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Research article

On idempotent-fine group rings

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Abstract: Let A be an associative ring. A nonzero element $t \in A$ is called *fine* if it can be written as $t = n + v$, where v is a unit and n is a nilpotent element. A ring A is called an *idempotent-fine ring* if every nonzero idempotent in A is fine. Let A be a ring (respectively, an integral domain) of characteristic p^m for some prime p and positive integer m, and let G be a locally finite nilpotent group (respectively, a locally finite group). We proved that $A[G]$ is an idempotent-fine ring if and only if G is a p-group. Moreover, if F is a field of characteristic p and $F[G]$ is an idempotent-fine ring, then every nontrivial element g in the group G of finite order is a p-element. Conversely, if G is a locally finite p-group, then $F[G]$ is an idempotent-fine ring.

Keywords: group ring; fine ring; idempotent-fine ring
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1. Introduction

One of the classic exercises in teaching commutative ring theory to undergraduate students is to prove that if n is nilpotent and v is a unit, then $v + n$ is also a unit. Remarkably, this simple observation has evolved into a significant concept in noncommutative rings: A nonzero element that can be decomposed as the sum of a unit and a nilpotent is now called a fine element [1].

G. Călugăreanu and T. Y. Lam [1] formally introduced the concept of fine elements. This joins a family of similar decomposition notions in ring theory, including clean elements (an element can be written as the sum of a unit and an idempotent), nil-clean elements (an element can be written as the sum of a nilpotent and an idempotent), and semiclean elements [2, 3].

Based on this foundation, G. Călugăreanu and Y. Zhou [4] initiated the study of idempotent-fine rings in which every nonzero idempotent is fine. Notably, this class is related to but distinct from clean rings. Fine rings constitute a subclass of simple rings that properly contains all simple Artinian rings [1]. Matrix rings over fine rings remain fine, and a local ring is fine precisely when it is a division ring. Unlike clean or nil-clean rings, fine rings are always simple, making them particularly interesting from a structural perspective.

The class of idempotent-fine rings forms a subclass of idempotent-simple rings (i.e., $AeA = A$ for every nonzero idempotent e in the ring A) and properly contains the class of Artinian idempotent-simple rings [4, Proposition 8]. For commutative rings, being idempotent-fine is equivalent to being indecomposable (having no idempotents other than 0 and 1). However, this equivalence fails in the noncommutative case. The idempotent-fine property is preserved by centers but not generally by subrings [4].

From a group-theoretic perspective, we recall several fundamental concepts: A group G is called a torsion group if every element has finite order, and torsion-free if no non-identity element has finite order; for a fixed prime p, G is a p-group if every element has an order that is a power of p. A group is nilpotent if its lower central series $\gamma_1(G) = G, \gamma_2(G) = [G, G], \gamma_{n+1}(G) = [G, \gamma_n(G)]$ terminates at the trivial subgroup after finitely many steps, which implies the existence of a central series of finite length. Furthermore, G is solvable if it has a normal series with abelian quotients, and supersolvable if it has a normal series with cyclic quotients.

This paper is devoted to finding the necessary and sufficient conditions on a ring A and a group G so that the group ring $A[G]$ is an idempotent-fine ring, establishing conditions under which $A[G]$ is idempotent-fine in terms of both the ring A and the group G. Let A be a ring (respectively, an integral domain) of characteristic p^m for some prime p and positive integer m, and let G be a locally finite nilpotent group (respectively, a locally finite group). We prove that $A[G]$ is an idempotent-fine ring if and only if G is a p-group. Furthermore, if D is an integral domain with characteristic 0 and G is a finite group, then we prove that $D[G]$ is an idempotent-fine ring if and only if every prime divisor of $|G|$ is nonunit in D.

Our key result is at the end of the paper. If F is a field of characteristic p and $F[G]$ is an idempotent-fine ring, then every nontrivial element g in the group G of finite order is a p-element. Conversely, if G is a locally finite p-group, then $F[G]$ is an idempotent-fine ring.

1.1. Preliminaries

Throughout, rings are unital associative, and groups are taken to be nontrivial except where indicated. Given a coefficient ring A and a group G, the corresponding group ring is denoted $A[G]$. There is a natural ring homomorphism:

$$\varepsilon\colon A[G] \longrightarrow A, \qquad \sum_{g\in G} a_g g \ \longmapsto\ \sum_{g\in G} a_g,$$

called the augmentation homomorphism. Its kernel,

$$\Delta(G) \;=\; \ker(\varepsilon),$$

is the augmentation ideal of $R[G]$. The main references for group rings include [5–7]. Our terminology in ring theory follows [5, 8] and in group theory follows [9].

We first establish a foundational understanding by reviewing key known properties and theorems concerning idempotent-fine rings. Building upon this established framework, we then introduce several novel results. These new findings provide crucial insights that significantly streamline and facilitate our primary objective: characterization of the group ring $A[G]$ as idempotent-fine. We recall from [1, 4]:

Definition 1.1.

i. *A nonzero element in a ring is said to be a fine element if it is the sum of a unit and a nilpotent element.*
ii. *A nonzero ring is called a fine ring if all nonzero elements are fine.*
iii. *A ring is called an idempotent-fine ring if every nonzero idempotent in it is fine.*
iv. *A ring A is called an idempotent-simple ring if $AeA = A$ for every nonzero idempotent e in A.*

Lemma 1.2. *[4, Proposition 7] Every nonzero idempotent-fine ring is idempotent-simple.*

2. Main results

Lemma 2.1. *Let A be a ring.*

(i) If A is an idempotent-fine ring, then A is indecomposable (i.e., A has no nontrivial central idempotents).
(ii) If A is abelian (that is, the idempotents in A are central), then A is an idempotent-fine ring if and only if A is indecomposable.
(iii) If A is an idempotent-fine ring, then the characteristic of A is zero or p^m for some prime number p and a positive integer m.
(iv) Let I be a nil ideal of A. Then, A is an idempotent-fine ring if and only if $\frac{A}{I}$ is an idempotent-fine ring.

Proof. (i) This follows directly from [4, Proposition 2].

(ii) Assume that A is an abelian ring. If A is an idempotent-fine ring, then by part (i), A is indecomposable. Conversely, if A is indecomposable and abelian, then A has only the trivial idempotents, and hence A is an idempotent-fine ring.

(iii) Let A be an idempotent-fine ring with $\text{char}(A) \neq 0$. Then $\text{char}(A) = n$ for some positive integer n. Clearly, $\mathbb{Z}_n$ is a subring of A and it is contained in the center of A. Since A an idempotent-fine ring by part (i), it is indecomposable, and then $\mathbb{Z}_n$ is indecomposable. Thus, $\mathbb{Z}_n$ has only two idempotents 0 and 1. This implies that there exists a prime p and a positive integer m such that $n = p^m$, and so $\text{char}(A) = p^m$.

(iv) This follows directly from [4, Proposition 3(2)].

□

Part (iii) of the above lemma is the most critical, as it has a direct bearing on our main results.

Although $\mathbb{Z}_6$ is a homomorphic image of $\mathbb{Z}$, which is an idempotent-fine ring, $\mathbb{Z}_6$ itself is not. This observation leads us to the following theorem:

Theorem 2.2. *Let A be a ring and G a group. If $A[G]$ is an idempotent-fine ring, then A is an idempotent-fine ring.*

Proof. Assume that $A[G]$ is idempotent-fine and let $\omega : A[G] \longrightarrow A$ be the augmentation homomorphism. If e is an idempotent in A, then e is also an idempotent in $A[G]$. Since $A[G]$ is idempotent-fine, it has a fine decomposition, meaning that $e = n + u$, where n is nilpotent in $A[G]$ and u is a unit in $A[G]$. We know that ω is unital, so we conclude that $\omega(u)$ is a unit and $\omega(n)$ is nilpotent in A. Thus,

$$e = \omega(e) = \omega(n) + \omega(u)$$

is a fine decomposition in A. This implies that A is an idempotent-fine ring. □

We now proceed to our main results, beginning with the establishment of necessary conditions.

Theorem 2.3. *Let A be a ring and G a nontrivial group. If $A[G]$ is an idempotent-fine ring, then for every finite normal subgroup N of G, the order of N is non-unit in A.*

Proof. Assume that $A[G]$ is an idempotent-fine ring, and let N be a nontrivial finite normal subgroup of G. Suppose that $|N|$ is a unit in A. Let

$$e_N = \frac{1}{|N|} \sum_{h \in N} h.$$

Since N is a finite nontrivial normal subgroup of G, e_N is a nontrivial central idempotent of $A[G]$ by [10, Lemma 3.6.6]. Thus, $A[G]$ is not indecomposable, contradicting our assumption by Lemma 2.1(i). □

Theorem 2.4. *Let A be a ring of characteristic p^m for some prime p and m a positive integer. If $A[G]$ is an idempotent-fine ring, then p divides $|N|$ for every finite normal subgroup N of G.*

Proof. Assume that $A[G]$ is an idempotent-fine ring, and let N be a nontrivial finite normal subgroup of G. We will show that p divides $|N|$. Suppose, for contradiction, that $p \nmid |N|$. Then $|N|$ is relatively prime to p, and hence $|N|$ is a unit in A. This contradicts the conclusion of Theorem 2.3. Therefore, p must divide $|N|$. □

From this point onward, we find that there are always two different cases to consider: The case where the ring A has characteristic zero, and the case where the characteristic is nonzero.

Theorem 2.5. *Let A be an idempotent-fine ring of characteristic p^m for some prime p, m a positive integer, and G a locally finite nilpotent group. Then, $A[G]$ is an idempotent-fine ring if and only if G is a p-group.*

Proof. Since G is locally finite, for every $x \in A[G]$, there exists a finite subgroup H of G such that $x \in A[H]$. Thus, without loss of generality, we can assume that G is a finite nilpotent group.

Assume that $A[G]$ is an idempotent-fine ring, and let $g \in G$. We show that g is a p-element. Suppose not; that is, there exists a prime $q \neq p$ such that q divides $|g|$. Then q divides $|G|$. Since G is a finite nilpotent group, the q-Sylow subgroup N_q of G is normal (see [9, Proposition 5.2.4]). By Theorem 2.4, p divides $|N_q|$ where $|N_q| = q^t$ for some positive integer t, which is impossible because p and q are distinct primes. Therefore, G is a p-group. Conversely, since G is a locally finite p-group, as A has the charactaristic p^m, then p is nilpotent. So, by [6, Theorem 9], $\Delta(A[G])$ is nil. As $A[G]/\Delta(A[G]) \cong A$, it follows from Lemma 1(iv) that $A[G]$ is an idempotent-fine ring. □

Theorem 2.6. *Let D be an integral domain and G is a nontrivial group. Then $D[G]$ is an idempotent-fine ring if and only if $D[G]$ has only trivial idempotents.*

Proof. Assume that $D[G]$ is an idempotent-fine ring. If $e \in D[G]$ and $e \neq 0, e \neq 1$ is an arbitrary idempotent, and e has a fine decomposition, that is $e = n + u$ where n is a nilpotent in $D[G]$ and u is a unit in $D[G]$. Now $\omega(e) = \omega(n) + \omega(u)$. Since D is an integral domain, it has no nonzero nilpotents. Hence, $\omega(n) = 0$ and $\omega(u)$ is a unit because ω is a unital ring homomorphism. But we have $\omega(e) = \omega(n) + \omega(u) = \omega(u)$ and this implies that $\omega(e) = 1$ because $\omega(e)$ is both a unit and idempotent. Now apply the same argument to $f = 1 - e$, which is also a nontrivial idempotent. We find $\omega(f) = 1$ and $1 = \omega(1 - e) = \omega(1) - \omega(e) = 0$, which is a contradiction. Therefore, $D[G]$ has only trivial idempotents. Conversely, if $D[G]$ does not have nontrivial idempotents, then it is clear that $D[G]$ is an idempotent-fine ring. □

Lemma 2.7. *Let D be an integral domain and G a nontrivial group. Then $D[G]$ is an idempotent-fine ring if and only if $D[H]$ is an idempotent-fine ring for every $H \leq G$.*

Proof. Assume that $D[G]$ is an idempotent-fine ring, and let $H \leq G$. Suppose that $D[H]$ is not an idempotent-fine ring, as Theorem 2.6 implies that $D[H]$ has an idempotent e different from 0 and 1, and so $e \in D[H] \leq D[G]$ which means that $D[G]$ is not an idempotent-fine, which is a contradiction. The converse follows directly by taking $H = G$. □

Theorem 2.8. *Let D be an integral domain characteristic 0 and G a finite group. Then, $D[G]$ is an idempotent-fine ring if and only if every prime divisor of $|G|$ is non-unit in D.*

Proof. Let D be an integral domain of characteristic 0, and assume that $D[G]$ is an idempotent-fine ring. Let p be a prime divisor of $|G|$. By Cauchy's theorem, there exists a subgroup H of G such that $|H| = p$. If p is a unit in D, then we find a nontrivial idempotent $e_H \in D[H] \subseteq D[G]$, and this means that $D[G]$ is not idempotent-fine, which is a contradiction. Thus, p is a non-unit in D.

Conversely, if no prime divisor of $|G|$ is a unit in D, then by [11, Theorem 1], $D[G]$ has no idempotents different from 0 and 1. Thus, $D[G]$ is an idempotent-fine ring. □

Corollary 2.9. *For any finite group G, the integral group ring $Z[G]$ is an idempotent-fine ring.*

Proof. This follows directly from Theorem 2.8. □

Theorem 2.10. *Let D be an integral domain of characteristic p^m for some prime p, m a positive integer, and G a locally finite group. Then, $D[G]$ is an idempotent-fine ring if and only if G is a p-group.*

Proof. Since G is locally finite, for every $x \in D[G]$, there exists a finite subgroup H of G such that $x \in D[H]$. Thus, without loss of generality, we can assume that G is a finite group. Assume that $D[G]$ is an idempotent-fine ring, and let $g \in G$. We show that g is a p-element. Suppose not; that is, there exists a prime $q \neq p$ such that q divides $|g|$. Then q divides $|G|$. Since G is a finite group, by Cauchy's theorem, there exists a subgroup H of G such that $q = |H|$. Since p and q are distinct primes, q is a unit in D. So easily we can construct a nontrivial idempotent

$$e_H = \frac{1}{q} \sum_{h \in H} h.$$

As D is an integral domain and $D[G]$ is an idempotent-fine ring, Theorem 2.6 implies that $D[G]$ has only trivial idempotents. Thus, G cannot have such elements. Therefore, G is a p-group.

Conversely, we have that G is a locally finite p-group and A has charactaristic p^m, so p is nilpotent. Hence by [6, Theorem 9], $\Delta([DG])$ is nil. As $D[G]/\Delta(D[G]) \cong A$, it follows from Lemma 1(iv) that $D[G]$ is an idempotent-fine ring. □

Since this result establishes that there are no general necessary conditions on a group G such that the group ring $A[G]$ is idempotent-fine, we must shift our focus to other ring-theoretic settings. In particular, we will explore classes of rings that are not integral domains.

Theorem 2.11. *Let F be a field of characteristic* 0 *and G a nontrivial group. If* $F[G]$ *is an idempotent-fine ring, then G is a torsion-free group.*

Proof. Let F be a field of characteristic 0 and G a nontrivial group. Assume that $F[G]$ is an idempotent-fine ring. Let $g \in G$ and assume that g has finite order. Then $H =< g >$ is a finite subgroup, and since F has the characteristic 0, $|H|$ is a unit in F. This implies that $e = \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent. By Theorem 2.6, e should be trivial and hence $g = 1$. Therefore, every non-identity element in G has infinite order, and this means that G is torsion-free. □

In light of Theorem 2.6, it is evident that the converse of the above theorem is parallel to solving the famous Kaplansky idempotent conjecture.

Theorem 2.12. *Let F be a field of characteristic* 0 *and G a nilpotent group. Then,* $F[G]$ *is an idempotent-fine ring if and only if G torsion-free.*

Proof. If $F[G]$ is idempotent-fine then by Theorem 2.11, G is torsion-free. Conversely, since G is a nilpotent torsion-free group, $F[G]$ has no zero divisors and hence has only two idempotents 0 and 1 [10, Exercise 9, p. 214]. Thus $F[G]$ is idempotent-fine. □

The following theorem establishes the optimal necessary condition for a group algebra $F[G]$ to be idempotent-fine. The question of sufficiency, however, leads us—by virtue of Theorem 2.6—into the domain of the famous Kaplansky idempotent conjecture. Our characterization problem is thus parallel to this conjecture, which, though still open in general, has been settled for a wide range of significant groups, such as abelian, nilpotent, and ordered groups [7, 12].

Theorem 2.13. *Let F be a field of characteristic p and G a nontrivial group. If* $F[G]$ *is an idempotent-fine ring, then every element* $g \in G$ *of finite order is a p-element. Conversely, if G is a locally finite p-group, then* $F[G]$ *is an idempotent-fine ring.*

Proof. Let F be a field of characteristic p and G a nontrivial group. Assume that $F[G]$ is an idempotent-fine ring. Let $g \neq 1$ be an element of G with finite order $|g| = m$, and let $H = \langle g \rangle$. Theorem 2.7 implies that $F[H]$ is an idempotent-fine ring. By Theorem 2.10, we find that H is a p-group. Hence $m = p^t$ for some positive integer t. Therefore, g is a p-element, and so every torsion element in G is a p-element. The converse follows directly from Theorem 2.10. □

Example 2.14. *Let* $\mathbb{F}_4$ *be the field with four elements (which has characteristic* 2*).*

1. *Let* C_6 *denote the cyclic group of order* 6*. The group algebra* $\mathbb{F}_4[C_6]$ *is not an idempotent-fine ring. Indeed, one can easily construct a nontrivial idempotent in this algebra, which implies that* $\mathbb{F}_4[C_6]$ *is not indecomposable.*
2. *Let* Q_8 *denote the quaternion group of order* 8*. Then the group algebra* $\mathbb{F}_4[Q_8]$ *is an idempotent-fine ring; this follows directly from the preceding theorem.*

3. Conclusions

For a torsion-free group G and a field k, the Kaplansky idempotent conjecture asserts that the group algebra $k[G]$ contains no idempotents other than 0 and 1. This deceptively simple statement has stood for decades as one of the most resilient open problems in the theory of group rings, resisting numerous attempts at a general proof.

However, progress has been steady and illuminating. In characteristic zero, the combined work of Linnell, Burger, and Valette established the conjecture for all amenable groups (a broad class of groups that encompasses solvable and locally finite groups) [13]. Meanwhile, in arbitrary characteristic, Bass's celebrated 1976 theorem settled the case for linear groups (those embeddable in some $\mathrm{GL}_n(K)$), thereby covering free groups and many others [14]. Subsequent years saw the conjecture verified for various geometrically defined groups, including hyperbolic groups and many Artin groups. Yet despite these advances, the general conjecture remains open.

Our work contributes to this narrative by developing a new approach that handles the class of locally nilpotent groups over fields of any characteristic. For a group algebra $F[G]$ over a field F and a group G, we proved that $F[G]$ has no nontrivial idempotents if and only if $F[G]$ is idempotent-fine, see Theorem 2.6.

In this way, our work provides a new lens through which to view an old problem and perhaps a step toward the ultimate resolution of Kaplansky's enduring question.

Author contributions

Omar Al-Mallah and Mohammed Abu-Saleem: Formal analysis, writing–original draft; Noômen Jarboui: Methodology, results analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no competing interests to disclose.

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