AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 322–344.
DOI: 10.3934/math.2026013
Received: 23 August 2025
Revised: 10 November 2025
Accepted: 21 November 2025
Published: 05 January 2026

Research article

Generalized probabilistic hesitant Pythagorean fuzzy aggregation operators and their application in teaching equipment procurement

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Abstract: Probabilistic hesitant Pythagorean fuzzy sets (PrHPyFSs) provide a robust framework for modeling decision-makers' preferences by simultaneously capturing probabilistic uncertainty, hesitation degrees, and the independent support/non-support relationships, thereby enabling a more accurate representation of real-world decision-making compared to traditional fuzzy sets. This study explores the aggregation of probabilistic hesitant Pythagorean fuzzy information in complex environments and its application to multi-criteria decision-making (MCDM). The research includes four main components: (1) developing an arithmetic operation system for probabilistic hesitant Pythagorean fuzzy elements (PrHPyFEs); (2) proposing four types of generalized aggregation operators for PrHPyFEs; (3) constructing a PrHPyFS-based MCDM framework using these operators, with effectiveness validated through a teaching equipment procurement case study; and (4) demonstrating the method's advantages via comparative analysis. The results confirm that the proposed solution effectively bridges the gap between theoretical foundations and practical decision-making applications.

Keywords: generalized aggregation operator; multi-criteria decision-making (MCDM); probabilistic hesitant Pythagorean fuzzy element (PrHPyFE); probabilistic hesitant Pythagorean fuzzy set (PrHPyFS)
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

In complex real-world decision-making scenarios, processing ambiguous information holds significant theoretical and practical value. Zadeh [1] pioneered fuzzy set (FS) theory, achieving the first quantitative representation of ambiguity through membership functions—an innovative mathematical tool that transcends classical set theory's binary constraints (0/1) by employing continuous membership degrees within [0,1]. This multi-valued logic aligns with human cognition's continuity, demonstrating exceptional potential in expert systems, intelligent control, and pattern recognition. Driven by increasingly complex application demands, FS theory has undergone continuous innovation and evolution. Intuitionistic fuzzy sets proposed by Atanassov [2] establish a more refined binary characterization framework through the introduction of non-membership functions. Pythagorean fuzzy set (PyFS), developed by Yager [3], further relaxes membership and non-membership constraint conditions and enhances the model's expressive capacity. To characterize uncertainty in decision-making processes or the hesitation psychology of decision-makers, Torra [4] proposed the hesitant fuzzy set, which subsequently led to the derivation of hybrid models such as the dual hesitant fuzzy set [5] and hesitant Pythagorean fuzzy set [6, 7]. Xu and Zhou [8] innovatively introduced a probabilistic dimension, proposing a probabilistic hesitant fuzzy set (PrHFS) to enhance the model's realism in decision-making scenarios. The evolution of fuzzy information processing theory exhibits three distinct characteristics: (1) Transitioning from Zadeh's single membership degree to a multi-dimensional representation system; (2) Shifting from deterministic descriptions to probabilistic modeling; (3) Advancing from static processing to dynamic decision-making support. These theoretical breakthroughs continually expand application boundaries, exerting profound impacts on fields including engineering optimization, economic forecasting, and social decision-making. In the AI and big data era, integrating fuzzy information processing theory with cutting-edge technologies such as deep learning and reinforcement learning will provide enhanced analytical tools for complex system modeling, propelling intelligent decision-making to more sophisticated levels.

Probabilistic hesitant Pythagorean fuzzy set (PrHPyFS) [9–11] is an important theoretical innovation in the field of fuzzy mathematics. By integrating the advantages of PyFS and PrHFS, a comprehensive mathematical framework that can simultaneously handle fuzziness, hesitancy, and randomness is constructed. This theory has significant core innovative features: (1) Mathematical structural innovation: using probability-weighted sets of membership and non-membership degrees to describe element attributes; strictly satisfying the Pythagorean constraint, that is, the sum of the squares of membership and non-membership values does not exceed 1; representing hesitation characteristics in the decision-making process through a set of multi-valued membership degrees; Using probability distribution to reflect the credibility of evaluation results. (2) Theoretical advantage: improving the accuracy of characterizing complex fuzzy phenomena; reflecting the uncertainty in the real world more accurately; providing a more reliable theoretical foundation for intelligent decision-making systems. (3) Application value: achieving collaborative processing of fuzzy and probabilistic information or providing more accurate analysis and modeling tools in fields such as risk assessment, medical diagnosis, and financial prediction. As a cutting-edge theory in fuzzy information processing, PrHPyFS not only expands the research boundaries of fuzzy mathematics but also provides new methodological support for complex system analysis, which is of great significance for promoting the development and application of innovation in uncertainty mathematical theory.

The current academic research on PrHPyFS mainly focuses on the following core directions.

(1) Basic theoretical construction: Mainly including basic concepts, arithmetic operations and their properties, comparative indicators and rules, distance measures, information aggregation operators, etc.

- Basic concepts: Luo and Liu [9], Batool et al. [10], and Ji et al. [11] respectively provide different formal definitions for PrHPyFS, or probabilistic hesitant Pythagorean fuzzy element (PrHPyFE).

- Arithmetic operations and properties: Luo and Liu [9], Batool et al. [10], and Ji et al. [11] defined basic arithmetic operations based on the corresponding PrHPyFS or PrHPyFE, and further provided some basic operation properties. Afterwards, Sun et al. [12], Ashraf et al. [13], and Liao et al. [14] made some meaningful supplements and improvements to the properties of arithmetic operations.
- Comparison indicators and rules: Luo and Liu [9], Batool et al. [10], Ji et al. [11], Sun et al. [12], Liao et al. [14], Rasheed et al. [15], and Sarkar et al. [16] proposed different forms of the PrHPyFE score function and accuracy function definitions and further provided the PrHPyFEs comparison rules based on these indicators. Sun and Wang [17] pointed out that relying solely on score functions and accuracy functions cannot achieve a complete comparison of PrHPyFEs and developed a three-indicator PrHPyFEs, or probabilistic hesitant Pythagorean fuzzy vectors, complete comparison method based on a new score function, accuracy function, and variance functions.
- Distance measures: Batool et al. [10], Ji et al. [11], and Sun et al. [12] provided different definitions of distance measures to characterize the relationship between PrHPyFEs. In addition, Sun et al. [12] also proposed a similarity function based on the presented distance measure.
- Information aggregation operators: Fuzzy information aggregation is a research hotspot in this field, and different forms of PrHPyFEs aggregation operators have been continuously proposed, such as the Hamacher Choquet integral geometric operator [9], power Hamy mean operators [11], weighted operators [18, 19], Einstein operators [13], triangular fuzzy aggregation operators [14], Choquet integral operators [15], and the improved power weighted averaging operator [16].

(2) Innovation in decision-making methods: Significant progress has been made in the research of MCDM and multi-criteria group decision-making (MCGDM) models based on PrHPyFS theory. Batool et al. [10] presented an extended MCDM TOPSIS approach. Ji et al. [11] proposed a novel MCGDM method. Sun et al. [12] developed an interactive method to solve the MCGDM problem. Tang et al. [20] proposed an MCGDM method to solve the problem of unknown weight information. Liao et al. [14] developed an MCDM model in probabilistic hesitant Pythagorean triangular fuzzy environments. Qahtan et al. [21] presented an integrated MCDM method. In addition, Luo and Liu [9], Batool et al. [18], Batool et al. [19], Ashraf et al. [13], Rasheed et al. [15], Sarkar et al. [16], and Sun and Wang [17] have also made theoretical contributions in this field, jointly promoting the application and development of PrHPyFS in complex decision-making scenarios.

(3) Frontier application exploration: Some researchers have successfully extended the probabilistic hesitant Pythagorean fuzzy MCDM/MCGDM models to multiple interdisciplinary practice areas. Luo and Liu [9] applied the proposed MCDM method to the selection of project private partners. Batool et al. [10] developed a comprehensive evaluation system for the fog-haze factor. Tang et al. [20] achieved intelligent decision-making for the commercialization path of breakthrough inventions. Qahtan et al. [21] applied the proposed MCDM method to solve the optimization modeling problem of a fuel supply system for electric vehicles. Sun and Wang [17] developed a multidimensional evaluation framework for the level of professional development in universities. These empirical studies not only validate the theoretical advantages of PrHPyFS in dealing with uncertain decision-making problems, but also provide new decision-making support tools for major practical issues such as environmental governance, technological innovation, and educational evaluation.

The current research trend indicates that the study of PrHPyFS is undergoing a paradigm shift from basic theoretical construction to complex decision-making applications, exhibiting dual-driven characteristics of theoretical innovation and practical verification. As a cutting-edge approach in uncertain decision-making, PrHPyFS demonstrates unique theoretical advantages in fuzzy clustering analysis, intelligent pattern recognition, and multi-criteria evaluation, while providing methodological innovations for smart manufacturing and urban governance. This transition not only enriches fuzzy mathematical theory but also addresses real-world challenges in dynamic prediction and complex information processing.

Notwithstanding the breakthroughs achieved in specific application scenarios, extant research on PrHPyFSs exhibits persistent limitations across three critical dimensions:

(1) Theoretical completeness limitations

- Algebraic operation deficiencies: Current arithmetic operation systems lack comprehensive closure verification mechanisms, particularly regarding normalized probability distribution processing.
- Comparative metric shortcomings: Existing score/accuracy functions fail to achieve complete comparability.
- Dynamic regulation limitations: Absence of coordinated mechanisms for synchronizing membership degrees, non-membership degrees, and probability distributions.

(2) Technological universality challenges

- Operator architecture rigidity: Conventional linear weighted operators inadequately capture nonlinear criterion interactions.
- Probability distribution handling: Certain comparison functions exhibit measurement deficiencies in probability differences.

(3) Engineering applicability constraints

- Decision-making context adaptation: Limited generalization capacity for dynamic prediction and complex information processing scenarios.
- Information aggregation bias: Traditional operator structures risk information distortion in group decision-making environments.

The aforementioned limitations not only highlight key areas for future research breakthroughs but also call for deep collaborative innovation between academia and industry to collectively advance the field.

This study is driven by three key motivations for advancing the theory and applications of PrHPyFSs, as outlined below:

(1) Theoretical refinement necessity

Current research reveals systematic limitations in algebraic operation systems and aggregation operator frameworks. Conventional linear weighted operators inadequately capture nonlinear interactions among criteria, while existing operator structures lack dynamic mechanisms for coordinated regulation of membership degrees, non-membership degrees, and probability distributions. Moreover, certain comparison functions exhibit deficiencies in probability difference measurement, directly affecting decision-making system reliability and effectiveness.

(2) Methodological innovation potential

PrHPyFS's research is currently undergoing a paradigm shift from fundamental theory to complex decision-making applications. Developing parameter-tunable generalized aggregation operators, refining PrHPyFEs' algebraic operation systems, and establishing a comprehensive mathematical framework integrating fuzziness, hesitancy, and randomness are expected to address existing technical challenges.

(3) Practical value realization

As a frontier achievement in fuzzy mathematics, PrHPyFSs have demonstrated significant potential in risk assessment and medical diagnosis. Methodological innovations promise to deliver more precise decision-making tools for critical challenges in smart manufacturing and urban governance, thereby advancing uncertainty mathematical theory.

In the fields of MCDM/MCGDM, the fuzzy information aggregation method, as the core link in dealing with uncertainty and subjective preferences, directly determines the reliability and effectiveness of the decision-making system based on its performance. Although the research on information aggregation operators based on PrHPyFSs has achieved phased results, there are still several key scientific issues to be addressed. Firstly, traditional linear weighted averaging and geometric weighted averaging operators are difficult to effectively characterize the complex nonlinear interactions between criteria, which can lead to information distortion when dealing with some complex decision-making problems. Secondly, some existing operator architectures lack dynamic adjustment mechanisms for the synergistic effects of membership degree, non-membership degree, and probability distribution, which can easily lead to information aggregation bias in group decision-making environments. In addition, some existing comparison indicator functions have deficiencies in measuring the differences in probability distributions, which may lead to counterintuitive comparison results. In response to these challenges, this study aims to systematically address them by constructing generalized aggregation operators with adjustable parameters. The specific research content includes four progressive levels. (1) Improve the algebraic operation system of PrHPyFEs, verify the applicable boundaries of operation laws, and lay a theoretical foundation for operators' construction. (2) Propose generalized aggregation operators. (3) Develop an MCDM decision-making framework based on the proposed operators and conduct empirical research using equipment procurement issues as a typical scenario. (4) Verify the superiority of the proposed method in terms of decision-making accuracy, robustness, and computational efficiency through comparative experiments.

This paper adopts the systematic research path of "theoretical construction - methodology design - verification analysis": Section 2 rigorously constructs the formal definition framework of PrHPyFS and PrHPyFE from mathematical foundations and establishes arithmetic operation laws; Section 3 systematically demonstrates the arithmetic operation properties of PrHPyFEs through strict mathematical derivation; Section 4 innovatively proposes four types of generalized aggregation operators, including generalized weighted averaging operator, generalized weighted geometric operator, generalized hybrid averaging operator, and generalized hybrid geometric operator, providing new mathematical tools for complex decision-making problems; Section 5 transforms theoretical achievements into an application model by constructing a multi-criteria decision-making framework based on the proposed operators, and verifies the model's practicality through numerical simulation of an educational equipment procurement case; Section 6 designs systematic comparative experiments to quantitatively validate the advantages of the developed method over existing aggregation operators; Section 7 summarizes the theoretical innovations and application value while outlining future research directions such as operator extensions and cross-domain applications.

2. Preliminaries

This section reviews the definitions of PrHPyFS and PrHPyFE, as well as the arithmetic operations of PrHPyFEs.

Definition 1 [10] Let universe of discourse X be a nonempty set. Then a PrHPyFS P on X is defined as

$$P = \{\langle x, \mu(x)/p(x), \nu(x)/q(x)\rangle \,|x \in X\},$$

where $\mu(x)/p(x) = \{\mu_i(x)/p_i(x)|i = 1, 2, ..., m\}$ is a membership degree set with probabilistic characteristics, and $\nu(x)/q(x) = \left\{\nu_j(x)/q_j(x)|j = 1, 2, ..., n\right\}$ is a non-membership degree set with probabilistic characteristics, and satisfy $\mu_i(x) \in [0, 1]$, $\nu_j(x) \in [0, 1]$, $\mu_i^2(x) + \nu_j^2(x) \leq 1$, $p_i(x) \in (0, 1]$, $q_j(x) \in (0, 1]$, $\sum_{i=1}^m p_i(x) = 1$, and $\sum_{j=1}^n q_j(x) = 1$.

A PrHPyFE has the form of $\left(\{\mu_i/p_i|i = 1, 2, ..., m\}, \left\{\nu_j/q_j|j = 1, 2, ..., n\right\}\right)$, where $\mu_i \in [0, 1]$, $\nu_j \in [0, 1]$, $\mu_i^2 + \nu_j^2 \leq 1$, $p_i \in (0, 1]$, $q_j \in (0, 1]$, $\sum_{i=1}^m p_i = 1$, and $\sum_{j=1}^n q_j = 1$. For simplicity, PrHPyFE(X) is used in the following text to represent the set consisting of all PrHPyFEs.

Definition 2 [10] For α_1, α_2, $\alpha \in$ PrHPyFE(X) and $\lambda > 0$, where

$$\begin{aligned}
\alpha_1 &= \left(\left\{\mu'_{i_1}/p'_{i_1}|i_1 = 1, 2, ..., m_1\right\}, \left\{\nu'_{j_1}/q'_{j_1}|j_1 = 1, 2, ..., n_1\right\}\right),\\
\alpha_2 &= \left(\left\{\mu''_{i_2}/p''_{i_2}|i_2 = 1, 2, ..., m_2\right\}, \left\{\nu''_{j_2}/q''_{j_2}|j_2 = 1, 2, ..., n_2\right\}\right),\\
\alpha &= \left(\{\mu_i/p_i|i = 1, 2, ..., m\}, \left\{\nu_j/q_j|j = 1, 2, ..., n\right\}\right),
\end{aligned}$$

some arithmetic operations are defined as follows:

(1) $\alpha_1 \oplus \alpha_2 = \left(\begin{array}{l} \left\{\sqrt{\mu'^{\,2}_{i_1} + \mu''^{\,2}_{i_2} - \mu'^{\,2}_{i_1}\mu''^{\,2}_{i_2}}/p'_{i_1}p''_{i_2}|i_k = 1, 2, ..., m_k, k = 1, 2\right\}, \\ \left\{\nu'_{j_1}\nu''_{j_2}/q'_{j_1}q''_{j_2}|j_k = 1, 2, ..., n_k, k = 1, 2\right\} \end{array}\right)$;

(2) $\alpha_1 \otimes \alpha_2 = \left(\begin{array}{l} \left\{\mu'_{i_1}\mu''_{i_2}/p'_{i_1}p''_{i_2}|i_k = 1, 2, ..., m_k, k = 1, 2\right\}, \\ \left\{\sqrt{\nu'^{\,2}_{j_1} + \nu''^{\,2}_{j_2} - \nu'^{\,2}_{j_1}\nu''^{\,2}_{j_2}}/q'_{j_1}q''_{j_2}|j_k = 1, 2, ..., n_k, k = 1, 2\right\} \end{array}\right)$;

(3) $\lambda\alpha = \left(\left\{\sqrt{1 - \left(1 - \mu_i^2\right)^\lambda}/p_i|i = 1, 2, ..., m\right\}, \left\{\nu_j^\lambda/q_j|j = 1, 2, ..., n\right\}\right)$;

(4) $\alpha^\lambda = \left(\left\{\mu_i^\lambda/p_i|i = 1, 2, ..., m\right\}, \left\{\sqrt{1 - \left(1 - \nu_j^2\right)^\lambda}/q_j|j = 1, 2, ..., n\right\}\right)$;

(5) $\bar{\alpha} = \left(\left\{\nu_j/q_j|j = 1, 2, ..., n\right\}, \{\mu_i/p_i|i = 1, 2, ..., m\}\right)$.

3. Some properties of arithmetic operations

In subsequent research on PrHPyFE aggregation operators, the arithmetic operations established in Definition 2 serve as fundamental building blocks for operator construction, forming the cornerstone for developing more complex computational models. As we know, closure is a core requirement in defining operations, ensuring internal consistency and logical coherence within a specific set, thereby simplifying compliance verification for complex operations. However, for relatively complex

PrHPyFEs, the closure properties of these arithmetic operations in Definition 2 are not immediately evident. Therefore, we provide a supplemental analysis of their closure properties.

Theorem 1 For α_1, α_2, $\alpha \in$ PrHPyFE(X) and $\lambda > 0$, there are $\alpha_1 \oplus \alpha_2$, $\alpha_1 \otimes \alpha_2$, $\lambda\alpha$, α^λ, $\bar{\alpha} \in$ PrHPyFE(X).

Proof: Let α_1, α_2, and α be as shown in Definition 2.

(1) According to

$$\left(\sqrt{\mu_{i_1}'^{2} + \mu_{i_2}''^{2} - \mu_{i_1}'^{2}\mu_{i_2}''^{2}}\right)^2 + \left(\nu_{j_1}'\nu_{j_2}''\right)^2 \leq \mu_{i_1}'^{2} + \mu_{i_2}''^{2} - \mu_{i_1}'^{2}\mu_{i_2}''^{2} + \left(1 - \mu_{i_1}'^{2}\right)\left(1 - \mu_{i_2}''^{2}\right) = 1,$$

$$\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2} p_{i_1}' p_{i_2}'' = \sum_{i_1=1}^{m_1}\left(p_{i_1}' \sum_{i_2=1}^{m_2} p_{i_2}''\right) = \sum_{i_1=1}^{m_1} p_{i_1}' = 1,$$

and

$$\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2} q_{j_1}' q_{j_2}'' = \sum_{j_1=1}^{n_1}\left(q_{j_1}' \sum_{j_2=1}^{n_2} q_{j_2}''\right) = \sum_{j_1=1}^{n_1} q_{j_1}' = 1,$$

it can be obtained that $\alpha_1 \oplus \alpha_2 \in$ PrHPyFE(X).

(2) According to

$$\left(\mu_{i_1}'\mu_{i_2}''\right)^2 + \left(\sqrt{\nu_{j_1}'^{2} + \nu_{j_2}''^{2} - \nu_{j_1}'^{2}\nu_{j_2}''^{2}}\right)^2 \leq \left(1 - \nu_{j_1}'^{2}\right)\left(1 - \nu_{j_2}''^{2}\right) + \nu_{j_1}'^{2} + \nu_{j_2}''^{2} - \nu_{j_1}'^{2}\nu_{j_2}''^{2} = 1,$$

$$\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2} p_{i_1}' p_{i_2}'' = 1,$$

and

$$\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2} q_{j_1}' q_{j_2}'' = 1,$$

it can be obtained that $\alpha_1 \otimes \alpha_2 \in$ PrHPyFE(X).

(3) According to

$$\left(\sqrt{1 - \left(1 - \mu_i^2\right)^\lambda}\right)^2 + \left(\nu_j^\lambda\right)^2 = 1 - \left(1 - \mu_i^2\right)^\lambda + \left(\nu_j^2\right)^\lambda \leq 1 - \left(1 - \mu_i^2\right)^\lambda + \left(1 - \mu_i^2\right)^\lambda = 1,$$

$$\sum_{i=1}^{m} p_i = 1,$$

and

$$\sum_{j=1}^{n} q_j = 1,$$

it can be obtained that $\lambda\alpha \in$ PrHPyFE(X).

(4) According to

$$\left(\mu_j^\lambda\right)^2 + \left(\sqrt{1 - \left(1 - \nu_i^2\right)^\lambda}\right)^2 = \left(\mu_j^2\right)^\lambda + 1 - \left(1 - \nu_i^2\right)^\lambda \leq \left(1 - \nu_i^2\right)^\lambda + 1 - \left(1 - \nu_i^2\right)^\lambda = 1,$$

$$\sum_{i=1}^{m} p_i = 1,$$

and

$$\sum_{j=1}^{n} q_j = 1,$$

it can be obtained that $\alpha^{\lambda} \in \text{PrHPyFE}(X)$.

(5) Obviously, $\bar{\alpha} \in \text{PrHPyFE}(X)$. ■

In the probabilistic hesitant Pythagorean fuzzy environment, the arithmetic operations of PrHPyFEs adhere to specific operational laws, which establish a theoretical foundation for the flexible application of aggregation operators. By clarifying the commutativities, associativities, distributivities, and dualities of arithmetic operations, it is possible to efficiently aggregate probabilistic hesitant Pythagorean fuzzy information. These operation laws not only ensure the mathematical rigor of the calculation process, but also enhance the reliability of complex information aggregation through the fusion of probability and fuzzy logic.

In [12], some operation laws are provided for the arithmetic operations given in Definition 2: For $\alpha_1, \alpha_2, \alpha_3, \alpha \in \text{PrHPyFE}(X)$ and $\lambda > 0$, there are

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;

(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;

(3) $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$;

(4) $(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 = \alpha_1 \otimes (\alpha_2 \otimes \alpha_3)$;

(5) $\lambda (\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$;

(6) $(\alpha_1 \otimes \alpha_2)^{\lambda} = \alpha_1^{\lambda} \otimes \alpha_2^{\lambda}$.

The following will provide some other operation laws that PrHPyFE arithmetic operations satisfy.

Firstly, two stipulations are given: For $\alpha \in \text{PrHPyFE}(X)$, let

(1) $0\alpha = (\{0/1\}, \{1/1\})$;

(2) $\alpha^0 = (\{1/1\}, \{0/1\})$, when $\alpha \neq (\{0/1\}, \{1/1\})$.

It is obvious that the above stipulations are reasonable.

Theorem 2 For $\alpha_1, \alpha_2, \alpha \in \text{PrHPyFE}(X)$ and $\lambda, \lambda_1, \lambda_2 \geq 0$, there are

(1) $\overline{\alpha_1 \oplus \alpha_2} = \bar{\alpha}_1 \otimes \bar{\alpha}_2$;

(2) $\overline{\alpha_1 \otimes \alpha_2} = \bar{\alpha}_1 \oplus \bar{\alpha}_2$;

(3) $\overline{\lambda\alpha} = \bar{\alpha}^{\lambda}$;

(4) $\overline{\alpha^{\lambda}} = \lambda\bar{\alpha}$;

(5) $\lambda_1\lambda_2\alpha = \lambda_1 (\lambda_2\alpha) = \lambda_2 (\lambda_1\alpha)$;

(6) $\alpha^{\lambda_1\lambda_2} = \left(\alpha^{\lambda_1}\right)^{\lambda_2} = \left(\alpha^{\lambda_2}\right)^{\lambda_1}$.

Proof: Let α_1, α_2, and α be as shown in Definition 2.

(1)

$$\overline{\alpha_1 \oplus \alpha_2} = \overline{\begin{pmatrix} \left\{ \sqrt{\mu'^2_{i_1} + \mu''^2_{i_2} - \mu'^2_{i_1}\mu''^2_{i_2}} / p'_{i_1} p''_{i_2} | i_k = 1,2,...,m_k, k = 1,2 \right\}, \\ \left\{ \nu'_{j_1}\nu''_{j_2} / q'_{j_1} q''_{j_2} | j_k = 1,2,...,n_k, k = 1,2 \right\} \end{pmatrix}}$$

$$= \begin{pmatrix} \left\{ \nu'_{j_1}\nu''_{j_2} / q'_{j_1} q''_{j_2} | j_k = 1,2,...,n_k, k = 1,2 \right\}, \\ \left\{ \sqrt{\mu'^2_{i_1} + \mu''^2_{i_2} - \mu'^2_{i_1}\mu''^2_{i_2}} / p'_{i_1} p''_{i_2} | i_k = 1,2,...,m_k, k = 1,2 \right\} \end{pmatrix}$$

$$= \left(\left\{ \nu'_{j_1} / q'_{j_1} | j_1 = 1,2,...,n_1 \right\}, \left\{ \mu'_{i_1} / p'_{i_1} | i_1 = 1,2,...,m_1 \right\} \right)$$
$$\otimes \left(\left\{ \nu''_{j_2} / q''_{j_2} | j_2 = 1,2,...,n_2 \right\}, \left\{ \mu''_{i_2} / p''_{i_2} | i_2 = 1,2,...,m_2 \right\} \right)$$
$$= \bar{\alpha}_1 \otimes \bar{\alpha}_2.$$

(2)

$$\overline{\alpha_1 \otimes \alpha_2} = \overline{\begin{pmatrix} \left\{ \mu'_{i_1}\mu''_{i_2} / p'_{i_1} p''_{i_2} | i_k = 1,2,...,m_k, k = 1,2 \right\}, \\ \left\{ \sqrt{\nu'^2_{j_1} + \nu''^2_{j_2} - \nu'^2_{j_1}\nu''^2_{j_2}} / q'_{j_1} q''_{j_2} | j_k = 1,2,...,n_k, k = 1,2 \right\} \end{pmatrix}}$$

$$= \begin{pmatrix} \left\{ \sqrt{\nu'^2_{j_1} + \nu''^2_{j_2} - \nu'^2_{j_1}\nu''^2_{j_2}} / q'_{j_1} q''_{j_2} | j_k = 1,2,...,n_k, k = 1,2 \right\}, \\ \left\{ \mu'_{i_1}\mu''_{i_2} / p'_{i_1} p''_{i_2} | i_k = 1,2,...,m_k, k = 1,2 \right\} \end{pmatrix}$$

$$= \left(\left\{ \nu'_{j_1} / q'_{j_1} | j_1 = 1,2,...,n_1 \right\}, \left\{ \mu'_{i_1} / p'_{i_1} | i_1 = 1,2,...,m_1 \right\} \right)$$
$$\oplus \left(\left\{ \nu''_{j_2} / q''_{j_2} | j_2 = 1,2,...,n_2 \right\}, \left\{ \mu''_{i_2} / p''_{i_2} | i_2 = 1,2,...,m_2 \right\} \right)$$
$$= \bar{\alpha}_1 \oplus \bar{\alpha}_2.$$

(3) When $\lambda = 0$, it is clearly true. When $\lambda > 0$, there are

$$\overline{\lambda\alpha} = \overline{\left(\left\{ \sqrt{1 - \left(1 - \mu_i^2\right)^{\lambda}} / p_i | i = 1,2,...,m \right\}, \left\{ \nu_j^{\lambda} / q_j | j = 1,2,...,n \right\} \right)}$$
$$= \left(\left\{ \nu_j^{\lambda} / q_j | j = 1,2,...,n \right\}, \left\{ \sqrt{1 - \left(1 - \mu_i^2\right)^{\lambda}} / p_i | i = 1,2,...,m \right\} \right)$$
$$= \left(\left\{ \nu_j / q_j | j = 1,2,...,n \right\}, \{ \mu_i / p_i | i = 1,2,...,m \} \right)^{\lambda}$$
$$= \bar{\alpha}^{\lambda}.$$

(4) When $\lambda = 0$, it is clearly true. When $\lambda > 0$, there are

$$\overline{\alpha^{\lambda}} = \overline{\left(\left\{ \mu_i^{\lambda} / p_i | i = 1,2,...,m \right\}, \left\{ \sqrt{1 - \left(1 - \nu_j^2\right)^{\lambda}} \nu_j^{\lambda} / q_j | j = 1,2,...,n \right\} \right)}$$
$$= \left(\left\{ \sqrt{1 - \left(1 - \nu_j^2\right)^{\lambda}} \nu_j^{\lambda} / q_j | j = 1,2,...,n \right\}, \left\{ \mu_i^{\lambda} / p_i | i = 1,2,...,m \right\} \right)$$
$$= \lambda \left(\left\{ \nu_j / q_j | j = 1,2,...,n \right\}, \{ \mu_i / p_i | i = 1,2,...,m \} \right)$$
$$= \lambda\bar{\alpha}.$$

(5) When $\lambda_1 = 0$ or $\lambda_2 = 0$, it is clearly true. When $\lambda_1 > 0$ and $\lambda_2 > 0$, there are

$$\begin{aligned}
\lambda_1\lambda_2\alpha &= \left(\left\{\sqrt{1-\left(1-\mu_i^2\right)^{\lambda_1\lambda_2}}/p_i|i=1,2,...,m\right\},\left\{\nu_j^{\lambda_1\lambda_2}/q_j|j=1,2,...,n\right\}\right)\\
&= \left(\left\{\sqrt{1-\left(\left(1-\mu_i^2\right)^{\lambda_2}\right)^{\lambda_1}}/p_i|i=1,2,...,m\right\},\left\{\left(\nu_j^{\lambda_2}\right)^{\lambda_1}/q_j|j=1,2,...,n\right\}\right)\\
&= \lambda_1\left(\left\{\sqrt{1-\left(1-\mu_i^2\right)^{\lambda_2}}/p_i|i=1,2,...,m\right\},\left\{\nu_j^{\lambda_2}/q_j|j=1,2,...,n\right\}\right)\\
&= \lambda_1\left(\lambda_2\left(\{\mu_i/p_i|i=1,2,...,m\},\left\{\nu_j/q_j|j=1,2,...,n\right\}\right)\right)\\
&= \lambda_1\left(\lambda_2\alpha\right).
\end{aligned}$$

Similarly, it can be proven that $\lambda_1\lambda_2\alpha = \lambda_2\left(\lambda_1\alpha\right)$.
(6) When $\lambda_1 = 0$ or $\lambda_2 = 0$, it is clearly true. When $\lambda_1 > 0$ and $\lambda_2 > 0$, there are

$$\begin{aligned}
\alpha^{\lambda_1\lambda_2} &= \left(\left\{\mu_i^{\lambda_1\lambda_2}/p_i|i=1,2,...,m\right\},\left\{\sqrt{1-\left(1-\nu_j^2\right)^{\lambda_1\lambda_2}}/q_j|j=1,2,...,n\right\}\right)\\
&= \left(\left\{\left(\mu_i^{\lambda_1}\right)^{\lambda_2}/p_i|i=1,2,...,m\right\},\left\{\sqrt{1-\left(\left(1-\nu_j^2\right)^{\lambda_1}\right)^{\lambda_2}}/q_j|j=1,2,...,n\right\}\right)\\
&= \left(\left\{\mu_i^{\lambda_1}/p_i|i=1,2,...,m\right\},\left\{\sqrt{1-\left(1-\nu_j^2\right)^{\lambda_1}}/q_j|j=1,2,...,n\right\}\right)^{\lambda_2}\\
&= \left(\left(\{\mu_i/p_i|i=1,2,...,m\},\left\{\nu_j/q_j|j=1,2,...,n\right\}\right)^{\lambda_1}\right)^{\lambda_2}\\
&= \left(\alpha^{\lambda_1}\right)^{\lambda_2}.
\end{aligned}$$

Similarly, it can be proven that $\alpha^{\lambda_1\lambda_2} = \left(\alpha^{\lambda_2}\right)^{\lambda_1}$. ■
According to Theorem 2, the following generalizations can be obtained.
Corollary 1 For $\alpha_t \in$ PrHPyFE(X) and $\lambda_t \geq 0$, $t = 1, 2, ..., l$, there are

$$\begin{aligned}
\overline{\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_l} &= \bar{\alpha}_1 \otimes \bar{\alpha}_2 \otimes \cdots \otimes \bar{\alpha}_l,\\
\overline{\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_l} &= \bar{\alpha}_1 \oplus \bar{\alpha}_2 \oplus \cdots \oplus \bar{\alpha}_l,\\
\overline{\lambda_1\alpha_1 \oplus \lambda_2\alpha_2 \oplus \cdots \oplus \lambda_l\alpha_l} &= \bar{\alpha}_1^{\lambda_1} \otimes \bar{\alpha}_2^{\lambda_2} \otimes \cdots \otimes \bar{\alpha}_l^{\lambda_l},\\
\overline{\alpha_1^{\lambda_1} \otimes \alpha_2^{\lambda_2} \otimes \cdots \otimes \alpha_l^{\lambda_l}} &= \lambda_1\bar{\alpha}_1 \oplus \lambda_2\bar{\alpha}_2 \oplus \cdots \oplus \lambda_l\bar{\alpha}_l.
\end{aligned}$$

It should be noted that due to the formal complexity of PrHPyFE, some commonly used operation laws no longer apply to its arithmetic operations. Please refer to the following example.

Example 1 Let $\alpha_1 = (\{0.7/0.6, 0.6/0.4\}, \{0.4/0.7, 0.5/0.3\})$, $\alpha_2 = (\{0.8/0.5, 0.7/0.5\}, \{0.3/1\})$, $\alpha_3 = (\{0.7/1\}, \{0.4/1\})$, $\alpha = (\{0.7/0.5, 0.6/0.5\}, \{0.2/0.4, 0.3/0.6\})$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda = 3$. It can be calculated that

$$\begin{cases} (\alpha_1 \oplus \alpha_2) \otimes \alpha_3 = \begin{pmatrix} \{0.6325/0.3, 0.6141/0.2, 0.6021/0.3, 0.5745/0.2\}, \\ \{0.4148/0.7, 0.4230/0.3\} \end{pmatrix}, \\ (\alpha_1 \otimes \alpha_3) \oplus (\alpha_2 \otimes \alpha_3) = \begin{pmatrix} \{0.6917/0.3, 0.6593/0.2, 0.6500/0.3, 0.6117/0.2\}, \\ \{0.2634/0.7, 0.2952/0.3\} \end{pmatrix}; \end{cases}$$

$$\begin{cases} (\alpha_1 \otimes \alpha_2) \oplus \alpha_3 = \begin{pmatrix} \{0.8062/0.3, 0.7826/0.3, 0.7794/0.2, 0.7616/0.2\}, \\ \{0.1942/0.7, 0.2254/0.3\} \end{pmatrix}, \\ (\alpha_1 \oplus \alpha_3) \otimes (\alpha_2 \oplus \alpha_3) = \begin{pmatrix} \{0.7772/0.3, 0.7416/0.2, 0.7399/0.3, 0.7060/0.2\}, \\ \{0.1991/0.7, 0.2320/0.3\} \end{pmatrix}; \end{cases}$$

$$\begin{cases} (\lambda_1 + \lambda_2)\, \alpha = (\{0.9313/0.5, 0.8590/0.5\}, \{0.0080/0.4, 0.0270/0.6\}), \\ \lambda_1 \alpha \oplus \lambda_2 \alpha = \begin{pmatrix} \{0.9313/0.25, 0.9130/0.25, 0.8894/0.25, 0.8590/0.25\}, \\ \{0.0080/0.16, 0.0120/0.24, 0.0180/0.24, 0.0270/0.36\} \end{pmatrix}; \end{cases}$$

$$\begin{cases} \alpha^{\lambda_1+\lambda_2} = (\{0.3430/0.5, 0.2160/0.5\}, \{0.3395/0.4, 0.4964/0.6\}), \\ \alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \begin{pmatrix} \{0.3430/0.25, 0.2940/0.25, 0.2520/0.25, 0.2160/0.25\}, \\ \{0.3395/0.16, 0.4017/0.24, 0.4528/0.24, 0.4964/0.36\} \end{pmatrix}; \end{cases}$$

$$\begin{cases} \lambda\,(\alpha_1 \otimes \alpha_2) = (\{0.8226/0.3, 0.7491/0.3, 0.7377/0.2, 0.6643/0.2\}, \{0.1144/0.7, 0.1789/0.3\}), \\ \lambda \alpha_1 \otimes \alpha_2 = (\{0.7451/0.3, 0.6872/0.2, 0.6519/0.3, 0.6013/0.2\}, \{0.3061/0.7, 0.3228/0.3\}), \\ \alpha_1 \otimes \lambda \alpha_2 = (\{0.6835/0.3, 0.6519/0.3, 0.5858/0.2, 0.5588/0.2\}, \{0.4008/0.7, 0.5005/0.3\}). \end{cases}$$

According to the above calculation results, it can be concluded that

$$(\alpha_1 \oplus \alpha_2) \otimes \alpha_3 \not\equiv (\alpha_1 \otimes \alpha_3) \oplus (\alpha_2 \otimes \alpha_3),$$

$$(\alpha_1 \otimes \alpha_2) \oplus \alpha_3 \not\equiv (\alpha_1 \oplus \alpha_3) \otimes (\alpha_2 \oplus \alpha_3),$$

$$(\lambda_1 + \lambda_2)\, \alpha \not\equiv \lambda_1 \alpha \oplus \lambda_2 \alpha,$$

$$\alpha^{\lambda_1+\lambda_2} \not\equiv \alpha^{\lambda_1} \otimes \alpha^{\lambda_2},$$

$$\lambda\,(\alpha_1 \otimes \alpha_2) \not\equiv \lambda \alpha_1 \otimes \alpha_2 \not\equiv \alpha_1 \otimes \lambda \alpha_2.$$

Remark 1 The empirical analysis in Example 1 reveals that PrHPyFE's inherent formal complexity, combined with the non-standard arithmetic operations introduced by Definition 2, leads to the breakdown of certain conventional operational laws (e.g., the distributive law) under specific conditions. The direct application of unverified operational laws in prior work (e.g., [12]) risks introducing logical inconsistencies in derivations. Furthermore, as these arithmetic operations form the foundation of aggregation operators, the failure of these operational laws may compromise fundamental operator properties, including idempotency and boundedness.

4. Generalized aggregation operators

In this section, we will discuss some generalized aggregation operators of PrHPyFEs. Due to the fact that some definitions of aggregation operators involve the sorting of PrHPyFEs, it is necessary to first provide a score function and an accuracy function, as well as a comparison rule based on them for PrHPyFEs.

Definition 3 For $\alpha = \left(\{\mu_i/p_i | i = 1, 2, ..., m\}, \left\{\nu_j/q_j | j = 1, 2, ..., n\right\}\right) \in$ PrHPyFE(X), the score function $S(\alpha)$ and accuracy function $H(\alpha)$ of α are defined as follows:

$$S(\alpha) = \sum_{i=1}^{m} \mu_i^2 p_i - \sum_{j=1}^{n} \nu_j^2 q_j, \tag{4.1}$$

$$H(\alpha) = \sum_{i=1}^{m} \mu_i^2 p_i + \sum_{j=1}^{n} \nu_j^2 q_j. \tag{4.2}$$

Notation 1 For α_1, $\alpha_2 \in$ PrHPyFE(X), $\alpha_1 \succ \alpha_2$ represents that α_1 is stronger than α_2; $\alpha_1 \prec \alpha_2$ represents that α_1 is weaker than α_2; $\alpha_1 \sim \alpha_2$ represents that α_1 is equivalent to α_2.

Comparison rule 1 For α_1, $\alpha_2 \in$ PrHPyFE(X), let $S(\alpha_1)$ and $S(\alpha_2)$ be score values, and $H(\alpha_1)$ and $H(\alpha_2)$ are accuracy values. Then

(1) if $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 \succ \alpha_2$;
(2) if $S(\alpha_1) < S(\alpha_2)$, then $\alpha_1 \prec \alpha_2$;
(3) if $S(\alpha_1) = S(\alpha_2)$ and
  1) $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 \succ \alpha_2$;
  2) $H(\alpha_1) < H(\alpha_2)$, then $\alpha_1 \prec \alpha_2$;
  3) $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 \sim \alpha_2$.

Definition 4 For $\alpha_t \in$ PrHPyFE(X), $t = 1, 2, ..., l$, the generalized probabilistic hesitant Pythagorean fuzzy weighted averaging (GPrHPyFWA) operator is defined as follows:

$$\text{GPrHPyFWA}_w^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) = \left(\mathop{\oplus}_{t=1}^{l} w_t \alpha_t^{\kappa}\right)^{1/\kappa} = (w_1\alpha_1^{\kappa} \oplus w_2\alpha_2^{\kappa} \oplus \cdots \oplus w_l\alpha_l^{\kappa})^{1/\kappa}, \tag{4.3}$$

where κ is the generalized aggregation operator parameter that satisfies $\kappa > 0$; $w = (w_1, w_2, ..., w_l)$ and w_t is the weight of α_t that satisfies $w_t \in [0, 1]$ and $\sum_{t=1}^{l} w_t = 1$.

Definition 5 For $\alpha_t \in$ PrHPyFE(X), $t = 1, 2, ..., l$, the generalized probabilistic hesitant Pythagorean fuzzy weighted geometric (GPrHPyFWG) operator is defined as follows:

$$\text{GPrHPyFWG}_w^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) = \frac{1}{\kappa} \mathop{\otimes}_{t=1}^{l} (\kappa\alpha_t)^{w_t} = \frac{1}{\kappa} \left((\kappa\alpha_1)^{w_1} \otimes (\kappa\alpha_2)^{w_2} \otimes \cdots \otimes (\kappa\alpha_l)^{w_l}\right), \tag{4.4}$$

where κ is the generalized aggregation operator parameter that satisfies $\kappa > 0$; $w = (w_1, w_2, ..., w_l)$ and w_t is the weight of α_t that satisfies $w_t \in [0, 1]$ and $\sum_{t=1}^{l} w_t = 1$.

Definition 6 For $\alpha_t \in$ PrHPyFE(X), $t = 1, 2, ..., l$, the generalized probabilistic hesitant Pythagorean fuzzy hybrid averaging (GPrHPyFHA) operator is defined as follows:

$$\text{GPrHPyFHA}_{w,\omega}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) = \left(\mathop{\oplus}_{t=1}^{l} \omega_t \beta_{\tau(t)}^{\kappa}\right)^{1/\kappa} = \left(\omega_1\beta_{\tau(1)}^{\kappa} \oplus \omega_2\beta_{\tau(2)}^{\kappa} \oplus \cdots \oplus \omega_l\beta_{\tau(l)}^{\kappa}\right)^{1/\kappa}, \tag{4.5}$$

where κ is the generalized aggregation operator parameter that satisfies $\kappa > 0$; $w = (w_1, w_2, ..., w_l)$ and w_t is the weight of α_t that satisfies $w_t \in [0, 1]$ and $\sum_{t=1}^{l} w_t = 1$; $\omega = (\omega_1, \omega_2, ..., \omega_l)$ and ω_t is the weight of the t-th position that satisfies $\omega_t \in [0, 1]$ and $\sum_{t=1}^{l} \omega_t = 1$; $\beta_t = lw_t\alpha_t$ and $\beta_{\tau(t)}$ is the t-th strongest PrHPyFE in $\{\beta_1, \beta_2, ..., \beta_l\}$.

Definition 7 For $\alpha_t \in$ PrHPyFE(X), $t = 1, 2, ..., l$, the generalized probabilistic hesitant Pythagorean fuzzy hybrid geometric (GPrHPyFHG) operator is defined as follows:

$$\mathrm{GPrHPyFHG}_{w,\omega}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) = \frac{1}{\kappa} \overset{l}{\underset{t=1}{\otimes}} (\kappa\gamma_{\tau(t)})^{\omega_t} = \frac{1}{\kappa}\left((\kappa\gamma_{\tau(1)})^{\omega_1} \otimes (\kappa\gamma_{\tau(2)})^{\omega_2} \otimes \cdots \otimes (\kappa\gamma_{\tau(l)})^{\omega_l}\right), \tag{4.6}$$

where κ is the generalized aggregation operator parameter that satisfies $\kappa > 0$; $w = (w_1, w_2, ..., w_l)$ and w_t is the weight of α_t that satisfies $w_t \in [0, 1]$ and $\sum_{t=1}^{l} w_t = 1$; $\omega = (\omega_1, \omega_2, ..., \omega_l)$ and ω_t is the weight of the t-th position that satisfies $\omega_t \in [0, 1]$ and $\sum_{t=1}^{l} \omega_t = 1$; $\gamma_t = \alpha_t^{lw_t}$ and $\gamma_{\tau(t)}$ is the t-th strongest PrHPyFE in $\{\gamma_1, \gamma_2, ..., \gamma_l\}$.

Theorem 3 For $\alpha_t \in$ PrHPyFE(X), $t = 1, 2, ..., l$, $\kappa > 0$, $w_t \in [0, 1]$, $\omega_t \in [0, 1]$, $\sum_{t=1}^{l} w_t = 1$, and $\sum_{t=1}^{l} \omega_t = 1$, there are

(1) $\mathrm{GPrHPyFWA}_{w}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) \in$ PrHPyFE(X);

(2) $\mathrm{GPrHPyFWG}_{w}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) \in$ PrHPyFE(X);

(3) $\mathrm{GPrHPyFHA}_{w,\omega}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) \in$ PrHPyFE(X);

(4) $\mathrm{GPrHPyFHG}_{w,\omega}^{\kappa}(\alpha_1, \alpha_2, ..., \alpha_l) \in$ PrHPyFE(X).

Proof: According to the construction methods outlined in Definitions 4, 5, 6, and 7, the generalized aggregation operators GPrHPyFWA, GPrHPyFWG, GPrHPyFHA, and GPrHPyFHG are all constructed through the arithmetic operations of PrHPyFE as defined in Definition 2. Given the closure property of PrHPyFE under arithmetic operations (as established in Theorem 1), it can be concluded that the operation results of these operators are all PrHPyFEs. ■

Existing research on PrHPyFE aggregation operators has proposed multiple implementation approaches, including probabilistic hesitate Pythagorean fuzzy weighted averaging (PrHPyFWA) operator, probabilistic hesitate Pythagorean fuzzy weighted geometric (PrHPyFWG) operator, probabilistic hesitant Pythagorean fuzzy hybrid averaging (PrHPyFHA) operator, and probabilistic hesitant Pythagorean fuzzy hybrid geometric (PrHPyFHG) operator developed by Batool et al. [18, 19]; probabilistic dual-hesitant Pythagorean fuzzy power Hamy mean operator and probabilistic dual-hesitant Pythagorean fuzzy power weighted Hamy mean operator designed by Ji et al. [11], and Pythagorean probabilistic hesitant fuzzy Einstein weighted averaging operator, Pythagorean probabilistic hesitant fuzzy Einstein ordered weighted averaging operator, Pythagorean probabilistic hesitant fuzzy Einstein weighted geometric operator, and Pythagorean probabilistic hesitant fuzzy Einstein ordered weighted geometric operator constructed by Ashraf et al. [13]. Notably, the operators proposed in this study exhibit formal similarity with the PrHPyFWA, PrHPyFWG, PrHPyFHA, and PrHPyFHG operators introduced by Batool et al. [18, 19], as shown in Table 1.

Table 1. The PrHPyFE aggregation operators introduced by Batool et al. [18, 19].

Operators	Forms
PrHPyFWA	$\text{PrHPyFWA}(\alpha_1, \alpha_2, ..., \alpha_l) = w_1\alpha_1 \oplus w_2\alpha_2 \oplus \cdots \oplus w_l\alpha_l$
PrHPyFWG	$\text{PrHPyFWG}(\alpha_1, \alpha_2, ..., \alpha_l) = \alpha_1^{w_1} \otimes \alpha_2^{w_2} \otimes \cdots \otimes \alpha_l^{w_l}$
PrHPyFHA	$\text{PrHPyFHA}(\alpha_1, \alpha_2, ..., \alpha_l) = \omega_1\beta_{\tau(1)} \oplus \omega_2\beta_{\tau(2)} \oplus \cdots \oplus \omega_l\beta_{\tau(l)}$, where $\beta_t = lw_t\alpha_t$ $(t = 1, 2, ..., l)$ and $\beta_{\tau(t)}$ is the t-th strongest PrHPyFE in $\{\beta_1, \beta_2, ..., \beta_l\}$.
PrHPyFHG	$\text{PrHPyFHG}(\alpha_1, \alpha_2, ..., \alpha_l) = \gamma_{\tau(1)}^{\omega_1} \otimes \gamma_{\tau(2)}^{\omega_2} \otimes \cdots \otimes \gamma_{\tau(l)}^{\omega_l}$, where $\gamma_t = \alpha_t^{lw_t}$ $(t = 1, 2, ..., l)$ and $\gamma_{\tau(t)}$ is the t-th strongest PrHPyFE in $\{\gamma_1, \gamma_2, ..., \gamma_l\}$.

Note: As demonstrated in Table 1, operators PrHPyFHA and PrHPyFHG rely on the sorting of sets $\{\beta_1, \beta_2, ..., \beta_l\}$ and $\{\gamma_1, \gamma_2, ..., \gamma_l\}$, respectively. The sorting indicator functions employed by these operators differ from those proposed in this study, specifically in the score function $s(\alpha) = \left(\frac{1}{m}\sum_{i=1}^{m}\mu_i p_i\right)^2 - \left(\frac{1}{n}\sum_{j=1}^{n}\nu_j q_j\right)^2$ and accuracy function $h(\alpha) = \left(\frac{1}{m}\sum_{i=1}^{m}\mu_i p_i\right)^2 + \left(\frac{1}{n}\sum_{j=1}^{n}\nu_j q_j\right)^2$, for $\alpha = \left(\{\mu_i/p_i | i = 1, 2, ..., m\}, \left\{\nu_j/q_j | j = 1, 2, ..., n\right\}\right) \in \text{PrHPyFE}(X)$. However, the sorting rule adopted by both operators are fully consistent with comparison rule 1 established in this research.

Through comparative analysis, it is evident that the generalized aggregation operator system established in Definitions 4–7 constitutes a rigorous extension of the aggregation operators presented in Table 1. Specifically, when the parameter κ of the generalized aggregation operators GPrHPyFWA, GPrHPyFWG, GPrHPyFHA, and GPrHPyFHG is set to 1, they exactly reduce to their corresponding counterparts PrHPyFWA, PrHPyFWG, PrHPyFHA, and PrHPyFHG, respectively. Notably, this parameterized design bestows upon the system substantial operational flexibility: in complex information processing scenarios, by dynamically adjusting the parameter κ, optimal operator configurations can be selected based on specific application requirements, thereby enabling scenario-specific optimization of information processing adaptability and computational accuracy.

5. Application in procurement of teaching equipment in universities

Nowadays, teaching based on information technology is becoming increasingly popular in universities. With the continuous improvement of the utilization rate of modern teaching equipment, they play an increasingly important role in improving teaching effectiveness. In this situation, how to choose teaching equipment with good comprehensive performance has become a primary consideration in equipment procurement.

A university wants to purchase a batch of equipment for information-based teaching. Through open bidding, products $\{A_1, A_2, ..., A_H\}$ from H companies have been identified as candidates. In order to comprehensively consider the opinions of all relevant personnel, the procurement department organized an expert group consisting of frontline teachers, experimental teachers, laboratory managers, logistics support personnel, and financial managers to evaluate these candidate products. The expert group will evaluate the candidate products based on L criteria $\{C_1, C_2, ..., C_L\}$ with an evaluation weight vector of $w = (w_1, w_2, ..., w_L)$ that satisfies $w_l \in (0, 1]$, $l = 1, 2, ..., L$, and $\sum_{l=1}^{L} w_l = 1$. Suppose that PrHPyFE α_{hl} is the evaluation value given by the expert group for product A_h subject to criterion C_l, where $h = 1, 2, ..., H$ and $l = 1, 2, ..., L$. Equipment procurement decision-makers need to determine which equipment is the most optimal to choose.

In order to select the comprehensive optimal product, the equipment procurement department needs to develop a reasonable and effective MCDM method. This problem can be solved using the generalized aggregation operators and comparison rule proposed in Section 4. The main operational steps and executive process are shown in Table 2 and Figure 1, respectively.

Table 2. The main operational steps of the proposed MCDM method.

Steps	Operating
1	Convert the cost criterion values into the benefit criterion values: if C_l is a cost criterion, then set $\alpha_{hl} = \bar{\alpha}_{hl}$, $l = 1, 2, ..., L$, $h = 1, 2, ..., H$.
2	Calculate aggregation value $\alpha_h = \text{GPrHPyFWA}_w^\kappa (\alpha_{h1}, \alpha_{h2}, ..., \alpha_{hL})$ ($\alpha_h = \text{GPrHPyFWG}_w^\kappa (\alpha_{h1}, \alpha_{h2}, ..., \alpha_{hL})$, $\alpha_h = \text{GPrHPyFHA}_{w,\omega}^\kappa (\alpha_{h1}, \alpha_{h2}, ..., \alpha_{hL})$, $\alpha_h = \text{GPrHPyFHG}_{w,\omega}^\kappa (\alpha_{h1}, \alpha_{h2}, ..., \alpha_{hL})$), $h = 1, 2, ..., H$.
3	Calculate score value $S(\alpha_h)$ and accuracy value $H(\alpha_h)$, $h = 1, 2, ..., H$.
4	Sort $\alpha_1, \alpha_2, ..., \alpha_H$ according to Comparison rule 1.
5	Provide the comprehensive optimal product.

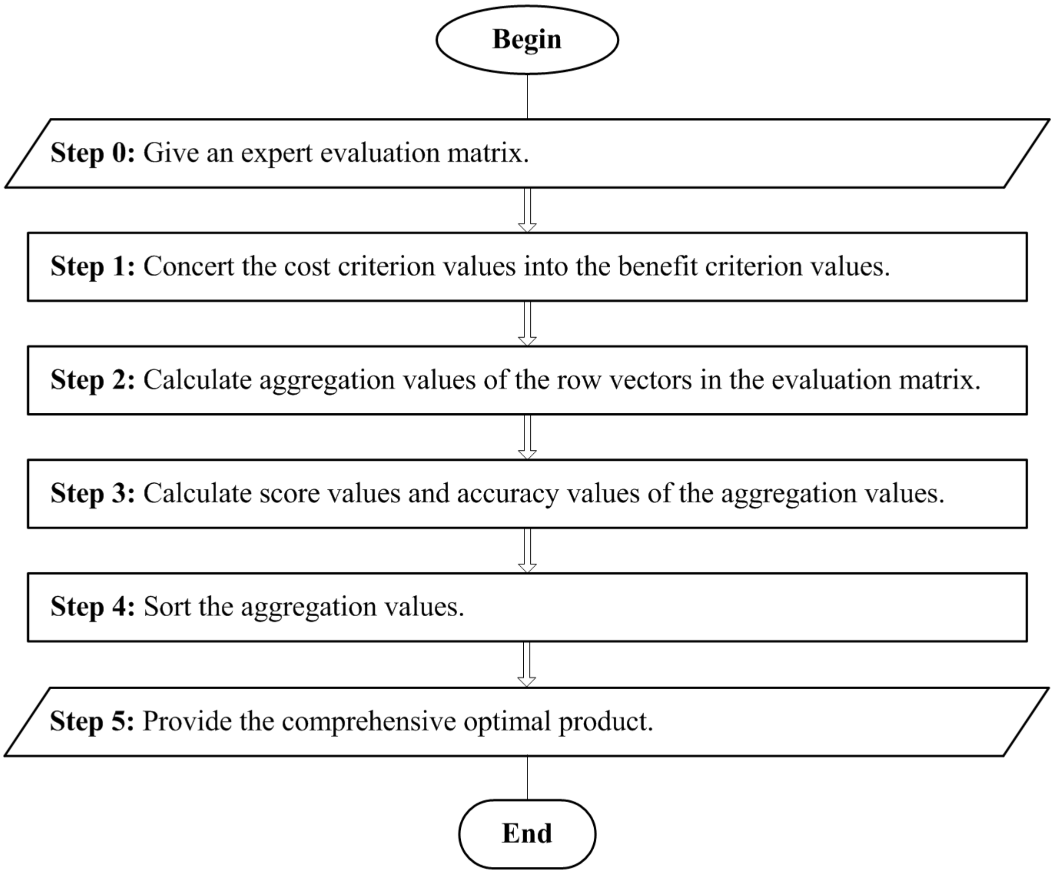

Figure 1. Executive process of the proposed MCDM method.

As shown in Table 2 and Figure 1, the proposed MCDM method based on PrHPyFE generalized aggregation operators is systematic. Next, the effectiveness of the method will be tested through a numerical example.

Example 2 Five products $\{A_1, A_2, A_3, A_4, A_5\}$ participated in the bidding process. Evaluation will utilize four criteria: easy operation (C_1), high quality (C_2), low maintenance cost (C_3), and good after-sales service (C_4), with $w = (0.35, 0.35, 0.2, 0.1)$ representing the criteria weight vector.

In multi-expert decision-making scenarios, achieving complete consensus on the same evaluation object is often challenging due to variations in experts' professional backgrounds, cognitive frameworks, and subjective preferences. To more comprehensively and objectively represent group assessment information, PrHPyFS/PrHPyFE can be utilized for modeling.

Consider an expert group of ten members evaluating product performance across multiple criteria from two dimensions: "degree of conformity" and "degree of non-conformity". For instance, regarding product A_1 under criterion C_1, the expert opinions are distributed as follows: There are six experts who assign a conformity degree of 0.7 and a non-conformity degree of 0.2; the remaining four experts assign 0.6 and 0.2, respectively. The aggregated group opinion is expressed as $\alpha_{11} = (\{0.7/0.6, 0.6/0.4\}, \{0.2/1\})$, where $\{0.7/0.6, 0.6/0.4\}$ indicates a probability of 0.6 for the conformity degree being 0.7 and a probability of 0.4 for the conformity degree being 0.6; $\{0.2/1\}$ reflects unanimous agreement (a probability of 1) among experts that the non-conformity degree is 0.2. This representation effectively captures the distributional characteristics of expert opinions, balancing uncertainty and consensus in the assessment.

Similarly, other evaluation values in the form of PrHPyFE can be obtained, as shown in Table 3.

Table 3. Evaluation values given by the expert group.

	Criteria			
Products	C_1	C_2	C_3	C_4
A_1	$\left(\begin{array}{l}\{0.7/0.6, 0.6/0.4\},\\ \{0.2/1\}\end{array}\right)$	$\left(\begin{array}{l}\{0.7/1\},\\ \{0.3/0.2, 0.4/0.8\}\end{array}\right)$	$(\{0.4/1\}, \{0.5/1\})$	$\left(\begin{array}{l}\{0.8/0.6, 0.7/0.4\},\\ \{0.2/0.7, 0.3/0.3\}\end{array}\right)$
A_2	$\left(\begin{array}{l}\{0.8/0.7, 0.7/0.3\},\\ \{0.3/1\}\end{array}\right)$	$\left(\begin{array}{l}\{0.7/0.7, 0.6/0.3\},\\ \{0.6/1\}\end{array}\right)$	$\left(\begin{array}{l}\{0.6/1\},\\ \{0.5/0.7, 0.6/0.3\}\end{array}\right)$	$(\{0.7/1\}, \{0.4/1\})$
A_3	$(\{0.6/1\}, \{0.5/1\})$	$\left(\begin{array}{l}\{0.7/1\},\\ \{0.3/0.3, 0.4/0.7\}\end{array}\right)$	$\left(\begin{array}{l}\{0.8/0.3, 0.7/0.7\},\\ \{0.3/1\}\end{array}\right)$	$\left(\begin{array}{l}\{0.7/1\},\\ \{0.4/0.8, 0.5/0.2\}\end{array}\right)$
A_4	$\left(\begin{array}{l}\{0.7/1\},\\ \{0.2/0.8, 0.3/0.2\}\end{array}\right)$	$(\{0.8/1\}, \{0.2/1\})$	$\left(\begin{array}{l}\{0.6/1\},\\ \{0.4/0.7, 0.5/0.3\}\end{array}\right)$	$\left(\begin{array}{l}\{0.6/1\},\\ \{0.2/0.5, 0.3/0.5\}\end{array}\right)$
A_5	$(\{0.6/1\}, \{0.2/1\})$	$\left(\begin{array}{l}\{0.8/0.5, 0.7/0.5\},\\ \{0.3/1\}\end{array}\right)$	$(\{0.8/1\}, \{0.3/1\})$	$(\{0.7/1\}, \{0.3/1\})$

Notes: (1) In order to compare and analyze the impact of parameter values on decision-making results, different values of the generalized aggregation operator parameter κ are used in experiments. (2) For GPrHPyFHA and GPrHPyFHG operators, the normal distribution weighting method [22] is used to determine the position weight vector as $\omega = (0.1550, 0.3450, 0.3450, 0.1550)$.

According to the decision-making process in Table 2 and Fig. 1, the implementation steps are given as follows:

Step 1: There is no need to perform inverse operations on the evaluation values, as all criteria are beneficial.

Step 2: Use the generalized aggregation operators proposed in Section 4 to perform aggregation operations on each row of PrHPyFEs in Table 3.

Step 3: Calculate the score values and accuracy values for the aggregation results according to Equations (4.1) and (4.2), respectively. The results are shown in column 3 of Table 4.

Step 4: Sort the aggregation results according to Comparison rule 1. The results are shown in column 4 of Table 4.

Step 5: Provide the optimal choice. See column 5 of Table 4.

Table 4. Relevant data and results in the decision-making process.

Aggregation operators	Parameter values	Score values and accuracy values $\begin{cases} S = (S(\alpha_1), S(\alpha_2), S(\alpha_3), S(\alpha_4), S(\alpha_5)) \\ H = (H(\alpha_1), H(\alpha_2), H(\alpha_3), H(\alpha_4), H(\alpha_5)) \end{cases}$	Sorting results	The optimal choice
GPrHPyFWA	$\kappa = 1$	$\begin{cases} S = (0.3372, 0.3027, 0.2999, 0.4564, 0.4468) \\ H = (0.5224, 0.6913, 0.6180, 0.5770, 0.5823) \end{cases}$	$\alpha_4 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$	A_4
GPrHPyFWA	$\kappa = 2$	$\begin{cases} S = (0.3532, 0.3139, 0.3054, 0.4651, 0.4553) \\ H = (0.5354, 0.6929, 0.6209, 0.5846, 0.5905) \end{cases}$	$\alpha_4 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2 \succ \alpha_3$	A_4
GPrHPyFWG	$\kappa = 1$	$\begin{cases} S = (0.2574, 0.2378, 0.2726, 0.4130, 0.4139) \\ H = (0.5062, 0.7165, 0.6216, 0.5680, 0.5595) \end{cases}$	$\alpha_5 \succ \alpha_4 \succ \alpha_3 \succ \alpha_1 \succ \alpha_2$	A_5
GPrHPyFWG	$\kappa = 2$	$\begin{cases} S = (0.2237, 0.2090, 0.2590, 0.3863, 0.4015) \\ H = (0.5159, 0.7333, 0.6284, 0.5782, 0.5542) \end{cases}$	$\alpha_5 \succ \alpha_4 \succ \alpha_3 \succ \alpha_1 \succ \alpha_2$	A_5
GPrHPyFHA	$\kappa = 1$	$\begin{cases} S = (0.3565, 0.2528, 0.3073, 0.4417, 0.4589) \\ H = (0.5115, 0.7077, 0.6138, 0.5629, 0.5700) \end{cases}$	$\alpha_5 \succ \alpha_4 \succ \alpha_1 \succ \alpha_3 \succ \alpha_2$	A_5
GPrHPyFHA	$\kappa = 2$	$\begin{cases} S = (0.3846, 0.2855, 0.3195, 0.4736, 0.4724) \\ H = (0.5312, 0.7210, 0.6189, 0.5889, 0.5794) \end{cases}$	$\alpha_4 \succ \alpha_5 \succ \alpha_1 \succ \alpha_3 \succ \alpha_2$	A_4
GPrHPyFHG	$\kappa = 1$	$\begin{cases} S = (0.2386, 0.2532, 0.2973, 0.4005, 0.4305) \\ H = (0.4916, 0.6886, 0.6152, 0.5441, 0.5959) \end{cases}$	$\alpha_5 \succ \alpha_4 \succ \alpha_3 \succ \alpha_2 \succ \alpha_1$	A_5
GPrHPyFHG	$\kappa = 2$	$\begin{cases} S = (0.2057, 0.2128, 0.2558, 0.3837, 0.3999) \\ H = (0.5000, 0.7094, 0.6203, 0.5479, 0.5766) \end{cases}$	$\alpha_5 \succ \alpha_4 \succ \alpha_3 \succ \alpha_2 \succ \alpha_1$	A_5

Due to the fact that complete sorting of alternative products can be achieved solely based on score values, an intuitive comparison among them is provided. See Figure 2.

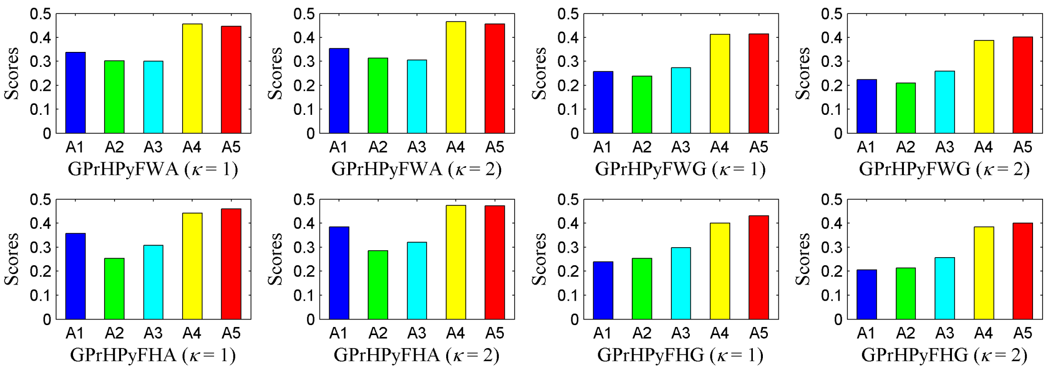

Figure 2. Score values of the aggregation results under different parameter values.

From Table 4 and Figure 2, it can be seen that the decision-making results obtained vary when different aggregation operators or different parameter values are used. Sometimes A_4 is the optimal choice, while other times A_5 is the optimal choice. In addition, the sorting results obtained also differ in the order of A_1, A_2, and A_3. Through analysis, it can be inferred that the main reasons for the differences come from three aspects. (1) The parameter selection of generalized aggregation operators has a certain impact on the decision-making results. (2) The averaging operators and geometric operators have different sensitivities to data, especially for some extreme data. (3) The actual differences between alternatives A_4 and A_5, as well as the actual differences among alternatives A_1, A_2, and A_3, are not significant. For example, from the data in Table 3, it can be roughly inferred that A_4 is slightly stronger than A_5 on criterion C_2; A_5 is slightly stronger than A_4 on criterion C_3; there are not many differences in strength between A_4 and A_5 on criteria C_1 and C_4. A similar situation exists among A_1, A_2, and A_3. Overall, the obtained sorting results and final decision-making results are within a reasonable range. In applications, suitable generalized aggregation operators and parameter values can be selected according to specific situations to obtain decision-making results that are more in line with reality.

6. Comparative analysis

This section conducts numerical computations of Example 2 using the following aggregation operators (PrHPyFWA, PrHPyFWG, PrHPyFHA, and PrHPyFHG) within the PrHPyFE framework, with a subsequent comparative analysis. The specific forms of these operators are presented in Table 1.

The detailed results obtained from the calculations are summarized in Table 5.

Table 5. Data and results corresponding to the aggregation operators for comparison.

Aggregation operators	Values of sorting indicators $\begin{cases} s = (s(\alpha_1), s(\alpha_2), s(\alpha_3), s(\alpha_4), s(\alpha_5)) \\ h = (h(\alpha_1), h(\alpha_2), h(\alpha_3), h(\alpha_4), h(\alpha_5)) \end{cases}$	Sorting results	The optimal choice
PrHPyFWA	$\begin{cases} s = (0.0211, -0.0175, 0.1048, 0.5158, 0.0608) \\ h = (0.0326, 0.0796, 0.1246, 0.5176, 0.1963) \end{cases}$	$\alpha_4 \succ \alpha_3 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2$	α_4
PrHPyFWG	$\begin{cases} s = (0.0161, -0.0300, 0.1009, 0.4893, 0.0488) \\ h = (0.0316, 0.0896, 0.1227, 0.4917, 0.1944) \end{cases}$	$\alpha_4 \succ \alpha_3 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2$	α_4
PrHPyFHA	$\begin{cases} s = (0.0222, -0.0012, 0.0938, 0.5014, 0.0716) \\ h = (0.0319, 0.0715, 0.1180, 0.5032, 0.2093) \end{cases}$	$\alpha_4 \succ \alpha_3 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2$	α_4
PrHPyFHG	$\begin{cases} s = (0.0173, -0.0250, 0.1041, 0.4875, 0.0443) \\ h = (0.0326, 0.0838, 0.1239, 0.4902, 0.1834) \end{cases}$	$\alpha_4 \succ \alpha_3 \succ \alpha_5 \succ \alpha_1 \succ \alpha_2$	α_4

Similarly, by utilizing score values, a complete sorting of the alternatives can be achieved, with the intuitive comparison among them presented in Figure 3.

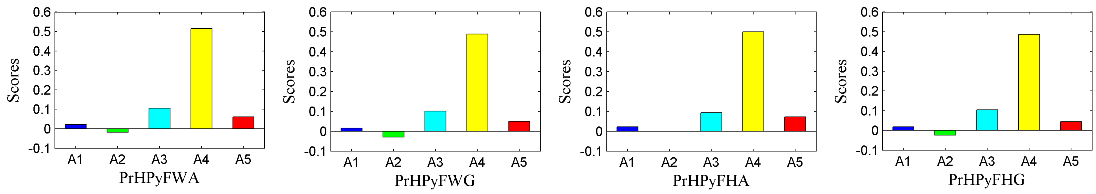

Figure 3. Score values based on some existing aggregation operations.

From Table 5 and Figure 3, it can be seen that the optimal choice obtained is A_4, which is consistent with some of the decision-making results obtained in Section 5. However, if they are compared comprehensively, there are many differences. This can be easily seen from Figures 2 and 3. (1) From Table 2, it can be seen that the difference between alternatives A_4 and A_5 is not significant, but Figure 3 shows that there is indeed a significant difference in their score values. This seems somewhat inconsistent with the actual situation. (2) According to Table 3, A_2 satisfies that the membership values are greater than or equal to the non-membership values for each criterion. From this, it can be roughly inferred that there should be a corresponding score value greater than or equal to zero. However, the calculated score is negative, which is somewhat inconsistent with the actual situation. (3) From Table 2, it can be seen that regardless of which criterion, the evaluation value of A_5 is stronger than A_3. Therefore, after aggregating information, it is common sense that the score value corresponding to A_5 should still be greater than the score value corresponding to A_3. However, opposite results are observed in Table 5 and Figure 3. The sorting results obtained in Section 5 all maintain that A_5 is stronger than A_3, which is consistent with the actual situation. The main reason for the above problems is that the score function used in this section has certain defects. That is, since probability value information is already used in the score function, there should no longer be coefficients $1/m$ and $1/n$ in the score function. Otherwise, the more elements a membership set contains, the more disadvantageous it is for the score value. Similarly, the more elements a non-membership set contains, the more advantageous the score will be.

Through the quantitative comparison of Tables 4 and 5 and the visual analysis of Figures 2 and 3, combined with systematic experimental validation, it has been found that existing decision-making models suffer from systematic bias due to information redundancy in their scoring functions, which simultaneously incorporate probability values and specific coefficients. This bias manifests in several critical issues: counterintuitive decision-making under data distribution imbalance, abnormal ranking results where theoretically superior alternatives are positioned unfavorably. In contrast, the proposed model addresses these limitations by: (1) Reconstructing the scoring function to eliminate redundant design elements, (2) establishing a data distribution calibration framework, and (3) employing parameterized operators for information aggregation. These innovations enable the model to demonstrate stable performance in imbalanced data scenarios, effectively reducing systematic bias and enhancing alignment with decision-making commonsense. The proposed approach provides a new technical pathway for improving the reliability of intelligent decision-making systems in complex data environments, particularly in addressing counterintuitive decision problems prevalent in real-world applications.

7. Conclusions

7.1. *Foundational theoretical innovations*

This research establishes three cornerstone theoretical advancements:

(1) Complete operational framework – A comprehensive arithmetic operation system for PrHPyFEs is developed, enabling precise mathematical characterization of hybrid uncertainties through rigorously defined algebraic structures.

(2) Generalized operator architecture – Four novel parameter-driven aggregation operators (GPrHPyFWA, GPrHPyFWG, GPrHPyFHA, and GPrHPyFHG) are proposed, featuring dynamic adjustment capabilities across membership degrees, non-membership degrees, and probability distributions.

(3) Integrated decision-making methodology – A unified MCDM framework is constructed, systematically bridging the gap between theoretical constructs and practical implementation scenarios through rigorous algorithmic integration and scenario-based validation.

7.2. *Empirical validation and comparative advantages*

Through systematic case studies and comparative analysis, the proposed methodology demonstrates three core advantages:

(1) Decision-making effectiveness – The framework enhances accuracy in processing probabilistic and hesitant information via dynamic adjustment mechanisms that adaptively optimize information aggregation based on uncertainty characteristics.

(2) Algorithmic superiority – It achieves improved computational efficiency and robustness compared to conventional aggregation operators.

(3) Practical applicability – It demonstrates validated effectiveness in real-world scenarios through educational equipment procurement cases.

7.3. *Methodological limitations and constraints*

The present framework exhibits three principal limitations:

(1) Scope restriction – The framework is currently confined to Pythagorean fuzzy environments, without extension to q-rung orthopair fuzzy sets (q-ROFSs) that provide superior modeling capabilities for complex uncertainty.

(2) Weighting constraint – Dependence on fixed weight vectors may introduce subjective bias, especially in high-dimensional decision spaces where expert judgments might fail to fully capture intricate interdependencies.

(3) Criterion interaction limitation – The aggregation operators assume non-interactive criteria, inherently limiting their capacity to model complex nonlinear interdependencies in real-world decision-making.

7.4. *Strategic research trajectory*

Future investigations will prioritize four critical dimensions:

(1) Cross-domain implementation: Deploying the methodology in supply chain optimization and clinical decision support systems to validate its applicability.

(2) Theoretical expansion: Extending the framework to q-ROFSs and other advanced uncertainty modeling paradigms.

(3) Collaborative decision enhancement: Investigating PrHPyFSs in MCGDM contexts, incorporating attribute interdependence modeling.

(4) Adaptive weighting systems: Developing intelligent aggregation operators for scenarios with unknown or partially known weight vectors by integrating machine learning techniques.

7.5. Concluding synthesis and forward vision

This research presents theoretical tools for intelligent decision analysis in complex uncertainty environments. Through the synergistic integration of generalized aggregation operators with the PrHPyFS theoretical framework, a computational platform is established to address contemporary decision-making challenges. The developed framework not only provides immediate solutions to current complex decision problems but also delineates structured pathways for future theoretical development and cross-disciplinary practical applications.

Author contributions

Mingxin Wang: Investigation, Conceptualization, Formal analysis, Validation, Writing – original draft, Writing – revised draft. Luping Liu: Methodology, Writing – original draft.

Acknowledgments

This work has been partially supported by the Natural Science Foundation of Guizhou Province (Grant No. MS(2025)047) and the Research Foundation of Guizhou University of Commerce (Grant No. 2024XJSDYB01).

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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