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Research article

Generalizations of higher-order Taylor method: Fractional and q-fractional approaches for initial value problems

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Abstract: Motivated by the challenges faced by standard methods in solving nonlinear fractional and q-fractional models with strong memory effects, this study develops numerical approaches capable of handling these complex behaviors more effectively. The proposed techniques are tested on four representative fractional and q-fractional initial value problems for several values of the order $\alpha \in [0.5, 1]$ and $q \in (0, 1)$. In particular, the major aim of this work is to propose two generalizations of the higher-order Taylor method: The first one is the fractional Taylor method, and the second one is the q-fractional Taylor method. These methods will then be used to find approximate solutions for several fractional and q-fractional initial value problems. Numerous numerical comparisons will be performed to verify the effectiveness of our proposed generalizations.

Keywords: fractional calculus; fractional q-calculus; fractional Taylor method; fractional q-Taylor method; numerical solutions
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

Fractional calculus illustrates integration or differentiation in fractional order and is considered a potent tool in mathematical analysis [1–3]. Numerous scholars have exhibited interest in this topic, with some focusing on the analytical side of solving fractional differential equations, which encompasses uniqueness, existence, stability, and other features; for example, see the study work [4–6].

Furthermore, because many fractional-order problems are difficult to address analytically, a large amount of research has been done with an applied focus [7, 8]. Nowadays, the majority of research focuses on this topic since generating a numerical approximation for a given nonlinear fractional problem is less expensive than obtaining an analytical approximation. In this regard, a number of numerical approaches for solving fractional differential equations have been proposed and used. A variety of numerical methods for solving fractional differential equations have been proposed and used in this context (see [9–11]). We mention the homotopy perturbation and matrix approach techniques [12], the Adomian decomposition method [13], the fractional difference method [15], the fractional Euler method [14], and others [16].

Fractional q-calculus is an intriguing topic and an important area of mathematical analysis that was pioneered and developed in the 20th century by [17–19]. Many researchers are interested in it because of its relevance to mathematical modeling in a variety of domains, including biomathematics, engineering, physics, and technical sciences. Furthermore, fractional q-difference equations have played an important role in modeling a variety of events in several fields; for further information, see [13]. Scholars have investigated and debated solutions to initial and boundary value problems of fractional q-differential equations employing Caputo's fractional q-derivative in recent years. There are two types of solutions: The first type is analytical solution, which involves using traditional and analytical techniques to solve problems, but scientists faced difficulties and barriers in doing so; for details, see reference [20]. This stimulated the investigation of the second category, numerical solutions, which incorporate numerical techniques and applications. However, they did not make much headway in them since studying fractional q-difference equations is a current and contemporary technique; see [17] and the references therein. However, Stempin and Sumelka [21] proposed an approximation for the fractional Caputo derivative with variable order and terminals by expressing it as a short series of higher-order classical derivatives. Their method focuses on approximating the operator itself, while our approaches build generalized time-stepping schemes called the fractional higher-order Taylor method (FHOTM) and the fractional higher-order q-Taylor method (FHOqTM) that directly compute the solution for both fractional and q-fractional cases, respectively.

This work primarily aims to suggest two generalizations of the higher-order Taylor technique: The q-fractional Taylor method and the fractional Taylor method. The approximate solutions to a number of fractional and q-fractional initial value problems will then be found using these techniques. We shall do a number of numerical comparisons to confirm the efficacy of our suggested generalizations.

2. Basic fundamentals

This section discusses several fundamental definitions and notations of the Riemann–Liouville integral and the Caputo derivative, q-derivative, q-integral, q-Gamma function, fractional q-integral, and fractional q-derivative. This will set the way for the main results later on.

Definition 2.1. Let α be a real non-negative number. Then, J_a^α is defined on $L_1[a, b]$, where $L_1[a, b]$ is the set of all functions such that their absolute values are Lebesgue integrable on $[a, b]$ by [22].

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ a \le t \le b \tag{2.1}$$

is called the Riemann–Liouville fractional-order integral operator of order α.

Definition 2.2. If $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$ such that $n-1 < \alpha \leq n$, then the following defines the Caputo fractional-order derivative operator of order α:

$$ {}^{C}D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x f^{(n)}(t)(x-t)^{n-\alpha-1} dt, \ x > a. \tag{2.2} $$

Definition 2.3. The Caputo fractional-order derivative operator is defined as follows: Where $\alpha \in \mathbb{R}^+$ and $m-1 < \alpha \leq n$ such that $n \in \mathbb{N}$,

$$ D_a^{\alpha} f(x) = J_a^{n-\alpha} D^n f(x). \tag{2.3} $$

In the same regard, the power rule property of the Caputo operator can be obtained as follows:

$$ D^{\alpha} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, & n-1 < \alpha < n, \ p > n-1, \ p \in \mathbb{R}, \\ 0, & n-1 < \alpha < n, \ p \leq n-1, \ p \in \mathbb{N}. \end{cases} \tag{2.4} $$

Definition 2.4. (q-Derivative) The q-derivative of order $n \in \mathbb{N}$ of a function $f : [0, x] \to \mathbb{R}$ is defined by

$$ (D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t}, \ t \neq 0. \tag{2.5} $$

Definition 2.5. A function $f : [0, b] \to \mathbb{R}$ has a q-integral that is defined as

$$ (I_q f)(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n), $$

provided that the series converges.

Definition 2.6. (q-Gamma function) We define the q-Gamma function by

$$ \Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \ \alpha > 0. \tag{2.6} $$

The q-Gamma function has a q-integral expression,which is defined by

$$ \Gamma_q(\alpha) = \int_0^{\infty} x^{\alpha-1} E_q^{-qx} d_q x, \ \alpha > 0, \tag{2.7} $$

where

$$ E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]!}. $$

Definition 2.7. The fractional q-integral of order $\alpha \in \mathbb{R}^+$ in Riemann–Liouville for a function $f : [0, b] \to \mathbb{R}$ is defined by

$$ (I_q^{\alpha} f)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s, \ t \in [0, b]. \tag{2.8} $$

Definition 2.8. The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}^+$ of a function $f : [0, b] \to \mathbb{R}$ is defined by

$$ ({}^{C}D_q^{\alpha} f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha} f)(t), \ t \in \mathbb{J}, $$

where $[\alpha]$ is the integer part of α and $\mathbb{J}$ is a given interval.

3. Generalizations of higher-order Taylor method

In this section, we will propose two generalizations for solving fractional differential equations and fractional q-differential equations: The fractional higher-order Taylor method (FHOTM) and the fractional higher-order q-Taylor method (FHOqTM). These generalizations, based on Taylor's theorem, extend higher-order Taylor techniques. Next, some suitable theoretical results will be illustrated in order to evaluate the local truncation errors caused by the suggested methods. In order to explain how the suggested techniques can be applied to their exact solutions, numerous numerical examples will be presented at the end.

To provide a more transparent formulation of the proposed approach, we outline the construction of the fractional and q-fractional Taylor expansions in a step-by-step manner. For a fractional differential equation of the form $D^{\alpha}y(t) = f(t, y(t))$ with $0 < \alpha \leq 1$, the fractional Taylor approximation about $t = t_0$ is obtained by repeatedly applying the fractional derivative operator to the unknown solution. In practice, the coefficients are computed sequentially as

$$c_0 = y(t_0), \qquad c_1 = D^{\alpha}y(t_0), \qquad c_2 = D^{2\alpha}y(t_0), \quad \text{etc.}$$

The approximate solution is then written as the truncated series

$$y_N(t) = \sum_{k=0}^{N} c_k (t - t_0)^{k\alpha},$$

where N is chosen to balance accuracy and computational cost. A similar construction is used for the q-fractional Taylor method, in which the classical powers $(t - t_0)^{k\alpha}$ are replaced by their q-analogs, and the coefficients are obtained via the q-fractional derivative operator D_q^{α}. In both cases, the required derivatives of the right-hand side $f(t, y)$ are evaluated recursively, and the resulting truncated expansions provide efficient approximations to the exact solution.

3.1. Some useful theorems

In this part, we will recall some important theorems that will be very useful for proposing the first generalization of the higher-order Taylor method called the FHOTM.

Theorem 3.1. (Second mean value theorem for integrals) Suppose that f(x) is continuous on [a,b] and g(x) does not change sign on [a,b]. Then, there exists c in (a, b) such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Theorem 3.2. (Generalized mean value theorem) Suppose that $f(x) \in C([0, b])$ and ${}^{C}D_*^{\alpha} f(x) \in C([0, b])$, for $0 < \alpha \leq 1$. Then, we have

$$f(x) = f(0_+) + \frac{1}{\Gamma(\alpha + 1)} (D^{\alpha} f)(\xi) x^{\alpha},$$

where $0 \leq \xi \leq \alpha$, for all $x \in (0, b]$.

Proof. By operating the Riemann–Linville fractional integral operator on the Caputo fractional derivative operator, we get

$$(J^{\alpha} D_*^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} D_*^{\alpha} f(t) dt.$$

By using the classical second mean value theorem for integrals, we get

$$\begin{aligned}
(J^{\alpha} D_*^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)} (D_*^{\alpha} f)(\xi) \int_0^x (x-t)^{\alpha-1} dt \\
&= \frac{1}{\Gamma(\alpha)} (D_*^{\alpha} f)(\xi) \left(\frac{-(x-t)^{\alpha}}{\alpha} \Big|_0^{\alpha} \right) \\
&= \frac{1}{\alpha\Gamma(\alpha)} (D^{\alpha} f)(\xi) x^{\alpha} \\
&= \frac{1}{\Gamma(\alpha+1)} (D^{\alpha} f)(\xi) x^{\alpha},
\end{aligned} \tag{3.1}$$

where $0 \leq \xi \leq x$. Now, by using the following relation:

$$(J^{\alpha} D_*^{\alpha} f)(x) = f(x) - f(0_+),$$

for $0 \leq \alpha \leq 1$, the equality (3.1) becomes

$$f(x) - f(0_+) = \frac{1}{\Gamma(\alpha+1)} (D^{\alpha} f)(\xi) x^{\alpha},$$

which implies

$$f(x) = f(0_+) + \frac{1}{\Gamma(\alpha+1)} (D^{\alpha} f)(\xi) x^{\alpha},$$

for $0 \leq \xi \leq x$. □

Theorem 3.3. Consider $D_*^{n\alpha} f(x), D_*^{(n+1)\alpha} f(x) \in C((0,b])$, for $\alpha \in (0,b]$. Then, we have

$$(J^{n\alpha} D_*^{n\alpha} f)(x) - (J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) = \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} (D_*^{n\alpha} f)(0_+), \tag{3.2}$$

where

$$D_*^{n\alpha} = D_*^{\alpha} D_*^{\alpha} \cdots D_*^{\alpha} \quad (n - times).$$

Proof. By taking the left–hand side, we get

$$\begin{aligned}
(J^{n\alpha} D_*^{n\alpha} f)(x) - (J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) &= J^{n\alpha}\Big((D_*^{n\alpha} f)(x) - (J^{\alpha} D_*^{\alpha})(D_*^{n\alpha} f)(x) \Big) \\
&= J^{n\alpha}\Big((D_*^{n\alpha} f)(x) - \big((D_*^{n\alpha} f)(x) - D_*^{n\alpha} f(0_+) \big) \Big) \\
&= J^{n\alpha} D_*^{n\alpha} f(0_+) \\
&= D_*^{n\alpha} f(0_+) J^{n\alpha} x^0 \\
&= D_*^{n\alpha} f(0_+) \frac{\Gamma(1)}{\Gamma(1+n\alpha)} x^{n\alpha}.
\end{aligned}$$

This implies

$$(J^{n\alpha}D_*^{n\alpha}f)(x) - (J^{(n+1)\alpha}D_*^{(n+1)\alpha}f)(x) = \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}(D_*^{n\alpha}f)(0_+).$$

□

Theorem 3.4. (Generalized Taylor's formula) Suppose $D_*^{k\alpha}f(x) \in C((0,b])$, for $k = 0, 1, 2, \cdots, n+1$, where $0 < \alpha \leq 1$. Then,

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+) + \frac{D^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha},$$

where $0 \leq \xi \leq x$, for all $x \in (0, b]$.

Proof. Based on the previous theorem, we can get:

$$\sum_{i=0}^{n}\left[(J^{i\alpha}D_*^{i\alpha}f)(x) - (J^{(i+1)\alpha}D_*^{(i+1)}f)(x)\right] = \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+).$$

This means

$$\begin{aligned}&(J^0D_*^0f)(x) + (J^{\alpha}D_*^{\alpha}f)(x) + (J^{2\alpha}D_*^{2\alpha}f)(x) + (J^{3\alpha}D_*^{3\alpha}f)(x) + \cdots +\\&(J^{n\alpha}D_*^{n\alpha}f)(x) - (J^{\alpha}D_*^{\alpha}f)(x) + (J^{2\alpha}D_*^{2\alpha}f)(x) - (J^{3\alpha}D_*^{3\alpha}f)(x) - \cdots -\\&(J^{n\alpha}D_*^{n\alpha}f)(x) - J^{(n+1)}D_*^{(n+1)\alpha}f(x) = \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+).\end{aligned}$$

Then, we have

$$f(x) - (J^{(n+1)}D_*^{(n+1)}f)(x) = \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+),$$

$$f(x) - \frac{1}{\Gamma((n+1)\alpha+1)}\int_0^x (x-t)^{(n+1)\alpha}D_*^{(n+1)\alpha}f(t).dt = \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+).$$

By the second mean value theorem for integrals, we obtain

$$f(x) - \frac{D^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}\int_0^x (x-t)^{(n+1)\alpha}dt = \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+).$$

This implies

$$\begin{aligned}f(x) &= \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+) + \frac{D_*^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}\left(\frac{(x-t)^{(n+1)\alpha}}{-((n+1)\alpha+1)}\Bigg|_0^x\right)\\&= \sum_{i=0}^{n}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D_*^{i\alpha}f)(0_+) + \frac{D_*^{(n+1)\alpha}f(\xi)}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha},\end{aligned}$$

which represents the desired result. □

Theorem 3.5. (q-Mean value theorem for q-integrals) Suppose that $f(x)$ and $g(x)$ are two continuous functions on $[a, b]$. Then, there exists $q^* \in (0, 1)$ such that

$$I_q(fg) = g(\xi)I_q f,$$

for some $\xi \in (a, b)$ and for all $q \in (q^*, 1)$.

Proof. Based on the classical form of Theorem 3.1, we can have

$$I(fg) = g(c)If,$$

for some $c \in (a, b)$. This implies

$$\lim_{q\to 1} I_q(fg) = g(c)If = g(c) = \lim_{q\to 1} I_q(f),$$

or

$$\lim_{q\to 1} \frac{I_q(fg)}{I_q(f)} = g(c).$$

Then, there exists $q^* \in (0, 1)$ such that

$$g(c) - \epsilon < \frac{I_q(fg)}{I_q(f)} < g(c) + \epsilon,$$

for all $q \in (q^*, 1)$ and for some $\epsilon > 0$. Since $g(x)$ is a continuous function on $[a, b]$, it attains its minimum m_g and maximum M_g. Now, assume that

$$\epsilon \leq \min(M_g - g(c), g(c) - m_g).$$

Then, we obtain

$$m_g < g(c) - \epsilon < \frac{I_q(fg)}{I_q(f)} < g(c) + \epsilon < M_g,$$

i.e.,

$$m_g < \frac{I_q(fg)}{I_q(f)} < M_g,$$

for all $q \in (q^*, 1)$ such that $q^* \in (0, 1)$. Due to $f(x)$ taking all values between m_g and M_g, we deduce

$$\frac{I_q(fg)}{I_q(f)} = g(\xi),$$

for some $\xi \in (a, b)$. □

Theorem 3.6. Suppose $f(t)$, ${}^C D_q^\alpha f(t) \in C[a, b]$, and $\alpha \in (0, 1]$. Then, there exists $c \in (a, b)$ such that

$$f(t) = f(a) + \frac{1}{\Gamma_q(\alpha + 1)} \, {}^C D_q^\alpha f(c)(t - a)^{(\alpha)},$$

for all $q \in (q^*, 1)$, where $q^* \in (0, 1)$.

Proof. If one applies I_q^α to the Caputo q-fractional derivative, we get

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - xq)^{(\alpha-1)} \, {}^C D_q^\alpha f(x) d_q x.$$

Now, by using the q-mean value theorem for q-integrals, we obtain

$$I_q(fg) = g(\xi) I_q(f).$$

This implies

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \, {}^C D_q^\alpha f(c) \int_a^t (t - xq)^{(\alpha-1)} d_q x,$$

for some $c \in (a, t)$. This consequently implies

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = \, {}^C D_q^\alpha f(c). \frac{\Gamma_q(0+1)}{\Gamma_q(\alpha+1)} (t-a)^{(\alpha)},$$

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha+1)} \, {}^C D_q^\alpha f(c)(t-a)^{(\alpha)}. \tag{3.3}$$

By using the relation between the Caputo derivative and Riemann–Liouvelle integral operators, we get

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(D_q^k f(a))}{[k]_q!} (t-a)^{(k)}.$$

So, using $n = 1$ in the above equality yields

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = f(t) - \frac{(D_q^0 f(a))}{[0]_q!} (t-a)^{(0)}.$$

This implies

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = f(t) - f(a). \tag{3.4}$$

But (3.3) implies

$$I_q^\alpha \, {}^C D_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha+1)} \, {}^C D_q^\alpha f(c)(t-a)^{(\alpha)}. \tag{3.5}$$

So, by combining (3.4) and (3.5), we get

$$f(t) = f(a) + \frac{1}{\Gamma_q(\alpha+1)} \, {}^C D_q^\alpha f(c)(t-a)^{(\alpha)}.$$

□

Lemma 3.7. Suppose $({}^C D_q^\alpha)^i f(t)$ and $({}^C D_q^\alpha)^{i+1} f(t) \in C[a, b]$. Then, we have

$$I_q^{i\alpha} ({}^C D_q^\alpha)^i f(t) - I_q^{(i+1)\alpha} ({}^C D_q^\alpha)^{i+1} f(t) = \frac{({}^C D_q^\alpha)^i f(a)}{\Gamma_q(i\alpha+1)} (t-a)^{i\alpha},$$

for $\alpha \in (0, 1]$.

Proof. We can have

$$
\begin{aligned}
I_q^{i\alpha}({}^C D_q^\alpha)^i f(t) - I_q^{(i+1)\alpha}({}^C D_q^\alpha)^{i+1} f(t) &= I^{i\alpha}\Big(({}^C D_q^\alpha)^i f(t) - I_q^\alpha({}^C D_q^\alpha)^{i+1} f(t)\Big) \\
&= I^{i\alpha}\Big(({}^C D_q^\alpha)^i f(t) - (I_q^\alpha . {}^C D_q^\alpha)({}^C D_q^\alpha)^i f(t)\Big) \\
&= I^{i\alpha}\bigg(({}^C D_q^\alpha)^i f(t) - \Big(({}^C D_q^\alpha)^i f(t) - ({}^C D_q^\alpha)^i f(a)\Big)\bigg) \\
&= I^{i\alpha}\bigg(({}^C D_q^\alpha)^i f(t) - ({}^C D_q^\alpha)^i f(t) + ({}^C D_q^\alpha)^i f(a)\bigg) \\
&= I^{i\alpha}\bigg(({}^C D_q^\alpha)^i f(a)\bigg) \\
&= ({}^C D_q^\alpha)^i f(a) I^{i\alpha}(t-a)^0 \\
&= ({}^C D_q^\alpha)^i f(a) \frac{\Gamma_q(0+1)}{\Gamma_q(1+i\alpha)}(t-a)^{i\alpha},
\end{aligned}
$$

or

$$
I_q^{i\alpha}({}^C D_q^\alpha)^i f(t) - I_q^{(i+1)\alpha}({}^C D_q^\alpha)^{i+1} f(t) = \frac{({}^C D_q^\alpha)^i f(a)}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)}.
$$

□

Theorem 3.8. (q-Taylor theorem) Suppose that ${}^C D_q^{\alpha k} f(t) \in [a,b]$, for $k = 0,1,2,3,\cdots,n+1$, where $n \in \mathbb{N}$. For $\alpha \in (0,1]$, there exists $c \in (a,b)$ and $q^* \in (0,1)$ such that

$$
f(t) = \sum_{i=1}^{n} \frac{({}^C D_q^i)^i f(a)}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)} + \frac{({}^C D_q^i)^{n+1} f(c)}{\Gamma_q((n+1)\alpha+1)}(t-a)^{((n+1)\alpha)},
$$

for all $q \in (q^*, 1)$, where $({}^C D_q^i) = {}^C D_q \, {}^C D_q \, {}^C D_q \cdots {}^C D_q \quad$ (i-times).

Proof. By Theorem 3.7, we can have

$$
\sum_{i=0}^{n}\left[I_q^{i\alpha}({}^C D_q^\alpha)^i f(t) - I_q^{(i+1)\alpha}({}^C D_q^{i+1}) f(t)\right] = \sum_{i=0}^{n} \frac{({}^C D_q^\alpha)^i}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)}.
$$

This implies

$$
f(t) - I_q^{(n+1)\alpha}({}^C D_q^{n+1}) f(t) = \sum_{i=0}^{n} \frac{({}^C D_q^\alpha)^i}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)}.
$$

By using the Riemann–Liouivelle q-integral, we get

$$
I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-xq)^{(\alpha-1)} f(x) d_q x,
$$

which gives

$$
f(t) - \frac{1}{\Gamma_q((n+1)\alpha)} \int_a^t (t-xq)^{((n+1)\alpha-1)}({}^C D_q^\alpha)^{(n+1)} f(x) d_q x = \sum_{i=0}^{n} \frac{({}^C D_q^\alpha)^i}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)}.
$$

This implies

$$f(t)=\sum_{i=0}^{n}\frac{({}^{C}D_q^{\alpha})^{i}}{\Gamma_q(i\alpha+1)}(t-a)^{(i\alpha)}+\frac{1}{\Gamma_q((n+1)\alpha)}\int_a^t(t-xq)^{((n+1)\alpha-1)}({}^{C}D_q^{\alpha})^{(n+1)}f(x)d_qx.$$

By using the classical form of Theorem 3.5, $I_q(fg)=g(\xi)I_q(f)$, for $\xi\in(a,b)$, there exists $q\in(q^*,1)$, where $q^*\in(0,1)$. Then, we get

$$\begin{aligned}
f(t)&=\sum_{i=0}^{n}\frac{(t-a)^{(i\alpha)}}{\Gamma_q(i\alpha+1)}({}^{C}D_q^{\alpha})^{i}f(a)+\frac{1}{\Gamma_q((n+1)\alpha)}({}^{C}D_q^{\alpha})^{(n+1)f(c)}\int_a^t(t-xq)^{((n+1)\alpha-1)}f(c)d_qx\\
&=\sum_{i=0}^{n}\frac{(t-a)^{(i\alpha)}}{\Gamma_q(i\alpha+1)}({}^{C}D_q^{\alpha})^{i}f(a)+\frac{({}^{C}D_q^{\alpha})^{(n+1)}f(c)}{\Gamma_q((n+1)\alpha)}I_q^{(n+1)\alpha}(t-a)^{0}\\
&=\sum_{i=0}^{n}\frac{(t-a)^{(i\alpha)}}{\Gamma_q(i\alpha+1)}({}^{C}D_q^{\alpha})^{i}f(a)+\frac{({}^{C}D_q^{\alpha})^{(n+1)}f(c)}{\Gamma_q((n+1)\alpha)}\frac{\Gamma_q((n+1)\alpha)}{\Gamma_q((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}\\
&=\sum_{i=0}^{n}\frac{(t-a)^{(i\alpha)}}{\Gamma_q(i\alpha+1)}({}^{C}D_q^{\alpha})^{i}f(a)+\frac{({}^{C}D_q^{\alpha})^{(n+1)}f(c)}{\Gamma_q((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}.
\end{aligned}$$

□

3.2. FHOTM and FHOqTM

In this part, we will propose two generalizations of higher-order Taylor methods: the FHOTM and the FHOqTM. The first one will be carried out by utilizing the so-called generalized Taylor theorem and a few simple computations. We then determine the local truncation error of the FHOTM through a theoretical result. The proposed method can be used to find the approximate solution of the following fractional initial value problem (FIVP):

$$D^{\alpha}y(t)=f(t,y(t)),\ a\le t\le b, \tag{3.6}$$

with initial condition

$$y(0)=y_0. \tag{3.7}$$

To deal with problem (3.6) and (3.7), we discertize the interval $[a,b]$ as $a=t_0<t_1<\cdots<t_n=b$, with $h=\frac{b-a}{n}$, for a positive integer n. Suppose that

$$D^{(n+1)\alpha}y(t)\in C^{n+1}[a,b].$$

Now, by expanding the solution $y(t)$ in terms of its corresponding Taylor's formula (Theorem 3.4) about t_i, we get

$$\begin{aligned}
y(t)=y(t_i)&+\frac{D^{\alpha}y(t_i)}{\Gamma(\alpha+1)}(t-t_i)^{\alpha}+\frac{D^{2\alpha}y(t_i)}{\Gamma(2\alpha+1)}(t-t_i)^{2\alpha}+\frac{D^{3\alpha}y(t_i)}{\Gamma(3\alpha+1)}(t-t_i)^{3\alpha}\\
&+\cdots+\frac{D^{n\alpha}y(t_i)}{\Gamma(n\alpha+1)}(t-t_i)^{n\alpha}+\frac{D^{(n+1)\alpha}y(\xi)}{\Gamma((n+1)\alpha+1)}(t-t_i)^{(n+1)\alpha},
\end{aligned}$$

where $\xi \in (t_i, t_{i+1})$. By replacing t with t_{i+1} in the above equality, we obtain

$$\begin{aligned} y(t_{i+1}) = y(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} D^\alpha y(t_i) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} y(t_i) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{3\alpha} y(t_i) \\ + \cdots + \frac{h^{n\alpha}}{\Gamma(n\alpha+1)} D^{n\alpha} y(t_i) + \frac{h^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D^{(n+1)\alpha} y(\xi). \end{aligned} \tag{3.8}$$

Now, due to

$$\begin{aligned} D^\alpha y(t) &= f(t, y(t)) \\ D^{2\alpha} y(t) &= D^\alpha f(t, y(t)) \\ D^{3\alpha} y(t) &= D^{2\alpha} f(t, y(t)) \\ &\vdots \\ D^{n\alpha} y(t) &= D^{(n+1)\alpha} f(t, y(t)), \end{aligned}$$

equality (3.8) becomes

$$\begin{aligned} y(t_{i+1}) = y(t_i) + \Bigg[\frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, y(t_i)) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^\alpha f(t_i, y(t_i)) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{2\alpha} f(t_i, y(t_i)) \\ + \cdots + \frac{h^{n\alpha}}{\Gamma(n\alpha+1)} D^{(n+1)\alpha} f(t_i, y(t_i)) \Bigg] + \frac{h^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D^{n\alpha} f(\xi_i, y(\xi_i)). \end{aligned} \tag{3.9}$$

In fact, formula (3.9) can be approximately expressed as follows:

$$\begin{aligned} \omega_0 &= y_0, \\ \omega_{i+1} &= \omega_i + h^\alpha T(t_i, \omega_i), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} T(t_i, \omega_i) = \frac{1}{\Gamma(\alpha+1)} f(t_i, \omega_i) + \frac{h^\alpha}{\Gamma(2\alpha+1)} D^\alpha f(t_i, \omega_i) + \frac{h^\alpha}{\Gamma(3\alpha+1)} D^{2\alpha} f(t_i, \omega_i) \\ + \cdots + \frac{h^{(n-1)\alpha}}{\Gamma(n\alpha+1)} D^{(n+1)\alpha} f(t_i, y(t_i)), \end{aligned} \tag{3.11}$$

for $i = 0, 1, 2, 3, \cdots, n-1$.

Theorem 3.9. Suppose that the FHOTM of order α is applied to find an approximation to the fractional initial value problem (3.6) and (3.7) with step size h, and assume $D^{k\alpha} y(t) \in C[a, b]$, for $k = 0, 1, 2, \cdots, n+1$, where $0 < \alpha \leq 1$. Then, the local truncation error is $O(h^{n\alpha})$.

Proof. Based on Eq (3.9), we can obtain

$$\begin{aligned} y(t_{i+1}) - y(t_i) - \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_i, y(t_i)) - \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^\alpha f(t_i, y(t_i)) - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{2\alpha} f(t_i, y(t_i)) \\ - \cdots - \frac{h^{n\alpha}}{\Gamma(n\alpha+1)} D^{(n+1)\alpha} f(t_i, y(t_i)) = \frac{h^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D^{n\alpha} f(\xi_i, y(\xi_i)), \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Thus, the local truncation error is of the form

$$\psi_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h^\alpha} - T(t_i, y(t_i)),$$

where

$$T(t_i, y(t_i)) = \frac{1}{\Gamma(\alpha+1)} f(t_i, y(t_i)) + \frac{h^{\alpha}}{\Gamma(2\alpha+1)} D^{\alpha} f(t_i, y(t_i)) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{2\alpha} f(t_i, y(t_i))$$
$$+ \cdots + \frac{h^{(n-1)\alpha}}{\Gamma(n\alpha+1)} D^{(n+1)\alpha} f(t_i, y(t_i)),$$

for $i = 0, 1, 2. \cdots, n-1$. In other words, we have the following local truncation error:

$$\psi_{i+1}(h) = \frac{h^{n\alpha}}{\Gamma((n+1)\alpha+1)} D^{n\alpha} f(\xi_i, y(\xi_i)).$$

Now, due to $D^{k\alpha} y(t) \in C[a, b]$, for $k = 0, 1, 2, \cdots, n+1$, we have

$$D^{(n+1)\alpha} y(t) = D^{n\alpha} f(t, y(t)),$$

which is bounded on $[a, b]$. Hence, $\psi_{i+1}(h) = O(h^{n\alpha})$. □

Now, we will propose the FHOqTM. Consider the fractional initial value problem

$${}^{C}D_q^{\alpha} y(t) = f(t, y(t)), \quad 0 < \alpha \leq 1,$$

with initial condition

$$y(0) = y_0, \tag{3.12}$$

where $q \in (0, 1)$, $y_0 \in \mathbb{R}$, and $t \in [0, T]$ such that $T > 0$, and where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

Remark 3.10. Theorem 3.9 shows that the local truncation error of the FHOTM is $O(h^{n\alpha})$, which indicates that the convergence rate depends on both the step size h and the fractional order α. For $\alpha = 1$, the scheme recovers the classical Taylor method of order n, while for $0 < \alpha < 1$, the rate decreases proportionally with α because of the memory property of the fractional derivative. This agrees with the findings reported by Blaszczyk et al. [23], who demonstrated that the convergence rate of numerical methods for the Riesz–Caputo operator decreases as α becomes smaller.

Lemma 3.11. There is an approximate solution for the fractional initial value problem (3.12) given as follows:

$$\begin{aligned} z_0 &= y_0, \\ z_{i+1} &= z_i + h^{\alpha} H(t_i, z_i), \end{aligned} \tag{3.13}$$

where

$$H(t_i, z_i) = \frac{1}{\Gamma_q(\alpha+1)} f(t_i, z_i) + \frac{h^{\alpha}}{\Gamma_q(2\alpha+1)}\, {}^{C}D_q^{\alpha} f(t_i, z_i) + \cdots + \frac{h^{(n-1)\alpha}}{\Gamma_q(n\alpha+1)}\, {}^{C}D_q^{(n+1)\alpha} f(t_i, z_i), \tag{3.14}$$

for $i = 0, 1, \cdots, n-1$ such that z_i represents the approximate solution of the exact solution y at t_i, h represents proper step size, and ${}^{C}D_q^{(n-1)\alpha} y(t) \in C^{n+1}([0, T])$.

Proof. For the aim of showing this result, we divide the interval $[0, T]$ in the following manner:

$$0 = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b,$$

for which the mesh points are $t_i = t_0 + ih$, for $i = 1, 2, \cdots, n$ and $h = \frac{b}{n}$ is the step size. Now, with the use of the fractional q-Taylor's formula, we can expand $y(t)$ about $t = t_i$ as follows:

$$\begin{aligned} y(t) =& y(t_i) + \frac{{}^C D_q^{\alpha} y(t_i)}{\Gamma_q(\alpha+1)}(t-t_i)^{(\alpha)} + \frac{{}^C D_q^{\alpha} y(t_i)}{\Gamma_q(2\alpha+1)}(t-t_i)^{(2\alpha)} \\ &+ \cdots + \frac{{}^C D_q^{n\alpha} y(t_i)}{\Gamma_q(n\alpha+1)}(t-t_i)^{(n\alpha)} + \frac{{}^C D_q^{(n+1)\alpha} y(\xi)}{\Gamma_q((n+1)\alpha+1)}(t-t_i)^{(n+1)\alpha}, \end{aligned}$$

where $t_i < \xi < t_{i+1}$. By replacing t_{i+1} instead of t_i in the above equality, we obtain

$$\begin{aligned} y(t_{in}) = y(t_i) &+ \frac{h^{\alpha}}{\Gamma_q(\alpha+1)}\, {}^C D_q^{\alpha} y(t_i) + \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)}\, {}^C D_q^{2\alpha} y(t_i) \\ &+ \cdots + \frac{h^{n\alpha}}{\Gamma_q(n\alpha+1)}\, {}^C D_q^{n\alpha} y(t_i) + \frac{h^{(n+1)\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^C D_q^{(n+1)\alpha} y(\xi). \end{aligned} \tag{3.15}$$

Now, due to

$$\begin{aligned} {}^C D_q^{\alpha} y(t) &= f(t, y(t)) \\ {}^C D_q^{2\alpha} y(t) &= {}^C D_q^{\alpha} f(t, y(t)) \\ &\vdots \\ {}^C D_q^{\alpha} y(t) &= {}^C D_q^{(n-1)\alpha} f(t, y(t)), \end{aligned}$$

formula (3.14) can be then transformed to the following form:

$$\begin{aligned} y(t_{i+1}) = y(t_i) + \Bigg[&\frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, y(t_i) + \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)}\, {}^C D_q^{\alpha} f(t_i, y(t_i) + \cdots + \\ &\frac{h^{n\alpha}}{\Gamma_q(n\alpha+1)}\, {}^C D_q^{(n+1)\alpha} f(t_i, y(t_i)) \Bigg] + \frac{h^{(n+1)\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^C D_q^{n\alpha} f(\xi, y(\xi)). \end{aligned}$$

This formula can be represented by the form

$$\begin{aligned} z_0 &= y_0, \\ z_{i+1} &= z_i + h^{\alpha} H(t_i, z_i), \end{aligned}$$

where

$$H(t_i, z_i) = \frac{1}{\Gamma_q(\alpha+1)} f(t_i, z_i) + \frac{h^{\alpha}}{\Gamma_q(2\alpha+1)}\, {}^C D_q^{\alpha} f(t_i, z_i) + \cdots + \frac{h^{(n+1)\alpha}}{\Gamma_q(n\alpha+1)}\, {}^C D_q^{(n-1)\alpha} f(t_i, z_i),$$

for $i = 0, 1, \cdots, n-1$, where

$$\psi(h) = \frac{{}^C D_q^{(n+1)\alpha} y(\xi)}{\Gamma_q((n+1)\alpha+1)} h^{((n+1)\alpha)}$$

is the truncation error of the obtained formula. □

Theorem 3.12. Consider the fractional initial value problem (3.12) with an approximate solution reported in (3.13) with step size h, for which ${}^{C}D_q^{j\alpha}y(t) \in C([0,t])$, for $j = 0, 1, 2, \cdots, n+1$, where $q \in (0,1)$ and $\alpha \in (0,1]$. Then, the local truncation error is given by $O(h^{n\alpha})$.

Proof. Based on formula (3.13), we can obtain

$$\begin{aligned} y(t_{i+1}) - y(t_i) &= \frac{h^{\alpha}}{\Gamma_q(\alpha+1)} f(t_i, y(t_i)) - \frac{h^{2\alpha}}{\Gamma_q(2\alpha+1)}\, {}^{C}D_q^{\alpha} f(t_i, y(t_i)) \\ &\quad - \cdots - \frac{h^{n\alpha}}{\Gamma_q(n\alpha+1)}\, {}^{C}D_q^{(n-1)\alpha} f(t_i, y(t_i)) \\ &= \frac{h^{(n+1)\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^{C}D_q^{n\alpha} f(\xi, y(\xi)), \end{aligned}$$

for $\xi \in (t_i, t_{i+1})$. This implies

$$c - h^{\alpha} H(t_i, y(t_i)) = \frac{h^{(n+1)\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^{C}D_q^{n\alpha} f(\xi, y(\xi)), \tag{3.16}$$

where

$$\begin{aligned} H(t_i, y(t_i)) &= \frac{1}{\Gamma_q(\alpha+1)} f(t_i, y(t_i)) + \frac{h^{\alpha}}{\Gamma_q(2\alpha+1)}\, {}^{C}D_q^{\alpha} f(t_i, y(t_i)) \\ &\quad + \cdots + \frac{h^{(n-1)\alpha}}{\Gamma_q(n\alpha+1)}\, {}^{C}D_q^{(n-1)\alpha} f(t_i, y(t_i)), \end{aligned}$$

for $i = 0, 1, 2, \cdots, n-1$. Formula (3.16) can be written as

$$\frac{y(t_{i+1}) - y(t_i)}{h^{\alpha}} - H(t_i, y(t_i)) = \frac{h^{n\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^{C}D_q^{n\alpha} f(\xi, y(\xi)).$$

This gives the local truncation error, which is

$$\xi_{i+1}^{H} = \frac{h^{n\alpha}}{\Gamma_q((n+1)\alpha+1)}\, {}^{C}D_q^{n\alpha} f(\xi, y(\xi)).$$

Now, because ${}^{C}D_q^{j\alpha}y(t) \in C([0,T])$, for $i = 0, 1, \cdots, n+1$, then

$${}^{C}D_q^{(n+1)\alpha}y(t) = {}^{C}D_q^{n\alpha} f(t, y(t)),$$

which means that ${}^{C}D_q^{n\alpha} f(t, y(t))$ is bounded on $[a,b]$. Hence, $\xi_{i+1}^{H}(h) = O(h^{n\alpha})$. □

Remark 3.13. Similarly, Theorem 3.12 confirms that the local truncation error of the FHOqTM is $O(h^{n\alpha})$. Hence, the rate of convergence is directly related to the order of the q-fractional derivative α. As α decreases, the convergence slightly reduces but remains consistent with the theoretical order, which is in line with the numerical observations in [23].

Remark 3.14. It is important to note that the fractional and q-fractional initial value problems considered in this study involve smooth right-hand sides, which ensure that their solutions possess the regularity required for the construction of higher-order Taylor-type expansions. Therefore, the smooth approximations produced by the proposed Taylor methods are mathematically justified and consistent with the well-known regularity theory for fractional and q-fractional differential equations.

4. Numerical solutions

In this section, we will give some numerical examples to demonstrate the difference between FHOTM and FHOqTM. All numerical simulations and graphical outputs presented in this study were generated using MATLAB (R2018a).

Example 4.1. Consider the following initial value problems:

$$D^{\alpha}y(t) = -y + t + 2, \quad 0 \leq t \leq 1, \tag{4.1}$$

with initial condition

$$y(0) = \frac{1}{3},$$

and

$$D_q^{\alpha}y(t) = -y + t + 2, \quad 0 \leq t \leq 1, \tag{4.2}$$

with initial condition

$$y(0) = \frac{1}{3}.$$

It should be noted that the exact solution of the classical case of the previous problem is $y(t) = t+1-\frac{2}{3}e^{-t}$. To deal with problem (4.1) by using the FHOTM of 4α-order, we use formulas (3.10) and (3.11) with step size $h = 0.1$. This implies to find the following items:

$$f(t, y(t)) = -y + t + 2, D^{\alpha}f(t, y(t)) = y - t - 2 + \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha},$$
$$D^{2\alpha}f(t, y(t)) = -y + t + 2 - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha},$$

and

$$D^{3\alpha}f(t, y(t)) = y - t - 2 + \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}.$$

Then, (3.11) becomes

$$\begin{aligned} T(t_i, \omega_i) = {} & \frac{1}{\Gamma(\alpha+1)}(-\omega_i + t_i + 2) + \frac{h^{\alpha}}{\Gamma(2\alpha+1)}(\omega_i - t_i - 2 + \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)}) \\ & + \frac{h^{2\alpha}}{\Gamma(3\alpha+1)}(-\omega_i + t_i + 2 - \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)}) + \frac{h^3\alpha}{\Gamma(4\alpha+1)}(\omega_i - t_i - 2 + \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)}). \end{aligned}$$

Consequently, (3.10) gives

$$\begin{aligned} \omega_{i+1} = \omega_i + h^{\alpha}\Bigg(& \frac{1}{\Gamma(\alpha+1)}(-\omega_i + t_i + 2) + \frac{h^{\alpha}}{\Gamma(2\alpha+1)}(\omega_i - t_i - 2 + \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)}) \\ & + \frac{h^{2\alpha}}{\Gamma(3\alpha+1)}(-\omega_i + t_i + 2 - \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)}) + \frac{h^3\alpha}{\Gamma(4\alpha+1)}(\omega_i - t_i - 2 + \frac{t_i^{1-\alpha}}{\Gamma(2-\alpha)})\Bigg). \end{aligned} \tag{4.3}$$

In fact, formula (4.3) gives an approximate solution for the FIVP (4.1).

On the other hand, to address problem (4.2) by using the FHOqTM of 4α-order, we use formula (3.13) and (3.14) with step size $h = 0.1$. This implies to find the following items:

$$\begin{aligned}
&f(t, y(t)) = -y + t + 2,\\
&D_q^{\alpha} f(t, y(t)) = -y - t - 2 + \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} t^{1-\alpha},\\
&D_q^{2\alpha} f(t, y(t)) = y - t - 2 - \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} t^{1-\alpha} + \frac{\Gamma_q(2)}{\Gamma_q(2-2\alpha)} t^{1-2\alpha},
\end{aligned}$$

and

$$D_q^{3\alpha} f(t, y(t)) = -y + t + 2 - \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} t^{1-\alpha} - \frac{\Gamma_q(2)}{\Gamma_q(2-2\alpha)} t^{1-2\alpha} + \frac{\Gamma_q(2)}{\Gamma_q(2-3\alpha)} t^{1-3\alpha}.$$

In a similar manner of the previous discussion, we can obtain

$$\begin{aligned}
\omega_{i+1} = \omega_i + h^{\alpha}\Bigg(&\frac{1}{\Gamma_q(\alpha+1)}(-\omega_i + t_i + 2) + \frac{h^{\alpha}}{\Gamma_q(2\alpha+1)}\left(-\omega_i - t_i - 2 + \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} t^{1-\alpha}\right)\\
&+ \frac{h^{2\alpha}}{\Gamma_q(3\alpha+1)}(\omega_i - t_i - 2 - \frac{\Gamma_q(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + \frac{\Gamma_q(2)}{\Gamma_q(2-2\alpha)} t^{1-2\alpha})\\
&+ \frac{h^3\alpha}{\Gamma_q(4\alpha+1)}(-\omega_i + t_i + 2 - \frac{\Gamma_q(2)}{\Gamma_q(2-\alpha)} t^{1-\alpha} - \frac{\Gamma_q(2)}{\Gamma_q(2-3\alpha)} t^{1-2\alpha} + \frac{\Gamma_q(2)}{\Gamma_q(2-3\alpha)} t^{1-3\alpha})\Bigg).
\end{aligned} \tag{4.4}$$

This, however, represents an approximate solution for q-FIVP (4.2).

To verify the two approximate solutions given in (4.3) and (4.4), we plot these solutions and compare them with the exact one as shown in Figure 1. In the same regard, we also display the absolute errors of such solutions in Figure 2 and Table 1.

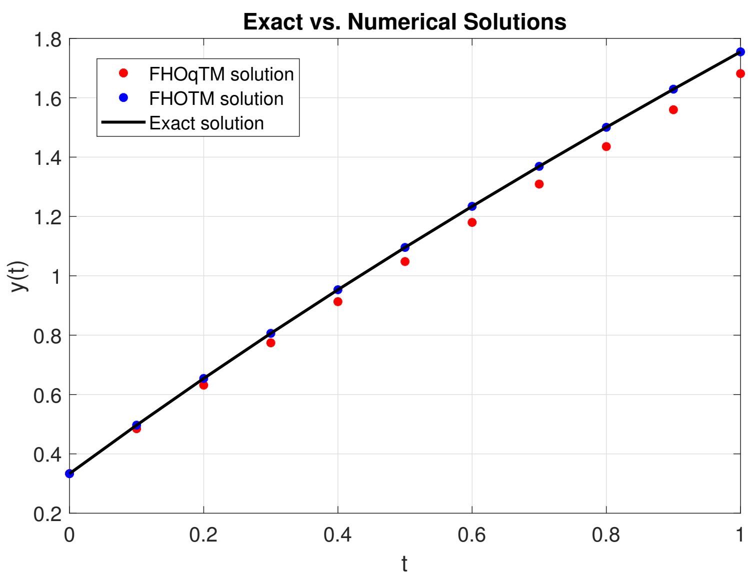

Figure 1. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) with $h = 0.1$, $\alpha = 1$, and $q \to 1$.

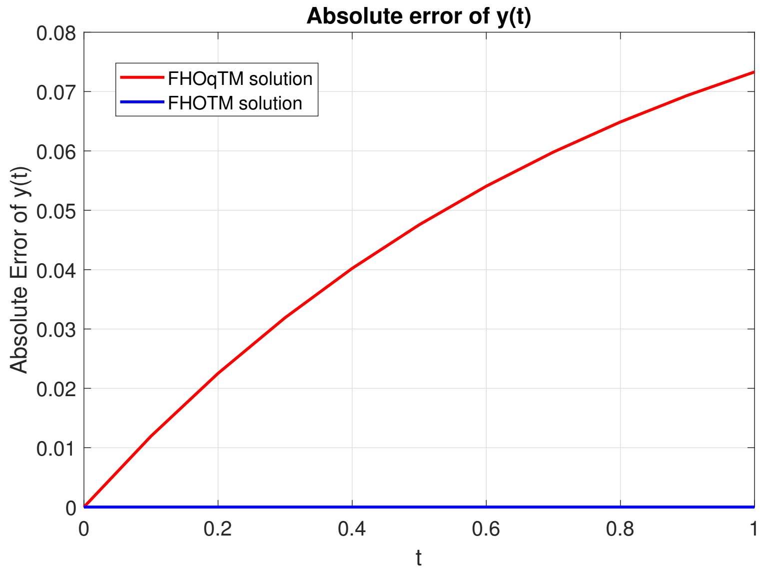

Figure 2. Absolute error between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2), respectively.

Table 1. Absolute error between the exact and numerical solutions of problems (4.1) and (4.2), respectively.

t	FHOqTM	FHOTM
0	0	0
0.1	0.0546×10^{-6}	0.0119×10^{-6}
0.2	0.0989×10^{-6}	0.0225×10^{-6}
0.3	0.1342×10^{-6}	0.0319×10^{-6}
0.4	0.1619×10^{-6}	0.0402×10^{-6}
0.5	0.1831×10^{-6}	0.0476×10^{-6}
0.6	0.1989×10^{-6}	0.0541×10^{-6}
0.7	0.2099×10^{-6}	0.0598×10^{-6}
0.8	0.2171×10^{-6}	0.0649×10^{-6}
0.9	0.2210×10^{-6}	0.0694×10^{-6}
1	0.2222×10^{-6}	0.0733×10^{-6}

From the previous numerical results, we can clearly observe that the FHOTM is more accurate than the FHOqTM. For more illustration, we furthermore plot the two approximate solutions given in (4.3) and (4.4) in Figure 3 with step size $h = 0.01$. Moreover, we plot in Figure 4 the absolute errors of these solutions with the same step size.

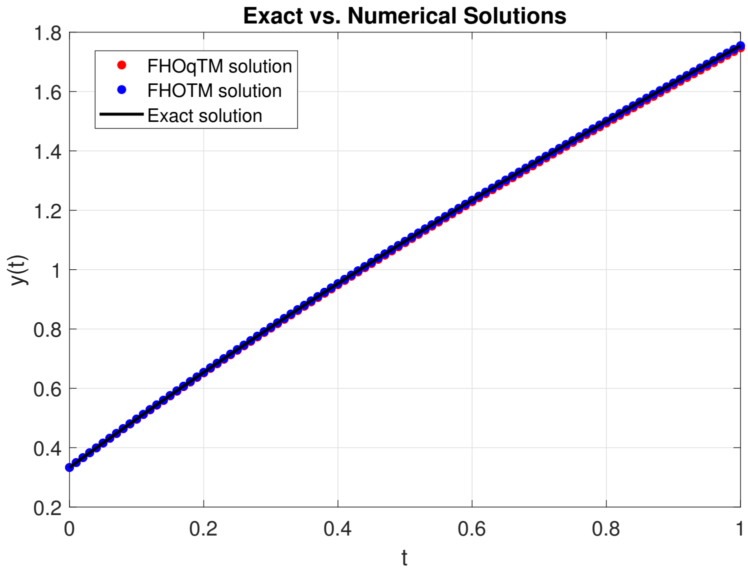

Figure 3. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) with $h = 0.01$, $\alpha = 1$, and $q \to 1$.

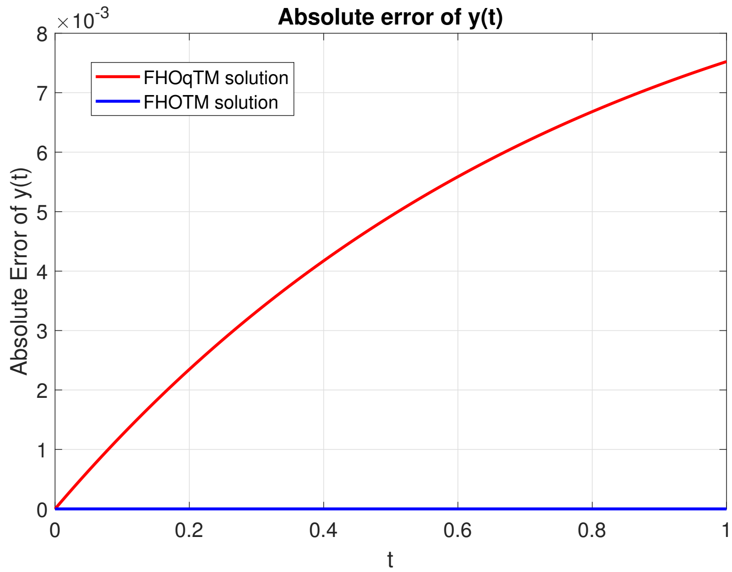

Figure 4. Absolute error between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2), respectively.

Based on the previous numerical results, we observe that the FHOTM is also better than the FHOqTM. In addition, we note that the less step size of h, the more accurate approximate solution. In order to display the behaviour of the approximate solutions (4.3) and (4.4) of problems (4.1) and (4.2), respectively, we plot these solutions according to different values of α and q as shown in Figures 5–7.

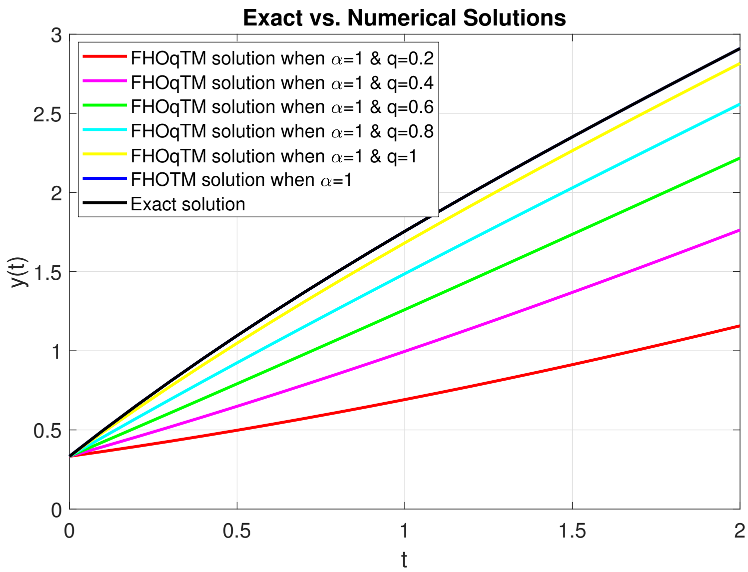

Figure 5. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by fixing $\alpha = 1$ and varying q with $h = 0.1$.

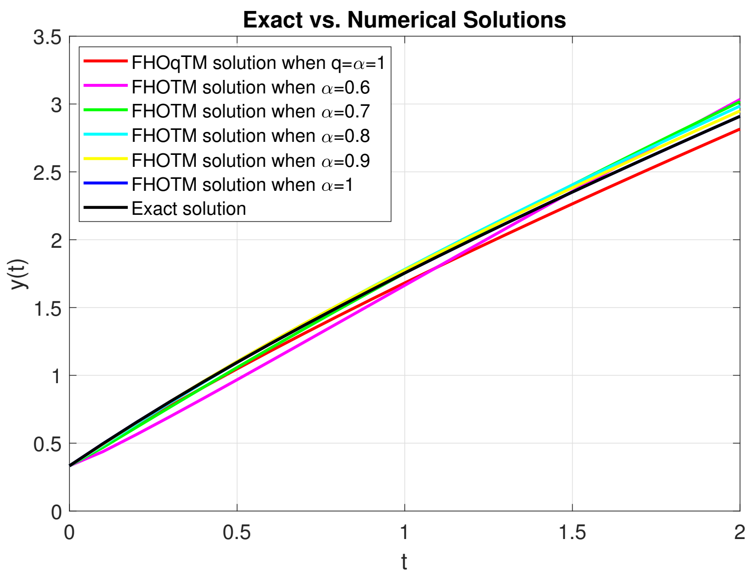

Figure 6. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by fixing $q \to 1$ and varying α with $h = 0.1$.

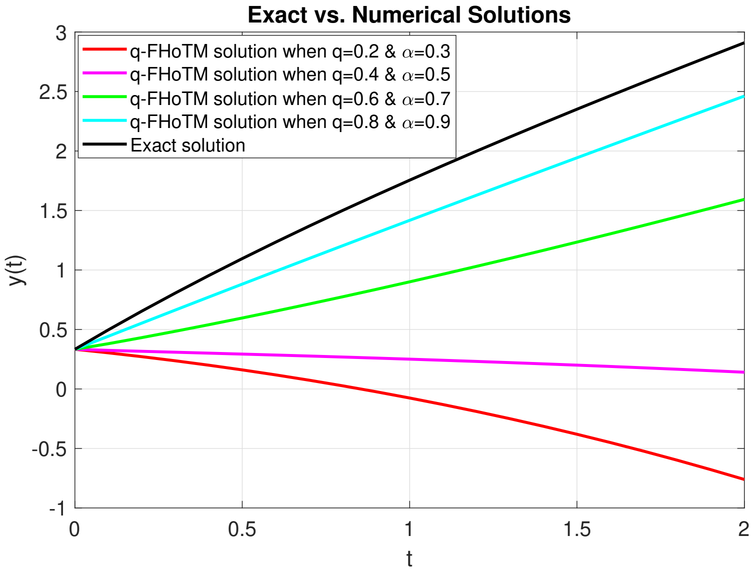

Figure 7. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by varying q and varying α with $h = 0.1$.

Example 4.2. Consider the following IVPs:

$$D^{\alpha} y(t) = y + t^2, \quad 0 \le t \le 1, \tag{4.5}$$

with initial condition

$$y(0) = 1,$$

and

$$D_q^{\alpha} y(t) = y + t^2, \quad 0 \le t \le 1, \tag{4.6}$$

with initial condition

$$y(0) = 1.$$

It should be noted that the exact solution of the classical case of the previous problem is $y(t) = 7e^t - t^3 - 3t^2 - 6t - 6$. To deal with problem (4.5) by using the FHOTM of 4α-order, we use formulas (3.10) and (3.11) with step size $h = 0.1$. This implies to find the following items:

$$\begin{aligned}
&f(t, y(t)) = y + t^2,\\
&D^{\alpha} f(t, y(t)) = y + t^2 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha},\\
&D^{2\alpha} f(t, y(t)) = y + t^2 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{2}{\Gamma(3-2\alpha)} t^{2-2\alpha},
\end{aligned}$$

and

$$D^{3\alpha} f(t, y(t)) = y + t^2 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{2}{\Gamma(3-2\alpha)} t^{2-2\alpha} + \frac{2}{\Gamma(3-3\alpha)} t^{2-3\alpha}.$$

Then, (3.11) becomes

$$\begin{aligned}T(t_i,\omega_i) &= \frac{1}{\Gamma(\alpha+1)}(\omega_i+t_i^2)+\frac{h^\alpha}{\Gamma(2\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha})\\ &+\frac{h^{2\alpha}}{\Gamma(3\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{2}{\Gamma(3-2\alpha)}t^{2-2\alpha})\\ &+\frac{h^3\alpha}{\Gamma(4\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{2}{\Gamma(3-2\alpha)}t^{2-2\alpha}+\frac{2}{\Gamma(3-3\alpha)}t^{2-3\alpha}).\end{aligned}$$

Consequently, (3.10) gives

$$\begin{aligned}\omega_{i+1} &= \omega_i+h^\alpha\Big(\frac{1}{\Gamma(\alpha+1)}(\omega_i+t_i^2)+\frac{h^\alpha}{\Gamma(2\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha})\\ &+\frac{h^{2\alpha}}{\Gamma(3\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{2}{\Gamma(3-2\alpha)}t^{2-2\alpha})\\ &+\frac{h^3\alpha}{\Gamma(4\alpha+1)}(\omega_i+t_i^2+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{2}{\Gamma(3-2\alpha)}t^{2-2\alpha}+\frac{2}{\Gamma(3-3\alpha)}t^{2-3\alpha})\Big).\end{aligned} \tag{4.7}$$

In fact, formula (4.7) gives an approximate solution for the FIVP (4.4). On the other hand, to address problem (4.5) by using the FHOqTM of 4α-order, we use formula (3.13) and (3.14) with step size $h = 0.1$. This implies to find the following items:

$$\begin{aligned}f(t,y(t)) &= y+t^2,\\ D_q^\alpha f(t,y(t)) &= y+t^2+\frac{\Gamma_q(3)}{\Gamma_q(3-\alpha)}t^{2-\alpha},\\ D_q^{2\alpha} f(t,y(t)) &= y+t^2+\frac{\Gamma_q(3)}{\Gamma_q(3-\alpha)}t^{2-\alpha}+\frac{\Gamma_q(3)}{\Gamma_q(3-2\alpha)}t^{2-2\alpha},\end{aligned}$$

and

$$D_q^{3\alpha} f(t,y(t)) = y+t^2+\frac{\Gamma_q(3)}{\Gamma_q(3-3\alpha)}t^{2-3\alpha}.$$

$$Therefore \quad \begin{aligned}\omega_{i+1} &= \omega_i+h^\alpha\Big(\frac{1}{\Gamma_q(\alpha+1)}(\omega_i+t_i^2)+\frac{h^\alpha}{\Gamma_q(2\alpha+1)}(\omega_i+t_i^2+\frac{\Gamma_q(3)}{\Gamma_q(3-\alpha)}t^{2-\alpha})\\ &+\frac{h^{2\alpha}}{\Gamma_q(3\alpha+1)}(\omega_i+t_i^2+\frac{\Gamma_q(3)}{\Gamma_q(3-\alpha)}t^{2-\alpha}+\frac{\Gamma_q(3)}{\Gamma_q(3-2\alpha)}t^{2-2\alpha})\\ &+\frac{h^3\alpha}{\Gamma_q(4\alpha+1)}(\omega_i+t_i^2+\frac{\Gamma_q(3)}{\Gamma_q(3-\alpha)}t^{2-\alpha}+\frac{\Gamma_q(3)}{\Gamma_q(3-2\alpha)}t^{2-2\alpha}+\frac{\Gamma_q(3)}{\Gamma_q(3-3\alpha)}t^{2-3\alpha})\Big)\end{aligned} \tag{4.8}$$

is the approximate solution for q-FIVP (4.5).

To verify the two approximate solutions given in (4.7) and (4.4), we plot these solutions and compare them with the exact one as shown in Figure 8. In the same regard, we also display the absolute errors of such solutions in Figure 9 and Table 2.

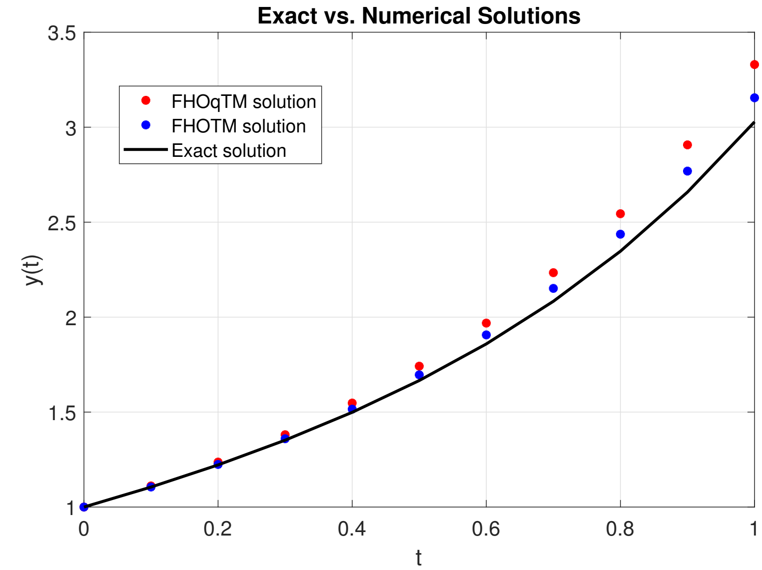

Figure 8. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.7) and (4.8) with $h = 0.1$, $\alpha = 1$, and $q \to 1$.

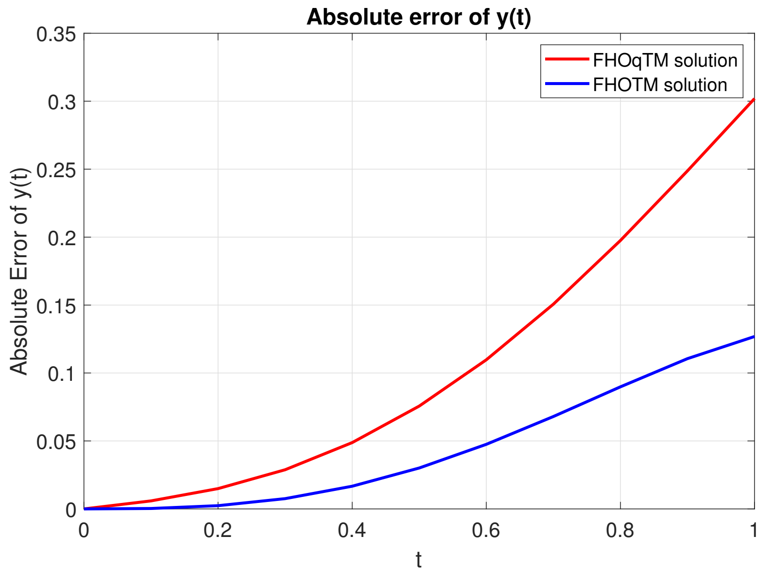

Figure 9. Absolute error between the FHOTM and FHOqTM solutions of problems (4.3) and (4.4), respectively.

Table 2. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.7) and (4.8) with $h = 0.1$, $\alpha = 1$, and $q \to 1$.

t	FHOTM	FHOqTM
0	0.000000	0.000000
0.1	0.0003	0.0059
0.2	0.0024	0.0149
0.3	0.0076	0.0288
0.4	0.0167	0.0488
0.5	0.0301	0.0757
0.6	0.0475	0.1097
0.7	0.0680	0.1506
0.8	0.0898	0.1975
0.9	0.1106	0.2487
1	0.1269	0.3019

From the previous numerical results, we can clearly observe that the FHOTM is more accurate than the FHOqTM. For more illustration, we furthermore plot the two approximate solutions given in (4.7) and (4.8) in Figure 10 with step size $h = 0.01$. Moreover, we plot in Figure 11 the absolute errors of these solutions with the same step size.

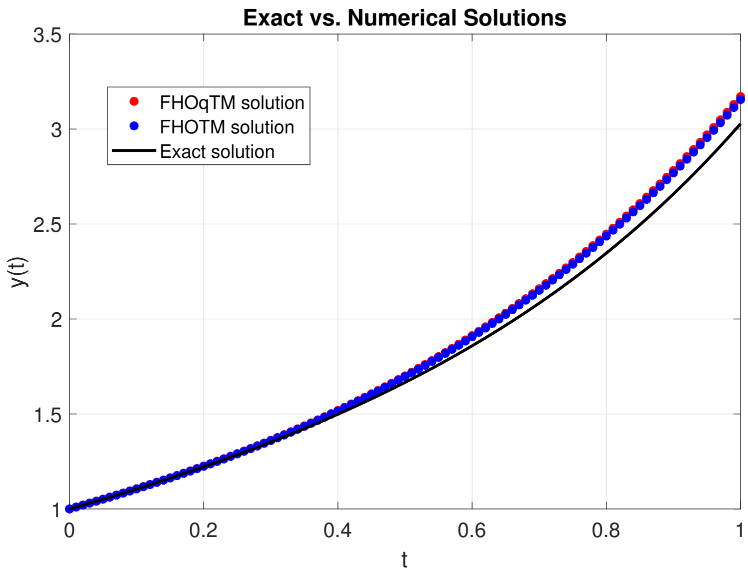

Figure 10. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.3) and (4.4) with $h = 0.01$, $\alpha = 1$, and $q \to 1$.

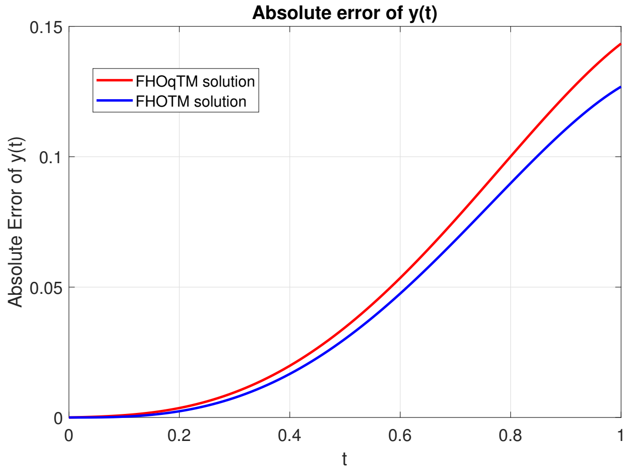

Figure 11. Absolute error between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2), respectively.

Based on the previous numerical results, we observe that the FHOTM is also better than the FHOqTM. In addition, we note that the less step size of h, the more accurate approximate solution. In order to display the behaviour of the approximate solutions (4.7) and (4.8) of problems (4.4) and (4.5), respectively, we plot these solutions according to different values of α and q as shown in Figures 12–14.

It is worth noting that fractional and q-fractional initial value problems of the form $D^{\alpha}y(t) = y + t^2$ and $D_q^{\alpha}y(t) = y + t^2$ arise in several real-world applications. Such equations appear in models that incorporate memory and hereditary effects, including anomalous diffusion, viscoelastic materials, and population growth with delayed response to environmental changes. The q-fractional formulation is also relevant in problems where the underlying dynamics evolve on discrete or nonuniform time scales, such as in quantum calculus, lattice-based models, and scale-dependent physical systems. These connections highlight the practical relevance of the considered problems and justify the use of fractional and q-fractional techniques in their analysis.

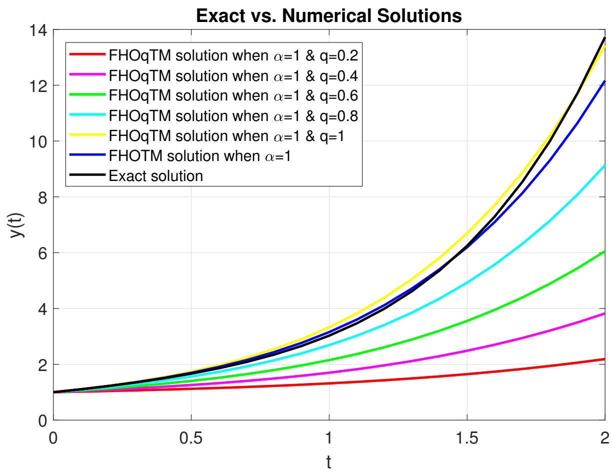

Figure 12. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by fixing $\alpha = 1$ and varying q with $h = 0.1$.

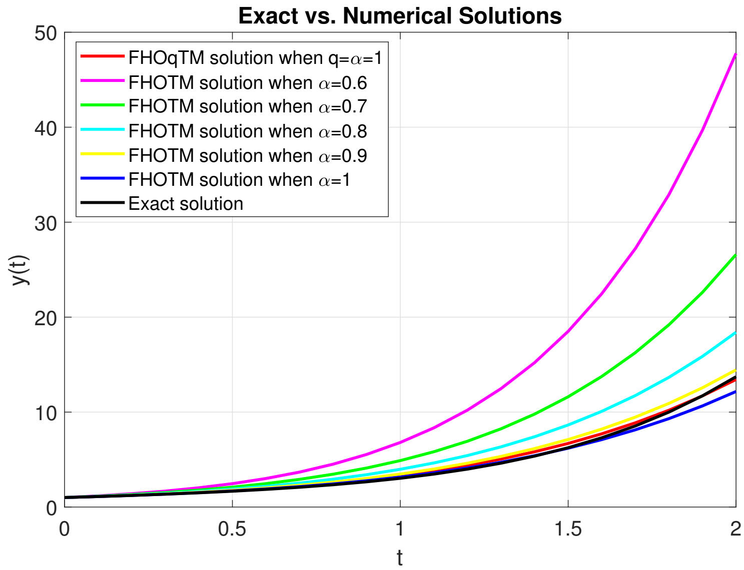

Figure 13. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by fixing $q \to 1$ and varying α with $h = 0.1$.

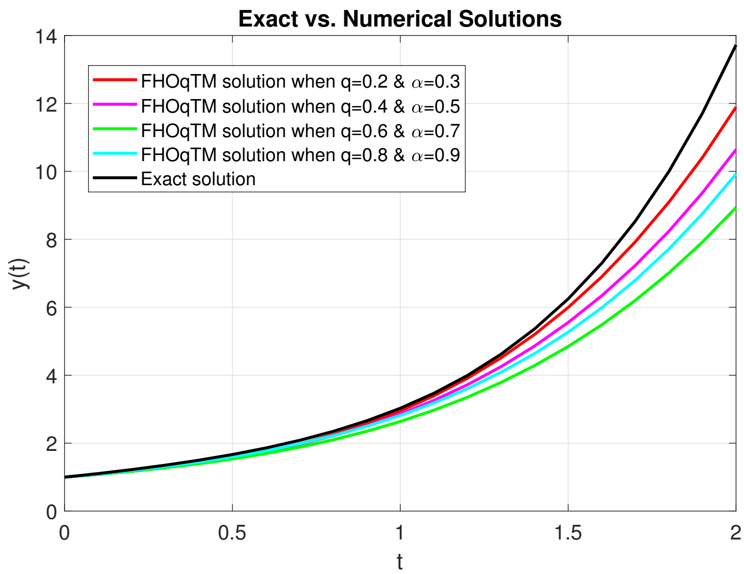

Figure 14. Numerical comparison between the FHOTM and FHOqTM solutions of problems (4.1) and (4.2) by varying q and varying α with $h = 0.1$.

From a physical perspective, fractional and q-fractional differential equations provide a versatile framework for modeling systems with memory, nonlocal interactions, and scale-dependent dynamics. Fractional derivatives naturally appear in the study of viscoelastic materials, anomalous diffusion processes, and relaxation phenomena in complex media, where the present time state depends on the entire past evolution. The q-fractional formulation is also relevant for describing systems evolving on discrete or nonuniform time scales, which arise in quantum calculus, lattice-based transport, and certain optical and mechanical models. The numerical results obtained in this work illustrate how the proposed Taylor-type schemes can capture such physical effects by producing smooth trajectories for subdiffusive ($\alpha < 1$) behavior and reflecting the influence of the deformation parameter q on the dynamic response. These observations underline the physical relevance of the developed methods and justify their applicability to problems governed by memory and nonlocality.

We note that many time-fractional diffusion and parabolic problems may exhibit weak singularities near $t = 0$, typically of order $t^{\alpha-1}$ when the forcing term or initial data are non-smooth. For such problems, the use of graded meshes, smoothing transformations, or specially designed discretizations (see, e.g., the recent works in [24–27]) becomes essential. In contrast, the fractional and q-fractional initial value problems studied in this work involve smooth right-hand sides and therefore do not generate initial weak singular behavior. Nevertheless, the treatment of weakly singular fractional dynamics remains an important topic and will be considered in future work.

5. Conclusions

This work introduces two generalized forms of the higher-order Taylor technique, namely the q-fractional Taylor method and the fractional Taylor method. These approaches are applied to compute approximate solutions for a range of fractional and q-fractional initial value problems. The numerical results confirm that the proposed generalizations yield dependable approximations and are capable of handling models characterized by memory effects and nonlocal dynamics. The flexibility observed in the tested examples reflects the potential of these techniques for broader classes of fractional differential equations. Future work may consider extending these formulations to higher-dimensional systems, variable-order models, or alternative fractional operators.

Author contributions

Iqbal M. Batiha: Conceived the main idea, developed the mathematical framework, and write the initial draft; Sharaf Aldeen O. Albteiha and Hamzah O. Al-Khawaldeh: Contributed to the theoretical analysis and numerical experiments; Shaher Momani: Supervised the study, reviewed the results, and improved the final version of the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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