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Research article

Adaptive error feedback tracking for a wave equation with a nonlocal term and multi-channel unknown harmonic disturbances

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Abstract: This paper addressed the output regulation problem for a one-dimensional (1-D) wave equation subject to a nonlocal term and multi-channel unknown disturbances. Motivated by the combined challenge of structural instability from nonlocal coupling and the realistic presence of multi-channel harmonic disturbances with unknown frequencies, this work aimed to integrate and simultaneously address both issues to meet more complex application scenarios. The nonlocal term caused energy growth and open-loop instability, requiring sequential stabilization and output regulation. The disturbances consisted of sinusoidal signals with unknown amplitudes and frequencies, where only an upper bound on the frequencies was known. Our approach constructed an auxiliary system to eliminate the nonlocal effect and employed a coordinate transformation that concentrated disturbances into the tracking error channel. An adaptive observer was then developed for online frequency identification, enabling output-feedback control using the tracking error and its derivative. Theoretical analysis established the well-posedness and state boundedness of the closed-loop system, while numerical simulations confirmed the effectiveness of the proposed approach and demonstrated exponential convergence of the tracking error to zero.

Keywords: wave equation; output regulation; adaptive observer; disturbance rejection
Mathematics Subject Classification: 93C20

1. Introduction

The output regulation of distributed parameter systems under external disturbances has attracted considerable research attention in recent years for various types of partial differential equations (PDEs), including the heat equations [1, 2], wave equations [3–5], beam equations [6], and Schrödinger equations [7, 8], among others. Driven by rapid developments in mechanical manufacturing, marine engineering, robotics, and related fields, the control of disturbances in PDEs systems has been addressed through a variety of developed methods. For systems where reference signals and disturbances originate from a linear autonomous exosystem, foundational work on finite-dimensional systems dates back to the internal model principle of the 1970s [9, 10], with applications shown in [11, 12]. Moreover, active disturbance rejection control [13, 14], introduced in the 1980s, has found broad application in output regulation, including scenarios involving the heat equations [15], wave equations [16], and Euler-Bernoulli beam equations [17]. A common limitation in these studies, however, is that the exosystem-generated disturbance only acts through a single channel, and the reference signals are fully known. In [18], an adaptive method was employed to achieve output regulation for a one-dimensional (1-D) wave equation. Both the disturbance and the reference signal were sinusoidal with known frequencies. More recently, in the case of a 1-D wave equation under mismatched disturbances, a controller was designed in [19] using a trajectory planning approach, where the sinusoidal disturbance generated by the exosystem was also known. In contrast, the problem of output tracking under unknown frequencies remains less explored. Recent studies have begun to address this challenge. In [20], a riser-cable elevator was abstracted as a wave equation subject to an unknown-frequency disturbance, with the excitation confined to a sole system channel. In [21], a 1-D non-collocated wave equation was examined under the condition of a known exosystem while the disturbance frequency was unknown. The work in [22] leverages the adaptive internal model principle to devise a controller. Authors in [23] investigated a 1-D anti-stable wave equation along with unknown harmonic disturbances and reference trajectories. Notably, the disturbance was applied exclusively through one channel of the system.

In practical engineering systems such as offshore platform structures, large-scale flexible manipulators, vibration control of oil pipelines, and wind-induced vibration suppression in high-rise buildings, distributed parameter systems described by PDEs (e.g., wave equations) are often subject to multi-source external harmonic disturbances. The frequencies and amplitudes of these disturbances are typically unknown and time-varying. Moreover, nonlocal coupling terms in the system (such as boundary velocity recirculation) can lead to open-loop instability, further complicating the control design. The wave equation model studied in this paper can be regarded as an abstract representation of the above engineering systems. The control objective is to achieve accurate output tracking under multi-channel disturbances with unknown frequencies and structural instability induced by the nonlocal term. This problem has clear engineering relevance, including wave compensation for offshore structures, trajectory tracking of robotic arm endpoints, and pipeline vibration suppression. Therefore, the study of adaptive control strategies for such systems holds significant theoretical and practical value.

The main contributions of this paper are summarized as follows: (i) Unlike the conservative plant studied in [22], the system in this work contains a nonlocal term $\gamma y_t(0, t)$ that induces open-loop instability. This structural difference is critical: the coordinate transformation employed in [22], designed to decouple the system into an exponentially stable PDE subsystem, is not directly applicable to our unstable plant. Therefore, our control design necessarily follows a sequential procedure: We first stabilize the inherent instability induced by the nonlocal term, and then incorporate adaptive regulation to reject unknown multi-channel disturbances. This approach successfully extends the adaptive internal model framework to a class of non-conservative, unstable wave equations. (ii) A key difference concerns what is assumed about the exosystem's dimension. Authors in [24] require the exact number of distinct frequencies (four) to be known a priori, enabling a fixed-order internal model design. Our work only requires an upper bound N on this number, treating the exosystem's true order (and internal structure) as unknown. This less restrictive assumption, which is crucial when dealing with disturbances whose harmonic composition may change with operating conditions or system modifications, demands an adaptive scheme capable of online order identification, not merely parameter estimation, thereby addressing the practical limitation of fixed-order designs in evolving environments.

The following provides an overview of the paper's organization. Section 2 presents the problem statement of a 1-D wave equation with a nonlocal term and introduces the preliminary mathematical framework, establishing the groundwork for further analysis. Section 3 simplifies the studied system and designs a feedforward plus state-feedback controller for it. Section 4 recovers the system states and develops an error-based feedback regulator. Section 5 establishes the well-posedness and stability of the closed-loop system as well as the exponential convergence of the tracking error. Section 6 provides numerical simulations to validate the proposed methodology. The main findings are summarized in Section 7.

2. Problem statement

This paper focuses on a 1-D wave equation with a nonlocal term modeled by

$$\begin{cases} y_{tt}(s,t) = y_{ss}(s,t) + \gamma y_t(0,t) + \mathcal{F}(s)\rho_1(t), \quad s \in (0,1), t > 0, \\ y_s(0,t) = \rho_2(t), \quad t \geq 0, \\ y_s(1,t) = U(t) + \rho_3(t), \quad t \geq 0, \\ y(s,0) = y_0(s), \;\; y_t(s,0) = y_1(s), \;\; s \in (0,1), \\ Y_{out}(t) = y(0,t), \quad t \geq 0, \end{cases} \tag{2.1}$$

in this formulation, $\gamma > 0$, $U(t)$ denotes the control input, $Y_{out}(t)$ serves as the target output for regulation, and $\mathcal{F}(s)$ corresponds to an uncertain function, which is assumed to belong to $L^2(0,1)$. The signals $\rho_k(t), k = 1,2,3$ denote disturbances from the following exosystem:

$$\begin{cases} \dot{\alpha}(t) = G\alpha(t), \;\; \alpha(0) = \alpha_0 \in \mathbb{R}^n, \\ \rho_k(t) = D_k\alpha(t), \;\; k = 1,2,3, \end{cases} \tag{2.2}$$

where $G \in \mathbb{R}^{n\times n}$, $D_k \in \mathbb{R}^{1\times n}$ ($k = 1,2,3$), and α_0 are all unknown. Let

$$r(t) = D_4\alpha(t)$$

as reference signal, in which D_4 is likewise unknown. The tracking error is

$$\mathcal{E}(t) = Y_{out}(t) - r(t).$$

This work develops a feedback control scheme derived from the tracking error to ensure

$$\lim_{t\to\infty} |\mathcal{E}(t)| = \lim_{t\to\infty} |Y_{out}(t) - r(t)| = 0. \tag{2.3}$$

First, the Laplace transformation is applied to the disturbance-free version of (2.1). Defining $\hat{y}(s,\epsilon)$, $\hat{U}(\epsilon)$, and $\hat{Y}(\epsilon)$ as the Laplace transforms of $y(s,t)$, $U(t)$, and $Y_{out}(t)$, it leads to

$$\begin{cases} \epsilon^2\hat{y}(s,\epsilon) = \hat{y}_{ss}(s,\epsilon) + \gamma\epsilon\hat{y}(0,\epsilon), \\ \hat{y}_s(0,\epsilon) = 0, \\ \hat{y}_s(1,\epsilon) = \hat{u}(\epsilon), \\ \hat{Y}(\epsilon) = \hat{y}(0,\epsilon). \end{cases} \tag{2.4}$$

Consequently, the input-output dynamic is characterized by

$$H(\epsilon) = \frac{1}{(\epsilon-\gamma)\sinh(\epsilon)}, \tag{2.5}$$

which has no zeros. The following assumption is now postulated.

Assumption 2.1. *The spectrum of G is*

$$\sigma(G) = \{0, \pm l\rho_k,\ 1 \le k \le p\},\ n = 2p+1,$$

where $\{\rho_k\}_{k=1}^{p}$ are unknown positive parameters that are different from each other. Furthermore, we assume that G is diagonalizable over $\mathbb{C}$, i.e., there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of G. The unknown integer p is assumed to have a known upper bound N.

Hence, the exosystem (2.2) has sinusoidal signals with less than N unknown frequencies, which are specified by the eigenvalues of G. The exogenous disturbances and reference trajectory are explicitly characterized as

$$\begin{cases} \rho_l(t) = \sum_{k=1}^{N}[c_{kl}\cos\rho_k t + o_{kl}\sin\rho_k t] + h_l, \quad l = 1,2,3, \\ r(t) = \sum_{k=1}^{N}[c_{k4}\cos\rho_k t + o_{k4}\sin\rho_k t] + h_4, \end{cases}$$

with the parameters $\{\rho_k\}$, $\{c_{kl}\}$,$\{o_{kl}\}$, and h_l being unknown.

This assumption is motivated by several practical considerations. First, the purely imaginary spectrum of G corresponds to harmonic and constant signals, which accurately model many real-world disturbances such as mechanical vibrations, electrical harmonics, and periodic reference trajectories. Second, the shared exosystem state $\alpha(t)$ captures the common scenario where disturbances at different locations contain correlated frequency components, as often observed in multi-sensor systems or structurally coupled environments. Third, requiring only an upper bound N on the number of frequencies reflects realistic situations where the exact harmonic composition may be unknown or time-varying due to operating condition changes. The diagonalizability condition ensures analytical tractability while covering most practical harmonic systems. The main limitation of this assumption is its restriction to harmonic-type signals; it does not encompass non-periodic disturbances or rapidly time-varying frequency content. This limitation also points to potential research directions for extending the framework to broader signal classes in future work.

Define the Hilbert space

$$\mathcal{H} = H^1(0,1) \times L^2(0,1),$$

with the inner product defined for all $(\zeta_1, \eta_1), (\zeta_2, \eta_2) \in \mathcal{H}$ by

$$\langle(\zeta_1, \eta_1), (\zeta_2, \eta_2)\rangle_{\mathcal{H}} = \int_0^1 [\zeta_1'(s)\overline{\zeta_2'(s)} + \eta_1(s)\overline{\eta_2(s)}]ds + \hbar_2\zeta_1(0)\overline{\zeta_2(0)},$$

where $\hbar_2 > 0$.

3. Feedforward plus state feedback regulator design

The control design follows a sequential procedure to address the coupled challenges of open-loop instability and unknown disturbances. First, the destabilizing nonlocal term is removed through a transformation that constructs an auxiliary system. Second, the coupled PDE-ODE (ordinary differential equation) dynamics are decoupled via a coordinate change, which concentrates all exogenous effects into the tracking error. Finally, a composite controller comprising feedforward compensation and stabilizing state feedback is synthesized.

The difficulties arising from the nonlocal term are addressed by the transformation

$$z(s,t) = y(s,t) - m(s,t), \tag{3.1}$$

where $m(s,t)$ is governed by

$$\begin{cases} m_t(s,t) = -m_s(s,t) + \frac{\gamma}{\hbar_1}m(0,t), \\ m(0,t) = \hbar_1\mathcal{E}(t), m(s,0) = m_0(s). \end{cases} \tag{3.2}$$

Here, $m_0(s)$ is an arbitrary initial condition, and the parameter $\hbar_1$ lies in the interval $(0,1)$ with $\hbar_1 \neq \frac{1}{2}$. Thus, (2.1) becomes

$$\begin{cases} z_{tt}(s,t) = z_{ss}(s,t) + \gamma\dot{r}(t) + \mathcal{F}(s)\rho_1(t), \\ z_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}z_t(0,t) - \dfrac{\gamma}{1-\hbar_1}z(0,t) - \dfrac{\hbar_1}{1-\hbar_1}\dot{r}(t) + \dfrac{\gamma}{1-\hbar_1}r(t) + \rho_2(t), \\ z_s(1,t) = U(t) + \rho_3(t) - m_s(1,t), \\ z_t(s,0) = y_1(s) + m_0'(s) - \dfrac{\gamma}{\hbar_1}m_0(0). \end{cases} \tag{3.3}$$

This transformation isolates the nonlocal coupling $\gamma y_t(0,t)$ into the auxiliary variable $m(s,t)$. Consequently, the z-subsystem (3.3) represents a wave equation freed from the internal velocity feedback, though still subject to distributed and boundary disturbances. This step prepares the system for the subsequent regulation design. Next, a feedback regulator is designed to regulate the tracking error. This requires decoupling the coupled systems (2.1) and (2.2) via coordinate transformation

$$\xi(s,t) = z(s,t) + \varkappa(s)\alpha(t), \tag{3.4}$$

where $\varkappa$: $[0,1] \to \mathbb{R}^{1\times(2p+1)}$ satisfies

$$
\begin{cases}
\varkappa''(s) = \varkappa(s)G^2 + D_1\mathcal{F}(s) + \gamma D_4 G, \\
\varkappa'(0) = \varkappa(0)(\hbar_2 + \hbar_3 G + \dfrac{G\hbar_1 - \gamma}{1-\hbar_1}) + \dfrac{\hbar_1 G}{1-\hbar_1}D_4 \\
\qquad\qquad + \hbar_3 D_4 G - \dfrac{\gamma}{1-\hbar_1}D_4 + \hbar_2 D_4 - D_2, \\
\varkappa'(1) = -D_3,
\end{cases}
\tag{3.5}
$$

with

$$\hbar_2 > \frac{\gamma}{1-\hbar_1} > 0, \ \ \hbar_3 > 0.$$

Lemma 3.1. *The boundary value problem* (3.5) *admits a unique solution* $\varkappa^\top \in H^2((0,1);\mathbb{R}^{2p+1})$.

Proof. Let $\varkappa_1(\cdot)$ satisfy the boundary value problem

$$
\begin{cases}
\varkappa_1''(s) = 0, \\
\varkappa_1'(0) = \varkappa_1(0)(\hbar_2 + \hbar_3 G + \dfrac{G\hbar_1 - \gamma}{1-\hbar_1}) + \dfrac{\hbar_1 G}{1-\hbar_1}D_4 + \hbar_3 D_4 G - \dfrac{\gamma}{1-\hbar_1}D_4 + \hbar_2 D_4 - D_2, \\
\varkappa_1'(1) = -D_3.
\end{cases}
\tag{3.6}
$$

This equation admits a unique solution $\varkappa_1 \in C^\infty([0,1];\mathbb{R}^{2p+1})$. Consider the following additional boundary value problem:

$$
\begin{cases}
q''(s) = q(s)G^2 + \varkappa_1(s)G^2 + D_1\mathcal{F}(s) + \gamma D_4 G, \\
q'(0) = q(0)(\hbar_2 + \hbar_3 G + \dfrac{G\hbar_1 - \gamma}{1-\hbar_1}), \\
q'(1) = 0.
\end{cases}
\tag{3.7}
$$

Let $\{\psi_k\}_{k=1}^{2p+1}$ be eigenvectors of G corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^{2p+1}$, respectively. Right-multiplying both sides of (3.7) by ψ_k, we obtain

$$
\begin{cases}
q_k''(s) = q_k(s)\lambda_k^2 + \mathcal{F}_k(s), \\
q_k'(0) = q_k(0)(\hbar_2 + \hbar_3\lambda_k + \dfrac{\hbar_1\lambda_k - \gamma}{1-\hbar_1}), \\
q_k'(1) = 0,
\end{cases}
\tag{3.8}
$$

where

$$q_k(s) = q(s)\psi_k$$

and

$$\mathcal{F}_k(s) = (\varkappa_1(s)G^2 + D_1\mathcal{F}(s) + \gamma D_4 G)\psi_k.$$

Since $\varkappa_1(\cdot)$ is smooth, $\mathcal{F}(\cdot) \in L^2(0,1)$, and D_1, D_4, G are constant matrices, it follows that the source term $\mathcal{F}_k(\cdot)$ belongs to $L^2(0,1)$ for each k.

Now, we analyze the regularity for each mode.

For $\lambda_k \neq 0$, the solution of (3.8) is given by

$$q_k(s) = \frac{C_1 e^{\lambda_k s} + e^{-\lambda_k s}}{2\lambda_k (C_1 e^{\lambda_k} + e^{-\lambda_k})} \int_0^1 (e^{\lambda_k(1-\epsilon)} + e^{-\lambda_k(1-\epsilon)}) \mathcal{F}_k(\epsilon) d\epsilon - \frac{1}{2\lambda_k} \int_0^1 (e^{\lambda_k(s-\epsilon)} + e^{-\lambda_k(s-\epsilon)}) \mathcal{F}_k(\epsilon) d\epsilon$$

with

$$C_1 = \frac{\lambda_k + \hbar_2 + \hbar_3 \lambda_k + \frac{\hbar_1 \lambda_k - \gamma}{1-\hbar_1}}{\lambda_k - \hbar_2 - \hbar_3 \lambda_k - \frac{\hbar_1 \lambda_k - \gamma}{1-\hbar_1}}.$$

For the eigenvalue $\lambda_k = 0$, the system (3.8) reduces to

$$\begin{cases} q_k''(s) = \mathcal{F}_k(s), \\ q_k'(0) = q_k(0)(\hbar_2 - \frac{\gamma}{1-\hbar_1}), \\ q_k'(1) = 0, \end{cases}$$

which also admits a solution in $H^2(0, 1)$.

In both cases, since the source term $\mathcal{F}_k(\cdot) \in L^2(0, 1)$ and the boundary conditions are homogeneous, each solution q_k belongs to $H^2(0, 1)$.

Therefore, the vector-valued function

$$q(x) = (q_1(x), \ldots, q_{2p+1}(x))[\psi_1, \ldots, \psi_{2p+1}]^{-1}$$

is in $H^2((0, 1); \mathbb{R}^{2p+1})$. Consequently,

$$\varkappa(s) = \varkappa_1(s) + q(s)$$

is the unique solution of (3.5) and belongs to $H^2((0, 1); \mathbb{R}^{2p+1})$. □

By (3.4), the extended system $(\xi(\cdot, \cdot), \alpha(\cdot))$ admits the following representation:

$$\begin{cases} \xi_{tt}(s,t) = \xi_{ss}(s,t), \\ \xi_s(0,t) = \frac{\hbar_1}{1-\hbar_1} \xi_t(0,t) - \frac{\gamma}{1-\hbar_1} \xi(0,t) + \hbar_2(\xi(0,t) + m(0,t) - \mathcal{E}(t)) \\ \qquad + \hbar_3(\xi_t(0,t) + m_t(0,t) - \dot{\mathcal{E}}(t)), \\ \xi_s(1,t) = U(t) - m_s(1,t), \\ \mathcal{E}(t) = \xi(0,t) + m(0,t) - (\varkappa(0) + D_4)\alpha(t), \\ \dot{\mathcal{E}}(t) = \xi_t(0,t) + m_t(0,t) - (\varkappa(0) + D_4) G\alpha(t), \\ \dot{\alpha}(t) = G\alpha(t). \end{cases} \tag{3.9}$$

As stated in the text, the transformed system (3.9) exhibits two key features that justify the preceding coordinate transformation from z to ξ. First, it achieves a decoupled structure with an independent PDE subsystem and an independent ODE subsystem, which significantly reduces the design complexity; second, it ensures that all exogenous disturbances appear exclusively in the tracking error dynamics, concentrating the challenge of output regulation into the estimation and rejection of the signal $(\varkappa(0) + D_4)\alpha(t)$.

By Assumption 2.1, we know that $(\varkappa(0)+D_4)\alpha(t)$ represents a sinusoidal signal containing no more than N distinct frequency components. For generality, it may be formulated as

$$(\varkappa(0)+D_4)\alpha(t)=\sum_{k=1}^{j}(C_k\cos\rho_k t+O_k\sin\rho_k t)+H,\quad j\le p\le N, \tag{3.10}$$

C_k, O_k, H represent unknown parameters, $C_k^2+O_k^2>0,\ k=1,\cdots,j.$

Lemma 3.2. *There exists $\ell_0\in\mathbb{R}^{2N+1}$, such that $(\varkappa(0)+D_4)\alpha(t)$ admits the realization*

$$\begin{cases}\dot{\ell}(t)=F_c(\vartheta)\ell(t)=I_c\ell(t)-\sum_{k=1}^{N}\vartheta_k E_{2k}\ell_1(t),\quad \ell(0)=\ell_0,\\ (\varkappa(0)+D_4)\alpha(t)=\ell_1(t),\end{cases} \tag{3.11}$$

where

$$\ell(t)=(\ell_1(t),\ell_2(t),\cdots,\ell_{2N+1}(t))^\top\in\mathbb{R}^{2N+1},$$

and E_{2k} is the column vector extracted from the $(2N+1)$-dimensional identity matrix, specifically at the $2k$-th position. Here, I_c and $F_c(\vartheta)$ are defined as

$$I_c=\begin{bmatrix}0&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&1\\ 0&0&\cdots&0\end{bmatrix},$$

$$F_c(\vartheta)=\begin{bmatrix}0&1&0&\cdots&0&0\\ -\vartheta_1&0&1&\cdots&0&0\\ 0&0&0&\cdots&0&0\\ -\vartheta_2&0&0&\cdots&0&0\\ \vdots&\vdots&\vdots&&\vdots&\vdots\\ 0&0&0&\cdots&0&1\\ -\vartheta_N&0&0&\cdots&0&0\end{bmatrix}$$

with the parameter vector

$$\vartheta=[\vartheta_1,\vartheta_2,\ldots,\vartheta_N]^\top=[\vartheta_1,\ldots,\vartheta_j,0,\ldots,0]^\top\in\mathbb{R}^N.$$

The nonzero coefficients $\vartheta_1,\ldots,\vartheta_j$ are determined by the polynomial identity

$$\kappa^{2j}+\vartheta_1\kappa^{2(j-1)}+\cdots+\vartheta_j\triangleq\prod_{k=1}^{j}(\kappa^2+\rho_k^2).$$

Proof. Clearly, the term $(\varkappa(0)+D_4)\alpha(t)$ is expressible as

$$\begin{cases}\dot{\phi}(t)=G_\phi\phi(t),\ \phi(t)\in\mathbb{R}^{(2j+1)},\\ (\varkappa(0)+D_4)\alpha(t)=\nu_\phi\phi(t),\end{cases} \tag{3.12}$$

with

$$\begin{cases} G_\phi = \operatorname{diag}\left\{\mathrm{G}(\rho_1), \mathrm{G}(\rho_2), \cdots, \mathrm{G}(\rho_\mathrm{j}), 0_{1\times 1}\right\}, \\ G_(\rho_k) = \begin{bmatrix} 0 & \rho_k \\ -\rho_k & 0 \end{bmatrix}, \\ \nu_\phi = [1, 0, \cdots, 1, 0, 1], \\ \phi(0) = (C_1, O_1, \cdots, C_j, O_j, H)^\top. \end{cases} \tag{3.13}$$

The observability of (G_ϕ, ν_ϕ) can be directly established. Therefore, we may introduce the following coordinate transformation

$$\phi^E(t) = B_1\phi(t), \ \ \phi^E(t) = (\phi_1^E(t), \cdots, \phi_{2j+1}^E(t))^\top, \tag{3.14}$$

with B_1 being a nonsingular $(2s + 1)$-dimensional square matrix. This observability-preserving transformation brings (G_ϕ, ν_ϕ) into the canonical form

$$\begin{cases} \dot{\phi}^E(t) = G_E(\vartheta)\phi^E(t), \\ (\varkappa(0) + D_4)\alpha(t) = \phi_1^E(t), \end{cases} \tag{3.15}$$

where

$$G_E(\vartheta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\vartheta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\vartheta_j & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since matrices G_ϕ and G_E share identical characteristic polynomials, the parameters $\vartheta_1, \ldots, \vartheta_j$ satisfy

$$\kappa^{2j+1} + \vartheta_1\kappa^{2j-1} + \cdots + \vartheta_{j-1}\kappa^3 + \vartheta_j\kappa \triangleq \prod_{k=1}^{j}(\kappa^2 + \rho_k^2),$$

by defining

$$B_2 = \left[I_{2j+1} \ 0_{(2j+1)\times(2N-2j)}\right]^\top$$

and

$$\ell(t) = B_2\phi^E(t),$$

so $\ell(\cdot)$ satisfies (3.11), and

$$\ell(0) = B_2B_1\phi(0).$$

This completes the proof. □

It follows that system (3.9) can be reformulated as

$$
\begin{cases}
\xi_{tt}(s,t) = \xi_{ss}(s,t), \\
\xi_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\xi_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\xi(0,t) + \hbar_2(\xi(0,t) + m(0,t) - \mathcal{E}(t)) \\
\qquad\qquad +\hbar_3(\xi_t(0,t) + m_t(0,t) - \dot{\mathcal{E}}(t)), \\
\xi_s(1,t) = U(t) - m_s(1,t), \\
\quad \dot{\ell}(t) = F_c(\vartheta)\ell(t), \\
\quad \mathcal{E}(t) = \xi(0,t) + m(0,t) - A_c\ell(t), \\
\quad \dot{\mathcal{E}}(t) = \xi_t(0,t) + m_t(0,t) - A_cF_c(\vartheta)\ell(t),
\end{cases}
\tag{3.16}
$$

in which

$$
A_c = [1, 0, \cdots, 0] \in \mathbb{R}^{1\times(2N+1)}.
$$

Then, we develop a feedforward plus state feedback control scheme for (3.16).

Let

$$
f_0(s,\vartheta) = f_0(s) \in \mathbb{R}^{1\times(2N+1)}
$$

satisfy the initial-value problem

$$
\begin{cases}
f_0''(s) = f_0(s)F_c(\vartheta), \\
f_0'(0) = f_0(0)\left(\dfrac{\hbar_1}{1-\hbar_1}F_c(\vartheta) - \dfrac{\gamma}{1-\hbar_1}\right) + \hbar_2A_c + \hbar_3A_cF_c(\vartheta), \\
f_0(0) = A_c.
\end{cases}
\tag{3.17}
$$

Lemma 3.3. *Problem* (3.17) *possesses a unique solution with continuous differentiability in both the state s and the parameter ϑ.*

Proof. There exists a unique solution to (3.17)

$$
(f_0(s,\vartheta), f_0'(s,\vartheta)) = (f_0(0), f_0'(0))\, e^{\begin{pmatrix} 0 & F_c(\vartheta) \\ I & 0 \end{pmatrix} s}.
$$

Hence the solution to (3.17) is continuously differentiable. □

Let

$$
\Omega(s,t) = \xi(s,t) - f_0(s)\ell(t),
\tag{3.18}
$$

thus,

$$
\begin{cases}
\Omega_{tt}(s,t) = \Omega_{ss}(s,t), \\
\Omega_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\Omega_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\Omega(0,t), \\
\Omega_s(1,t) = U(t) - m_s(1,t) - f_0'(1)\ell(t), \\
\quad \dot{\ell}(t) = F_c(\vartheta)\ell(t), \\
\quad \mathcal{E}(t) = \Omega(0,t) + m(0,t).
\end{cases}
\tag{3.19}
$$

At this stage, the output regulation objective of driving $\xi(\cdot,t) \to f_0(\cdot)\ell(t)$ becomes the stabilization of $\Omega(\cdot,t) \to 0$ in $\mathcal{H}$, as $t \to \infty$. Therefore, under the assumption that $f_0'(1)\ell(t)$ is known, the following feedforward observer can be designed

$$\begin{cases} \hat{\Omega}_{tt}(s,t) = \hat{\Omega}_{ss}(s,t), \\ \hat{\Omega}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\hat{\Omega}_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\Omega(0,t) + \hbar_4(\hat{\Omega}(0,t) - \Omega(0,t)), \\ \hat{\Omega}_s(1,t) = U(t) - m_s(1,t) - f_0'(1)\ell(t), \\ \hat{\Omega}(s,0) = \hat{\Omega}_0(s),\ \hat{\Omega}_t(s,0) = \hat{\Omega}_1(s), \end{cases} \tag{3.20}$$

where $\hbar_4 > 0$, and $(\hat{\Omega}_0(s), \hat{\Omega}_1(s)) \in \mathcal{H}$ is any given initial value.

Let

$$\tilde{\Omega}(s,t) = \hat{\Omega}(s,t) - \Omega(s,t), \tag{3.21}$$

therefore,

$$\begin{cases} \tilde{\Omega}_{tt}(s,t) = \tilde{\Omega}_{ss}(s,t), \\ \tilde{\Omega}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\tilde{\Omega}_t(0,t) + \hbar_4\tilde{\Omega}(0,t), \\ \tilde{\Omega}_s(1,t) = 0. \end{cases} \tag{3.22}$$

Lemma 2.1 of [25] establishes the exponential stability of (3.22). Consequently, to stabilize (3.19), it suffices to design a controller that stabilizes (3.20). Toward this end, by introducing the variable $\tilde{\Omega}$, system (3.20) becomes

$$\begin{cases} \hat{\Omega}_{tt}(s,t) = \hat{\Omega}_{ss}(s,t), \\ \hat{\Omega}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\hat{\Omega}_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\hat{\Omega}(0,t) + (\hbar_4 + \dfrac{\gamma}{1-\hbar_1})\tilde{\Omega}(0,t), \\ \hat{\Omega}_s(1,t) = U(t) - m_s(1,t) - f_0'(1)\ell(t). \end{cases} \tag{3.23}$$

The observer system (3.23) contains the term $-\frac{\gamma}{1-\hbar_1}\hat{\Omega}(0,t)$ at the left boundary. Since $\gamma > 0$, this term introduces a destabilizing effect that complicates direct boundary control synthesis. To convert the second-order wave dynamics into a form amenable to cascade stabilization, we decompose the system into its characteristic components via the Riemann transformation. This transformation is natural for the wave operator $\partial_{tt} - \partial_{ss}$ and serves three key purposes. First, it reduces the second-order PDE to two first-order transport equations. Besides, it decouples the boundary dynamics, isolating the destabilizing term and it yields a cascade structure suitable for backstepping design. Therefore, we introduce the Riemann variables

$$\begin{cases} \mathrm{I}(s,t) = \hat{\Omega}_t(s,t) + \hat{\Omega}_s(s,t), \\ \mathrm{K}(s,t) = \dfrac{1}{1-2\hbar_1}\Big[\hat{\Omega}_t(s,t) - \hat{\Omega}_s(s,t)\Big], \end{cases} \tag{3.24}$$

and its inverse transformation is

$$\begin{cases} \hat{\Omega}_t(s,t) = \dfrac{1}{2}[\mathrm{I}(s,t) + (1-2\hbar_1)\mathrm{K}(s,t)], \\ \hat{\Omega}_s(s,t) = \dfrac{1}{2}[\mathrm{I}(s,t) - (1-2\hbar_1)\mathrm{K}(s,t)]. \end{cases} \tag{3.25}$$

Through the newly introduced transformation (3.24), system (3.23) takes the form

$$\begin{cases} \mathrm{I}_t(s,t) = \mathrm{I}_s(s,t), \quad \mathrm{K}_t(s,t) = -\mathrm{K}_s(s,t), \\ \mathrm{K}(0,t) = \mathrm{I}(0,t) + \dfrac{2\gamma}{1-2\hbar_1}\hat{\Omega}(0,t) - \dfrac{2(\hbar_4+\gamma-\hbar_1\hbar_4)}{1-2\hbar_1}\tilde{\Omega}(0,t), \\ \mathrm{I}(1,t) = \hat{\Omega}_t(1,t) + U(t) - m_s(1,t) - f_0'(1)\ell(t), \\ \hat{\Omega}_t(0,t) = (1-\hbar_1)\mathrm{I}(0,t) + \gamma\hat{\Omega}(0,t) - (\hbar_4+\gamma-\hbar_1\hbar_4)\tilde{\Omega}(0,t). \end{cases} \tag{3.26}$$

Define

$$U_1(t) = \hat{\Omega}_t(1,t) + U(t) - m_s(1,t) - f_0'(1)\ell(t).$$

Clearly, Eq (3.26) consists of two cascaded transport equations coupled with an ODE. Since the $(\mathrm{I}, \hat{\Omega}(0,t))$-subsystem is decoupled from the K-subsystem, we may first consider the $(\mathrm{I}, \hat{\Omega}(0,t))$-subsystem dynamics

$$\begin{cases} \hat{\Omega}_t(0,t) = \gamma\hat{\Omega}(0,t) + (1-\hbar_1)\mathrm{I}(0,t) - (\hbar_4+\gamma-\hbar_1\hbar_4)\tilde{\Omega}(0,t), \\ \mathrm{I}_t(s,t) = \mathrm{I}_s(s,t), \\ \mathrm{I}(1,t) = U_1(t). \end{cases} \tag{3.27}$$

The subsystem (3.27) still contains the destabilizing term $\gamma\hat{\Omega}(0,t)$. To achieve exponential stabilization, we employ an integral backstepping transformation that maps (3.27) into a target system whose stability is straightforward to establish. The transformation is designed to absorb the destabilizing term and introduce additional damping through a tunable parameter $\hbar_5$. Then, we introduce an invertible backstepping transformation on $\mathbb{R} \times L^2(0,1)$

$$\begin{cases} \hat{\Omega}(0,t) = \hat{\Omega}(0,t), \\ \varphi(s,t) = \mathrm{I}(s,t) + 2\hbar_5\displaystyle\int_0^s e^{2\gamma(s-\epsilon)}\mathrm{I}(\epsilon,t)d\epsilon + \dfrac{\hbar_5}{1-\hbar_1}e^{2\gamma s}\hat{\Omega}(0,t), \end{cases} \tag{3.28}$$

and its inverse transformation is

$$\begin{cases} \hat{\Omega}(0,t) = \hat{\Omega}(0,t), \\ \mathrm{I}(s,t) = \varphi(s,t) - 2\hbar_5\displaystyle\int_0^s e^{-2(\hbar_5-\gamma)(s-\epsilon)}\mathrm{I}(\epsilon,t)d\epsilon + \dfrac{\hbar_5}{1-\hbar_1}e^{-2(\hbar_5-\gamma)s}\hat{\Omega}(0,t), \end{cases} \tag{3.29}$$

where $\hbar_5 > |\gamma|$ is a tuning parameter. Under the transformations (3.28) and (3.29), (3.27) is equivalent to

$$\begin{cases} \varphi_t(s,t) = \varphi_s(s,t), \\ \varphi(1,t) = U_1(t) + 2\hbar_5\displaystyle\int_0^1 e^{2\gamma(1-\epsilon)}\mathrm{I}(\epsilon,t)d\epsilon + \dfrac{\hbar_5}{1-\hbar_1}e^{2\gamma}\hat{\Omega}(0,t), \\ \hat{\Omega}_t(0,t) = (\gamma+\hbar_5)\hat{\Omega}(0,t) + (1-\hbar_1)\varphi(0,t) - (\hbar_4+\gamma-\hbar_1\hbar_4)\tilde{\Omega}(0,t), \end{cases} \tag{3.30}$$

therefore, the controller

$$U_1(t) = -2\hbar_5 \int_0^1 e^{2\gamma(1-\epsilon)} \mathrm{I}(\epsilon, t) d\epsilon - \tfrac{\hbar_5}{1-\hbar_1} e^{2\gamma} \hat{\Omega}(0, t). \tag{3.31}$$

Under the controller (3.31), (3.30) becomes

$$\begin{cases} \varphi_t(s, t) = \varphi_s(s, t), \\ \varphi(1, t) = 0, \\ \hat{\Omega}_t(0, t) = (\gamma + \hbar_5)\hat{\Omega}(0, t) + (1 - \hbar_1)\varphi(0, t) - (\hbar_4 + \gamma - \hbar_1\hbar_4)\tilde{\Omega}(0, t). \end{cases} \tag{3.32}$$

Besides, Eq (3.26) becomes

$$\begin{cases} \mathrm{I}_t(s, t) = \mathrm{I}_s(s, t), \quad \mathrm{K}_t(s, t) = -\mathrm{K}_s(s, t), \\ \mathrm{K}(0, t) = \mathrm{I}(0, t) + \dfrac{2\gamma}{1 - 2\hbar_1}\hat{\Omega}(0, t) - \dfrac{2(\hbar_4 + \gamma - \hbar_1\hbar_4)}{1 - 2\hbar_1}\tilde{\Omega}(0, t), \\ \mathrm{I}(1, t) = -2\hbar_5 \displaystyle\int_0^1 e^{2\gamma(1-\epsilon)} \mathrm{I}(\epsilon, t) d\epsilon - \dfrac{\hbar_5}{1 - \hbar_1} e^{2\gamma}\hat{\Omega}(0, t), \\ \hat{\Omega}_t(0, t) = (1 - \hbar_1)\mathrm{I}(0, t) + \gamma\hat{\Omega}(0, t) - (\hbar_4 + \gamma - \hbar_1\hbar_4)\tilde{\Omega}(0, t). \end{cases} \tag{3.33}$$

Lemma 2.2 of [25] guarantees that system (3.33) is well-posed in $\left[L^2(0,1)\right]^2 \times \mathbb{R}$, possessing a unique solution for arbitrary initial data $\left(\mathrm{I}(\cdot, 0), \mathrm{K}(\cdot, 0), \hat{\Omega}(0, 0)\right)$. Furthermore, when $t \geq 0$, one can find positive constants M_1 and δ satisfying

$$\left\|\mathrm{I}(\cdot, t), \mathrm{K}(\cdot, t), \hat{\Omega}(0, t)\right\|_{[L^2(0,1)]^2\times\mathbb{R}} \leq M_1 e^{-\delta t} \left\|\mathrm{I}(\cdot, 0), \mathrm{K}(\cdot, 0), \hat{\Omega}(0, 0), \tilde{\Omega}(\cdot, 0), \tilde{\Omega}_t(\cdot, 0)\right\|_{[L^2(0,1)]^2\times\mathbb{R}\times\mathcal{H}}.$$

Because

$$U_1(t) = \hat{\Omega}_t(1, t) + U(t) - m_s(1, t) - f_0'(1)\ell(t),$$

consequently,

$$U(t) = -\hat{\Omega}_t(1, t) + m_s(1, t) + f_0'(1)\ell(t) - \frac{\hbar_5}{1 - \hbar_1} e^{2\gamma}\hat{\Omega}(0, t) - 2\hbar_5 \int_0^1 e^{2\gamma(1-\epsilon)} (\hat{\Omega}_t(\epsilon, t) + \hat{\Omega}_s(\epsilon, t)) d\epsilon.$$

Thus, the closed-loop system comprising (3.20) and (3.22) is

$$\begin{cases} \hat{\Omega}_{tt}(s, t) = \hat{\Omega}_{ss}(s, t), \\ \hat{\Omega}_s(0, t) = \dfrac{\hbar_1}{1 - \hbar_1}\hat{\Omega}_t(0, t) - \dfrac{\gamma}{1 - \hbar_1}\Omega(0, t) + (\hbar_4 + \dfrac{\gamma}{1 - \hbar_1})\tilde{\Omega}(0, t), \\ \hat{\Omega}_s(1, t) = -\hat{\Omega}_t(1, t) - \dfrac{\hbar_5}{1 - \hbar_1} e^{2\gamma}\hat{\Omega}(0, t) - 2\hbar_5 \displaystyle\int_0^1 e^{2\gamma(1-\epsilon)} (\hat{\Omega}_t(\epsilon, t) + \hat{\Omega}_s(\epsilon, t)) d\epsilon, \\ \tilde{\Omega}_{tt}(s, t) = \tilde{\Omega}_{ss}(s, t), \\ \tilde{\Omega}_s(0, t) = \dfrac{\hbar_1}{1 - \hbar_1}\tilde{\Omega}_t(0, t) + \hbar_4\tilde{\Omega}(0, t), \\ \tilde{\Omega}_s(1, t) = 0. \end{cases} \tag{3.34}$$

Lemma 3.4. *Under the standard inner product of the $\mathcal{H}^2$ space, system* (3.34) *possesses a unique solution* $(\hat{\Omega}, \hat{\Omega}_t, \tilde{\Omega}, \tilde{\Omega}_t) \in C(0, \infty; \mathcal{H}^2)$ *for arbitrary initial data* $(\hat{\Omega}(\cdot, 0), \hat{\Omega}_t(\cdot, 0), \tilde{\Omega}(\cdot, 0), \tilde{\Omega}_t(\cdot, 0))$. *Furthermore, as* $t \geq 0$, *this solution exhibits the property*

$$\left\|\hat{\Omega}(\cdot, t), \hat{\Omega}_t(\cdot, t), \tilde{\Omega}(\cdot, t), \tilde{\Omega}_t(\cdot, t)\right\|_{\mathcal{H}^2} \leq M_2 e^{-\delta t} \left\|\hat{\Omega}(\cdot, 0), \hat{\Omega}_t(\cdot, 0), \tilde{\Omega}(\cdot, 0), \tilde{\Omega}_t(\cdot, 0)\right\|_{\mathcal{H}^2},$$

where M_2, δ *are positive constants.*

This implies that

$$\begin{cases} \Omega_{tt}(s, t) = \Omega_{ss}(s, t), \\ \Omega_s(0, t) = \dfrac{\hbar_1}{1 - \hbar_1}\Omega_t(0, t) - \dfrac{\gamma}{1 - \hbar_1}\Omega(0, t), \\ \Omega_s(1, t) = -\hat{\Omega}_t(1, t) - \dfrac{\hbar_5}{1 - \hbar_1} e^{2\gamma}\hat{\Omega}(0, t) - 2\hbar_5 \displaystyle\int_0^1 e^{2\gamma(1-\epsilon)}(\hat{\Omega}_t(\epsilon, t) + \hat{\Omega}_s(\epsilon, t))d\epsilon, \end{cases} \tag{3.35}$$

is exponentially stable in the space $\mathcal{H}$.

4. Tracking error-based controller design

To reconstruct the state $(\xi(\cdot, t), \ell(t))$, this section designs an observer for system (3.16) and utilizes the measurement of $\mathcal{E}(t)$ and $\dot{\mathcal{E}}(t)$ to estimate ϑ online. Since the initial conditions of system (3.16) may be unknown, the observer for the ξ- subsystem is designed by directly replicating its dynamics

$$\begin{cases} \hat{\xi}_{tt}(s, t) = \hat{\xi}_{ss}(s, t), \\ \hat{\xi}_s(0, t) = \dfrac{\hbar_1}{1 - \hbar_1}\hat{\xi}_t(0, t) - \dfrac{\gamma}{1 - \hbar_1}\hat{\xi}(0, t) + \hbar_2(\hat{\xi}(0, t) + m(0, t) - \mathcal{E}(t)) \\ \qquad + \hbar_3(\hat{\xi}_t(0, t) + m_t(0, t) - \dot{\mathcal{E}}(t)), \\ \hat{\xi}_s(1, t) = U(t) - m_s(1, t), \\ (\hat{\xi}(\cdot, 0), \hat{\xi}_t(\cdot, 0)) = (\hat{\xi}_0(\cdot), \hat{\xi}_1(\cdot)) \in \mathcal{H}. \end{cases} \tag{4.1}$$

Set the observer error as

$$\tilde{\xi}(s, t) = \xi(s, t) - \hat{\xi}(s, t),$$

then

$$\begin{cases} \tilde{\xi}_{tt}(s, t) = \tilde{\xi}_{ss}(s, t), \\ \tilde{\xi}_s(0, t) = (\hbar_2 - \dfrac{\gamma}{1 - \hbar_1})\tilde{\xi}(0, t) + (\hbar_3 + \dfrac{\hbar_1}{1 - \hbar_1})\tilde{\xi}_t(0, t), \\ \tilde{\xi}_s(1, t) = 0. \end{cases} \tag{4.2}$$

Similar to (3.22), the system (4.2) is exponentially stable on the space $\mathcal{H}$. Additionally,

$$\tilde{\xi}(0, \cdot), \tilde{\xi}(1, \cdot) \in C([0, \infty); \mathbb{R})$$

with both $|\tilde{\xi}(0,t)|$ and $|\tilde{\xi}(1,t)|$ converging exponentially to zero when $t \to \infty$. Moreover, for some $\varsigma > 0$,

$$\int_0^\infty e^{\varsigma t} \left|\tilde{\xi}_t(1,t)\right|^2 dt < \infty.$$

Define

$$y_\ell(t) = -\mathcal{E}(t) + m(0,t) + \hat{\xi}(0,t) = A_c \ell(t) - \tilde{\xi}(0,t), \tag{4.3}$$

which is fully measurable. Now consider the system

$$\begin{cases} \dot{\ell}(t) = F_c(\vartheta)\ell(t) = I_c \ell(t) - \sum_{k=1}^{N} \vartheta_k E_{2k} \ell_1(t), \\ y_\ell(t) = A_c \ell(t) - \tilde{\xi}(0,t). \end{cases} \tag{4.4}$$

Here, $\ell(t) \in \mathbb{R}^{2N+1}$, I_c, E_{2k}, and $F_c(\vartheta)$ are as defined in Lemma 3.2, and A_c is given in (3.16).

Motivated by [26], two cascaded filters are implemented for frequency detection in the exosystem

$$\begin{cases} \dot{\zeta}_k(t) = \mathsf{H}(A)\zeta_k(t) - \mathcal{P} E_{2k} y_\ell(t), \;\; \zeta_k(t) \in \mathbb{R}^{2N}, \\ \varrho_k(t) = [1, 0, \cdots, 0]\zeta_k(t), \;\; 1 \le k \le N, \\ \dot{\Theta}(t) = -\varepsilon_b \Theta(t) + \varepsilon_c \varrho(t)\varrho(t)^\top, \;\; \Theta(t) \in \mathbb{R}^{N\times N}, \\ \Theta_k(t) = \mathcal{S}\Theta(t)\mathcal{S}^\top, \end{cases} \tag{4.5}$$

in which

$$\mathcal{P} = [0\ I_{2N}], \;\; \mathcal{S} = [I_k, 0_{k\times(N-k)}], \;\; 1 \le k \le N, \;\; \varrho(t) = [\varrho_1(t), \varrho_2(t), \cdots, \varrho_N(t)]^\top \in \mathbb{R}^N, \;\; \varepsilon_b, \varepsilon_c > 0,$$

and

$$\mathsf{H}(A) = \begin{bmatrix} -b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -b_{2N-1} & 0 & \cdots & 1 \\ -b_{2N} & 0 & \cdots & 0 \end{bmatrix},$$

and the vector

$$b = [b_1, \cdots, b_{2N}]^\top$$

is chosen to make sure that $\mathsf{H}(A)$ is Hurwitz. By the transformation

$$\begin{bmatrix} \mathcal{J}_1(t) \\ Q(t) \end{bmatrix} = \ell(t) - \begin{bmatrix} 0 \\ \sum_{k=1}^{N} \zeta_k(t)\vartheta_k + bA_c\ell(t) \end{bmatrix}, \tag{4.6}$$

in which

$$Q(t) \in \mathbb{R}^{2N}, \;\; \mathcal{J}_1(t) = A_c \ell(t) \in \mathbb{R},$$

one obtains

$$\begin{cases} \dot{\mathcal{J}}_1(t) = Q_1(t) + b_1 \mathcal{J}_1(t) + \vartheta^\top \varrho(t), \\ \dot{Q}(t) = \mathsf{H}(A)Q(t) + \mu \mathcal{J}_1(t) - \tilde{\xi}(0,t)\mathcal{D}\vartheta, \end{cases}$$

where $\mathcal{D}$ and μ are $2N \times N$ and $2N \times 1$ constant matrices, respectively. Specifically,

$$\mu = [b_2 - b_1^2, \cdots, b_{2N} - b_{2N-1}b_1, -b_{2N}b_1]^\top.$$

Building upon the framework of [27], when $1 \leq k \leq N$, the following adaptive observer for system (4.4), utilizing the output measurement $y_\ell(t)$, can be proposed

$$\begin{cases} \dot{\hat{\mathcal{J}}}_1(t) = \hat{Q}_1(t) + b_1 y_\ell(t) + \sum_{k=1}^{N} \varrho_k(t)\hat{\vartheta}_k(t) + \chi(y_\ell(t) - \hat{\mathcal{J}}_1(t)), \\ \dot{\hat{Q}}(t) = \mathsf{H}(A)\hat{Q}(t) + \mu y_\ell, \ \ \hat{Q}(t) \in \mathbb{R}^{2N}, \\ \hat{\ell}(t) = \begin{bmatrix} \hat{\mathcal{J}}_1(t) \\ \hat{Q}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{k=1}^{N} \zeta_k(t)\hat{\vartheta}_k(t) + b\hat{\mathcal{J}}_1(t) \end{bmatrix}, \end{cases} \tag{4.7}$$

with the parameter vector evolving according to

$$\begin{cases} \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)), \ e^{-|\det(\Theta_k)|^{\frac{1}{k}}t} \leq \frac{1}{2}, \\ \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)) - a\hat{\vartheta}_k(t), \ else, \end{cases} \tag{4.8}$$

where a, g, χ are arbitrary positive numbers. As previously mentioned,

$$\vartheta = [\vartheta_1, \cdots, \vartheta_N]^\top = [\vartheta_1, \cdots, \vartheta_j, 0, \cdots, 0]^\top \in \mathbb{R}^N,$$

where $\vartheta_1, \cdots, \vartheta_j$ represent the coefficients of the polynomial:

$$\kappa^{2j+1} + \vartheta_1\kappa^{2j-1} + \cdots + \vartheta_{j-1}\kappa^3 + \vartheta_j\kappa \triangleq \prod_{k=1}^{j}(\kappa^2 + \rho_k^2).$$

Lemma 4.1. *For arbitrary initial conditions*

$$(\hat{\mathcal{J}}_1(0), \hat{Q}(0), \hat{\vartheta}(0), \hat{\Theta}(0), \{\zeta_k(0)\}_{k=1}^N) \in \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}^N \times \mathbb{U} \times \mathbb{R}^{2N\times N}$$

with

$$\mathbb{U} = \{T \in \mathbb{R}^{N\times N} : T \succ 0\}$$

denoting the set of positive definite matrices, it holds that

$$\lim_{t\to\infty} e^{-|\det(\Theta_k)|^{\frac{1}{k}}t} = \begin{cases} 0, & k = 1, \cdots, j, \\ 1, & k = j+1, \cdots, N, \end{cases} \tag{4.9}$$

with exponential rate, and

$$\lim_{t\to\infty} \|\hat{\vartheta}(t) - \vartheta(t)\| = 0, \quad \lim_{t\to\infty} \|\hat{\ell}(t) - \ell(t)\| = 0, \tag{4.10}$$

also exponentially.

Proof. Let

$$\varrho_k(t) = \varrho_{kp}(t) + \varrho_{ke}(t),$$

where $\varrho_{kp}(\cdot)$ satisfies

$$\begin{cases} \dot{\zeta}_{kp}(t) = \mathsf{H}(A)\zeta_{kp}(t) - [0\ I_{2N}]E_{2k}\ell_1(t), \\ \varrho_{kp}(t) = [1, 0, \cdots, 0]\zeta_{kp}(t), \quad k = 1, 2, \cdots, N, \end{cases} \tag{4.11}$$

meanwhile, $\varrho_{ke}(\cdot)$ is generated by

$$\begin{cases} \dot{\zeta}_{ke}(t) = \mathsf{H}(A)\zeta_{ke}(t) - [0\ I_{2N}]E_{2k}\tilde{\xi}(0, t), \\ \varrho_{ke}(0) = \varrho_k(0) - \varrho_{kp}(0), \\ \varrho_{ke}(t) = [1, 0, \cdots, 0]\zeta_{ke}(t), \quad k = 1, 2, \cdots, N. \end{cases} \tag{4.12}$$

Since $\ell_1(\cdot)$ is bounded with respect to time and $\mathsf{H}(A)$ is Hurwitz, $\varrho_{kp}(\cdot)$ is also bounded. As shown in [28, Lemma 5.3.2], given that $\ell_1(\cdot)$ consists of j sinusoidal frequencies, the vector $[\varrho_{1p}(t), \varrho_{2p}(t), \cdots, \varrho_{jp}(t)]^\top$ exhibits persistent excitation; i.e., there exist constants $T, \alpha > 0$ such that

$$\int_t^{t+T} [\varrho_{1p}(s), \cdots, \varrho_{jp}(s)]^\top [\varrho_{1p}(s), \cdots, \varrho_{jp}(s)]\, ds \geq \alpha I_j, \quad \text{for all } t \geq 0.$$

In contrast, $[\varrho_{1p}(t), \varrho_{2p}(t), \cdots, \varrho_{kp}(t)]^\top$ fails to be persistent excitation when $k \geq j + 1$. Given the Hurwitz property of $\mathsf{H}(A)$ and when approaches zero, $\left|\tilde{\xi}(0, t)\right|$ tends to zero, which implies $|\varrho_{ke}(t)| \to 0$ accordingly. Lemma 4.8.3 in [28] states that $[\varrho_1(t), \cdots, \varrho_j(t)]^\top$ is persistently exciting. That is, there exist constants $Q, T > 0$ satisfying

$$\int_t^{t+T} [\varrho_1(s), \cdots, \varrho_k(s)]^\top [\varrho_1(s), \cdots, \varrho_k(s)] ds \geq Q I_k, \tag{4.13}$$

for every $t \geq 0,\ 1 \leq k \leq j$.

From (4.5),

$$\Theta_k(t) = e^{-\varepsilon_b t}\Theta_k(0) + \varepsilon_c \int_0^t e^{-\varepsilon_b(t-s)} [\varrho_1(s), \cdots, \varrho_k(s)]^\top [\varrho_1(s), \cdots, \varrho_k(s)] ds. \tag{4.14}$$

Since $\Theta_k(0)$ is positive definite, it's inspired by [26, Lemma 3.1] that

$$|\det(\Theta_k)|^{\frac{1}{k}} \geq \rho > 0$$

holds for all $t \geq 0,\ k = 1, \cdots, j$. Meanwhile, when $k = j + 1, \cdots, N$, $|\det(\Theta_k)|^{\frac{1}{k}}$ decays exponentially to zero, as $t \to \infty$. This behavior is captured by the limit

$$\lim_{t \to \infty} e^{-|\det(\Theta_k)|^{\frac{1}{k}} t} = \begin{cases} 0, & k = 1, \cdots, j, \\ 1, & k = j + 1, \cdots, N. \end{cases}$$

Therefore, for a constant $\Lambda > 0$, and whenever $t \geq \Lambda$, one obtains

$$\begin{cases} \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)), & 1 \leq k \leq j, \\ \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)) - a\hat{\vartheta}_k(t), & j + 1 \leq k \leq N. \end{cases}$$

Introduce the estimation errors:

$$\tilde{\mathcal{J}}_1(t) = \mathcal{J}_1(t) - \hat{\mathcal{J}}_1(t), \tilde{Q}(t) = Q(t) - \hat{Q}(t)$$

and

$$\tilde{\vartheta}(t) = \vartheta(t) - \hat{\vartheta}(t),$$

which yields

$$\begin{cases} \dot{\tilde{Q}}(t) = \mathsf{H}(A)\tilde{Q}(t) + \tilde{\xi}(0,t)(\mu - \mathcal{D}\vartheta), \\ \dot{\tilde{\mathcal{J}}}_1(t) = -\chi\tilde{\mathcal{J}}_1(t) + \tilde{Q}_1(t) + \varrho(t)^\top\tilde{\vartheta}(t) + \tilde{\xi}(0,t)(b_1 + \chi), \\ \dot{\tilde{\vartheta}}_k(t) = -g\varrho_k(t)\tilde{\mathcal{J}}_1(t) + g\varrho_k(t)\tilde{\xi}(0,t), \quad 1 \le k \le j, \\ \dot{\tilde{\vartheta}}_k(t) = -g\varrho_k(t)\tilde{\mathcal{J}}_1(t) - a\tilde{\vartheta}_k(t) + g\varrho_k(t)\tilde{\xi}(0,t), \quad j+1 \le k \le N. \end{cases} \tag{4.15}$$

Owing to the Hurwitz property of $\mathsf{H}(A)$ and the exponential decay of $\left|\tilde{\xi}(0,t)\right|$ to zero, we drive $\left\|\tilde{Q}(t)\right\|$, which also exhibits exponential convergence to zero. This isolates the $(\tilde{\mathcal{J}}_1, \tilde{\vartheta})$-subsystem. For stability analysis, we first consider the nominal system obtained by setting $\tilde{Q}_1(t) \equiv 0$ and $\tilde{\xi}(0,t) \equiv 0$ in (4.15)

$$\begin{cases} \dot{\tilde{\mathcal{J}}}_1(t) = -\chi\tilde{\mathcal{J}}_1(t) + \varrho(t)^\top\tilde{\vartheta}(t), \\ \dot{\tilde{\vartheta}}_k(t) = -g\varrho_k(t)\tilde{\mathcal{J}}_1(t), \quad 1 \le k \le j, \\ \dot{\tilde{\vartheta}}_k(t) = -g\varrho_k(t)\tilde{\mathcal{J}}_1(t) - a\tilde{\vartheta}_k(t), \quad j+1 \le k \le N. \end{cases} \tag{4.16}$$

Defining the Lyapunov function

$$\Xi(t) = \frac{1}{2}(\tilde{\mathcal{J}}_1^2 + \frac{1}{g}\tilde{\vartheta}^\top\tilde{\vartheta} + \tau(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)^\top(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)), \tag{4.17}$$

with τ being a positive real number to be specified,

$$\varrho[j] = [\varrho_1, \varrho_2, \cdots, \varrho_j]^\top, \quad \tilde{\vartheta}[j] = [\tilde{\vartheta}_1, \tilde{\vartheta}_2, \cdots, \tilde{\vartheta}_j]^\top,$$

while $\mathcal{V}(t)$ is given by

$$\begin{cases} \dot{\mathcal{V}} = -\mathcal{V} + \varrho[j]\varrho^\top[j](t), \quad \mathcal{V}(0) = e^{-T}QI_j, \\ \int_t^{t+T} \varrho[j](\epsilon)\varrho^\top[j](\epsilon)d\epsilon \ge QI_j, \quad \forall t \ge 0. \end{cases} \tag{4.18}$$

Because of the boundedness of $\varrho[j](t)$, $\|\varrho[j](t)\| \le \varrho_M$, $\forall t \ge 0$, combining (4.18), we have

$$Qe^{-2T}I \le \mathcal{V}(t) \le \varrho_M^2 I, \quad \forall t \ge 0. \tag{4.19}$$

From (4.16), $\forall t > \Lambda$, the time derivative of $\Xi(t)$ is

$$\begin{aligned}\dot{\Xi}(t) &= -\chi\tilde{\mathcal{J}}_1^2 + \varrho^\top \tilde{\vartheta}\mathcal{J}_1 - \varrho^\top \tilde{\vartheta}\mathcal{J}_1 - \sum_{k=j+1}^{N} \frac{a}{g}\tilde{\vartheta}_k^2 + \tau(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)^\top \\ &\quad (\dot{\mathcal{V}}\tilde{\vartheta}[j] - g\mathcal{V}\varrho[j]\tilde{\mathcal{J}}_1) + \tau(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)^\top(k\varrho[j]\tilde{\mathcal{J}}_1 - \varrho[j]\varrho^\top\tilde{\vartheta} - \dot{\varrho}[j]\tilde{\mathcal{J}}_1) \\ &= -\chi\tilde{\mathcal{J}}_1^2 - \sum_{k=j+1}^{N} \frac{a}{g}\tilde{\vartheta}_k^2 - \tau\left\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\right\|^2 + \tau(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)^\top \\ &\quad \left\{(k-1)\varrho[j]\tilde{\mathcal{J}}_1 - \varrho[j]\sum_{k=j+1}^{N}\varrho_k\tilde{\vartheta}_k - g\mathcal{V}\varrho[j]\tilde{\mathcal{J}}_1 - \dot{\varrho}[j]\tilde{\mathcal{J}}_1\right\}.\end{aligned} \tag{4.20}$$

Since $\varrho_k(t)$, $\dot{\varrho}_k(t)$, and $\mathcal{V}(t)$ are uniformly bounded, there exists a constant $C > 0$ such that

$$\|R(t)\| \leq C(\|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|)$$

for all $t \geq \Lambda$. Applying the Cauchy-Schwarz inequality,

$$(\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1)^\top R(t) \leq \|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\| \cdot \|R(t)\| \leq C\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|(\|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|).$$

Thus,

$$\dot{\Xi}(t) \leq -\chi\tilde{\mathcal{J}}_1^2 - \sum_{k=j+1}^{N}\frac{a}{g}\tilde{\vartheta}_k^2 - \tau\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|^2 + \tau C\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|(\|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|).$$

Applying Young's inequality

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$$

with

$$a = \|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|$$

and

$$b = \|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|,$$

we obtain, for any $\epsilon > 0$:

$$\dot{\Xi}(t) \leq -\chi\tilde{\mathcal{J}}_1^2 - \sum_{k=j+1}^{N}\frac{a}{g}\tilde{\vartheta}_k^2 - \tau\left(1 - \frac{\epsilon C}{2}\right)\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|^2 + \frac{\tau C}{2\epsilon}(\|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|)^2.$$

Since

$$(\|\tilde{\mathcal{J}}_1\| + \|\tilde{\vartheta}\|)^2 \leq 2(\|\tilde{\mathcal{J}}_1\|^2 + \|\tilde{\vartheta}\|^2),$$

we have

$$\dot{\Xi}(t) \leq -\left(\chi - \frac{\tau C}{\epsilon}\right)\tilde{\mathcal{J}}_1^2 - \sum_{k=j+1}^{N}\left(\frac{a}{g} - \frac{\tau C}{\epsilon}\right)\tilde{\vartheta}_k^2 - \tau\left(1 - \frac{\epsilon C}{2}\right)\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|^2. \tag{4.21}$$

Now, from the Lyapunov function definition (4.17) and the uniform positive definiteness of $\mathcal{V}(t)$ in (4.19), there exist constants $n_1, n_2 > 0$ such that

$$n_1(\tilde{\mathcal{J}}_1^2 + \|\tilde{\vartheta}\|^2) \leq \Xi(t) \leq n_2(\tilde{\mathcal{J}}_1^2 + \|\tilde{\vartheta}\|^2 + \|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|^2).$$

Choose $\epsilon = 1/C$, then select $\tau > 0$ sufficiently small such that

$$\chi - \frac{\tau C}{\epsilon} = \chi - \tau C^2 > 0$$

and

$$\frac{a}{g} - \frac{\tau C}{\epsilon} = \frac{a}{g} - \tau C^2 > 0.$$

With these choices, (4.21) becomes

$$\dot{\Xi}(t) \leq -\alpha_1\tilde{\mathcal{J}}_1^2 - \alpha_2\|\tilde{\vartheta}\|^2 - \alpha_3\|\mathcal{V}\tilde{\vartheta}[j] - \varrho[j]\tilde{\mathcal{J}}_1\|^2,$$

for some $\alpha_1, \alpha_2, \alpha_3 > 0$. Using the upper bound on $\Xi(t)$, we finally obtain

$$\dot{\Xi}(t) \leq -\alpha\Xi(t), \quad t \geq \Lambda, \tag{4.22}$$

for some $\alpha > 0$, establishing exponential stability of the nominal system (4.16).

The inequality (4.22) establishes that the origin of the nominal system (4.16) is exponentially stable. The full error dynamics (4.15) are this exponentially stable system perturbed by the terms $\tilde{Q}_1(t)$ and $\tilde{\xi}(0, t)$, both of which converge to zero exponentially. By the exponential stability lemma for linear time-varying systems (see [22, Lemma 1.2]), we conclude that

$$\|\tilde{\mathcal{J}}_1(t)\| + \|\tilde{\vartheta}(t)\| \leq Me^{-\mu t}, \quad t \geq 0,$$

for some $M, \mu > 0$. Since $\|\tilde{Q}(t)\|$ also decays exponentially, it follows from (4.6) and (4.7) that

$$\tilde{\ell}(t) = \ell(t) - \hat{\ell}(t) = \begin{bmatrix} \tilde{\mathcal{J}}_1(t) \\ \tilde{Q}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{k=1}^{N} \zeta_k(t)\tilde{\vartheta}_k + b\tilde{\mathcal{J}}_1(t) \end{bmatrix}, \tag{4.23}$$

It is evident that the boundedness of $\zeta_k(t)$ ensures exponential convergence of $\tilde{\ell}(t)$ to zero.

This completes the proof. □

At this stage,

$$\begin{aligned} U(t) = &-\hat{\Omega}_t(1, t) + m_s(1, t) + f_0'(1, \hat{\vartheta})\hat{\ell}(t) - \frac{\hbar_5}{1 - \hbar_1}e^{2\gamma}\hat{\Omega}(0, t) \\ &- 2\hbar_5\int_0^1 e^{2\gamma(1-\epsilon)}(\hat{\Omega}_t(\epsilon, t) + \hat{\Omega}_s(\epsilon, t))d\epsilon. \end{aligned} \tag{4.24}$$

5. Well-posedness and stability of the closed-loop system

The feedback interconnection of system (2.1) with controller (4.24) yields the following formulation

$$\begin{cases} y_{tt}(s,t) = y_{ss}(s,t) + \gamma y_t(0,t) + \mathcal{F}(s)D_1\alpha(t), \\ y_s(0,t) = D_2\alpha(t), \\ y_s(1,t) = U(t) + D_3\alpha(t), \\ \quad \dot{\alpha}(t) = G\alpha(t), \\ \quad \mathcal{E}(t) = y(0,t) - D_4\alpha(t), \\ \quad U(t) = -\hat{\Omega}_t(1,t) + m_s(1,t) + f_0'(1,\hat{\vartheta})\hat{\ell}(t) - \dfrac{\hbar_5}{1-\hbar_1}e^{2\gamma}\hat{\Omega}(0,t) \\ \qquad\qquad - 2\hbar_5\displaystyle\int_0^1 e^{2\gamma(1-\epsilon)}(\hat{\Omega}_t(\epsilon,t) + \hat{\Omega}_s(\epsilon,t))d\epsilon, \\ \hat{\Omega}_{tt}(s,t) = \hat{\Omega}_{ss}(s,t), \\ \hat{\Omega}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\hat{\Omega}_t(0,t) + \hbar_4\hat{\Omega}(0,t) - (\hbar_4 + \dfrac{\gamma}{1-\hbar_1})(\mathcal{E}(t) - m(0,t)), \\ \hat{\Omega}_s(1,t) = U(t) - m_s(1,t) - f_0'(1,\hat{\vartheta})\hat{\ell}(t), \\ \quad y_\ell(t) = -\mathcal{E}(t) + m(0,t) + \hat{\xi}(0,t), \\ \hat{\xi}_{tt}(s,t) = \hat{\xi}_{ss}(s,t), \\ \hat{\xi}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\hat{\xi}_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\hat{\xi}(0,t) + \hbar_2(\hat{\xi}(0,t) + m(0,t) - \mathcal{E}(t)) \\ \qquad\qquad + \hbar_3(\hat{\xi}_t(0,t) + m_t(0,t) - \dot{\mathcal{E}}(t)), \\ \hat{\xi}_s(1,t) = U(t) - m_s(1,t), \\ m_t(s,t) = -m_s(s,t) + \dfrac{\gamma}{\hbar_1}m(0,t), \\ m(0,t) = \hbar_1\mathcal{E}(t), m(s,0) = m_0(s), \\ \quad \dot{\hat{\mathcal{J}}}_1(t) = \hat{\mathcal{Q}}_1(t) + b_1 y_\ell(t) + \displaystyle\sum_{k=1}^{N}\varrho_k(t)\hat{\vartheta}_k(t) + \chi(y_\ell(t) - \hat{\mathcal{J}}_1(t)), \\ \quad \dot{\hat{\mathcal{Q}}}(t) = \mathsf{H}(A)\hat{\mathcal{Q}}(t) + \mu y_\ell, \\ \quad \hat{\ell}(t) = \begin{bmatrix} \hat{\mathcal{J}}_1(t) \\ \hat{\mathcal{Q}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \displaystyle\sum_{k=1}^{N}\zeta_k(t)\hat{\vartheta}_k(t) + b\hat{\mathcal{J}}_1(t) \end{bmatrix}, \end{cases} \tag{5.1}$$

with the filters (4.5) and the parameter update law

$$\begin{cases} \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)), \quad e^{-|\det(\Theta_k)|^{\frac{1}{k}}t} \le \frac{1}{2}, \\ \dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)) - a\hat{\vartheta}_k(t), \quad else, \end{cases} \tag{5.2}$$

where $1 \le k \le N$. Then, consider (5.1) in

$$\mathbb{X} = H^2(0,1) \times \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}^N \times \mathbb{U} \times \mathbb{R}^{2N\times N}.$$

Theorem 5.1. *The closed-loop system* (5.1) *is well-posed for all admissible uncertainties, including unknown coefficients* D_1, D_2, D_3, D_4, G*, uncertain function* $\mathcal{F}(\cdot)$*, and initial states* $(y(\cdot,0), y_t(\cdot,0),$ $\hat{\Omega}(\cdot,0), \hat{\Omega}_t(\cdot,0), \hat{\xi}(\cdot,0), \hat{\xi}_t(\cdot,0), \hat{\mathcal{J}}_1(0), \hat{Q}(0), \hat{\vartheta}(0), \Theta(0), \zeta_k(0)_{k=1}^N) \in \mathbb{X}$*. It possesses a unique solution in* $\mathbb{X}$ *that guarantees exponential convergence of* $\mathcal{E}(t)$ *to zero, along with the condition*

$$\int_0^\infty e^{\mu t} \left|\dot{\mathcal{E}}(t)\right|^2 dt < \infty.$$

Proof. Employing the variable set $\Omega(s,t), \tilde{\Omega}(s,t), \tilde{\xi}(s,t), \tilde{\mathcal{J}}(t), \tilde{Q}(t)$, and $\tilde{\vartheta}(t)$ defined by (3.18), (3.21), (4.2), (4.15), system (5.1) is equivalent to

$$\begin{cases}
\Omega_{tt}(s,t) = \Omega_{ss}(s,t), \\
\Omega_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\Omega_t(0,t) - \dfrac{\gamma}{1-\hbar_1}\Omega(0,t), \\
\Omega_s(1,t) = -\tilde{\Omega}_t(1,t) - \Omega_t(1,t) + \mathcal{R}(t) - \dfrac{\hbar_5}{1-\hbar_1}e^{2\gamma}(\tilde{\Omega}(0,t) + \Omega(0,t)) \\
\qquad - 2\hbar_5 \displaystyle\int_0^1 e^{2\gamma(1-\epsilon)}(\tilde{\Omega}_t(\epsilon,t) + \Omega_t(\epsilon,t) + \tilde{\Omega}_s(\epsilon,t) + \Omega_s(\epsilon,t))d\epsilon, \\
\tilde{\Omega}_{tt}(s,t) = \tilde{\Omega}_{ss}(s,t), \\
\tilde{\Omega}_s(0,t) = \dfrac{\hbar_1}{1-\hbar_1}\tilde{\Omega}_t(0,t) + \hbar_4\tilde{\Omega}(0,t), \\
\tilde{\Omega}_s(1,t) = 0, \\
y_\ell(t) = -\mathcal{E}(t) + m(0,t) + \Omega(0,t) + f_0(0)\ell(t) - \tilde{\xi}(0,t), \\
\tilde{\xi}_{tt}(s,t) = \tilde{\xi}_{ss}(s,t), \\
\tilde{\xi}_s(0,t) = (\hbar_2 - \dfrac{\gamma}{1-\hbar_1})\tilde{\xi}(0,t) + (\hbar_3 + \dfrac{\hbar_1}{1-\hbar_1})\tilde{\xi}_t(0,t), \\
\tilde{\xi}_s(1,t) = 0, \\
\mathcal{E}(t) = \Omega(0,t) + m(0,t) = \dfrac{1}{1-\hbar_1}\Omega(0,t), \\
m_t(s,t) = -m_s(s,t) + \dfrac{\gamma}{\hbar_1}m(0,t), \\
m(0,t) = \hbar_1\mathcal{E}(t), m(s,0) = m_0(s), \\
\dot{\hat{\mathcal{J}}}_1(t) = \hat{Q}_1(t) + b_1 y_\ell(t) + \displaystyle\sum_{k=1}^{j} \varrho_k(t)\hat{\vartheta}_k(t) + \chi(y_\ell(t) - \hat{\mathcal{J}}_1(t)), \\
\dot{\hat{Q}}(t) = \mathsf{H}(A)\hat{Q}(t) + \mu y_\ell, \\
\dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)),\ e^{-|\det(\Theta_k)|^{\frac{1}{k}}t} \le \dfrac{1}{2}, \\
\dot{\hat{\vartheta}}_k(t) = g\varrho_k(t)(y_\ell(t) - \hat{\mathcal{J}}_1(t)) - a\hat{\vartheta}_k(t),\ else, \\
\dot{\zeta}_k(t) = \mathsf{H}(A)\zeta_k(t) - [0\ I_{2N}]E_{2k}y_\ell(t),\ \ \zeta_k(t) \in \mathbb{R}^{2N}, \\
\dot{\Theta}(t) = -\varepsilon_b\Theta(t) + \varepsilon_c\varrho(t)\varrho(t)^\top, \Theta(t) \in \mathbb{R}^{N\times N},
\end{cases} \tag{5.3}$$

where

$$\mathcal{R}(t) = f_0'(1,\hat{\vartheta})\hat{\ell}(t) - f_0'(1,\vartheta)\ell(t). \tag{5.4}$$

The well-posedness and stability of the $(\tilde{\Omega}, \tilde{\xi}, \hat{\mathcal{J}}, \hat{\mathcal{Q}}, \hat{\vartheta}, \{\zeta_k\}_{k=1}^N, \Theta)$- subsystem in system (5.3) have been established in the preceding sections and Lemma 4.1. Consequently, the closed-loop system exhibits a cascade-coupling structure where the stable error subsystems $(\tilde{\Omega}, \tilde{\xi}, \hat{\mathcal{J}}, \hat{\mathcal{Q}}, \hat{\vartheta}, \{\zeta_k\}, \Theta)$ serve as external inputs to the Ω-subsystem. This structural property is crucial for the overall stability analysis.To establish the stability of the entire closed-loop system, it remains to analyze the Ω-subsystem in (5.3), which admits an operator formulation

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\Omega(\cdot,t)\\ \Omega_t(\cdot,t)\end{pmatrix} = \mathcal{K}\begin{pmatrix}\Omega(\cdot,t)\\ \Omega_t(\cdot,t)\end{pmatrix} + \mathcal{B}\Big(&\mathcal{R}(t) - \tilde{\Omega}_t(1,t)\\ &-\frac{\hbar_5}{1-\hbar_1}e^{2\gamma}\tilde{\Omega}(0,t) - 2\hbar_5\int_0^1 e^{2\gamma(1-\epsilon)}(\tilde{\Omega}_t(\epsilon,t)+\tilde{\Omega}_s(\epsilon,t))d\epsilon\Big),\end{aligned} \tag{5.5}$$

with the operator $\mathcal{K}$: $D(\mathcal{K}) (\subseteq \mathbb{X}) \to \mathbb{X}$

$$\begin{cases}\mathcal{K}(\varpi,\psi)^\top = (\psi,\varpi'')^\top, \quad \forall(\varpi,\psi)^\top \in D(\mathcal{K}),\\ D(\mathcal{K}) = \Big\{(\varpi,\psi)^\top \in \mathbb{X} : \mathcal{K}(\varpi,\psi)^\top \in \mathbb{X}\Big|\varpi'(0) = \dfrac{\hbar_1}{1-\hbar_1}\psi(0) - \dfrac{\gamma}{1-\hbar_1}\varpi(0),\\ \qquad \varpi'(1) = -\psi(1) - \frac{\hbar_5}{1-\hbar_1}e^{2\gamma}\varpi(0) - 2\hbar_5\int_0^1 e^{2\gamma(1-\epsilon)}(\psi(\epsilon)+\varpi'(\epsilon))d\epsilon\Big\},\end{cases} \tag{5.6}$$

and

$$\mathcal{B}_1 = (0, \delta(s-1))^\top.$$

It can be verified that the principal part of the Ω- subsystem coincides with the $\hat{w}$- subsystem in [25] and exhibits exponential decay. Furthermore,

$$\mathcal{R}(\cdot) \in C([0,\infty);\mathbb{R})$$

converges exponentially to zero as $t \to \infty$. To achieve this, it suffices to verify the time-domain continuity of $f_0'(1,\hat{\vartheta}(t))$ and the exponential convergence

$$\lim_{t\to\infty}\left|f_0'(1,\hat{\vartheta}) - f_0'(1,\vartheta)\right| = 0.$$

Thus,

$$\vartheta(\cdot) \in C([0,\infty);\mathbb{R}^m),$$

and $\hat{\vartheta}(t)$ converges exponentially to ϑ in norm; the boundedness of $\hat{\vartheta}$ follows as a result. Let

$$\vartheta, \hat{\vartheta}(t) \in [-D,D]^N, D > 0,$$

and according to Lemma 3.3, $f_0'(1,\vartheta)$ is continuously differentiable in ϑ, which implies its Lipschitz continuity over $[-D,D]^N$. Furthermore, it can be established that $f_0'(1,\hat{\vartheta}(t))$ is continuous in time, and

$$\lim_{t\to\infty}\left\|f_0'(1,\hat{\vartheta}) - f_0'(1,\vartheta)\right\| \le \lim_{t\to\infty} L\|\hat{\vartheta}(t) - \vartheta\| = 0$$

are both exponentially. By applying [25, Lemma 2.3], (which establishes well-posedness and exponential stability of a wave equation with analogous boundary structure and parameters $\hbar_1$, γ, $\hbar_5$ matching k_1, q, k_3 therein), the Ω-subsystem of (5.3) possesses a unique solution

$$(\Omega, \Omega_t) \in C([0,\infty); \mathcal{H})$$

satisfying

$$\|(\Omega(\cdot,t), \Omega_t(\cdot,t))\|_{\mathcal{H}} \le L' e^{-wt},$$

for some $L', w > 0$.

At this point, we have established the exponential stability of all subsystems. The overall closed-loop stability follows from the fact that the original system state can be recovered through invertible coordinate transformations from these exponentially stable subsystems. Under the transformations

$$\begin{aligned} y(s,t) &= \Omega(s,t) + f_0(s)\ell(t) - \varkappa(s)\alpha(t) + m(s,t), \\ \hat{\Omega}(s,t) &= \Omega(s,t) + \tilde{\Omega}(s,t), \\ \hat{\xi}(s,t) &= \Omega(s,t) + f_0(s)\ell(t) - \tilde{\xi}(s,t), \end{aligned}$$

the state vector $(y, y_t, \hat{\Omega}, \hat{\Omega}_t, \hat{\xi}, \hat{\xi}_t)$ is well-posed in

$$C([0,\infty); \mathcal{H}^2) \bigcap L^\infty(0,\infty; \mathcal{H}^2)$$

and exhibits uniform boundedness in time. Furthermore, by Sobolev trace theorem, $|\Omega(0,t)| \to 0$, as $t \to \infty$.

To complete the proof of Theorem 4.1, it remains to show that $\hat{\mathcal{E}}(t)$ is square-integrable with exponential weight. This is achieved through an energy estimate.

Let

$$\delta(t) = 2\int_0^1 (s-1)\Omega_t(s,t)\Omega_s(s,t)ds,$$

and it satisfies

$$|\ell(t)| \le \|(\Omega(\cdot,t), \Omega_t(\cdot,t))\|_{\mathcal{H}}^2.$$

Differentiation and subsequent integration by parts leads to

$$\begin{aligned} \dot{\delta}(t) &= 2\int_0^1 (s-1)\Omega_t(s,t)\Omega_{st}(s,t)ds + 2\int_0^1 (s-1)\Omega_{ss}(s,t)\Omega_s(s,t)ds \\ &= \Omega_t^2(0,t) + \Omega_s^2(0,t) - \int_0^1 [\Omega_t^2(s,t) + \Omega_s^2(s,t)]ds \\ &\ge \Omega_t^2(0,t) - \|\Omega(\cdot,t), \Omega_t(\cdot,t)\|_{\mathcal{H}}^2, \end{aligned}$$

$\forall\ 0 < \mu < 2w$,

$$\int_0^\infty e^{\mu t}\Omega_t^2(0,t)dt \le \int_0^\infty e^{\mu t}\|\Omega(\cdot,t), \Omega_t(\cdot,t)\|_{\mathcal{H}}^2 dt + |\delta(0)| + \mu\int_0^\infty e^{\mu t}|\delta(t)|dt < \infty.$$

□

6. Simulation results

To evaluate the effectiveness of the proposed controller (4.24), numerical simulations are performed on the resulting closed-loop system (5.1), which governs the dynamics of the original system (2.1). The disturbances are applied to all input channels according to the structure of the studied system, with

$$\rho_1(t) = \rho_2(t) = \rho_3(t) = 2\sin(0.6t),$$

while the reference signal is set as

$$D_4\alpha(t) = 2\sin(2t) + 1.$$

The remaining parameters are configured as

$$\begin{aligned} &\mathcal{F}(s) = s/6, \quad \gamma = 2, \quad \hbar_1 = 1, \quad \hbar_2 = 2, \\ &\hbar_3 = 2, \quad \hbar_4 = 0.5, \quad \hbar_5 = 3, \quad \varepsilon_b = \varepsilon_c = 0.5, \\ &b = [4, 6, 4, 1]^\top, \quad a = 0.7, \quad g = 1, \quad \chi = 0.05. \end{aligned}$$

All numerical simulations presented in this section were conducted using MATLAB 2020b. To comprehensively validate the accuracy of the proposed adaptive observer, the convergence of parameter estimates, and the effectiveness of the controller, this study draws on relevant simulation verification methods [29–31] to construct a numerical simulation design tailored to the system under investigation. The system is initialized with the following conditions

$$\begin{cases} y(s,0) = 0.7(\sin 5s) + s, \\ y_t(s,0) = 0, \\ \hat{\Omega}(s,0) = \sin 3s - 0.03s, \\ \hat{\Omega}_t(s,0) = \sin(2s) - s, \\ \hat{\xi}(s,0) = \sin 0.5s + \cos s, \\ \hat{\xi}_t(s,0) = \sin(2\pi s) - s, \\ \Theta(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ (\hat{Q}(0), \hat{\vartheta}(0), \hat{\mathcal{J}}_1(0), \{\zeta_k(0)\}_{k=1}^N) = (0,0,0,0). \end{cases} \tag{6.1}$$

In contrast to existing studies (e.g., [24]) where the disturbances and the reference share the same frequency, the numerical example here considers a more general scenario in which the disturbances on all channels have one frequency while the reference signal possesses a different frequency. This validates the controller's ability to handle multi-source disturbances with distinct frequencies.

The simulation results are now discussed with an emphasis on their physical interpretation, directly linking the observed phenomena to the theoretical challenges addressed in this paper.

Figure 1 depicts the spatiotemporal displacement $y(s,t)$ of the controlled wave system. The achieved bounded evolution, starting from the initial condition given by (6.1), provides a numerical verification of the theoretical result established in Section 4: The proposed controller successfully stabilizes a system that is open-loop unstable due to the nonlocal term $\gamma y_t(0,t)$. The stabilization is achieved despite the persistent action of distributed and boundary disturbances.

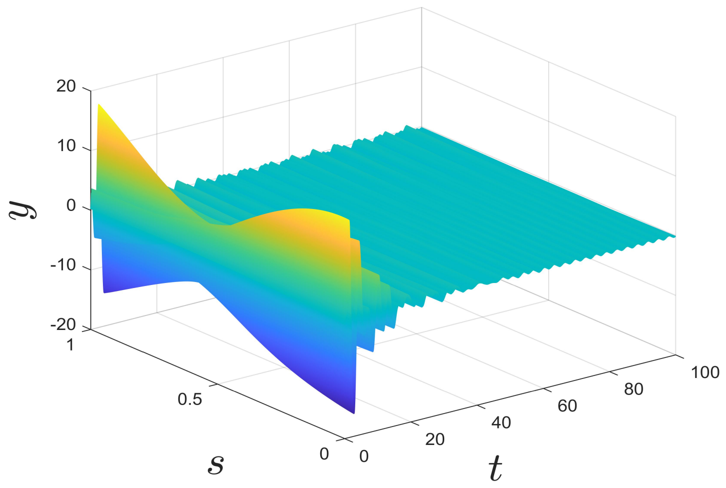

Figure 1. Solution of closed-loop system (5.1) (y-part).

Figures 2 and 3 display the states of the observers, $\hat{\Omega}$ and $\hat{\xi}$. Their bounded and well-behaved transients demonstrate the physical realizability and effectiveness of the observer designs presented before. Critically, the observer for ξ successfully operates using only the tracking error $\mathcal{E}(t)$ and its derivative $\dot{\mathcal{E}}(t)$, which are the only measured outputs. This validates the practical premise of output-feedback control. The observer for $\hat{\Omega}$, in turn, remains stable, confirming that the feedforward part of the control law is implemented based on a reliable internal state estimate.

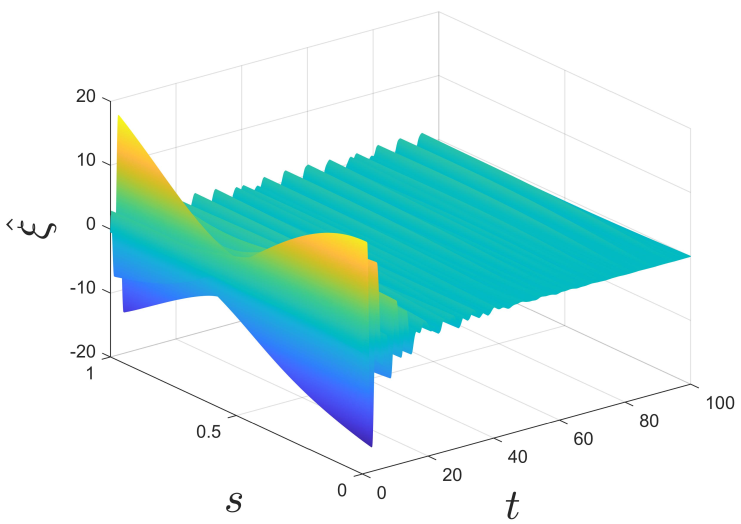

Figure 2. Solution of closed-loop system (5.1) ($\hat{\Omega}$-part).

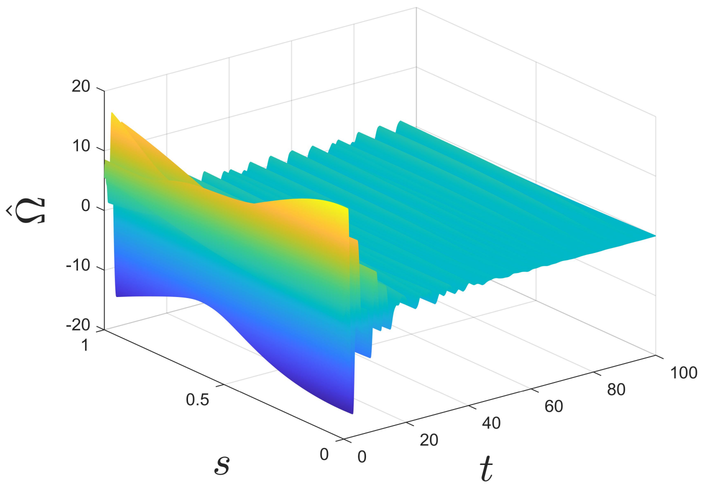

Figure 3. Solution of closed-loop system (5.1) ($\hat{\xi}$-part).

The primary physical achievement is evidenced in Figure 4. The exponential decay of the tracking error

$$\mathcal{E}(t) = y(0, t) - r(t)$$

to zero confirms that the physical output $y(0, t)$ asymptotically tracks the reference signal $r(t)$. This occurs under the combined difficulties of: (i) structural instability from the nonlocal term, (ii) multi-channel harmonic disturbances $\rho_k(t)$ with an unknown frequency acting on all input channels, and (iii) an unknown reference frequency. Therefore, the simulation validates that the adaptive error-feedback regulator not only stabilizes the plant but also achieves the core objective of output regulation in a realistic scenario where the exosystem generates both disturbances and the reference is fully unknown.

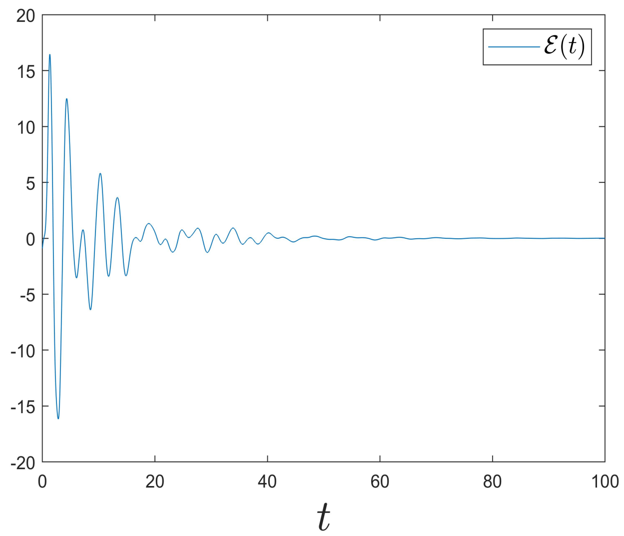

Figure 4. Tracking performance ($\mathcal{E}(t)$).

Although γ does not appear explicitly in $\mathcal{E}(t)$, it enters the Ω-dynamics via (3.19) and the controller via (3.31), thereby affecting the convergence speed. This influence can be compensated by tuning $\hbar_1$, $\hbar_2$, $\hbar_5$, as demonstrated in the simulation with $\gamma = 2$.

7. Conclusions

This paper has developed an adaptive error feedback tracking scheme for a 1-D wave equation subject to a nonlocal term and multi-channel unknown harmonic disturbances. The core of the work lies in providing a systematic and rigorous solution to the integrated problem of stabilizing an open-loop unstable wave system while achieving exact output regulation under multiple exogenous disturbances with unknown frequencies. To address these challenges sequentially, the control design first eliminated the nonlocal term involving boundary velocity through a coordinate transformation that constructed an auxiliary system. The controlled PDE was then decoupled from the exosystem ODE, thereby confining disturbances exclusively to the tracking error. Furthermore, a feedforward plus state feedback controller was developed, accompanied by a state observer for system state reconstruction and a parameter adaptation mechanism for uncertainty estimation. Within this framework, an error-feedback regulator was implemented using the error signal and its temporal derivative. Theoretical analysis has established the closed-loop system's well-posedness and stability, together with exponential convergence of the tracking error. This work thus demonstrates and validates a complete control framework for a class of unstable wave equations with complex, multi-source unknown harmonic disturbances. Future work may explore relaxing the assumption of a known upper bound on the number of disturbance frequencies, as well as extending the control architecture to higher-dimensional spatial domains or other types of unstable PDEs.

Author contributions

Xinting Xiao: conceptualization, methodology, writing—Original draft; Feng-Fei Jin: writing—review & editing, investigation, validation; Xiyu Liu: writing–review & editing, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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