t)\|_2 < \epsilon$, $\forall t \in [0, +\infty)$ whenever $\|\psi\|_\beta < \delta$;*

2. *globally uniformly stable (GUS) with respect to* $\sigma(t)$ *if* $\forall\epsilon > 0$, $\forall\delta > 0$ *such that* $\|x(t)\|_2 < \epsilon$, $\forall t \in [0, +\infty)$ *whenever* $\|\psi\|_\beta < \delta$*;*
3. *globally uniformly asymptotically stable (GUAS) with respect to* $\sigma(t)$ *if system (1.1) is GUS and satisfies* $\lim_{t\to+\infty} x(t) = 0$.

In general, actual systems can be modeled by NDSs in the form of interval uncertainties. Therefore, the definition and assumption of the interval uncertainties for studying the GRS of system (1.1) is stated as follows:

Definition 2.5. *[34, 36] For system (1.1),* A_i *and* D_i *are supposed to be interval uncertain; namely,*

$$\underline{A}_i \leq A_i \leq \overline{A}_i,$$

$$\underline{B}_i \leq B_i \leq \overline{B}_i,$$

and

$$\underline{C}_i \leq C_i \leq \overline{C}_i,$$

where $\underline{A}_i$, $\underline{B}_i$, $\underline{C}_i$, $\overline{A}_i$, $\overline{B}_i$, *and* $\overline{C}_i$ *are the given constant system matrices with appropriate dimensions for all* $i \in \underline{N}$.

Assumption 2.6. *For each* A_i, B_i, *and* C_i *in system (1.1), there are known Metzler matrices* $\underline{A}_i$, $\underline{C}_i$ *and matrices* $\underline{B}_i \geq 0$ *such that* $A_i \in [\underline{A}_i, \overline{A}_i]$, $C_i \in [\underline{C}_i, \overline{C}_i]$ *and* $B_i \in [\underline{B}_i, \overline{B}_i]$, *where* $\underline{A}_i$, $\underline{C}_i$, $\underline{B}_i$, $\overline{A}_i$, $\overline{C}_i$, $\overline{B}_i$ *are the given constant system matrices with appropriate dimensions for all* $i \in \underline{N}$.

To analyze the GRS of switched nonlinear positive time-varying delay systems with interval uncertainties, both DT and MDDT switching laws are typically applied. Specifically, when all unstable subsystems are active, we aim to use these switching laws to ensure the overall system operates stably over a long period. However, the MDDT switching rule is generally less conservative and more general than the DT switching rule. Therefore, the MDDT switching strategy is utilized in this article.

Definition 2.7. *[32, 34] For the length between successive switching moments during which the ith mode of system (1.1) is activated, if there exists a constant* $\tau_{i,l} > 0$ *such that* $\tau_{i,l} = t_{l+1} - t_l$ *holds for any* $i \in \underline{N}$, $l \in \mathbb{N}_0$, *then the constant* $\tau_{i,l}$ *is called MDDT of system (1.1).*

To avoid system instability owing to too small or too large DT switching, the MDDT switching technique is confined by a pair of upper and lower bounds defined in the next definition.

Definition 2.8. *[34]* $\tau_{i,l} \in [\tau_{i,min}, \tau_{i,max}]$, *where* $\tau_{i,min} = \inf_{l\in\mathbb{N}_0} \tau_{i,l}$, $\tau_{i,max} = \sup_{l\in\mathbb{N}_0} \tau_{i,l}$, $0 < \tau_{i,min} \leq \tau_{i,max}$, $i \in \underline{N}$, $l \in \mathbb{N}_0$. *In addition, the notation* $\Lambda_{[\tau_{i,min},\tau_{i,max}]}$ *is the set of all switching techniques with MDDT* $\tau_{i,l} \in [\tau_{i,min}, \tau_{i,max}]$, $i \in \underline{N}$, $l \in \mathbb{N}_0$.

3. Main results

For convenience, we first define important symbols used in our main theorem as follows:

$$\widetilde{B} = \left(\overline{b}_{rs}\right) \in \mathbb{R}^{n\times n}, \quad \overline{b}_{rs} = \max_{i\in\underline{N}} \left\{\overline{B}_i^{(rs)}\right\}, \tag{3.1}$$

where $\overline{B}_i^{(rs)}$ refers to the r^{th} row and s^{th} column element of system matrices $\overline{B}_i$, for all $i \in \underline{N}$.

Theorem 3.1. *Consider system (1.1) with AUSs satisfying Assumption 2.2 and Assumption 2.6. Given constants* $0 < \varepsilon_1 \leq \varepsilon_2$, $0 < \mu_i < 1$, $0 < \xi_i$, $0 < \tau_{i,min}$, $i \in \underline{N}$, *and* $K \in \mathbb{N}$. *System (1.1) containing AUSs is positive and GUAS with respect to* $\sigma(t) \in \Lambda_{[\tau_{i,min},\tau_{i,max}]}$ *if there exists constants* $\tau_{i,max} \geq \tau_{i,min}$ *and* $\nu_{i,q} \succ 0$, $i \in \underline{N}$, $q \in \underline{K}_0$ *satisfying the following conditions:*

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q} \prec 0, \tag{3.2}$$

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q+1} \prec 0, \tag{3.3}$$

$$\widetilde{B}^T\left(\Delta_{i,q} - \nu_{i,q}\right) \prec 0, \tag{3.4}$$

$$\widetilde{B}^T\left(\Delta_{i,q} - \nu_{i,q+1}\right) \prec 0, \tag{3.5}$$

$$\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,K} \prec 0, \tag{3.6}$$

$$\nu_{i,0} - \mu_i\nu_{j,K} \preceq 0, \tag{3.7}$$

$$\ln\mu_i + \xi_j\tau_{j,max} < 0, \tag{3.8}$$

for any $q = 0, 1, ..., K-1$, *and for any* $i, j \in \underline{N}$, $i \neq j$, *where*

$$\Delta_{i,q} = \frac{\left(\nu_{i,q+1} - \nu_{i,q}\right)K}{\tau_{i,min}}, \tag{3.9}$$

and $\widetilde{B}$ *is mentioned in (3.1).*

Proof. We divide the proof process into the following two steps.
Step 1. We will prove that system (1.1) is positive.

By Assumption 2.6, we obtain that there exist Metzler matrices $\underline{A}_i$, $\underline{C}_i$, and matrices $\underline{B}_i \geq 0$. These satisfy $A_i \in [\underline{A}_i, \overline{A}_i]$, $C_i \in [\underline{C}_i, \overline{C}_i]$, and $B_i \in [\underline{B}_i, \overline{B}_i]$ for all $i \in \underline{N}$. This implies that A_i and C_i are also Metzler matrices, and $B_i \geq 0$ for all $i \in \underline{N}$. For any $t \in [-\beta, 0]$, let $\psi(t)$ be the vector-valued initial function with $\psi(t) \geq 0$. By Assumption 2.2, according to the proof in [41, Lemma 2] and the detail in [21, Remark 1], the trajectory of system (1.1) satisfies $x(t) \geq 0$ for all $t \geq 0$ and for any switching signal $\sigma(t)$. Therefore, system (1.1) with $\psi(t) \geq 0$, $t \in [-\beta, 0]$, and any switching signal $\sigma(t)$ is positive.
Step 2. We will prove that system (1.1) is GUAS under the MDDT switching strategy satisfying condition (3.8).

In the literature [21, 41], switched nonlinear positive time-varying delay systems with all stable subsystems and partially unstable subsystems, respectively, are studied. These works employed the nonlinear Lyapunov-Krasovskii functional to analyze system stability. Nevertheless, this approach does not apply when all subsystems are unstable. In this paper, we establish the nonlinear multiple discretized copositive Lyapunov-Krasovskii functional described in the form of

$$V_i(t, x(t)) = (1-d)x^T(t)\nu_i(t) + \int_{t-\beta(t)}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_i(t)d\alpha + \int_{-\beta}^{0}\int_{t+w}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha dw, \tag{3.10}$$

for any $i \in \underline{N}$, and combine the MDDT switching rule to stabilize system (1.1) including AUSs. In (3.10), the vector function $\nu_i(t)$, $i \in \underline{N}$ is constructed by assuming that $t \in [t_l, t_{l+1}) = [t_l, t_l + \tau_{i,min}) \cup [t_l + \tau_{i,min}, t_{l+1})$, $l \in \mathbb{N}_0$, for any $t > 0$, and $\tau_{i,min}$, $i \in \underline{N}$ defined as in Definition 2.8. The interval $[t_l, t_l + \tau_{i,min})$ is divided into K segments with equal length $h_i = \frac{\tau_{i,min}}{K}$. Furthermore, $\Omega^i_{l,q} = [t_l + qh_i, t_l + (q+1)h_i)$, $q = 0, 1, ..., K-1$ and $[t_l, t_l + \tau_{i,min}) = \bigcup_{q=0}^{K-1} \Omega^i_{l,q}$ are also denoted. Namely, for any $i \in \underline{N}$,

$$\begin{aligned} \nu_i(t) &= \nu_i(t_l + qh_i + \gamma(t)h_i) \\ &= \begin{cases} (1-\gamma(t))\nu_{i,q} + \gamma(t)\nu_{i,q+1}, & t \in \Omega^i_{l,q}, \quad q = 0, 1, ..., K-1, \\ \nu_{i,K}, & t \in [t_l + \tau_{i,min}, t_{l+1}), \end{cases} \end{aligned} \tag{3.11}$$

where $l \in \mathbb{N}_0$, $\gamma(t) = \frac{t - t_l - qh_i}{h_i}$ with $0 \le \gamma(t) \le 1$, and $\nu_{i,q}$ are positive vectors for $i \in \underline{N}$, $q \in \underline{K}_0$. Based on the apportionment of the defined time interval for (3.10) and (3.11), we split the proof into the following two cases.

Case (i). When $t \in \Omega^i_{l,q} \subset [t_l, t_l + \tau_{i,min})$, it follows that

$$D^+\nu_i(t) = \frac{\nu_{i,q+1} - \nu_{i,q}}{h_i},$$

namely,

$$D^+\nu_i(t) = \Delta_{i,q},$$

where $\Delta_{i,q}$ is defined as in (3.9). According to (3.10), (3.11), and along the trajectory of system (1.1), we get

$$\begin{aligned} D^+V_i(t, x(t)) = (1-d)\Big[&x^T(t)\Delta_{i,q} + g^T(x(t))A_i^T\nu_i(t) + g^T(x(t-\beta(t)))B_i^T\nu_i(t) + x^T(t)C_i^T\nu_i(t)\Big] \\ &+ \int_{t-\beta(t)}^t g^T(x(\alpha))\widetilde{B}^T\Delta_{i,q}d\alpha + g^T(x(t))\widetilde{B}^T\nu_i(t) - g^T(x(t-\beta(t)))\widetilde{B}^T\nu_i(t)(1-\dot{\beta}(t)) \\ &+ \beta g^T(x(t))\widetilde{B}^T\nu_i(t) - \int_{t-\beta}^t g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha, \end{aligned}$$

for all $i \in \underline{N}$. By Assumption 2.6, we obtain

$$\begin{aligned} D^+V_i(t, x(t)) \le (1-d)\Big[&x^T(t)\Delta_{i,q} + g^T(x(t))\overline{A}_i^T\nu_i(t) + g^T(x(t-\beta(t)))\overline{B}_i^T\nu_i(t) + x^T(t)\overline{C}_i^T\nu_i(t)\Big] \\ &+ \int_{t-\beta(t)}^t g^T(x(\alpha))\widetilde{B}^T\Delta_{i,q}d\alpha + g^T(x(t))\widetilde{B}^T\nu_i(t) - g^T(x(t-\beta(t)))\widetilde{B}^T\nu_i(t)(1-\dot{\beta}(t)) \\ &+ \beta g^T(x(t))\widetilde{B}^T\nu_i(t) - \int_{t-\beta}^t g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha, \end{aligned}$$

for all $i \in \underline{N}$. From $\dot{\beta}(t) \le d$ and $\overline{B}_i \le \widetilde{B}$ mentioned as in (3.1) for all $i \in \underline{N}$, one has

$$\begin{aligned} D^+V_i(t, x(t)) \le (1-d)\Big[&x^T(t)\Delta_{i,q} + g^T(x(t))\overline{A}_i^T\nu_i(t) + x^T(t)\overline{C}_i^T\nu_i(t)\Big] + \int_{t-\beta(t)}^t g^T(x(\alpha))\widetilde{B}^T\Delta_{i,q}d\alpha \\ &+ g^T(x(t))\widetilde{B}^T\nu_i(t) + \beta g^T(x(t))\widetilde{B}^T\nu_i(t) - \int_{t-\beta}^t g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha. \end{aligned}$$

We observe that

$$
\begin{aligned}
D^+V_i(t,x(t)) - \xi_i V_i(t,x(t)) \le{}& (1-d)\left[x^T(t)\Delta_{i,q} + g^T(x(t))\overline{A}_i^T\nu_i(t) + x^T(t)\overline{C}_i^T\nu_i(t)\right] \\
&+ \int_{t-\beta(t)}^{t} g^T(x(\alpha))\widetilde{B}^T\Delta_{i,q}d\alpha + g^T(x(t))\widetilde{B}^T\nu_i(t) + \beta g^T(x(t))\widetilde{B}^T\nu_i(t) \\
&- \int_{t-\beta}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha - \xi_i(1-d)x^T(t)\nu_i(t),
\end{aligned}
$$

when $0 < \xi_i$ for all $i \in \underline{N}$. By Assumption 2.2, it is transformed into

$$
\begin{aligned}
D^+V_i(t,x(t)) - \xi_i V_i(t,x(t)) \le{}& (1-d)\left[\frac{1}{\varepsilon_1}g^T(x(t))\Delta_{i,q} + g^T(x(t))\overline{A}_i^T\nu_i(t) + \frac{1}{\varepsilon_1}g^T(x(t))\overline{C}_i^T\nu_i(t)\right] \\
&+ \int_{t-\beta(t)}^{t} g^T(x(\alpha))\widetilde{B}^T\Delta_{i,q}d\alpha + g^T(x(t))\widetilde{B}^T\nu_i(t) + \beta g^T(x(t))\widetilde{B}^T\nu_i(t) \\
&- \int_{t-\beta}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha - \frac{\xi_i(1-d)}{\varepsilon_2}g^T(x(t))\nu_i(t),
\end{aligned}
$$

then

$$
\begin{aligned}
D^+V_i(t,x(t)) - \xi_i V_i(t,x(t)) \le{}& g^T(x(t))\left\{\left(\frac{1-d}{\varepsilon_1}\right)\Delta_{i,q} + (1-d)\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_i(t)\right\} \\
&+ \int_{t-\beta(t)}^{t} g^T(x(\alpha))\widetilde{B}^T\left[\Delta_{i,q} - \nu_i(\alpha)\right]d\alpha,
\end{aligned}
$$

for all $i \in \underline{N}$. It leads to

$$
\begin{aligned}
&\left(\frac{1-d}{\varepsilon_1}\right)\Delta_{i,q} + (1-d)\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_i(t) \\
&\quad = (1-\gamma(t))\left\{\left(\frac{1-d}{\varepsilon_1}\right)\Delta_{i,q} + (1-d)\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q}\right\} \\
&\qquad + \gamma(t)\left\{\left(\frac{1-d}{\varepsilon_1}\right)\Delta_{i,q} + (1-d)\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q+1}\right\},
\end{aligned}
$$

and

$$
\widetilde{B}^T[\Delta_{i,q} - \nu_i(\alpha)] = (1-\gamma(\alpha))\widetilde{B}^T\left(\Delta_{i,q} - \nu_{i,q}\right) + \gamma(\alpha)\widetilde{B}^T\left(\Delta_{i,q} - \nu_{i,q+1}\right),
$$

for all $i \in \underline{N}$ and for any $q = 0, 1, ..., K-1$. According to conditions (3.2)–(3.5), we obtain

$$
D^+V_i(t,x(t)) - \xi_i V_i(t,x(t)) < 0,
$$

$t \in \Omega^i_{l,q}$, which implies

$$
D^+V_i(t,x(t)) < \xi_i V_i(t,x(t)), \tag{3.12}
$$

$t \in \bigcup_{q=0}^{K-1}\Omega^i_{l,q} = [t_l, t_l + \tau_{i,min})$.

Case (ii). When $t \in [t_l + \tau_{i,min}, t_{l+1})$, it yields that

$$
D^+\nu_i(t) = \mathbf{0}_n.
$$

By Assumption 2.6, $\dot{\beta}(t) \leq d$, $\overline{B}_i \leq \widetilde{B}$, Assumption 2.2, and along the trajectory of system (1.1), we have the following:

$$\begin{aligned} D^+V_i(t,x(t)) - \xi_i V_i(t,x(t)) &\leq (1-d)g^T(x(t))\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_i(t) \\ &\quad - \int_{t-\beta}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_i(\alpha)d\alpha \\ &= (1-d)g^T(x(t))\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,K} \\ &\quad - \int_{t-\beta}^{t} g^T(x(\alpha))\widetilde{B}^T\nu_{i,K}d\alpha \\ &\leq (1-d)g^T(x(t))\left[\overline{A}_i^T + \frac{1}{\varepsilon_1}\overline{C}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,K}, \end{aligned}$$

when $0 < \xi_i$ for all $i \in \underline{N}$. Utilizing condition (3.6), it is immediate that

$$D^+V_i(t,x(t)) < \xi_i V_i(t,x(t)), \tag{3.13}$$

$t \in [t_l + \tau_{i,min}, t_{l+1})$. Merging (3.12) with (3.13), it is obvious that

$$D^+V_i(t,x(t)) < \xi_i V_i(t,x(t)), \tag{3.14}$$

$t \in [t_l, t_{l+1}),\ \ l \in \mathbb{N}_0$. Taking the integral of both sides of (3.14) over $[t_l, t)$ for $t \in [t_l, t_{l+1}),\ l \in \mathbb{N}_0$, it follows that

$$V_i(t,x(t)) < \mathrm{e}^{\xi_i(t-t_l)}V_i(t_l, x(t_l)). \tag{3.15}$$

Applying condition (3.7), it implies that

$$\nu_i(t_l) \leq \mu_i \nu_j(t_l^-), \tag{3.16}$$

for all $i, j \in \underline{N},\ i \neq j$. Blending (3.10) with (3.16), it can be seen that

$$V_i(t_l, x(t_l)) \leq \mu_i V_j(t_l^-, x(t_l^-)), \tag{3.17}$$

for all $i, j \in \underline{N},\ i \neq j$. Combining (3.15) with (3.17), we can derive

$$\begin{aligned} V_{\sigma(t_l)}(t,x(t)) &< \mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}V_{\sigma(t_l)}(t_l, x(t_l)) \\ &\leq \mu_{\sigma(t_l)}\mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}V_{\sigma(t_{l-1})}(t_l^-, x(t_l^-)) \\ &< \mu_{\sigma(t_l)}\mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}\mathrm{e}^{\xi_{\sigma(t_{l-1})}(t_l-t_{l-1})}V_{\sigma(t_{l-1})}(t_{l-1}, x(t_{l-1})) \\ &\ \vdots \\ &< \mu_{\sigma(t_l)}\mu_{\sigma(t_{l-1})}\cdots\mu_{\sigma(t_1)}\mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}\mathrm{e}^{\xi_{\sigma(t_{l-1})}(t_l-t_{l-1})}\cdots\mathrm{e}^{\xi_{\sigma(t_0)}(t_1-t_0)}V_{\sigma(t_0)}(t_0, x(t_0)) \\ &= \mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}\left(\mu_{\sigma(t_l)}\mathrm{e}^{\xi_{\sigma(t_{l-1})}(t_l-t_{l-1})}\right)\cdots\left(\mu_{\sigma(t_1)}\mathrm{e}^{\xi_{\sigma(t_0)}(t_1-t_0)}\right)V_{\sigma(t_0)}(t_0, x(t_0)) \\ &= \mathrm{e}^{\xi_{\sigma(t_l)}(t-t_l)}\left(\prod_{k=0}^{l-1}\mu_{\sigma(t_{k+1})}e^{\xi_{\sigma(t_k)}(t_{k+1}-t_k)}\right)V_{\sigma(t_0)}(t_0, x(t_0)) \\ &\leq \mathrm{e}^{\xi_{\sigma(t_l)}\tau_{\sigma(t_l),max}}\left(\prod_{k=0}^{l-1}\mu_{\sigma(t_{k+1})}\mathrm{e}^{\xi_{\sigma(t_k)}\tau_{\sigma(t_k),max}}\right)V_{\sigma(t_0)}(t_0, x(t_0)). \end{aligned}$$

Employing condition (3.8), it is immediate that

$$\mu_i e^{\xi_j \tau_{j,max}} < 1, \tag{3.18}$$

for all $i, j \in \underline{N},\ i \neq j$. Therefore,

$$V_{\sigma(t_l)}(t, x(t)) < e^{\xi_{\sigma(t_l)} \tau_{\sigma(t_l),max}} V_{\sigma(t_0)}(t_0, x(t_0)).$$

Let $X = \max_{i \in \underline{N}} \{\xi_i\}$ and $T = \max_{i \in \underline{N}} \{\tau_{i,max}\}$, then

$$V_{\sigma(t_l)}(t, x(t)) < e^{XT} V_{\sigma(t_0)}(t_0, x(t_0)). \tag{3.19}$$

Without loss of generality, we let $V_{\sigma(t_0)}(t_0, x(t_0)) = V_{\sigma(0)}(0, x(0))$. Recalling the nonlinear multiple discretized copositive Lyapunov-Krasovskii functional (3.10) and vector function (3.11), we can derive

$$\begin{aligned}
V_{\sigma(0)}(0, x(0)) &\leq (1-d)\|x^T(0)\|_2 \|\nu_{\sigma(0)}(0)\|_2 + \int_{-\beta}^{0} d\alpha \varepsilon_2 \sup_{-\beta \leq \alpha \leq 0} \|x^T(\alpha)\|_2 \|\widetilde{B}^T\|_2 \|\nu_{\sigma(0)}(0)\|_2 \\
&\quad + \int_{-\beta}^{0}\int_{w}^{0} \|\nu_{\sigma(0)}(\alpha)\|_2 d\alpha dw \varepsilon_2 \sup_{-\beta \leq \alpha \leq 0} \|x^T(\alpha)\|_2 \|\widetilde{B}^T\|_2 \\
&\leq (1-d)\sqrt{n}\|\psi\|_\beta \sum_{\zeta \in \underline{K}_0} \|\nu_{\sigma(0),\zeta}\|_2 + \beta\varepsilon_2 \sqrt{n}\|\psi\|_\beta \|\widetilde{B}^T\|_2 \sum_{\zeta \in \underline{K}_0} \|\nu_{\sigma(0),\zeta}\|_2 \\
&\quad + \frac{\beta^2}{2}\varepsilon_2 \sqrt{n}\|\psi\|_\beta \|\widetilde{B}^T\|_2 \sum_{\zeta \in \underline{K}_0} \|\nu_{\sigma(0),\zeta}\|_2 \\
&= \left[(1-d) + \beta\varepsilon_2\left(1 + \frac{\beta}{2}\right)\|\widetilde{B}^T\|_2\right] \sqrt{n} \sum_{\zeta \in \underline{K}_0} \|\nu_{\sigma(0),\zeta}\|_2 \|\psi\|_\beta,
\end{aligned} \tag{3.20}$$

and

$$(1-d)\varsigma \|x(t)\|_2 \leq V_{\sigma(t_l)}(t, x(t)), \tag{3.21}$$

where $\varsigma = \min_{(a,b) \in \underline{N} \times \underline{K}_0} \left\{\underline{\omega}(\nu_{a,b})\right\}$. From (3.19)–(3.21), it implies that

$$\|x(t)\|_2 \leq \Xi \|\psi\|_\beta e^{XT}, \tag{3.22}$$

where $\Xi = \left[\frac{(1-d)+\beta\varepsilon_2\left(1+\frac{\beta}{2}\right)\|\widetilde{B}^T\|_2}{(1-d)\varsigma}\right] \sqrt{n} \sum_{\zeta \in \underline{K}_0} \|\nu_{\sigma(0),\zeta}\|_2$, for all $t \geq 0$. Next, for $\epsilon > 0$, we choose

$$\|\psi\|_\beta < \delta(\epsilon) = \frac{\epsilon}{\Xi e^{XT}}.$$

Hence,

$$\|x(t)\|_2 < \epsilon,$$

for all $t \geq 0$. This implies that system (1.1) with AUSs is US.

Next, we will show that $\lim_{t \to \infty} x(t) = 0$. From (3.17), it follows that

$$V_i(t_l, x(t_l)) < \mu_i e^{\xi_j (t_l - t_{l-1})} V_j(t_{l-1}, x(t_{l-1})) < \mu_i e^{\xi_j \tau_{j,max}} V_j(t_{l-1}, x(t_{l-1})),$$

for all $i, j \in \underline{N},\ i \neq j$. Setting $\lambda = \max_{i,j \in \underline{N},\ i \neq j} \left\{ \mu_i e^{\xi_j \tau_{j,max}} \right\}$ and from (3.18), it leads to

$$V_i(t_l, x(t_l)) < \lambda V_j(t_{l-1}, x(t_{l-1})),$$

and $0 < \lambda < 1$, which implies that the sequence $V_{\sigma(t_l)}(t_l, x(t_l)),\ l \in \mathbb{N}_0$ is strictly decreasing. Applying an iterative reduction yields to

$$0 < V_{\sigma(t_l)}(t_l, x(t_l)) < \lambda V_{\sigma(t_{l-1})}(t_{l-1}, x(t_{l-1})) < \cdots < \lambda^l V_{\sigma(0)}(0, x(0)).$$

Thus,

$$\lim_{l \to \infty} V_{\sigma(t_l)}(t_l, x(t_l)) = 0.$$

Because $\nu_{\sigma(t_l)}(t_l) = \nu_{\sigma(t_l),0}$ and by assumption that there exist $\nu_{i,q} \succ 0$, for any $i \in \underline{N}$ and $q \in \underline{K}_0$, it is immediate that

$$\nu_{\sigma(t_l)}(t_l) \succ 0.$$

Owing to the positivity of system (1.1), it can be seen that

$$\lim_{l \to \infty} x(t_l) = 0 \quad \text{and} \quad \lim_{l \to \infty} g(x(t_l)) = 0.$$

However, by Assumption 2.2 again, it implies that

$$\lim_{l \to \infty} x(t_l) = 0.$$

Similarly to the proof in [34, Theorem 1], it can be concluded that $\lim_{t \to \infty} x(t) = 0$, which will be omitted for the proof here. Consequently, system (1.1) with AUSs is GUAS with respect to switching signal $\sigma(t) \in \Lambda_{[\tau_{i,min}, \tau_{i,max}]}$. □

Remark 3.2. *Unlike from the systems studied in [21, 41] (even under similar assumptions for the nonlinear function), system (1.1) represents switched nonlinear positive time-varying delay systems with interval uncertainties and AUSs. The systems investigated for stability analysis in [21, 41] lack interval uncertainties, and their considered subsystems are all stable and partially unstable, respectively.*

Remark 3.3. *Inspired by the idea in [34–36], the nonlinear multiple discretized copositive Lyapunov-Krasovskii functional (3.10) and vector function (3.11) were constructed to analyze the GRS of system (1.1). This system includes interval uncertainties and AUSs under the suitable MDDT switching rule. The main idea of V_i, defined in (3.10), is that it can escalate with a bounded rate $\xi_i > 0$ for all $i \in \underline{N}$. Specifically, when each unstable subsystem is activated, $D^+ V_i(t, x(t)) < \xi_i V_i(t, x(t))$ where $\xi_i > 0$ for all $i \in \underline{N}$ and $t \in [t_l, t_{l+1})$, $l \in \mathbb{N}_0$. The value of V_i decreases according to $V_i(t_l, x(t_l)) \leq \mu_i V_j(t_l^-, x(t_l^-))$, where $0 < \mu_i < 1$ for all $i, j \in \underline{N},\ i \neq j$. As a result, robust stability of system (1.1) can be achieved by switching behavior, even though all subsystems are unstable. Our nonlinear multiple discretized copositive Lyapunov-Krasovskii functional, designed in Theorem 3.1, differs from that in [34–36]. While those Lyapunov functionals are sums of positive terms depending on $x(t)$, ours depends on both $x(t)$ and $g(x(t))$. Additionally, the nonlinear Lyapunov functional in [41] is not time-varying, which sets ours apart.*

Remark 3.4. *In this paper, we use the discretized Lyapunov function method to divide the domain of definition of vector function $v_i(t)$ for all $i \in \underline{N}$ defined in (3.11) into finite smaller regions. When the discretized Lyapunov function scheme is applied, the number of division segments K must be $K \geq 1$. And if $K = 0$, the discretized Lyapunov function is reduced to the multiple Lyapunov function, which is not applicable when all unstable subsystems work together. It has been mentioned in [34–36] that the number of discretized positive vectors $v_{i,q}$ for $i \in \underline{N}$, $q \in \underline{K}_0$ is $K + 1$, and K must be imposed in advance. When the larger K is selected, the denser division of $[t_l, t_l + \tau_{i,min})$ for all $i \in \underline{N}, l \in \mathbb{N}_0$ is produced.*

Remark 3.5. *As stated in Theorem 3.1, it is clear that the parameters ε_1, ε_2, K, μ_i, ξ_i and $\tau_{i,min}$ for all $i \in \underline{N}$ are set in advance. Namely, we have to assign that $0 < \varepsilon_1 \leq \varepsilon_2$, $K \in \mathbb{N}$, $0 < \mu_i < 1$, $\xi_i > 0$ and $\tau_{i,min} > 0$ for all $i \in \underline{N}$. Meanwhile, the parameters $\tau_{i,max}$ for all $i \in \underline{N}$ are calculated by conditions (3.2)–(3.8). One of the major aims of this work was to create the proper switching signal law for control the trajectory behavior of system (1.1) containing time-varying delays, interval uncertainties, and AUSs. These disturbances, which appear in various practical systems, may cause poor performance and instability of the system over a long time. Especially, when all unstable subsystems of system (1.1) work together, all conditions in Theorem 3.1 under suitable switching behavior are validly derived to compensate for the increment of the Lyapunov function described by condition 3.7 in Theorem 3.1. Thus, as stated in Theorem 3.1, we deal with the problem of GRS for system (1.1) even if the system includes time-varying delays, interval uncertainties, and AUSs.*

When $C_{\sigma(t)} \equiv 0$, $\sigma(t) \in \underline{N}$, system (1.1) can be reduced into the SNSs of the form:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}g(x(t)) + B_{\sigma(t)}g(x(t-\beta(t))), & t \geq 0, \\ x(t) = \psi(t), & t \in [-\beta, 0], \end{cases} \tag{3.23}$$

Similarly with system (1.1), A_i are the given Metzler matrices and $B_i \geq 0$, for all $i \in \underline{N}$. Both A_i and B_i are supposed to be interval uncertainties for all $i \in \underline{N}$. The assumptions of nonlinear function and the MDDT switching law in system (3.23) are similar to system (1.1). Moreover, all modes of system (3.23) are unstable.

Next, another result guaranteeing the positivity and the GRS of system (3.23) containing AUSs will be shown in the following corollary.

Corollary 3.6. *Consider system (3.23) with AUSs. Given constants $0 < \varepsilon_1 \leq \varepsilon_2$, $0 < \mu_i < 1$, $0 < \xi_i$, $0 < \tau_{i,min}$, $i \in \underline{N}$, and $K \in \mathbb{N}$. System (3.23) including AUSs is positive and GUAS with respect to $\sigma(t) \in \Lambda_{[\tau_{i,min},\tau_{i,max}]}$ if there exist constants $\tau_{i,max} \geq \tau_{i,min}$ and $v_{i,q} \succ 0$, $i \in \underline{N}$, $q \in \underline{K}_0$ satisfying the following conditions:*

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[\overline{A}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]v_{i,q} \prec 0,$$

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[\overline{A}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]v_{i,q+1} \prec 0,$$

$$\widetilde{B}^T\left(\Delta_{i,q} - v_{i,q}\right) \prec 0,$$

$$\widetilde{B}^T\left(\Delta_{i,q} - v_{i,q+1}\right) \prec 0,$$

$$\left[\overline{A}_i^T + \left(\frac{1+\beta}{1-d}\right)\widetilde{B}^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,K} \prec 0,$$

$$\nu_{i,0} - \mu_i \nu_{j,K} \preceq 0,$$

$$\ln \mu_i + \xi_j \tau_{j,max} < 0,$$

for any $q = 0, 1, ..., K-1$, and for any $i, j \in \underline{N}$, $i \neq j$, where $\Delta_{i,q}$ and $\widetilde{B}$ are defined in (3.9) and (3.1), respectively.

Proof. Under the same symbols, nonlinear multiple discretized copositive Lyapunov-Krasovskii functional (3.10), and vector function (3.11) in Theorem 3.1, this corollary can be easily proved. Hence, we omit here. □

Remark 3.7. *Compared with the systems studied in [34, 36] under the same MDDT switching rule, our system (3.23) is a nonlinear system while the considered systems in [34, 36] are linear systems. Besides, the switched system in [34] is also the delay-free case. It is worth noting that Corollary 3.6 covers the main theorem of both results, namely, when the given constants $\varepsilon_1 = \varepsilon_2 = 1$, all conditions in Corollary 3.6 are the same as in [36, Theorem 1] and in [34, Theorem 2] with $B_{\sigma(t)} \equiv 0$ for all $\sigma(t) \in \underline{N}$. Thus, our theoretical results are more general than those results.*

Sufficient criteria for GAS of system (1.1) with AUSs but without the interval uncertainties will be presented in the last result.

Corollary 3.8. *Consider system (1.1) with AUSs satisfying only Assumption 2.2. The matrices A_i and C_i are the Metzler matrices, and $B_i \succeq 0$, $\forall i \in \underline{N}$. Let the following constants be given: $0 < \varepsilon_1 \leq \varepsilon_2$, $0 < \mu_i < 1$, $0 < \xi_i$, $0 < \tau_{i,min}$, $i \in \underline{N}$, and $K \in \mathbb{N}$. Furthermore, let $B = (b_{rs}) \in \mathbb{R}^{n\times n}$ be a matrix where $b_{rs} = \max_{i\in\underline{N}}\left\{B_i^{(rs)}\right\}$, and $B_i^{(rs)}$ denotes the r^{th} row and s^{th} column element of B_i. The term $\Delta_{i,q}$ is defined in (3.9). System (1.1), without its interval uncertainty but with AUSs, is positive and GUAS with respect to $\sigma(t) \in \Lambda_{[\tau_{i,min},\tau_{i,max}]}$ if there exist constants $\tau_{i,max} \geq \tau_{i,min}$ and matrices $\nu_{i,q} \succ 0$, $i \in \underline{N}$, $q \in \underline{K}_0$ satisfying the following conditions:*

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[A_i^T + \frac{1}{\varepsilon_1}C_i^T + \left(\frac{1+\beta}{1-d}\right)B^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q} \prec 0,$$

$$\frac{1}{\varepsilon_1}\Delta_{i,q} + \left[A_i^T + \frac{1}{\varepsilon_1}C_i^T + \left(\frac{1+\beta}{1-d}\right)B^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,q+1} \prec 0,$$

$$B^T\left(\Delta_{i,q} - \nu_{i,q}\right) \prec 0,$$

$$B^T\left(\Delta_{i,q} - \nu_{i,q+1}\right) \prec 0,$$

$$\left[A_i^T + \frac{1}{\varepsilon_1}C_i^T + \left(\frac{1+\beta}{1-d}\right)B^T - \frac{\xi_i}{\varepsilon_2}I_n\right]\nu_{i,K} \prec 0,$$

$$\nu_{i,0} - \mu_i \nu_{j,K} \preceq 0,$$

$$\ln \mu_i + \xi_j \tau_{j,max} < 0,$$

for any $q = 0, 1, ..., K-1$, and for any $i, j \in \underline{N}$, $i \neq j$.

Proof. Based on the same symbols and vector function (3.11) in Theorem 3.1, this corollary can be proved by employing the nonlinear multiple discretized copositive Lyapunov-Krasovskii functional:

$$V_i(t, x(t)) = (1-d)x^T(t)v_i(t) + \int_{t-\beta(t)}^{t} f^T(x(\alpha))B^T v_i(t)d\alpha + \int_{-\beta}^{0}\int_{t+w}^{t} f^T(x(\alpha))B^T v_i(\alpha)d\alpha dw,$$

for any $i \in \underline{N}$. Thus, the proof details of Corollary 3.8 are similar to that of Theorem 3.1, which will be omitted here. □

The relationships between Theorem 3.1, Corollaries 3.6 and 3.8 are presented systematically in the following Table 1.

Table 1. A summary table for our theoretical results.

	system (1.1)	system (1.1) without $C_{\sigma(t)}$, $\forall\sigma(t) \in \underline{N}$	system (1.1) without the interval uncertainties
Theorem 3.1	✓		
Corollary 3.6		✓	
Corollary 3.8			✓

In Table 1, we have thoroughly summarized the theoretical results outlined in the article.

Remark 3.9. *It is worth noting that the switching law employed in Corollary 3.8 is based on the MDDT, where each subsystem has its own DT. In contrast, the switching technique used in [35] relies solely on the (single) DT. Furthermore, if we set* $\varepsilon_1 = \varepsilon_2 = 1$, $\mu = \mu_i$, $\xi = \xi_i$ $\tau_{min} = \tau_{i,min}$ *and all matrices* $C_i \equiv 0$ *for all* $i \in \underline{N}$ *in Corollary 3.8, then its criteria precisely match those of in [35, Theorem 3.1]. This demonstrates that our theoretical result is more general and practical than the findings in [35].*

4. Numerical simulations

In this section, we will present two numerical examples along with simulation results to demonstrate the correctness and effectiveness of our theoretical analysis proposed in the previous section.

Example 4.1. *Applied to delayed neural networks.*

Neural networks have found numerous applications in fields such as image processing, signal processing, industrial automation, and artificial intelligence. In particular, the stability of switched DNNs, whose parameters are governed by the switching signal, has been analyzed. The switched DNNs studied in [42] share properties similar to our system (1.1). Specifically, for $i \in \underline{N}$, *the constant matrices* C_i *defined by* $C(\cdot) \equiv diag\{c_1(\cdot), c_2(\cdot), ..., c_n(\cdot)\}$ *in [42] are viewed as the Metzler matrices* C_i *of system (1.1), with their off-diagonal elements being zero. For the switched DNNs proposed in [42],* $x(t)$ *and* $g(\cdot)$ *represent the neuron state vector and the nonlinear neuron activation function, respectively. In addition, for* $i \in \underline{N}$, *the constant matrices* $A(i)$ *and* $A_d(i)$, *as well as the properties of both nonlinear and time-varying delay functions, are similarly defined to our system (1.1). Thus, the GRS for switched DNNs can be studied and applied through system (1.1), even when its two unstable subsystems operate together. The first subsystem parameters are given as follows:*

$$\underline{A}_1 = \begin{bmatrix} -0.26 & 0.065 \\ 0.6 & -0.01 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} -0.24 & 0.075 \\ 0.6 & -0.01 \end{bmatrix},$$

$$\underline{B}_1 = \begin{bmatrix} 0.0003 & 0.0005 \\ 0.0002 & 0.0003 \end{bmatrix}, \ \overline{B}_1 = \begin{bmatrix} 0.0005 & 0.0007 \\ 0.0002 & 0.0005 \end{bmatrix},$$

$$\underline{C}_1 = \begin{bmatrix} -1.47 & 0 \\ 0 & -0.01 \end{bmatrix}, \ \overline{C}_1 = \begin{bmatrix} -1.45 & 0 \\ 0 & -0.01 \end{bmatrix}.$$

Also the second subsystem data are given as follows:

$$\underline{A}_2 = \begin{bmatrix} 0.03 & 0.025 \\ 0.065 & -2.6 \end{bmatrix}, \ \overline{A}_2 = \begin{bmatrix} 0.03 & 0.035 \\ 0.075 & -2.6 \end{bmatrix},$$

$$\underline{B}_2 = \begin{bmatrix} 0.0005 & 0.0004 \\ 0.0001 & 0.0002 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0.0005 & 0.0006 \\ 0.0001 & 0.0004 \end{bmatrix},$$

$$\underline{C}_2 = \begin{bmatrix} -0.01 & 0 \\ 0 & -3.6 \end{bmatrix}, \ \overline{C}_2 = \begin{bmatrix} -0.01 & 0 \\ 0 & -3.4 \end{bmatrix}.$$

It can be seen that $\underline{A}_1$, $\underline{A}_2$, $\underline{C}_1$ and $\underline{C}_2$ are Metzler matrices. It is obvious that $\underline{B}_1 \succeq 0$ and $\underline{B}_2 \succeq 0$. The time-varying delay function and nonlinear function of the underlying system are

$$\beta(t) = 0.15 + 0.05\sin(t),$$

and

$$g_p(x_p(t)) = x_p(t) + \frac{0.1x_p(t)}{x_p^2(t) + 1},$$

for $p = 1, 2$, respectively. Then, we can choose $\beta = 0.2$, $d = 0.05$ and set $\varepsilon_1 = 1$, $\varepsilon_2 = 1.1$. By Assumptions 2.2 and 2.6, the studied system is positive. Let $\psi = [5 \ \ 8]^T$ be the initial state for numerical simulation in this system. We set system matrices

$$A_1 = \begin{bmatrix} -0.25 & 0.07 \\ 0.6 & -0.01 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.0004 & 0.0006 \\ 0.0002 & 0.0004 \end{bmatrix}, \ C_1 = \begin{bmatrix} -1.46 & 0 \\ 0 & -0.01 \end{bmatrix},$$

for the first subsystem, and

$$A_2 = \begin{bmatrix} 0.03 & 0.03 \\ 0.07 & -2.6 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.0005 & 0.0005 \\ 0.0001 & 0.0003 \end{bmatrix}, \ C_2 = \begin{bmatrix} -0.01 & 0 \\ 0 & -3.5 \end{bmatrix},$$

for the second subsystem. We first present two figures for the corresponding state responses of two subsystems. From Figures 1 and 2, it can be seen that two subsystems are positive and unstable.

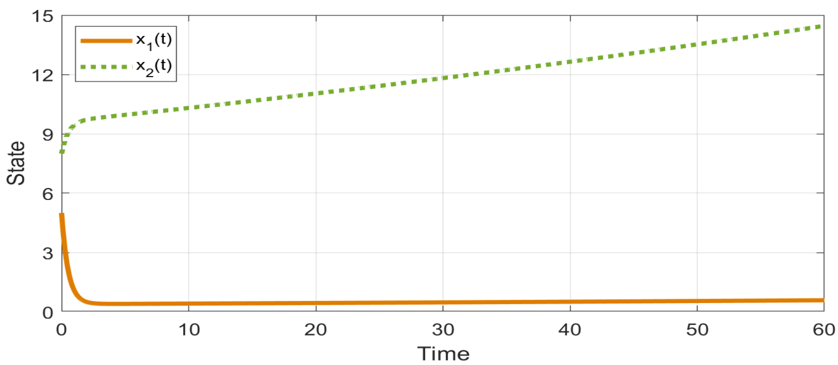

Figure 1. The state trajectory of the first subsystem in Example 4.1.

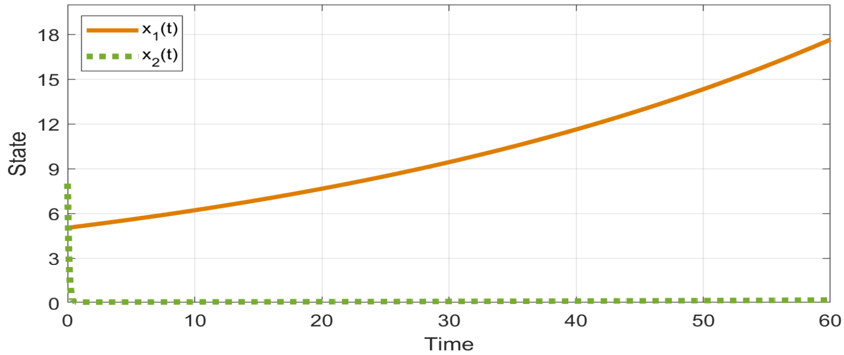

Figure 2. The state trajectory of the second subsystem in Example 4.1.

As defined in (3.1), we obtain that

$$\widetilde{B} = \begin{bmatrix} 0.0005 & 0.0007 \\ 0.0002 & 0.0005 \end{bmatrix}.$$

As stated in Remarks 3.3–3.5, the selection of following parameters is valid. For given parameters $K = 1$, $\mu_1 = 0.58$, $\mu_2 = 0.36$, $\xi_1 = 0.5$, $\xi_2 = 0.53$, $\tau_{1,min} = 2$ *and* $\tau_{2,min} = 1$, *we can get a set of feasible solutions for Theorem 3.1:*

$$\nu_{1,0} = \begin{bmatrix} 62.3296 \\ 68.2194 \end{bmatrix}, \quad \nu_{1,1} = \begin{bmatrix} 241.8569 \\ 67.1340 \end{bmatrix}, \quad \nu_{2,0} = \begin{bmatrix} 86.3039 \\ 23.2898 \end{bmatrix}, \quad \nu_{2,1} = \begin{bmatrix} 111.0557 \\ 119.2610 \end{bmatrix},$$

$\tau_{1,max} = 2.01$ *and* $\tau_{2,max} = 1.01$. *Thus, system (1.1), which serves as a model for switched DNNs, achieves GUAS with the MDDT switching signal satisfying* $\Lambda_{[2,2.01][1,1.01]}$. *The corresponding switching signal* $\sigma(t)$ *and the system's state response are illustrated in Figures 3 and 4, respectively. Figure 3,* $\sigma(t)$ *is designed as*

$$\sigma(t) = \begin{cases} 1, & t \in [1,3) \cup [4,6) \cup [7,9) \cup [10,12) \cup [13,15) \cup [16,18) \cup [19,21) \\ & \quad \cup [22,24) \cup [25,27) \cup [28,30) \cup [31,33) \cup [34,36) \cup [37,39) \\ & \quad \cup [40,42) \cup [43,45) \cup [46,48) \cup [49,51) \cup [52,54) \cup [55,57) \cup [58,60], \\ 2, & t \in [0,1) \cup [3,4) \cup [6,7) \cup [9,10) \cup [12,13) \cup [15,16) \cup [18,19) \\ & \quad \cup [21,22) \cup [24,25) \cup [27,28) \cup [30,31) \cup [33,34) \cup [36,37) \\ & \quad \cup [39,40) \cup [42,43) \cup [45,46) \cup [48,49) \cup [51,52) \cup [54,55) \cup [57,58). \end{cases}$$

This numerical simulation effectively confirms that the theoretical results ensure the positivity and GRS of the considered system.

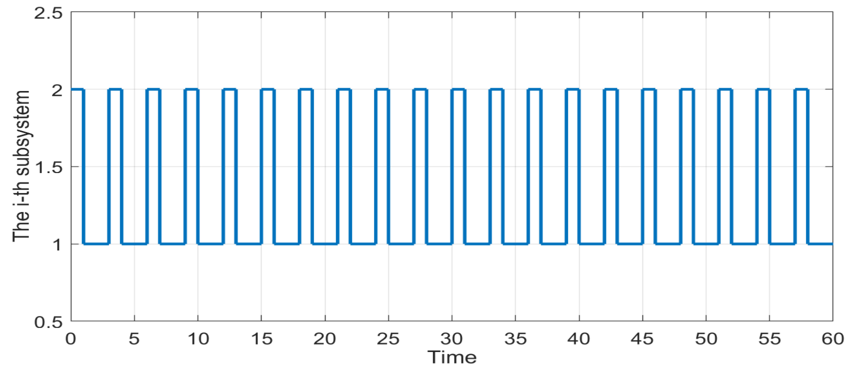

Figure 3. The switching sequence for the switched nonlinear system in Example 4.1.

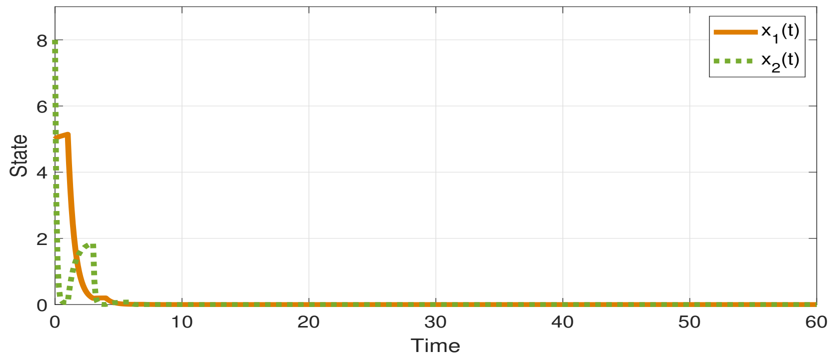

Figure 4. The state response of the switched nonlinear system in Example 4.1.

Example 4.2. *In this example, we will investigate the GRS problem for system (3.23) comprising of two unstable subsystems. The first subsystem parameters are given as follows:*

$$\underline{A}_1 = \begin{bmatrix} 0.06 & 0.29 \\ 0.1 & -5 \end{bmatrix}, \ \overline{A}_1 = \begin{bmatrix} 0.06 & 0.3 \\ 0.1 & -5 \end{bmatrix}, \ \underline{B}_1 = \begin{bmatrix} 0.0001 & 0.0002 \\ 0.0003 & 0.0002 \end{bmatrix}, \ \overline{B}_1 = \begin{bmatrix} 0.0003 & 0.0002 \\ 0.0005 & 0.0004 \end{bmatrix}.$$

And the second subsystem data are given as follows:

$$\underline{A}_2 = \begin{bmatrix} -3 & 0.05 \\ 0.01 & 0.06 \end{bmatrix}, \ \overline{A}_2 = \begin{bmatrix} -3 & 0.05 \\ 0.012 & 0.06 \end{bmatrix}, \ \underline{B}_2 = \begin{bmatrix} 0.0002 & 0.0002 \\ 0.0004 & 0.0001 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0.0004 & 0.0002 \\ 0.0004 & 0.0003 \end{bmatrix}.$$

The time-varying delay function and nonlinear function of the underlying system are

$$\beta(t) = 0.15 + 0.05\sin(t),$$

and

$$g_p(x_p(t)) = x_p(t) + \frac{0.2x_p(t)}{x_p^2(t) + 1},$$

for $p = 1, 2$*, respectively. From the above, we can choose* $\beta = 0.2$*,* $d = 0.05$ *and set* $\varepsilon_1 = 1$*,* $\varepsilon_2 = 1.2$*. It can be seen that* $\underline{A}_1$ *and* $\underline{A}_2$ *are Metzler matrices. Furthermore, it is obvious that* $\underline{B}_1 \geq 0$ *and* $\underline{B}_2 \geq 0$*.*

By Assumptions 2.2 and 2.6, the studied system is positive. For numerical simulation, let $\psi = [6 \;\; 3]^T$ *be the initial state for this system. Let*

$$A_1 = \begin{bmatrix} 0.06 & 0.295 \\ 0.1 & -5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0002 & 0.0002 \\ 0.0004 & 0.0003 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} -3 & 0.05 \\ 0.011 & 0.06 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0003 & 0.0002 \\ 0.0004 & 0.0002 \end{bmatrix},$$

be the system matrices for the first subsystem and the second subsystem, respectively. Then, we present two figures for the corresponding state responses of two subsystems. From Figures 5 and 6, it can be seen that two subsystems are positive and unstable.

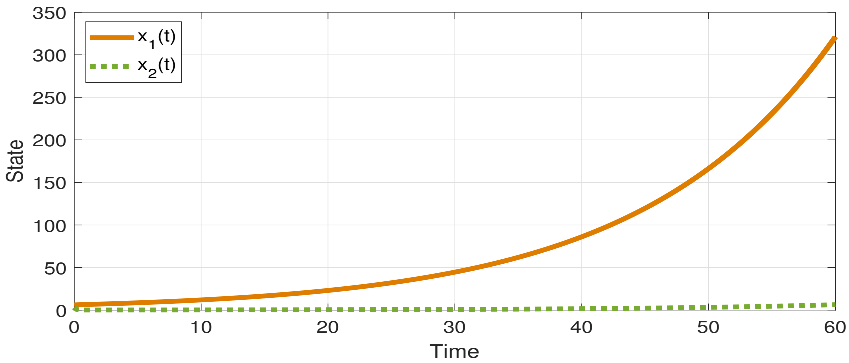

Figure 5. The state trajectory of the first subsystem in Example 4.2.

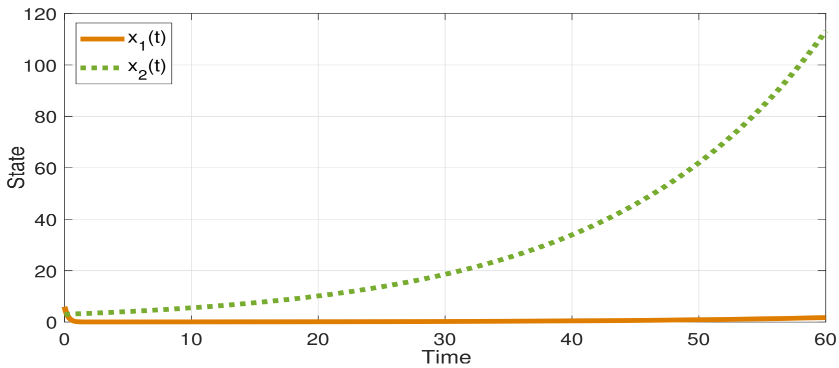

Figure 6. The state trajectory of the second subsystem in Example 4.2.

As defined in (3.1), we obtain that

$$\widetilde{B} = \begin{bmatrix} 0.0004 & 0.0002 \\ 0.0005 & 0.0004 \end{bmatrix}.$$

For given scalars $K = 1$, $\mu_1 = 0.36$, $\mu_2 = 0.58$, $\xi_1 = 0.53$, $\xi_2 = 0.5$, $\tau_{1,min} = 1$ *and* $\tau_{2,min} = 2$, *we can get a set of feasible solutions for Corollary 3.6:*

$$v_{1,0} = \begin{bmatrix} 80.0450 \\ 18.8277 \end{bmatrix}, \quad v_{1,1} = \begin{bmatrix} 78.9548 \\ 76.5135 \end{bmatrix}, \quad v_{2,0} = \begin{bmatrix} 45.3070 \\ 43.2352 \end{bmatrix}, \quad v_{2,1} = \begin{bmatrix} 227.9595 \\ 52.7598 \end{bmatrix},$$

$\tau_{1,max} = 1.01$ *and* $\tau_{2,max} = 2.01$. *Thus, system (3.23) achieves GUAS with the MDDT switching signal satisfying* $\Lambda_{[1,1.01][2,2.01]}$. *The corresponding switching signal* $\sigma(t)$ *and the system's state response are illustrated in Figures 7 and 8, respectively. Figure 7,* $\sigma(t)$ *is designed as*

$$\sigma(t) = \begin{cases} 1, & t \in [0,1) \cup [3,4) \cup [6,7) \cup [9,10) \cup [12,13) \cup [15,16) \cup [18,19) \\ & \cup [21,22) \cup [24,25) \cup [27,28) \cup [30,31) \cup [33,34) \cup [36,37) \\ & \cup [39,40) \cup [42,43) \cup [45,46) \cup [48,49) \cup [51,52) \cup [54,55) \cup [57,58), \\ 2, & t \in [1,3) \cup [4,6) \cup [7,9) \cup [10,12) \cup [13,15) \cup [16,18) \cup [19,21) \\ & \cup [22,24) \cup [25,27) \cup [28,30) \cup [31,33) \cup [34,36) \cup [37,39) \\ & \cup [40,42) \cup [43,45) \cup [46,48) \cup [49,51) \cup [52,54) \cup [55,57) \cup [58,60]. \end{cases}$$

It should be pointed out that even though the considered system comprises nonlinear SPSs with AUSs, our designed switching signal can still stabilize the system. When comparing our results to those of earlier studies, it is important to note that the GRS problem of system (3.23) with AUSs can also be effectively solved by the method presented in Corollary 3.6. Specifically, when $\varepsilon_1 = \varepsilon_2 = 1$, *all conditions in Corollary 3.6 become equivalent to in [36, Theorem 1] and in [34, Theorem 2] (with* $B_{\sigma(t)} \equiv 0$ *for all* $\sigma(t) \in \underline{N}$*). Consequently, our theoretical results are more general and applicable than those previous findings.*

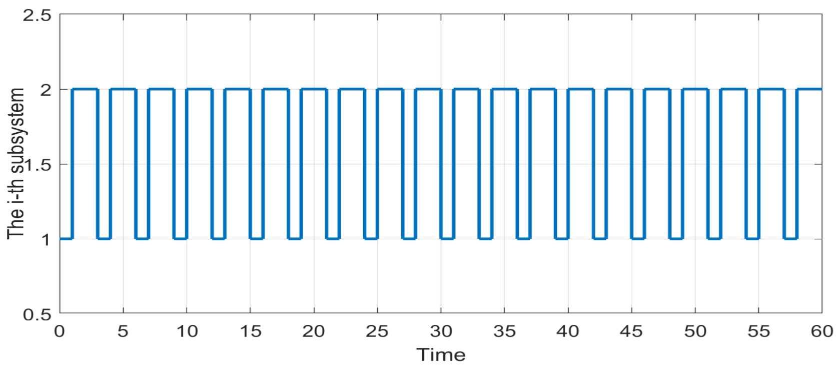

Figure 7. The switching sequence for the switched nonlinear system in Example 4.2.

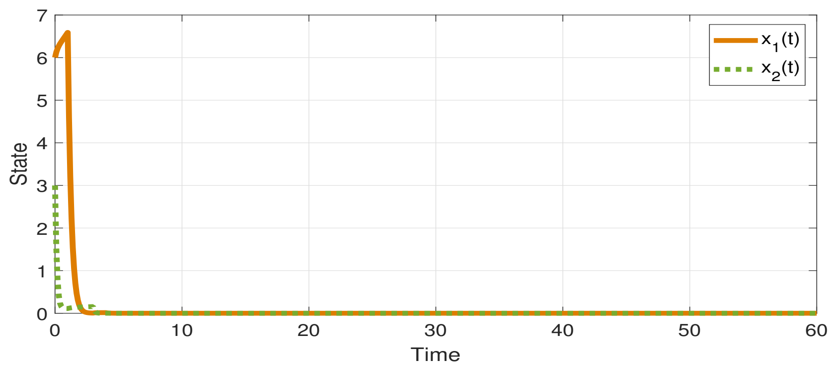

Figure 8. The state response of the switched nonlinear system in Example 4.2.

Example 4.3. *To show that our theoretical results are less conservative than the existing results, we conduct a thorough comparison of our proposed scheme with the method of Rojsiraphisal et al. [36]. The studied system in [36, Example 1] is a switched linear positive time-varying delay system with interval uncertainties and two unstable subsystems, as shown below.*

$$\underline{A}_1 = \begin{bmatrix} -3 & 0.04 \\ 0.03 & 0.1 \end{bmatrix}, \ \overline{A}_1 = \begin{bmatrix} -2.5 & 0.06 \\ 0.06 & 0.2 \end{bmatrix}, \ \underline{B}_1 = \begin{bmatrix} 0 & 0.0001 \\ 0.0004 & 0 \end{bmatrix}, \ \overline{B}_1 = \begin{bmatrix} 0.0001 & 0.0001 \\ 0.0004 & 0.0001 \end{bmatrix},$$

$$\underline{A}_2 = \begin{bmatrix} 0.1 & 0.02 \\ 0.01 & -2 \end{bmatrix}, \ \overline{A}_2 = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & -1.5 \end{bmatrix}, \ \underline{B}_2 = \begin{bmatrix} 0 & 0.0001 \\ 0.0004 & 0 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0 & 0.0001 \\ 0.0004 & 0.0001 \end{bmatrix}.$$

And the bound of the studied system matrices in [36, Example 1] is

$$\widetilde{B} = \begin{bmatrix} 0.0001 & 0.0001 \\ 0.0004 & 0.0001 \end{bmatrix}.$$

The considered time-varying delay is

$$\beta(t) = 0.1 - 0.1\sin(t),$$

and its constants are $\beta = 0.2$ *and* $d = 0.1$. *For given scalars* $K = 1$, $\mu_1 = 0.55$, $\mu_2 = 0.56$, $\xi_1 = 0.36$, $\xi_2 = 0.22$, $\tau_{1,min} = 1$ *and* $\tau_{2,min} = 2$, *we obtain a set of feasible solutions for [36, Theorem 1]:*

$$\nu_{1,0} = \begin{bmatrix} 47.1462 \\ 87.1190 \end{bmatrix}, \ \nu_{1,1} = \begin{bmatrix} 165.3132 \\ 91.0536 \end{bmatrix}, \ \nu_{2,0} = \begin{bmatrix} 90.1878 \\ 49.9822 \end{bmatrix}, \ \nu_{2,1} = \begin{bmatrix} 89.0055 \\ 160.1816 \end{bmatrix},$$

$\tau_{1,max} = 1.1$ *and* $\tau_{2,max} = 2.1$.

However, our system (3.23) studied in Corollary 3.6 is the switched nonlinear positive time-varying delay system with interval uncertainties and AUSs by the nonlinear function

$$g_p(x_p(t)) = x_p(t) + \frac{0.1x_p(t)}{x_p^2(t) + 1},$$

for $p = 1, 2$. *Thus, we can set* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = 1.1$. *Under the same system matrices and the same given constants in [36, Example 1], we get a set of feasible solutions for Corollary 3.6:*

$$\nu_{1,0} = \begin{bmatrix} 42.0688 \\ 71.4987 \end{bmatrix}, \ \nu_{1,1} = \begin{bmatrix} 146.0279 \\ 69.0954 \end{bmatrix}, \ \nu_{2,0} = \begin{bmatrix} 81.5949 \\ 38.0025 \end{bmatrix}, \ \nu_{2,1} = \begin{bmatrix} 77.3406 \\ 130.0713 \end{bmatrix},$$

$\tau_{1,max} = 1.6$ *and* $\tau_{2,max} = 2.6$. *To facilitate comparison of the results obtained from the different computational schemes above, the MDDT switching signals are presented in Table 2.*

Table 2. Computational results for the system with two different schemes.

References	Theoretical basis	The MDDT switching signals
[36]	Theorem 1	$\Lambda_{[1,1.1][2,2.1]}$
Our present result	Corollary 3.6	$\Lambda_{[1,1.6][2,2.6]}$

As shown in Table 2, the maximal MDDT for subsystem 1 and subsystem 2 of [36] is $\tau_{1,max} = 1.1$ *and* $\tau_{2,max} = 2.1$, *respectively. In contrast, the maximal MDDT for the first subsystem and the second subsystem of our present result is extended to* $\tau_{1,max} = 1.6$ *and* $\tau_{2,max} = 2.6$, *respectively. Therefore, our method yields less conservative results.*

5. Conclusions

The problems of both GRS and GAS have been investigated for switched nonlinear positive time-varying delay systems, including AUSs. Novel nonlinear multiple discretized copositive Lyapunov-Krasovskii functionals have been introduced. The theoretical results are based on the use of these nonlinear multiple discretized copositive Lyapunov-Krasovskii functionals and the MDDT switching approach. Sufficient conditions for GRS of the underlying systems have been obtained. Moreover, GAS criteria have been derived for the considered systems with AUSs but without interval uncertainty. Our theoretical results have been compared with previous results. Finally, the effectiveness of the proposed theories has been demonstrated through three numerical examples, and their applicability to switched DNNs has been shown. In future research, it will be interesting to investigate the stability analysis of discrete-time switched positive nonlinear systems with delays and AUSs.

Author contributions

Suriyon Yimnet: Methodology, conceptualization, writing, software, visualization; Kanyuta Poochinapan: Supervision, conceptualization, writing, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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