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Research article

Finite volume element discretization of optimal control of the parabolic equation using the discretize-then-optimize approach

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Abstract: This paper proposes a novel finite volume element (FVE) scheme for linear parabolic optimal control problems (OCPs) subject to integral control constraints. The state and co-state variables were approximated using continuous piecewise linear finite elements, while the control variable was discretized via piecewise constant functions. First, following the discretize-then-optimize approach, the FVE approximation of the parabolic OCP was formulated. Second, the first-order optimality conditions were derived, and corresponding error estimates in the $L^2(J; H^1(\Omega))$-norm for the state and co-state variables, as well as in the $L^2(J; L^2(\Omega))$-norm for the control variable, were established. These estimates quantify the deviation between the discrete solutions and the exact solutions over the time interval J and spatial domain Ω, providing rigorous bounds on the approximation errors. Third, some superclose results between the projection of the exact solution and the discrete solution for all variables were analyzed, leading to optimal-order error estimates in the $L^\infty(J; L^2(\Omega))$-norm for all variables. Finally, a numerical example was presented to validate the theoretical results. We believe that this is the first article to construct an FVE approximation based on the discretize-then-optimize approach for the parabolic OCP.

Keywords: parabolic problem; optimal control; finite volume; convergence; superclose
Mathematics Subject Classification: 49J20, 65N30

1. Introduction

Finite element approximation plays a crucial role in the numerical solution of OCPs. Extensive theoretical and numerical investigations have been conducted on the finite element approximation of various types of OCPs. Chen [1] employed the postprocessing projection operator, originally defined by Meyer and Rösch (see [2]), to establish quadratic superconvergence of the control approximation in mixed finite element methods. Chen and Dai [3] investigated superconvergence properties for semilinear elliptic OCPs discretized by the linear finite element method. Li et al. [4, 5] derived both residual-based and recovery-type a posteriori error estimates for finite element approximations of elliptic OCPs. Meyer and Rösch [6] analyzed L^∞-error estimates for elliptic OCPs discretized using finite element methods. Meidner and Vexler [7, 8] studied a priori error estimates for space-time finite element discretizations of parabolic OCPs, with and without control constraints, respectively. Zhou and Tan [9] examined finite element approximations for OCPs governed by space-fractional differential equations and discussed associated a priori error estimates. Liu et al. [10] developed a posteriori error estimates for discontinuous Galerkin time-stepping schemes applied to parabolic OCPs. Comprehensive introductions to finite element methods for partial differential equations and OCPs can be found in [11, 12].

For the discretization of the optimal control problems, there are two different approaches: optimize-then-discretize and discretize-then-optimize. In the optimize-then-discretize approach, first the necessary optimality conditions are established on the continuous level consisting of the state, co-state, and optimality equations, and then these equations are discretized by different numerical methods. In the discretize-then-optimize approach, the state equation is discretized and then the optimality system for the finite dimensional optimization problem is derived. Treating the control and state as independent of optimization variables, the discrete optimality conditions yield a system.

The FVE method is one of the most widely employed numerical techniques for solving partial differential equations. A key advantage of this method lies in its ability to preserve local physical conservation laws. Over the past decade, the FVE method has been increasingly applied to the computation of various OCPs. For instance, Luo et al. [13] utilized this method to solve distributed OCPs governed by hyperbolic equations and derived optimal a priori error estimates for all involved variables. Ge and Sun [14] proposed a hybrid approximation scheme for an OCP governed by an elliptic equation with random coefficients. The scheme approximates the optimality system by employing the FVE method for spatial discretization and a sparse grid stochastic collocation method based on Smolyak approximation for the probability space. Lin et al. [15] developed a Fourier FVE method to address distributed control and Dirichlet boundary control problems. In their approach, Fourier discretization was applied in the azimuthal direction using polar coordinates, while FVE approximation was used in the radial direction. Kumar et al. [16] introduced a discontinuous FVE method for distributed OCPs governed by the Brinkman equations, where the objective is to identify a force field that generates a desired velocity profile. Furthermore, Kumar et al. [17] presented a family of discretizations for OCPs governed by equations modeling immiscible displacement in porous media, combining mixed and discontinuous FVE methods within an optimize-then-discretize framework. It is noteworthy that all the aforementioned studies adopt the optimize-then-discretize approach to derive their discretization schemes.

The primary objective of this paper is to construct an FVE approximation for a parabolic OCP using the discretize-then-optimize approach. We consider the following OCP:

$$\min_{q\in Q^{ad}} \frac{1}{2}\left(\|y(x,t)-y_d(x,t)\|^2_{L^2(J;L^2(\Omega))}+\|q(x,t)\|^2_{L^2(J;L^2(\Omega))}\right), \tag{1.1}$$

$$y_t(x,t)-\mathrm{div}(A(x)\nabla y(x,t))+c(x)y(x,t)=f(x,t)+q(x,t),\ x\in\Omega,\ t\in J, \tag{1.2}$$

$$y(x,t)|_{\partial\Omega}=0,\ t\in J, \tag{1.3}$$

$$y(x,0)=y_0(x),\ x\in\Omega. \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^2$ is a convex polygon, $J = (0, T]$, and

$$Q^{ad} = \left\{ q \in L^2(J; L^2(\Omega)) : \int_0^T \int_\Omega q \, dxdt \geq 0 \right\}.$$

Assume that $y_d \in L^2(J; H^2(\Omega))$, $y_0 \in H^2(\Omega)$, and $0 \leq c(x) \in W^{1,\infty}(\Omega)$. The matrix function $A(x) = (a_{ij}(x))$ is symmetric. It satisfies $a_{ij}(x) \in W^{1,\infty}(\Omega)$ and

$$c_* |\zeta|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x) \zeta_i \zeta_j \leq c^* |\zeta|^2, \ \ \forall \ \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2, \ x \in \bar{\Omega},$$

where $|\zeta|^2 = \zeta_1^2 + \zeta_2^2$ and $c^* > c_* > 0$.

This paper is organized as follows. In Section 2, we formulate a new FVE scheme for the OCP (1.1)–(1.4) and present its equivalent optimality conditions. In Section 3, we establish error estimates in the $L^2(J; H^1(\Omega))$-norm for the state and co-state variables, and in the $L^2(J; L^2(\Omega))$-norm for the control variable. Section 4 investigates superclose results between the projection of the exact solution and the discrete solution for all variables, leading to optimal a priori error estimates in the $L^\infty(J; L^2(\Omega))$-norm for all variables. Section 5 presents a numerical example. Finally, Section 6 provides a summary of the main results and discusses potential directions for future research.

Standard Sobolev space notation $W^{m,k}(\Omega)$ is adopted throughout this paper, equipped with the norm $\|\cdot\|_{m,k}$ defined by $\|v\|_{m,k}^k = \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^k(\Omega)}^k$ and the semi-norm $|\cdot|_{m,k}$ given by $|v|_{m,k}^k = \sum_{|\alpha| = m} \|D^\alpha v\|_{L^k(\Omega)}^k$. We define $W_0^{m,k}(\Omega) = \{v \in W^{m,k}(\Omega) : v|_{\partial\Omega} = 0\}$. For the case $k = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and simplify the norms as $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(J; W^{m,k}(\Omega))$ the Banach space of functions that are L^s-integrable from J into $W^{m,k}(\Omega)$, with the norm defined as $\|v\|_{L^s(J;W^{m,k}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,k}(\Omega)}^s \, dt \right)^{1/s}$ for $s \in [1, \infty)$, with the standard modification for $s = \infty$. For brevity, we write $\|v\|_{L^s(W^{m,k})}$ to denote $\|v\|_{L^s(J;W^{m,k}(\Omega))}$. Analogously, spaces such as $H^1(W^{m,k})$ and $C^k(W^{m,k})$ are defined. Throughout the analysis, C represents a generic positive constant independent of the mesh size h, where h denotes the spatial discretization parameter.

2. FVE approximation

This section constructs a new FVE scheme of the problem (1.1)–(1.4) and introduces some necessary lemmas. Let $(L^2(\Omega))^2 := \{v = (v_1, v_2) | v_i \in L^2(\Omega), i = 1, 2\}$.

The weak form of (1.1)–(1.4) is to seek $(y, q) \in L^2(H_0^1) \times Q^{ad}$ such that

$$\min_{q \in Q^{ad}} \frac{1}{2} \left(\|y - y_d\|_{L^2(L^2)}^2 + \|q\|_{L^2(L^2)}^2 \right), \tag{2.1}$$

$$(y_t, v) + (A\nabla y, \nabla v) + (cy, v) = (f + q, v), \ \ \forall \ v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$.

By [12], we know that the OCP (2.1) and (2.2) has a unique solution $(y, q) \in L^2(H_0^1) \times Q^{ad}$. At the same time, for any $v \in H_0^1(\Omega)$, $x \in \Omega$, and $\dot{q} \in Q^{ad}$, there exists a co-state $p \in L^2(H_0^1)$ such that (y, p, q) satisfy

$$(y_t, v) + (A\nabla y, \nabla v) + (cy, v) = (f + q, v), \tag{2.3}$$

$$y(x,0) = y_0(x), \tag{2.4}$$
$$-(p_t, v) + (A\nabla p, \nabla v) + (cp, v) = (y - y_d, v), \tag{2.5}$$
$$p(x,T) = 0, \tag{2.6}$$
$$\int_0^T (q + p, \dot{q} - q)dt \geq 0. \tag{2.7}$$

By (2.7), we get

$$q = \max\{0, \bar{p}\} - p, \tag{2.8}$$

where $\bar{p} = \frac{\int_0^T \int_\Omega p dx dt}{\int_0^T \int_\Omega 1 dx dt}$.

For a polygonal domain Ω, we consider a quasi-uniform and regular triangulation T_h, composed of closed triangular elements K such that $\bar{\Omega} = \bigcup_{K \in T_h} K$. Let $\mathcal{N}_h$ denote the set of all nodes (vertices) of the triangulation T_h, and let $\mathcal{N}_h^0 = \mathcal{N}_h \cap \Omega$ represent the set of interior nodes.

We then construct a dual mesh T_h^* based on T_h. There are several approaches to defining the dual mesh, most of which can be described within the following general framework. For each element $K \in T_h$ with vertices x_i, x_j, x_k, select a point Q in the interior of K, and choose a point x_{ij} on each edge $\overline{x_i x_j}$ of K. Connect Q to each x_{ij} via straight-line segments r_{ij}. For a given vertex x_i, define V_i as the polygon formed by the union of all segments r_{ij} such that x_i is a vertex of K. The region V_i is referred to as the control volume centered at x_i. It follows that $\bar{\Omega} = \bigcup_{x_i \in \mathcal{N}_h} V_i$, and the dual mesh T_h^* is defined as the collection of all such control volumes.

The control volume mesh T_h^* is said to be quasi-uniform and regular if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq meas(V_i) \leq Ch^2, \ \forall \ V_i \in T_h^*,$$

where h is the maximum diameter of all elements $K \in T_h$.

Various approaches can be employed to construct a regular dual mesh T_h^*, depending on the selection of a point Q within each element $K \in T_h$ and points x_{ij} on its edges. In this work, we adopt a widely used configuration in which Q is chosen as the barycenter of each element $K \in T_h$, and the points x_{ij} are selected as the midpoints of the edges of K. This construction of control volumes applies to any triangulation T_h and facilitates relatively straightforward computations. Moreover, if T_h is locally regular, then the associated dual mesh T_h^* is also locally regular. Let S_h denote the standard piecewise linear finite element space defined over the triangulation T_h,

$$S_h = \{v \in C(\Omega) : v|_K \text{ is linear}, \ \forall \ K \in T_h; v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^* on T_h^*,

$$S_h^* = \{v \in L^2(\Omega) : v|_{V_i} \text{is constant for all } V_i \in T_h^*; v|_{V_i} = 0, \text{ if } x_i \in \partial\Omega\}.$$

Consequently, we obtain $S_h = span\{\phi_i(x) : x_i \in \mathcal{N}_h^0\}$ and $S_h^* = span\{\phi_i^*(x) : x_i \in \mathcal{N}_h^0\}$, where $\phi_i(x)$ represents the standard nodal basis function associated with node x_i, and $\phi_i^*(x)$ denotes the characteristic function of the control volume V_i.

The space of piecewise constant functions is defined as follows:

$$W_h := \{w_h \in L^2(\Omega) : w_h|_K \text{ is constant}, \forall \ K \in T_h\}.$$

We recall the linear interpolation operator $I_h : H_0^1(\Omega) \cap H^2(\Omega) \to S_h$, $\forall\, v \in H_0^1(\Omega) \cap H^2(\Omega)$, and we have

$$I_h v = \sum_{x_i \in \mathcal{N}_h} v(x_i)\phi_i(x)$$

and

$$\|\psi - I_h\psi\| + h\|\psi - I_h\psi\|_1 \le Ch^2\|\psi\|_2, \ \forall\, \psi \in H^2(\Omega). \tag{2.9}$$

We recall the interpolation operator $I_h^* : S_h \to S_h^*$, which is defined by

$$I_h^* v_h = \sum_{x_i \in \mathcal{N}_h} v_h(x_i)\phi_i^*(x).$$

For any $\psi \in L^2(\Omega)$, we recall the $L^2(\Omega)$-projection [18] $P_h : \ L^2(\Omega) \to W_h$, which satisfies:

$$(P_h\psi - \psi, w_h) = 0, \ \ \forall\, w_h \in W_h, \tag{2.10}$$

$$\|\psi - P_h\psi\|_{0,l} \le Ch\|\psi\|_{1,l}, \ \ 2 \le l \le \infty, \ \forall\, \psi \in W^{1,l}(\Omega). \tag{2.11}$$

For any $\psi \in H_0^1(\Omega)$, we define the elliptic volume projection (see [19]) $R_h : \ H_0^1(\Omega) \to S_h$, which satisfies:

$$(A\nabla\psi - \overline{A}\nabla R_h\psi, \nabla v_h) + (c\psi, v_h) - (\bar{c}R_h\psi, I_h^* v_h) = 0, \ \forall\, v_h \in S_h, \tag{2.12}$$

$$\|\psi - R_h\psi\| + h\|\psi - R_h\psi\|_1 \le Ch^2\|\psi\|_2, \ \forall\, \psi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.13}$$

$$\|(\psi - R_h\psi)_t\| + h\|(\psi - R_h\psi)_t\|_1 \le Ch^2\|\psi_t\|_2, \ \forall\, \psi_t \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.14}$$

where $\overline{A}|_K = A_K$, $\bar{c}|_K = c_K$, $A_K = \frac{1}{\mathrm{meas}(K)}\int_K A(x)dx$, and $c_K = \frac{1}{\mathrm{meas}(K)}\int_K c(x)dx$, $\forall\, K \in T_h$.

Some discrete inner products and norms defined on S_h and S_h^* are as follows:

$$(v_h, p_h)_{0,h} = \sum_{x_i \in N_h} \mathrm{meas}(V_i)v_{hi}p_{hi} = (I_h^* v_h, I_h^* p_h), \ |v_h|_{0,h}^2 = (v_h, v_h)_{0,h}, \ |||v_h|||_0^2 = (v_h, I_h^* v_h),$$

$$|v_h|_{1,h}^2 = \sum_{x_i \in N_h}\sum_{x_j \in \Pi(i)} \mathrm{meas}(V_i)((v_{hi} - v_{hj})/d_{ij})^2, \ \|v_h\|_{1,h}^2 = |v_h|_{0,h}^2 + |v_h|_{1,h}^2.$$

Here $d_{ij} = |x_i - x_j|$.

Let

$$a_h(y_h, I_h^* v_h) := -\sum_{x_i \in N_h}\int_{\partial V_i}\left(\overline{A}\nabla y_h\right)\cdot \mathbf{n} I_h^* v_h ds = -\sum_{x_i \in N_h} v_h(x_i)\int_{\partial V_i}\left(\overline{A}\nabla y_h\right)\cdot \mathbf{n} ds,$$

where $\mathbf{n}$ is the outer-normal vector.

By defining a new approximation for the target functional, we propose the following FVE scheme: Seek discrete solution $(y_h, q_h) \in L^2(S_h) \times Q_h^{ad}$, which satisfies

$$\min_{q_h \in Q_h^{ad}}\left\{\frac{1}{2}\int_0^T \left(|||y_h - I_h y_d|||_0^2 + \|q_h\|^2\right) dt\right\}, \tag{2.15}$$

$$(y_{ht}, I_h^* v_h) + (\overline{A}\nabla y_h, \nabla v_h) + (\bar{c}y_h, I_h^* v_h) = (f + q_h, I_h^* v_h), \ \forall\, v_h \in S_h, \tag{2.16}$$

where $Q_h^{ad} = L^2(W_h) \cap Q^{ad}$, $y_h(x,0) = R_h y_0(x)$, and $(\overline{A}\nabla y_h, \nabla v_h) = a_h(y_h, I_h^* v_h)$.

The solution $(y_h, q_h) \in L^2(S_h) \times Q_h^{ad}$ of the discretized problem (2.15) and (2.16) also is unique. When (y_h, q_h) is a solution to (2.15) and (2.16), there exists a discrete co-state $p_h \in L^2(S_h)$ which makes (y_h, p_h, q_h) satisfy the discrete optimality conditions:

$$(y_{ht}, I_h^* v_h) + (\overline{A}\nabla y_h, \nabla v_h) + (\bar{c} y_h, I_h^* v_h) = (f + q_h, I_h^* v_h), \tag{2.17}$$

$$y_h(x,0) = R_h y_0(x), \tag{2.18}$$

$$-(p_{ht}, I_h^* v_h) + (\overline{A}\nabla p_h, \nabla v_h) + (\bar{c} p_h, I_h^* v_h) = (y_h - I_h y_d, I_h^* v_h), \tag{2.19}$$

$$p_h(x,T) = 0, \tag{2.20}$$

$$\int_0^T (q_h + I_h^* p_h, \dot{q}_h - q_h) dt \geq 0, \tag{2.21}$$

where $v_h \in S_h$, $x \in \Omega$, and $\dot{q}_h \in Q_h^{ad}$.

From (2.21), q_h can be solved as

$$q_h = \max\{0, \overline{I_h^* p_h}\} - P_h(I_h^* p_h), \tag{2.22}$$

where $\overline{I_h^* p_h} = \frac{\int_0^T \int_\Omega I_h^* p_h dx dt}{\int_0^T \int_\Omega 1 dx dt}$.

For any $\ddot{q} \in L^2(L^2)$, we can define $(y_h(\ddot{q}), p_h(\ddot{q})) \in L^2(S_h) \times L^2(S_h)$, which satisfies

$$(y_{ht}(\ddot{q}), I_h^* v_h) + (\overline{A}\nabla y_h(\ddot{q}), \nabla v_h) + (\bar{c} y_h(\ddot{q}), I_h^* v_h) = (f + \ddot{q}, I_h^* v_h), \ \forall\, v_h \in S_h, \tag{2.23}$$

$$y_h(\ddot{q})(x,0) = R_h y_0(x), \ \forall x \in \Omega, \tag{2.24}$$

$$-(p_{ht}(\ddot{q}), I_h^* v_h) + (\overline{A}\nabla p_h(\ddot{q}), \nabla v_h) + (\bar{c} p_h(\ddot{q}), I_h^* v_h) = (y_h(\ddot{q}) - I_h y_d, I_h^* v_h), \ \forall\, v_h \in S_h, \tag{2.25}$$

$$p_h(\ddot{q})(x,T) = 0, \ \forall x \in \Omega. \tag{2.26}$$

Thus, the exact solution (y, p) can be further expressed as $(y(q), p(q))$. Similarly, the discrete solution (y_h, p_h) can be further expressed as $(y_h(q_h), p_h(q_h))$.

From [19], we recall some useful results.

Lemma 1. *There exists a positive constant C such that*

$$\|v_h - I_h^* v_h\| \leq Ch\|v_h\|_{1,h}, \forall\, v_h \in S_h.$$

Lemma 2. $\forall\, v_h \in S_h$, *we have*

$$|v_h|_{0,h} = \|I_h^* v_h\| \leq C_1\|v_h\|, \tag{2.27}$$

$$C_2\|v_h\| \leq |||v_h|||_0 \leq C_3\|v_h\|, \tag{2.28}$$

$$C_4\|v_h\|_1 \leq \|v_h\|_{1,h} \leq C_5\|v_h\|_1, \tag{2.29}$$

where C_1–C_5 are five positive constants.

Lemma 3. *We have*

$$\int_K v_h dx = \int_K I_h^* v_h dx, \ \forall\, K \in T_h, \ \forall\, v_h \in S_h.$$

Lemma 4. *We have*

$$(w_h, I_h^* v_h) = (v_h, I_h^* w_h), \ \forall\, w_h, v_h \in S_h.$$

3. Error analysis

In this part, we establish a priori $L^2(H^1)$-norm error estimates for the state and co-state variables, along with $L^2(L^2)$-norm error estimates for the control variable.

Lemma 5. *We choose $\ddot{q} = q$ and $\ddot{q} = q_h$ in (2.23)–(2.26), respectively. We can prove that*

$$\|y_h(q) - y_h\|_{L^\infty(L^2)} + \|\nabla(y_h(q) - y_h)\|_{L^2(L^2)} \leq C\|q - q_h\|_{L^2(L^2)}, \tag{3.1}$$

$$\|p_h(q) - p_h\|_{L^\infty(L^2)} + \|\nabla(p_h(q) - p_h)\|_{L^2(L^2)} \leq C\|q - q_h\|_{L^2(L^2)}. \tag{3.2}$$

Proof. By (2.23)–(2.26), we see that

$$(y_{ht}(q) - y_{ht}, I_h^* v_h) + (\overline{A}\nabla(y_h(q) - y_h), \nabla v_h) + (\bar{c}(y_h(q) - y_h), I_h^* v_h) = (q - q_h, I_h^* v_h),\ \forall\ v_h \in S_h, \tag{3.3}$$

$$\begin{aligned} &- (p_{ht}(q) - p_{ht}, I_h^* v_h) + (\overline{A}\nabla(p_h(q) - p_h), \nabla v_h) + (\bar{c}(p_h(q) - p_h), I_h^* v_h) \\ &\qquad = (y_h(q) - y_h, I_h^* v_h),\ \forall\ v_h \in S_h. \end{aligned} \tag{3.4}$$

We select $v_h = y_h(q) - y_h$ in (3.3) to get

$$\frac{1}{2}\frac{d}{dt}|||y_h(q) - y_h|||_0^2 + \|\overline{A}^{\frac{1}{2}}\nabla(y_h(q) - y_h)\|^2 + \|\bar{c}^{\frac{1}{2}}(y_h(q) - y_h)\|^2 = (q - q_h, I_h^*(y_h(q) - y_h)). \tag{3.5}$$

Integrating (3.5) and using $y_h(q)(x, 0) - y_h(x, 0) = 0$, we have

$$\begin{aligned} \frac{1}{2}|||y_h(q) - y_h|||_0^2 + \int_0^t \|\overline{A}^{\frac{1}{2}}\nabla(y_h(q) - y_h)\|^2 ds + \int_0^t \|\bar{c}^{\frac{1}{2}}(y_h(q) - y_h)\|^2 ds \\ = \int_0^t (q - q_h, I_h^*(y_h(q) - y_h)) ds. \end{aligned} \tag{3.6}$$

By the Cauchy-Schwarz inequality and (2.27), we have

$$\begin{aligned} (q - q_h, I_h^*(y_h(q) - y_h)) &\leq \|q - q_h\| \cdot \|I_h^*(y_h(q) - y_h)\| \\ &\leq C\|q - q_h\| \cdot \|y_h(q) - y_h\| \\ &\leq C(\|q - q_h\|^2 + \|y_h(q) - y_h\|^2). \end{aligned} \tag{3.7}$$

Applying Gronwall's lemma, (3.6), (3.7), (2.28), the assumption on A, and the assumption on c, we easily get

$$\|y_h(q) - y_h\|^2 + \int_0^t \|\nabla(y_h(q) - y_h)\|^2 ds \leq C \int_0^t \|q - q_h\|^2 ds. \tag{3.8}$$

Next, choosing $v_h = p_h(q) - p_h$ in (3.4), we have

$$-\frac{1}{2}\frac{d}{dt}|||p_h(q) - p_h|||_0^2 + \|\overline{A}^{\frac{1}{2}}\nabla(p_h(q) - p_h)\|^2 + \|\bar{c}^{\frac{1}{2}}(p_h(q) - p_h)\|^2 = (y_h(q) - y_h, I_h^*(p_h(q) - p_h)). \tag{3.9}$$

Integrating (3.9) over t, and then using $p_h(q)(x, T) = p_h(x, T) = 0$ and

$$(y_h(q) - y_h, I_h^*(p_h(q) - p_h) \leq \|y_h(q) - y_h\| \cdot \|I_h^*(p_h(q) - p_h)\| \leq \|y_h(q) - y_h\| \cdot \|p_h(q) - p_h\|,$$

we deduce that

$$\frac{1}{2}|||p_h(q) - p_h|||_0^2 + \int_t^T \|\overline{A}^{\frac{1}{2}}\nabla(p_h(q) - p_h)\|^2 ds + \int_t^T \|\bar{c}^{\frac{1}{2}}(p_h(q) - p_h)\|^2 ds \le \int_t^T \|y_h(q) - y_h\| \cdot \|p_h(q) - p_h\| ds.$$

Similar to (3.8), we arrive at

$$\|p_h(q) - p_h\|^2 + \int_t^T \|\nabla(p_h(q) - p_h)\|^2 ds \le C \int_t^T \|y_h(q) - y_h\|^2 ds. \tag{3.10}$$

Therefore, Lemma 5 can be completed by (3.8) and (3.10). □

Lemma 6. *Let y and p satisfy (2.3)–(2.7), while $(y_h(q), p_h(q))$ is the discrete solution of (2.23)–(2.26) by choosing $\ddot{q} = q$. Suppose $y, p \in L^\infty(H^2)$, $y_t, p_t \in L^2(H^2)$, and $f, q \in L^2(H^1)$. Then the following results hold:*

$$\|y - y_h(q)\|_{L^\infty(L^2)} + \|p - p_h(q)\|_{L^\infty(L^2)} \le Ch^2, \tag{3.11}$$

$$\|\nabla(R_h y - y_h(q))\|_{L^2(L^2)} + \|\nabla(R_h p - p_h(q))\|_{L^2(L^2)} \le Ch^2. \tag{3.12}$$

Proof. Let

$$y - y_h(q) = y - R_h y + R_h y - y_h(q) =: r_y + \xi_y,$$
$$p - p_h(q) = p - R_h p + R_h p - p_h(q) =: r_p + \xi_p.$$

We get the two error equations by subtracting (2.23) and (2.25) from (2.3) and (2.5):

$$(\xi_{yt}, I_h^* v_h) + (\overline{A}\nabla\xi_y, \nabla v_h) + (\bar{c}\xi_y, I_h^* v_h) = (f + q - y_t, v_h - I_h^* v_h) - (r_{yt}, I_h^* v_h), \tag{3.13}$$

$$\begin{aligned} -(\xi_{pt}, I_h^* v_h) + (\overline{A}\nabla\xi_p, \nabla v_h) + (\bar{c}\xi_p, I_h^* v_h) &= (y - y_d + p_t, v_h - I_h^* v_h) \\ &+ (y - y_h(q) + I_h y_d - y_d, I_h^* v_h) + (r_{pt}, I_h^* v_h), \end{aligned} \tag{3.14}$$

where $v_h \in S_h$.

Letting $v_h = \xi_y$ in (3.13), we have

$$\frac{1}{2}\frac{d}{dt}|||\xi_y|||_0^2 + \|\overline{A}^{\frac{1}{2}}\nabla\xi_y\|^2 + (\bar{c}\xi_y, I_h^*\xi_y) = (f + q - y_t, \xi_y - I_h^*\xi_y) - (r_{yt}, I_h^*\xi_y). \tag{3.15}$$

Integrating (3.15) and using $\xi_y(0) = 0$, we see that

$$\frac{1}{2}|||\xi_y|||_0^2 + \int_0^t \|A^{\frac{1}{2}}\nabla\xi_y\|^2 ds + \int_0^t (\bar{c}\xi_y, I_h^*\xi_y) ds = \int_0^t (f + q - y_t, \xi_y - I_h^*\xi_y) ds - \int_0^t (r_{yt}, I_h^*\xi_y) ds. \tag{3.16}$$

By (2.10) and (2.11), the Cauchy inequality, Lemma 1, (2.14), (2.27), (2.29), and Poincare's inequality, we conclude

$$\begin{aligned} (f + q - y_t, \xi_y - I_h^*\xi_y) &= (f + q - y_t - P_h(f + q - y_t), \xi_y - I_h^*\xi_y) \\ &\le C\|f + q - y_t - P_h(f + q - y_t)\| \cdot \|\xi_y - I_h^*\xi_y\| \end{aligned}$$

$$\leq Ch^2\|f+q-y_t\|_1\|\xi_y\|_{1,h}$$
$$\leq Ch^2\|f+q-y_t\|_1\|\xi_y\|_1$$
$$\leq Ch^2\|f+q-y_t\|_1\|\nabla\xi_y\|, \tag{3.17}$$

and

$$(r_{yt}, I_h^*\xi_y) \leq \|r_{yt}\|\cdot\|I_h^*\xi_y\| \leq Ch^2\|y_t\|_2\|\xi_y\|. \tag{3.18}$$

Using (2.28), the assumption on A, the assumption on c, (3.16)–(3.18), and the Schwarz inequality, we derive

$$\|\xi_y\|^2 + \int_0^t \|\nabla\xi_y\|^2 ds \leq Ch^4\int_0^t (\|f+q-y_t\|_1^2 + \|y_t\|_2^2)ds + \int_0^t \|\xi_y\|^2 ds. \tag{3.19}$$

Applying Gronwall's inequality to (3.19), it is easy to get

$$\|\xi_y\| + \|\nabla\xi_y\|_{L^2(L^2)} \leq Ch^2(\|f\|_{L^2(H^1)} + \|q\|_{L^2(H^1)} + \|y_t\|_{L^2(H^2)}). \tag{3.20}$$

We see from (2.13) and (3.20) that

$$\|y-y_h(q)\|_{L^\infty(L^2)} + \|\nabla(R_hy-y_h(q))\|_{L^2(L^2)} \leq Ch^2(\|f\|_{L^2(H^1)} + \|q\|_{L^2(H^1)} + \|y_t\|_{L^2(H^2)} + \|y\|_{L^\infty(H^2)}). \tag{3.21}$$

Setting $v_h = \xi_p$ in (3.14), we have

$$-\frac{1}{2}\frac{d}{dt}|||\xi_p|||_0^2 + \|A^{\frac{1}{2}}\nabla\xi_p\|^2 + (\bar{c}\xi_p, I_h^*\xi_p) = (y-y_d+p_t, \xi_p - I_h^*\xi_p) + (r_{pt}, I_h^*\xi_p)$$
$$+ (y-y_h(q)+I_hy_d-y_d, I_h^*\xi_p). \tag{3.22}$$

Similar to (3.17) and (3.18), we get

$$(y-y_d+p_t, \xi_p - I_h^*\xi_p) = (y-y_d+p_t-P_h(y-y_d+p_t), \xi_p - I_h^*\xi_p)$$
$$\leq Ch^2(\|y\|_1 + \|y_d\|_1 + \|p_t\|_1)\|\nabla\xi_p\|, \tag{3.23}$$

$$(y-y_h(q)+I_hy_d-y_d, I_h^*\xi_p) \leq Ch^2(\|y\|_2 + \|y_d\|_2)\|\xi_p\| + \|\xi_y\|\cdot\|\xi_p\|, \tag{3.24}$$

and

$$(r_{pt}, I_h^*\xi_p) \leq Ch^2\|p_t\|_2\|\xi_p\|. \tag{3.25}$$

Integrating (3.22) from t to T, and then using $\xi_p(T) = 0$, (3.23)–(3.25), the assumption on A, the assumption on c, Poincare's inequality, and the Schwarz inequality, we conclude

$$\|\xi_p\|^2 + \int_t^T \|\nabla\xi_p\|^2 ds + \int_t^T (\bar{c}\xi_p, I_h^*\xi_p)ds$$
$$\leq Ch^4\int_t^T (\|y\|_2^2 + \|y_d\|_2^2 + \|p_t\|_2^2)ds + C\int_t^T \|\xi_y\|^2 ds + C\int_t^T \|\xi_p\|^2 ds. \tag{3.26}$$

We apply Gronwall's inequality to (3.26) to get

$$\|\xi_p\|^2 + \|\nabla\xi_p\|_{L^2(L^2)}^2 \leq Ch^4(\|y\|_{L^2(H^2)}^2 + \|y_d\|_{L^2(H^2)}^2 + \|p_t\|_{L^2(H^2)}^2) + C\|\xi_y\|_{L^2(L^2)}^2. \tag{3.27}$$

Now, the proof can be completed by (2.13), (3.21), (3.27), and the triangle inequality. □

Theorem 1. *Let (q, y, p) satisfy (2.3)–(2.7) and (q_h, y_h, p_h) satisfy (2.17)–(2.21). Suppose $y, p \in L^\infty(H^2)$, $y_t, p_t \in L^2(H^2)$, and $f, q \in L^2(H^1)$. Now we prove that*

$$\|q - q_h\|_{L^2(L^2)} + \|\nabla(y - y_h)\|_{L^2(L^2)} + \|\nabla(p - p_h)\|_{L^2(L^2)} \leq Ch.$$

Proof. We choose $\dot{q} = q_h$ and $\dot{q}_h = P_h q$ in the inequalities (2.7) and (2.21) to get

$$\int_0^T (q + p, q_h - q) \geq 0, \tag{3.28}$$

and

$$\int_0^T (q_h + I_h^* p_h, P_h q - q_h) \geq 0. \tag{3.29}$$

By (3.28), (3.29), and (2.10), we obtain

$$\begin{aligned}
\|q - q_h\|^2_{L^2(L^2)} &= \int_0^T (q - q_h, q - q_h)dt \\
&= \int_0^T (q + p, q - q_h)dt + \int_0^T (I_h^* p_h - p, q - q_h)dt \\
&\quad + \int_0^T (I_h^* p_h + q_h, q_h - P_h q)dt + \int_0^T (I_h^* p_h + q_h, P_h q - q)dt \\
&\leq \int_0^T (I_h^* p_h - p, q - q_h)dt + \int_0^T (I_h^* p_h + q_h, P_h q - q)dt \\
&= \int_0^T (I_h^* p_h - I_h^* p_h(q), q - q_h)dt + \int_0^T (I_h^* p_h(q) - p_h(q), q - q_h)dt \\
&\quad + \int_0^T (p_h(q) - p, q - q_h)dt + \int_0^T (I_h^* p_h, P_h q - q)dt \\
&= \int_0^T (I_h^* p_h - I_h^* p_h(q), q - q_h)dt + \int_0^T (I_h^* p_h(q) - p_h(q), q - q_h)dt \\
&\quad + \int_0^T (p_h(q) - p, q - q_h)dt + \int_0^T (I_h^* p_h - p_h, P_h q - q)dt \\
&\quad + \int_0^T (p_h - P_h p_h, P_h q - q)dt.
\end{aligned} \tag{3.30}$$

Integrating the Eqs (3.3) and (3.4) over the interval $[0, T]$, and selecting $v_h = p_h(q) - p_h$ in the first equation and $v_h = y_h(q) - y_h$ in the second equation, gives

$$\begin{aligned}
&\int_0^T (I_h^* p_h - I_h^* p_h(q), q - q_h)dt \\
&= \int_0^T (y_{ht}(q) - y_{ht}, I_h^*(p_h - p_h(q)))dt + \int_0^T (\overline{A}\nabla(y_h(q) - y_h), \nabla(p_h - p_h(q)))dt \\
&\quad + \int_0^T (\bar{c}(y_h(q) - y_h), I_h^*(p_h - p_h(q)))dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \frac{d}{dt}(y_h(q) - y_h, I_h^*(p_h - p_h(q)))dt + \int_0^T (y_h(q) - y_h, I_h^*(p_{ht} - p_{ht}(q)))dt \\
&\quad + \int_0^T (\overline{A}\nabla(y_h(q) - y_h), \nabla(p_h - p_h(q)))dt + \int_0^T (\bar{c}(y_h(q) - y_h), I_h^*(p_h - p_h(q)))dt \\
&= (y_h(q) - y_h, I_h^*(p_h - p_h(q)))|_0^T - \int_0^T (p_{ht} - p_{ht}(q), I_h^*(y_h(q) - y_h))dt \\
&\quad + \int_0^T (\overline{A}\nabla(p_h - p_h(q)), \nabla(y_h(q) - y_h))dt + \int_0^T (\bar{c}(p_h - p_h(q)), I_h^*(y_h(q) - y_h))dt \\
&= - \int_0^T (p_{ht} - p_{ht}(q), I_h^*(y_h(q) - y_h))dt + \int_0^T (\overline{A}\nabla(p_h - p_h(q)), \nabla(y_h(q) - y_h))dt \\
&\quad + \int_0^T (\bar{c}(p_h - p_h(q)), I_h^*(y_h(q) - y_h))dt \\
&= - \int_0^T (y_h(q) - y_h, I_h^*(y_h(q) - y_h))dt \\
&= - \int_0^T |||y_h(q) - y_h|||_0^2 dt \\
&\leq - C_4 \int_0^T \|y_h(q) - y_h\|^2 dt \\
&\leq 0,
\end{aligned}
\tag{3.31}
$$

where we also used $y_h(q)(0) - y_h(0) = 0$, $p_h(q)(T) = p_h(T) = 0$, Lemma 4, and (2.28).

By Cauchy-Schwarz inequality, Lemma 1, (2.11), and (2.29), we respectively obtain

$$
\begin{aligned}
\int_0^T (I_h^* p_h(q) - p_h(q), q - q_h)dt &= \int_0^T (I_h^* p_h(q) - p_h(q), q - P_h q)dt \\
&\leq Ch^2(\|q\|_{L^2(H^1)}^2 + \|p_h(q)\|_{L^2(H^1)}^2),
\end{aligned}
\tag{3.32}
$$

$$
\int_0^T (p_h(q) - p, q - q_h)dt \leq C\|p - p_h(q)\|_{L^2(L^2)}^2 + \frac{1}{2}\|q - q_h\|_{L^2(L^2)}^2, \tag{3.33}
$$

$$
\int_0^T (I_h^* p_h - p_h, P_h q - q)dt \leq Ch^2(\|q\|_{L^2(H^1)}^2 + \|p_h\|_{L^2(H^1)}^2), \tag{3.34}
$$

and

$$
\int_0^T (p_h - P_h p_h, P_h q - q)dt \leq Ch^2(\|q\|_{L^2(H^1)}^2 + \|p_h\|_{L^2(H^1)}^2). \tag{3.35}
$$

Applying the stability analysis as in Lemma 5, (2.9), and (2.13) gives

$$
\begin{aligned}
\|p_h(q)\|_{L^2(H^1)} &\leq C(\|y_h(q)\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)}) \\
&\leq C(\|y_h(q)(0)\| + \|f\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)})
\end{aligned}
$$

$$\leq C(\|R_h y_0\| + \|f\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + \|y_d - I_h y_d\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)})$$
$$\leq C(\|R_h y_0 - y_0\| + \|y_0\| + \|f\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + h^2\|y_d\|_{L^2(H^2)} + \|y_d\|_{L^2(L^2)})$$
$$\leq C(h^2\|y_0\|_2 + \|y_0\| + \|f\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + h^2\|y_d\|_{L^2(H^2)} + \|y_d\|_{L^2(L^2)})$$
$$\leq C(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)}) \leq C. \tag{3.36}$$

Similarly, we obtain

$$\begin{aligned}
\|p_h\|_{L^2(H^1)} &\leq C(\|y_h\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)}) \\
&\leq C(\|R_h y_0\| + \|f\|_{L^2(L^2)} + \|q_h\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)}) \\
&\leq C(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|q_h\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)}) \\
&\leq C(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|q - q_h\|_{L^2(L^2)} + \|q\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)}) \\
&\leq C + C\|q - q_h\|_{L^2(L^2)}.
\end{aligned} \tag{3.37}$$

For a sufficiently small h, it can be proved by combining (3.30)–(3.37) that

$$\|q - q_h\|_{L^2(L^2)} \leq Ch + C\|p - p_h(q)\|_{L^2(L^2)}. \tag{3.38}$$

Thus, the main results of the theorem can be easily derived by (2.13), (3.38), Lemmas 5 and 6, and the triangle inequality. □

4. Superclose results

In this part, we first obtain some superclose results between the projection of the exact solution and the discrete solution, and then obtain the $L^\infty(L^2)$-norm error of the optimal convergence order for the all variables.

Lemma 7. *Choose $\ddot{q} = q$ and $\ddot{q} = P_h q$ in (2.23)–(2.26), respectively. Then the following results hold under the assumption $q \in L^2(H^1)$:*

$$\|y_h(q) - y_h(P_h q)\|_{L^\infty(L^2)} + \|\nabla(y_h(q) - y_h(P_h q))\|_{L^2(L^2)} \leq Ch^2,$$
$$\|p_h(q) - p_h(P_h q)\|_{L^\infty(L^2)} + \|\nabla(p_h(q) - p_h(P_h q))\|_{L^2(L^2)} \leq Ch^2.$$

Proof. Choosing $\ddot{q} = q$ and $\ddot{q} = P_h q$ in (2.23)–(2.26), we obtain

$$(y_{ht}(q) - y_{ht}(P_h q), I_h^* v_h) + (\overline{A}\nabla(y_h(q) - y_h(P_h q)), \nabla v_h) + (\bar{c}(y_h(q) - y_h(P_h q)), I_h^* v_h) = (q - P_h q, I_h^* v_h),\ \forall\, v_h \in S_h, \tag{4.1}$$

$$-(p_{ht}(q) - p_{ht}(P_h q), I_h^* v_h) + (\overline{A}\nabla(p_h(u) - p_h(P_h q)), \nabla v_h) + (\bar{c}(p_h(q) - p_h(P_h q)), I_h^* v_h) = (y_h(q) - y_h(P_h q), I_h^* v_h),\ \forall\, v_h \in S_h. \tag{4.2}$$

It follows from (2.10) and (2.11), Lemma 1, (2.29), and the Cauchy inequality that

$$(q - P_h q, I_h^* v_h) = (q - P_h q, I_h^* v_h - v_h) + (q - P_h q, v_h - P_h v_h) \leq Ch^2\|q\|_1\|v_h\|_1. \tag{4.3}$$

Using (4.2), (4.3), and the stability estimate, we can conclude

$$\|y_h(q) - y_h(P_h q)\| + \|\nabla(y_h(q) - y_h(P_h q))\|_{L^2(L^2)} \leq Ch^4\|q\|_1^2 \tag{4.4}$$

and

$$\|p_h(q) - p_h(P_hq)\| + \|\nabla(p_h(q) - p_h(P_hq))\|_{L^2(L^2)} \le C\|y_h(q) - y_h(P_hq)\|_{L^2(L^2)}. \tag{4.5}$$

Thus, the proof is complete with (4.4) and (4.5). □

Lemma 8. *For the discrete solutions* (y_h, p_h) *and* $(y_h(P_hq), p_h(P_hq))$*, the following results hold:*

$$\|y_h(P_hq) - y_h\|_{L^\infty(L^2)} + \|\nabla(y_h(P_hq) - y_h)\|_{L^2(L^2)} \le C\|P_hq - q_h\|_{L^2(L^2)},$$
$$\|p_h(P_hq) - p_h\|_{L^\infty(L^2)} + \|\nabla(p_h(P_hq) - p_h)\|_{L^2(L^2)} \le C\|P_hq - q_h\|_{L^2(L^2)}.$$

Proof. From (2.17), (2.19), (2.23), and (2.25), we have

$$\begin{aligned}(y_{ht}(P_hq) - y_{ht}, I_h^*v_h) + (\overline{A}\nabla(y_h(P_hq) - y_h), \nabla v_h) + (\bar{c}(y_h(P_hq) - y_h), I_h^*v_h)\\ = (P_hq - q_h, I_h^*v_h),\ \forall\ v_h \in S_h,\end{aligned} \tag{4.6}$$

$$\begin{aligned}-(p_{ht}(P_hq) - p_{ht}, I_h^*v_h) + (\overline{A}\nabla(p_h(P_hq) - p_h), \nabla v_h) + (\bar{c}(p_h(P_hq) - p_h), I_h^*v_h)\\ = (y_h(P_hq) - y_h, I_h^*v_h),\ \forall\ v_h \in S_h.\end{aligned} \tag{4.7}$$

By Lemma 3, we easily get

$$(P_hq - q_h, I_h^*v_h) = (P_hq - q_h, v_h),\ \forall\ v_h \in S_h.$$

Thus, the expected results can be obtained by the stability analysis. □

Theorem 2. *Suppose* $y, p \in L^\infty(H^2)$*,* $y_t, p_t \in L^2(H^2)$*, and* $f, q \in L^2(H^1)$*. Then we have*

$$\|P_hq - q_h\|_{L^2(L^2)} + \|\nabla(R_hy - y_h)\|_{L^2(L^2)} + \|\nabla(R_hy - y_h)\|_{L^2(L^2)} \le Ch^2.$$

Proof. Choose $\dot{q} = q_h$ and $\dot{q_h} = P_hq$ in the two inequalities (2.7) and (2.21) to get

$$\int_0^T (q + p, q_h - q)dt \ge 0 \tag{4.8}$$

and

$$\int_0^T (q_h + I_h^*p_h, P_hq - q_h)dt \ge 0. \tag{4.9}$$

We see from (4.8), (4.9), and (2.10) that

$$\begin{aligned}\|P_hq - q_h\|^2_{L^2(L^2)} &\le \int_0^T (I_h^*p_h - p, P_hq - q_h)dt + \int_0^T (q + p, P_hq - q)dt\\ &= \int_0^T (I_h^*p_h - I_h^*p_h(P_hq), P_hq - q_h)dt + \int_0^T (I_h^*p_h(P_hq) - I_h^*p_h(q), P_hq - q_h)dt\\ &\quad + \int_0^T (I_h^*p_h(q) - p, P_hq - q_h)dt + \int_0^T (q + p, P_hq - q)dt.\end{aligned} \tag{4.10}$$

We set $w_h = p_h(P_h q) - p_h$ in (4.6) and $w_h = y_h(P_h q) - y_h$ in (4.7), respectively. We use the proof technique in (3.31) to obtain

$$\int_0^T (I_h^* p_h - I_h^* p_h(P_h q), P_h q - q_h)dt = -\int_0^T |||y_h(P_h q) - y_h|||_0^2 dt \leq 0. \tag{4.11}$$

By Lemma 3 and the Cauchy inequality, we have

$$\begin{aligned}\int_0^T (I_h^* p_h(P_h q) - I_h^* p_h(q), P_h q - q_h)dt &= \int_0^T (p_h(P_h q) - p_h(q), P_h q - q_h)dt \\ &\leq \int_0^T \|p_h(q) - p_h(P_h q)\| \cdot \|P_h q - q_h\| dt \end{aligned}\tag{4.12}$$

and

$$\int_0^T (I_h^* p_h(q) - p, P_h q - q_h)dt = \int_0^T (p_h(q) - p, P_h q - q_h)dt \leq \int_0^T \|p - p_h(q)\| \cdot \|P_h q - q_h\| dt. \tag{4.13}$$

By (2.8) and (2.10), it is obvious that

$$\int_0^T (q + p, P_h q - q)dt = \int_0^T (\max\{0, \bar{p}\}, P_h q - q)dt = 0. \tag{4.14}$$

Thus, we complete the proof by (4.10)–(4.14), (3.11) and (3.12), and Lemmas 7 and 8. □

Now, we can get the following optimal $L^\infty(L^2)$-norm error estimates.

Theorem 3. *Suppose $y, p \in L^\infty(H^2)$, $y_t, p_t \in L^2(H^2)$, and $f, q \in L^2(H^1)$. Then we have*

$$\|q - q_h\|_{L^\infty(L^2)} \leq Ch, \tag{4.15}$$

$$\|y - y_h\|_{L^\infty(L^2)} + \|p - p_h\|_{L^\infty(L^2)} \leq Ch^2. \tag{4.16}$$

Proof. By (2.8) and (2.22), we conclude that

$$\begin{aligned}|q - q_h| &= |\max\{0, \bar{p}\} - p - \max\{0, \overline{I_h^* p_h}\} + P_h(I_h^* p_h)| \\ &\leq |\bar{p} - \overline{I_h^* p_h}| + |p - P_h(I_h^* p_h)| \\ &\leq C\|p - I_h^* p_h\|_{L^2(L^2)} + |p - p_h| + |p_h - P_h p_h| + |P_h(p_h - I_h^* p_h)|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|q - q_h\| &\leq C\|p - I_h^* p_h\|_{L^2(L^2)} + C\|p - p_h\| + C\|p_h - P_h p_h\| + C\|P_h(p_h - I_h^* p_h)\| \\ &\leq C\|p - p_h\|_{L^2(L^2)} + C\|p_h - I_h^* p_h\|_{L^2(L^2)} + C\|p - p_h\| \\ &\quad + C\|p_h - P_h p_h\| + C\|p_h - I_h^* p_h\|,\end{aligned}\tag{4.17}$$

where we used

$$\|P_h(p_h - I_h^* p_h)\| \leq C\|p_h - I_h^* p_h\|.$$

According to (4.17), (2.11), Lemma 1, and (2.29), we find that

$$\begin{aligned}\|q - q_h\|_{L^\infty(L^2)} &\leq C\|p - p_h\|_{L^2(L^2)} + C\|p_h - I_h^* p_h\|_{L^2(L^2)} + C\|p - p_h\|_{L^\infty(L^2)} \\ &\quad + C\|p_h - P_h p_h\|_{L^\infty(L^2)} + C\|p_h - I_h^* p_h\|_{L^\infty(L^2)} \\ &\leq C\|p - p_h\|_{L^\infty(L^2)} + Ch\|p_h\|_{L^\infty(H^1)}.\end{aligned}\tag{4.18}$$

Using (3.11), Lemmas 7 and 8, and Theorem 2, we derive the desired estimate (4.16). Then, (4.15) can be obtained by (4.16) and (4.18). The proof is complete. □

5. Numerical experiments

This section presents an example to illustrate the theoretical results. The discretization method, previously detailed, involves discretizing the control function q via piecewise constant functions, while the state variable y and co-state variable p are approximated using continuous piecewise linear finite element functions.

Let $\Omega = (0,1)^2$, $T = 1$, $c(x) = 0$, and $A(x) = \begin{pmatrix} 1 + x_1^2 & 0 \\ 0 & e^{x_2} \end{pmatrix}$. The data are as follows:

$$
\begin{aligned}
y &= e^t \sin(\pi x_1)\sin(\pi x_2),\\
p &= \frac{1-t}{\pi}\sin(2\pi x_1)\sin(2\pi x_2),\\
q &= \max\{0, \bar{p}\} - p,\\
f &= y_t - \operatorname{div}(A\nabla y) - q,\\
y_d &= y + p_t + \operatorname{div}(A\nabla p),
\end{aligned}
$$

where $\bar{p} = \frac{\int_0^T \int_\Omega p dx dt}{\int_0^T \int_\Omega 1 dx dt} = 0$ and $q = -p$.

We now consider the fully discrete scheme for the control problem. Let $\Delta t > 0$, $N = 1/\Delta t \in \mathbb{Z}$, and $t_n = n\Delta t$, $n \in \mathbb{Z}$. Moreover, let

$$
\psi^n = \psi^n(x) = \psi(x, t_n), \quad d_t\psi^n = \frac{\psi^n - \psi^{n-1}}{\Delta t}.
$$

For $1 \le s < \infty$ and $s = \infty$, we define the discrete norms

$$
|||\psi|||_{L^s(J;W^{m,k}(\Omega))} := \left(\sum_{n=1-l}^{N-l} \Delta t \|\psi^n\|_{m,k}^s \right)^{\frac{1}{s}}, \quad |||\psi|||_{L^\infty(J;W^{m,k}(\Omega))} := \max_{1-l \le n \le N-l} \|\psi^n\|_{m,k},
$$

where $l = 0$ for the control variable q and the state variable y, and $l = 1$ for the co-state variable p. Moreover, we denote $|||\psi|||_{L^s(J;W^{m,k}(\Omega))}$ and $|||\psi|||_{L^\infty(J;W^{m,k}(\Omega))}$ by $|||\psi|||_{L^s(W^{m,k})}$ and $|||\psi|||_{L^\infty(W^{m,k})}$ in this section.

We use the following fully discrete scheme: Find $(y_h^n, p_h^{n-1}, q_h^n) \in S_h \times S_h \times W_h$, $n = 1, 2, \cdots, N$, such that

$$
\begin{aligned}
(d_t y_h^n, I_h^* v_h) + (\overline{A}\nabla y_h^n, \nabla v_h) &= (f^n + q_h^n, I_h^* v_h), \ \forall\, v_h \in S_h,\\
y_h^0(x) &= I_h y_0, \ \forall\, x \in \Omega,\\
-(d_t p_h^n, I_h^* v_h) + (\overline{A}\nabla p_h^{n-1}, \nabla v_h) &= (y_h^n - I_h y_d^n, I_h^* v_h), \ \forall\, v_h \in S_h,\\
p_h^N(x) &= 0, \ \forall\, x \in \Omega,\\
q_h^n &= -P_h p_h^{n-1}.
\end{aligned}
$$

Let $\Delta t = \frac{1}{200}$, and we display the errors of $|||q - q_h|||_{L^\infty(L^2)}$, $|||y - y_h|||_{L^2(H^1)}$, $|||p - p_h|||_{L^2(H^1)}$, $|||y - y_h|||_{L^\infty(L^2)}$, and $|||p - p_h|||_{L^\infty(L^2)}$ with different h in Tables 1 and 2. The convergence orders of errors are also given in these tables. It can be seen that the numerical results are in agreement with the theoretical analysis.

Table 1. Errors and convergence orders of $|||q - q_h|||_{L^\infty(L^2)}$, $|||y - y_h|||_{L^2(H^1)}$, and $|||p - p_h|||_{L^2(H^1)}$.

h	$\|\|\|q - q_h\|\|\|_{L^\infty(L^2)}$	Rate	$\|\|\|y - y_h\|\|\|_{L^2(H^1)}$	Rate	$\|\|\|p - p_h\|\|\|_{L^2(H^1)}$	Rate
1/8	5.5976e-02	∼	1.4311e-01	∼	9.9985e-02	∼
1/16	2.8345e-02	0.9817	7.3034e-02	0.9704	5.1076e-02	0.9690
1/32	1.4237e-02	0.9934	3.6645e-02	0.9949	2.5632e-02	0.9946
1/64	7.2521e-03	0.9731	1.8410e-02	0.9931	1.2936e-02	0.9865
1/128	3.6344e-03	0.9966	9.2564e-03	0.9919	6.5178e-03	0.9889

Table 2. Errors and convergence orders of $|||y - y_h|||_{L^\infty(L^2)}$ and $|||p - p_h|||_{L^\infty(L^2)}$.

h	$\|\|\|y - y_h\|\|\|_{L^\infty(L^2)}$	Rate	$\|\|\|p - p_h\|\|\|_{L^\infty(L^2)}$	Rate
1/8	6.7556e-02	∼	3.1753e-02	∼
1/16	1.7188e-02	1.9746	7.8645e-03	2.0134
1/32	4.2554e-03	2.0140	1.9257e-03	2.0299
1/64	1.0743e-03	1.9858	4.8754e-04	1.9818
1/128	2.7141e-04	1.9848	1.2250e-04	1.9927

6. Conclusions

In this paper, we propose a novel FVE scheme for a linear parabolic OCP subject to integral control constraints. The discrete optimality conditions are derived through the discretize-then-optimize approach. A priori error estimates and superclose properties for all involved variables, including the state variable (representing the physical quantity being controlled), co-state variable (adjoint variable capturing sensitivity information), and control variable (the decision variable to be optimized), are rigorously established. These theoretical results provide quantitative bounds on the approximation errors, demonstrating that the proposed scheme achieves optimal convergence orders in appropriate norms and exhibits enhanced convergence (superconvergence) at specific points or for certain variables, thereby validating the robustness and efficiency of the method. To the best of our knowledge, these theoretical results on FVE methods for OCPs with integral constraints are original and have not been previously reported in the literature. In future work, we intend to investigate error estimates in the $L^\infty(H^1)$-norm for the state and co-state variables. Furthermore, we plan to explore a posteriori error estimates for this class of FVE approximations applied to the problem (1.1)–(1.4), which will enable adaptive mesh refinement strategies to dynamically adjust the computational grid based on local error indicators, enhancing both accuracy and computational efficiency in practical applications.

Author contributions

Chunjuan Hou: Writing original draft, formal analysis, project administration; Baitong Ma: Writing original draft, methodology, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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