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Research article

Statistical inference for a one-parameter lifetime model under unified progressive hybrid censoring with binomial random removals

Ehab M. Almetwally[1,2,*]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia

[2] Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt

* **Correspondence:** Email: emalmetwally@imamu.edu.sa, ehab.metwaly@deltauniv.edu.eg.

Abstract: A unified progressive hybrid censoring scheme is introduced by combining progressive and hybrid plans, allowing tests to be terminated either after a predetermined number of failures or at a fixed time. Both likelihood and Bayesian procedures are developed for estimating the parameter, reliability, and hazard rate of a one-parameter lifetime model when data are generated under this scheme. Maximum likelihood estimates are obtained via the Newton-Raphson algorithm, and asymptotic confidence intervals are constructed using the delta method with the Fisher information matrix. In addition, parametric bootstrap methods are employed for constructing confidence intervals. Within the Bayesian framework, Markov chain Monte Carlo techniques are employed under non-informative and informative independent gamma priors, with computational intractability addressed through the Metropolis-Hastings algorithm. Progressive censoring with binomial random removals has also been considered within this framework to enhance flexibility in test termination and data collection. Extensive Monte Carlo simulations are conducted to compare the efficiency of the likelihood and Bayesian estimators across multiple censoring designs, and the superiority of Bayesian inference with informative priors is demonstrated. The applicability of the proposed estimators is illustrated using three real datasets: tensile strength of polyester fibers, aircraft air-conditioning failures, and ordered failure times. The one-parameter model is further compared with ten standard unit distributions. The censoring framework is successfully applied to these datasets, confirming its practical value in modeling reliability and failure behavior.

Keywords: progressive hybrid censoring; binomial random removals; one-parameter lifetime model; maximum likelihood; Bayesian inference; MCMC; reliability
Mathematics Subject Classification: 62N02, 62F15, 62N05

1. Introduction

Censoring and life-testing are key for time-to-event analyses when complete follow-up isn't available. They let investigators extract valid conclusions from partial data common in medicine, engineering, and the social sciences while keeping studies efficient and affordable so decisions can be made sooner. Used well, they inform public health, strengthen product reliability, and guide policy. Hybrid censoring blends Type-I (time-limited) and Type-II (failure-count) schemes: the experiment stops at the earlier of a prechosen number of failures r or a fixed time t. Although this guarantees the test lasts no longer than t, a practical drawback is that too few failures may be observed by t to support strong inference.

Hybrid censoring which combines elements of Type-I and Type-II designs terminates a test either by time or by failure count. For information about Type-I and Type-II censored sample, see [1]. A common specification sets the stopping time as $T^{\bullet} = \min\{X_{k:n}, T\}$, where $k \in \{1, \ldots, n\}$ is the target number of failures and $T > 0$ is a preset time limit. While this caps the test duration at T, a drawback is that too few failures may be observed by T to support reliable inference. To distinguish variants, [2] named the min rule Type-I hybrid censoring (HCT1) and proposed the complementary Type-II hybrid censoring (HCT2) with $T^{\bullet} = \max\{X_{k:n}, T\}$.

Childs et al. [2] referred to this scheme as Type-I hybrid censoring (HCT1) and introduced a companion design, Type-II hybrid censoring (HCT2), where the test ends at $\max\{Y_{m:m:n}, t\}$. Although HCT2 guarantees at least m failures, it may considerably lengthen the experiment when failures are sparse. To overcome the drawbacks of both HCT1 and HCT2, Chandrasekar et al. [3] proposed two generalized extensions, namely generalized HCT1 (GHCT1) and generalized HCT2 (GHCT2), collectively termed generalized hybrid censoring schemes. Under the GHCT1 scheme, n items enter a life test at time zero, with three prespecified quantities: integers $r, m \in \{1, 2, \ldots, n\}$ (with $r \leq m$ indicating failure-count requirements), and a censoring time $t \in (0, \infty)$. The stopping rule is: if $Y_{r:m:n} < t$, terminate at $\min\{Y_{m:m:n}, t\}$; otherwise, stop at $Y_{r:m:n}$. This extends HCT1 by permitting the test to continue beyond t when needed while guaranteeing observation of at least r failures. In contrast, GHCT2 tests n items with a fixed failure threshold $r \in \{1, 2, \ldots, n\}$ and two predetermined time bounds $t_1, t_2 \in (0, \infty)$ satisfying $t_1 < t_2$.

Under GHCT2, the stopping rule is: stop at t_1 if $Y_{r:n} < t_1$; stop at $Y_{r:n}$ if $t_1 < Y_{r:n} < t_2$; otherwise, stop at t_2 when $Y_{r:n} \geq t_2$. Although this modification of HCT2 confines the experiment to the interval $[t_1, t_2]$, it can still yield fewer, possibly no failures before t_2, echoing the limitation seen in HCT1. Hence, residual drawbacks persist in both generalized plans: in GHCT1, reliance on a single censoring time t may be insufficient to guarantee observation of m failures, and in GHCT2, very few or no failures may still occur prior to t_2. To balance the respective strengths and limitations of the two generalized schemes, Balakrishnan et al. [4] proposed a unified hybrid censoring (UHC) strategy. While UHC integrates the core features of GHCT1 and GHCT2 into a single, more adaptable framework, it does not permit the interim (progressive) removal of test units prior to termination.

Type-II progressive censoring (P-CT2) is widely used in reliability studies because it strikes a balance between statistical efficiency and practical feasibility; it can shorten test duration and reduce cost, particularly in biomedical and engineering applications. In addition, P-CT2 yields richer and more flexible data, improves inference for lifetime parameters, and supports more effective experimental design under real-world constraints (see Balakrishnan and Cramer [5]). To address the inability to remove surviving units during UHC tests, Górny and Cramer [6] proposed a unified progressive hybrid censoring plan (UPHCT1) that merges features of UHC and P-CT2 to enhance flexibility in life-testing experiments.

In progressive Type-II censoring (PCT2) with m observations, when the ith failure occurs, a preassigned number R_i of surviving units is randomly removed, and the experiment continues until the m-th failure time. Consider a life test that begins at time 0 with n identical units. Fix integers $r, m \in \{1, \dots, n\}$ with $r < m$, two control times $t_1 < t_2$ $(t_1, t_2 > 0)$, and a progressive removal scheme $\boldsymbol{R} = (R_1, \dots, R_m)$ satisfying $n = m + \sum_{i=1}^{m} R_i$. Under the UPHCT1 mechanism, removals are carried out as in the standard PCT2 plan: as soon as the ith failure occurs, R_i surviving units are randomly withdrawn. Let $X_{i:m:n}$ denote the ith progressive order statistic (the ith observed failure time) and write $R(t; \boldsymbol{\theta}) = 1 - F(t; \boldsymbol{\theta})$ for the parent survival (reliability) function.

The experiment terminates at the stopping time $s^\star$ given by

$$s^\star = \begin{cases} X_{m:m:n}, & \text{if } X_{m:m:n} < t_1, \\ t_1, & \text{if } X_{r:m:n} < t_1 \le X_{m:m:n}, \\ X_{r:m:n}, & \text{if } t_1 < X_{r:m:n} \le t_2, \\ t_2, & \text{if } t_2 < X_{r:m:n}. \end{cases} \tag{1.1}$$

Let d_i denote the number of failures observed by time t_i $(i = 1, 2)$, and let ν be the number of failures actually observed up to $s^\star$ (that $\nu \in \{m, d_1, r, d_2\}$, depending on which case of (1.1) occurs). The number of units remaining on test at termination is

$$R^\star = \begin{cases} n - m - \sum_{i=1}^{m-1} R_i, & \text{if } s^\star = Y_{m:m:n}, \\ n - d_1 - \sum_{i=1}^{d_1} R_i, & \text{if } s^\star = t_1, \\ n - r - \sum_{i=1}^{r-1} R_i, & \text{if } s^\star = Y_{r:m:n}, \\ n - d_2 - \sum_{i=1}^{d_2} R_i, & \text{if } s^\star = t_2. \end{cases} \tag{1.2}$$

If $\mathbf{x} = (x_{1:m:n}, \dots, x_{\nu:m:n})$ denotes the observed UPHCT1 data from a population with probability density function (PDF) $g(\cdot; \boldsymbol{\gamma})$ and cumulative distribution function (CDF) $F(\cdot; \boldsymbol{\gamma})$, the likelihood takes the standard progressive form

$$\mathcal{L}(\boldsymbol{\gamma} \mid \mathbf{x}) \propto \left\{ \prod_{i=1}^{\nu} g(x_{i:m:n}; \boldsymbol{\gamma}) \left[R(x_{i:m:n}; \boldsymbol{\gamma}) \right]^{R_i} \right\} \left[R(s^\star; \boldsymbol{\gamma}) \right]^{R^\star}. \tag{1.3}$$

As emphasized by Górny and Cramer, UPHCT1 merges the strengths of three controls progressive (unit removals), time-controlled (t_1, t_2), and failure–controlled (at least r failures) into a single, efficient design. In practice, this plan mitigates two common risks of conventional GHCT1 and GHCT2 schemes:

- ending too early with too few failures for reliable inference;
- running far beyond acceptable test duration.

Enforcing both a lower bound on the number of failures and an upper bound on time typically improves the precision of parameter estimation and the robustness of subsequent analysis. Its ability to accommodate constraints on time, cost, and attrition makes UPHCT1 broadly applicable across many reliability settings.

This work makes several original contributions on both theoretical and applied fronts. A parsimonious one-parameter unit distribution is proposed and shown to outperform several competing unit models, including the new unit–Lindley, Topp-Leone, Beta, unit-Weibull, unit–Gompertz, unit–XLindley, Alpha–Power Topp–Leone, unit log–logistic, and unit Burr–XII in goodness-of-fit and reliability inference across complete and censored datasets (including UPHCT1 settings). It is motivated by the need for flexible, information-rich life-testing designs that can handle the complexities of modern reliability data, especially for high-value or rare materials (e.g., tensile strength of polyester fibers, aircraft air-conditioning failures, and ordered failure-time studies). While prior studies provide rigorous analyses of the proposed unit distribution across diverse reliability settings, they also underscore its suitability in the presence of UPHCT1 data. By leveraging UPHCT1, which guarantees a minimum number of observed failures while confining the experiment to a fixed time window, and the proposed unit model, which flexibly captures upside-down-bathtub and heavy-tailed hazard behaviors, this paper develops the corresponding inferential procedures.

To fill the identified gap in the current literature, the presented research works under the framework of the UPHCT1 scheme and combines it with an efficient single-parameter model of unit lifetime. The main intention was to cover two aspects: (i) to offer a useful censoring scheme that can distribute a minimum number of failures under the limitation of the total test time, and (ii) to introduce a simplified single-parameter model that can model diverse shapes of the hazard function like bathtub and increasing failure rate (IFR) curves so that exact reliability analysis can be obtained without much complexity.

To meet this objective within the proposed framework, maximum-likelihood and Bayesian inferential procedures are developed for the model parameters together with the associated reliability measures. Bayesian computations are implemented via Markov chain Monte Carlo (MCMC) under diffuse (noninformative) and gamma priors, using a Metropolis-Hastings algorithm. Within these setups, large-sample (asymptotic) confidence intervals and Bayesian credible intervals are constructed for the unknown quantities. Furthermore, estimator performance is evaluated through a comprehensive Monte Carlo simulation which covers multiple censoring configurations and assessed by several precision metrics by which the robustness of the Bayesian procedures is demonstrated.

Recently, using the proposed censored sample, different lifetime models are investigated via likelihood and Bayes methods of parameter estimations; see, for example, Dutta and Kayal [7] for Burr Type-XII, Yousef et al. [8] for truncated Cauchy power exponential, Dutta et al. [9] for Kumaraswamy-G family, Anwar et al. [10] for inverted exponentiated Rayleigh, Mohammed et al. [11] for inverted Nadarajah–Haghighi, Prakash et al. [12] for Lomax, Elsherpieny and Abdel-Hakim [13] for Alpha-Power exponential distribution, Alotaibi et al. [14] for Hjorth Competing Risk Data, and so forth.

For the purposes of this analysis, two different methods of parameter estimation will be employed and contrasted with one another, namely Maximum Likelihood Estimates (MLEs), and a Bayesian estimate generated with a Metropolis-Hastings algorithm. The use of these two methods is theoretically and pragmatically justifiable on a number of grounds. The first, MLEss, is a classical choice particularly because of its optimality properties within a large sample framework. The second, the Bayesian model is able to generate a full posterior probability function, which, in turn, has a number of convenient properties; namely, it is able to generate full probabilistic inferences about model parameters without needing further tests, because credible regions often have better performance characteristics, especially in terms of precision in small samples, and because it is able to integrate prior information about the parameter sets being estimated.

From a computational perspective, both MLEs can be obtained by means of common numerical algorithms such as Newton-Raphson, whereas the Bayesian Estimates can be obtained by means of Markov Chain Monte Carlo simulations with Metropolis-Hastings sampling for ensured exploration of the posterior space. To conduct a comparison among these approaches, we will be interested in considering some finite sample performance criteria such as bias, mean squared error (MSE), coverage probability, average width of the confidence sets, besides comparing the decision theoretic risks with different loss functions, with the linear–exponential (LINEX) loss function being used in the Monte Carlo comparison. The comparison among these approaches will be used to measure robustness against finite samples and censoring, with goal of understanding in what types of situations a certain approach is preferable over the other, and for providing information about which estimator may be used given a specific situation concerning the sample size, proportion of censored observations, and existing prior information.

The remainder of the paper is organized as follows. Section 2 introduces unit distribution with one parameter. Sections 3 discusses the likelihood estimation for model based on the UPHCT1 mechanism. Section 4 develops Bayesian estimators for the model parameters and associated reliability measures. Section 5 obtains bootstrap algorithms. Section 6 summarizes the Monte Carlo study. Section 7 analyzes the three data sets. Finally, Section 8 concludes the paper.

2. Unit distribution with one parameter

Finite–support probability models are common in applications, for instance, the uniform on $[\gamma, b]$, beta on $[0, 1]$, truncated normal on $[\gamma, b]$, and arcsine on $[\gamma, b]$; some families (e.g., generalized Pareto) admit bounded support depending on their parameters. In particular, Muhammad [15] introduced a two-parameter lifetime distribution supported on $(0, b]$. We adopt that model reparameterized with $\gamma > 0$ in place of γ and recall its CDF, and PDF function as given in Eqs (2.1) and (2.2).

The CDF of the proposed model with parameters $\gamma > 0$ and $b > 0$ is given by

$$F(x) = \frac{(\gamma + 1)(x/b)}{\gamma + (x/b)}, \qquad 0 < x \le b, \tag{2.1}$$

and the corresponding PDF is

$$f(x) = \frac{\gamma b(\gamma + 1)}{(\gamma b + x)^2}, \qquad 0 < x \le b. \tag{2.2}$$

Figure 1 depicts density and hazard plots for different value os parameter γ. As γ increases, the PDF on $(0, 1)$ evolves from sharply left-skewed (mass near 0) to a flatter, more even profile, while the hazard shifts from a pronounced bathtub shape (high early risk, mid-interval dip, late surge) toward a mainly increasing IFR pattern. Hence, γ controls both skewness and the early-versus-late failure risk, demonstrating strong shape flexibility.

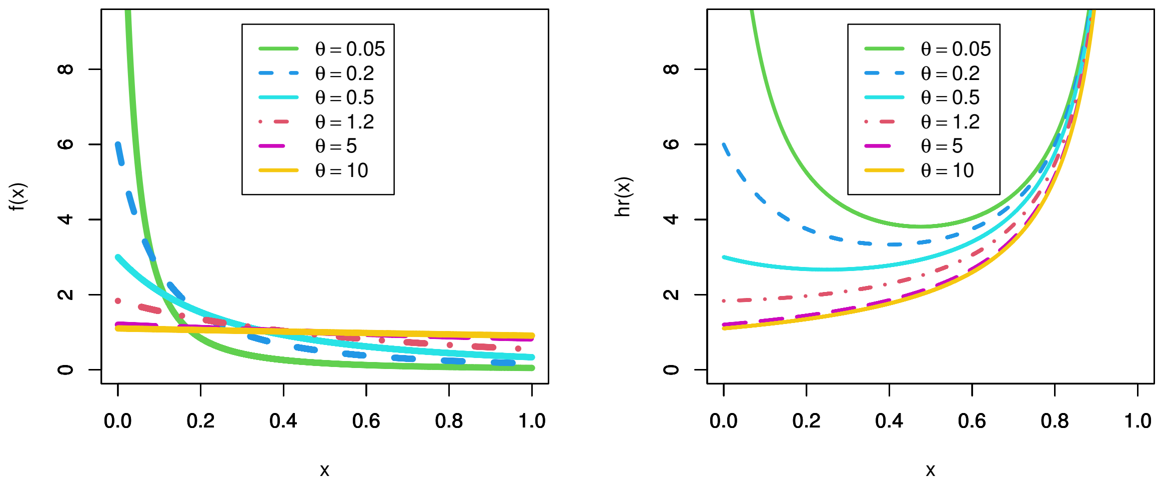

Figure 1. Density and hazard for different value os parameter γ.

Because b acts as a positive scale parameter in Eqs (2.1) and (2.2), it is convenient to study the unit–scale form. Setting $b = 1$ in the general expressions with shape $\gamma > 0$ yields the distribution on $(0, 1]$ with

$$F(x) = \frac{(\gamma + 1)\,x}{\gamma + x}, \qquad 0 < x \le 1, \tag{2.3}$$

$$f(x) = \frac{\gamma(\gamma + 1)}{(\gamma + x)^2}, \qquad 0 < x \le 1, \tag{2.4}$$

$$s(x) = \frac{\gamma(1 - x)}{\gamma + x}, \qquad 0 < x \le 1, \tag{2.5}$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{\gamma + 1}{(\gamma + x)(1 - x)}, \qquad 0 < x < 1. \tag{2.6}$$

Basic properties

Let F, f, S, and h denote the CDF, PDF, survival, and hazard functions of the unit-scale model $(b = 1)$.

- Support and endpoint limits: From (2.3) and (2.5), we obtain

$$F(0^+) = 0, \qquad F(1) = 1, \qquad s(0^+) = 1, \qquad s(1^-) = 0,$$

  so the distribution is supported on $(0, 1]$ and has the stated boundary behaviour.
- Monotonicity of the PDF (explicit): Differentiate the PDF in (2.4):

$$\frac{d}{dx} f(x) = -\frac{2\gamma(\gamma + 1)}{(\gamma + x)^3}.$$

Because $\gamma > 0$ and $\gamma + x > 0$ for $x \in (0, 1]$, it follows that

$$f'(x) < 0 \quad \text{for all } x \in (0, 1];$$

hence f is strictly decreasing on $(0, 1]$. The density therefore attains its largest value near the left endpoint; in fact,

$$\lim_{x \downarrow 0} f(x) = \frac{\gamma(\gamma + 1)}{\gamma^2} = \frac{\gamma + 1}{\gamma}.$$

- Hazard shape and its derivative (explicit): Write $h(x) = (\gamma + 1)/D(x)$ with $D(x) = (\gamma + x)(1 - x)$. Expand and differentiate $D(x)$:

$$D(x) = \gamma + x - \gamma x - x^2, \qquad D'(x) = 1 - \gamma - 2x.$$

Differentiating h gives

$$h'(x) = -(\gamma + 1)\frac{D'(x)}{D(x)^2} = -(\gamma + 1)\frac{1 - \gamma - 2x}{[(\gamma + x)(1 - x)]^2} = \frac{(\gamma + 1)(\gamma + 2x - 1)}{(\gamma + x)^2(1 - x)^2}.$$

Thus, the sign of $h'(x)$ is determined by $N(x) := \gamma + 2x - 1$:

  - If $\gamma \geq 1$, then $N(x) \geq \gamma - 1 \geq 0$ for all $x \in [0, 1)$, so $h'(x) \geq 0$. In this case the hazard is monotone increasing (IFR behaviour).
  - If $0 < \gamma < 1$, then $N(x)$ changes sign at

$$x^* = \frac{1 - \gamma}{2} \in (0, \tfrac{1}{2}).$$

For $0 < x < x^*$ we have $N(x) < 0$ and $h'(x) < 0$ (decreasing), while for $x > x^*$, we have $N(x) > 0$ and $h'(x) > 0$ (increasing). Therefore, for $0 < \gamma < 1$, the hazard is bathtub-shaped (decreases and then increases).

These analytic conclusions match the hazard plots provided in the manuscript.

- Quantile function and simulation: Solve $u = F(x)$ for x. From (2.3),

$$u = \frac{(\gamma + 1)x}{\gamma + x} \quad \Longrightarrow \quad x = \frac{\gamma u}{\gamma + 1 - u},$$

so the quantile function is

$$Q(u) = F^{-1}(u) = \frac{\gamma u}{\gamma + 1 - u}, \qquad 0 < u < 1.$$

Therefore, sampling is direct: if $U \sim \text{Unif}(0, 1)$, then $X = Q(U)$ has distribution F.

- First moment (closed form): The expected value can be computed in closed form by substitution $t = \gamma + x$:

$$\mathbb{E}[X] = \int_0^1 x f(x)\, dx = \gamma(\gamma + 1) \int_0^1 \frac{x}{(\gamma + x)^2}\, dx$$
$$= \gamma(\gamma + 1) \int_\gamma^{\gamma+1} \frac{t - \gamma}{t^2}\, dt = \gamma(\gamma + 1)\Big\{ \ln(\tfrac{\gamma+1}{\gamma}) + \frac{\gamma}{\gamma + 1} - 1 \Big\}.$$

Higher moments $\mathbb{E}[X^r]$ are obtained analogously by direct integration.

3. Likelihood estimation

Under the UPHCT1 scheme, with observed progressive failure times $\mathbf{x} = (x_{1:m:n}, \ldots, x_{\nu:m:n})$, removals $\{R_i\}_{i=1}^{\nu}$, terminal time $s^\star$, and terminal removals $R^\star$ (as in (1.1)-(1.2)), the likelihood in (1.3) specializes to

$$\begin{aligned}\mathcal{L}(\gamma \mid \mathbf{x}) &\propto \prod_{i=1}^{\nu} \frac{\gamma(\gamma+1)}{(\gamma + x_{i:m:n})^2} \left[\frac{\gamma(1-x_{i:m:n})}{\gamma + x_{i:m:n}}\right]^{R_i} \left[\frac{\gamma(1-s^\star)}{\gamma+s^\star}\right]^{R^\star} \\ &\propto (\gamma+1)^{\nu}\, \gamma^{\nu+\sum_{i=1}^{\nu} R_i + R^\star}\, (1-s^\star)^{R^\star} (\gamma+s^\star)^{-R^\star} \left\{\prod_{i=1}^{\nu} (1-x_{i:m:n})^{R_i} (\gamma + x_{i:m:n})^{-(2+R_i)}\right\}.\end{aligned} \tag{3.1}$$

Equivalently, the log-likelihood is

$$\begin{aligned}\ell(\gamma) &\propto \left(\nu + \textstyle\sum_{i=1}^{\nu} R_i + R^\star\right) \ln\gamma \;+\; \nu \ln(\gamma+1) \;+\; \textstyle\sum_{i=1}^{\nu} R_i \ln(1-x_{i:m:n}) + R^\star \ln(1-s^\star) \\ &\quad - \sum_{i=1}^{\nu} (2+R_i) \log(\gamma + x_{i:m:n}) \;-\; R^\star \ln(\gamma + s^\star),\end{aligned} \tag{3.2}$$

valid for $0 < x_{i:m:n} \leq 1$ and $0 < s^\star \leq 1$, with $\gamma > 0$.

Differentiating the log-likelihood in (3.2) with respect to γ and setting the result to zero yields the likelihood equation for the MLE $\hat{\gamma}$, i.e., $d\ell(\gamma)/d\gamma = 0$;

$$\frac{d\ell(\gamma)}{d\gamma} = \frac{\nu + \sum_{i=1}^{\nu} R_i + R^\star}{\gamma} + \frac{\nu}{\gamma+1} - \sum_{i=1}^{\nu} \frac{2+R_i}{\gamma + x_{i:m:n}} - \frac{R^\star}{\gamma+s^\star}. \tag{3.3}$$

From the score equation associated with (3.3), a closed-form solution for the MLE $\hat{\gamma}$ does not exist. Hence, $\hat{\gamma}$ must be obtained numerically, for example, via the Newton–Raphson algorithm, as implemented in the maxLik package [16] using the available UPHCT1 data. By the invariance principle, once $\hat{\gamma}$ is computed, the reliability and hazard functions on $0 < x \leq 1$ admit the plug-in estimators

$$\widehat{R}(x) = \frac{\hat{\gamma}(1-x)}{\hat{\gamma}+x}, \quad \text{and} \quad \widehat{h}(x) = \frac{\hat{\gamma}+1}{(\hat{\gamma}+x)(1-x)}, \tag{3.4}$$

(in particular at $x = s^\star$). These summaries are central to durability assessment, maintenance planning, and risk evaluation under operating constraints.

Next, confidence intervals for γ follow from the sample normality of the MLE $\hat{\gamma}$. In the one–parameter case, the asymptotic variance of $\hat{\gamma}$ is the reciprocal of the Fisher information,

$$\mathbb{I}(\gamma) \;=\; -\,\mathbb{E}\left[\frac{d^2\,\ell(\gamma)}{d\gamma^2}\right], \qquad \mathrm{Var}(\hat{\gamma}) \;\approx\; \mathbb{I}(\gamma)^{-1},$$

typically evaluated at $\gamma = \hat{\gamma}$ (or using the observed information $-\,d^2\ell(\gamma)/d\gamma^2\big|_{\gamma=\hat{\gamma}}$), where

$$\frac{d^2\ell(\gamma)}{d\gamma^2} = -\frac{\nu + \sum_{i=1}^{\nu} R_i + R^\star}{\gamma^2} - \frac{\nu}{(\gamma+1)^2} + \sum_{i=1}^{\nu} \frac{2+R_i}{(\gamma + x_{i:m:n})^2} + \frac{R^\star}{(\gamma+s^\star)^2}. \tag{3.5}$$

Accordingly, the $(1-\alpha)\times 100\%$ asymptotic maximum likelihood (ML) confidence intervals for γ is

$$\hat{\gamma} \;\pm\; z_{\alpha/2}\sqrt{\mathrm{Var}(\hat{\gamma})},$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution.

To construct ML-based asymptotic confidence intervals for $R(x)$ and $h(x)$, we first approximate their sampling variances via the delta method, a standard tool for functions of MLEs (see [17]). Let

$$\hat{\aleph}_1 = \nabla_\gamma R(x;\gamma)\Big|_{\gamma=\hat{\gamma}}, \text{ and } \hat{\aleph}_2 = \nabla_\gamma h(x;\gamma)\Big|_{\gamma=\hat{\gamma}}.$$

Then

$$\widehat{\Upsilon}_{\hat{R}} = \left(\frac{dR}{d\gamma}\Big|_{\hat{\gamma}}\right)^2 \mathbb{I}_L^{-1}(\hat{\gamma}), \qquad \widehat{\Upsilon}_{\hat{h}} = \left(\frac{dh}{d\gamma}\Big|_{\hat{\gamma}}\right)^2 \mathbb{I}_L^{-1}(\hat{\gamma}).$$

Accordingly, the $(1-\alpha)\times 100\%$ ML-based asymptotic confidence intervals for $R(t)$ and $h(t)$ are

$$\left(\hat{R}(t) \;\pm\; z_{\alpha/2}\sqrt{\widehat{\Upsilon}_{\hat{R}}}\right) \quad \text{and} \quad \left(\hat{h}(t) \;\pm\; z_{\alpha/2}\sqrt{\widehat{\Upsilon}_{\hat{h}}}\right),$$

respectively.

4. Bayes inference

Bayesian estimators of an arbitrary parametric function $\pi(\gamma)$ under squared–error loss (SEL) are obtained. An informative gamma(a,c) prior and, for comparison, a non-informative prior are adopted for γ. Under the gamma specification with $a>0$ and $c>0$, the prior density is

$$\pi(\gamma) \;\propto\; \gamma^{a-1}e^{-c\gamma}. \tag{4.1}$$

The gamma prior is used for positive one-parameter models due to its flexibility and computational convenience by adjusting (a,c). It can encode varied prior beliefs and often yields posteriors of manageable form; see [18] for details.

By combining the likelihood in (3.1) with the prior in (4.1), the posterior density (denoted π_L^*) for γ is obtained as

$$\pi_L^*(\gamma|\mathbf{x}) \propto (\gamma+1)^{\nu}\,\gamma^{\nu+\sum_{i=1}^{\nu}R_i+R^\star+a-1}\,e^{-c\gamma}(\gamma+s^\star)^{-R^\star}\left\{\prod_{i=1}^{\nu}(\gamma+x_{i:m:n})^{-(2+R_i)}\right\}. \tag{4.2}$$

From (4.2), the Bayes estimator $\tilde{g}_L$ of an arbitrary function $g(\gamma)$ under SEL is its posterior mean:

$$\tilde{\gamma}_L = E\left(g\left(\gamma\,|\mathbf{x}\right)\right) = \frac{\int_0^\infty \gamma\,\pi(\gamma)\,\mathcal{L}(\gamma\mid\mathbf{x})\,d\gamma}{\int_0^\infty \pi(\gamma)\,\mathcal{L}(\gamma\mid\mathbf{x})\,d\gamma}. \tag{4.3}$$

The Bayes estimators implied by (4.3) are the ratio of integral that lack closed–form solutions in general. Accordingly, they are approximated via the Metropolis-Hastings algorithm, and a standard Markov chain Monte Carlo (MCMC) method by simulating from the posterior distribution and computing Monte Carlo estimates [19, 20]. From (4.2), the posterior density of γ does not reduce to any standard family under analytic manipulation. Consequently, direct sampling by conventional closed-form methods is not feasible.

With a normal random–walk proposal on the log–scale, MCMC draws from the posterior $\pi_L^*(\gamma \mid \mathbf{x})$ are obtained as follows:

Step 1 Initialize $\gamma^{[0]} = \hat{\gamma}$ and choose a proposal variance $s^2 > 0$ (e.g., $s^2 = \widehat{\mathbb{I}}^{-1}(\hat{\gamma})$). Set $\varrho = 1$.

Step 2 Propose on the log–scale: let $\eta^{[\varrho-1]} = \log \gamma^{[\varrho-1]}$, draw $\eta^* \sim \mathcal{N}(\eta^{[\varrho-1]}, s^2)$, and set $\gamma^* = e^{\eta^*} > 0$.

Step 3 Compute the Metropolis–Hastings acceptance probability

$$q = \min\left\{1,\ \frac{\pi_L^*(\gamma^* \mid \mathbf{x})\, \gamma^*}{\pi_L^*(\gamma^{[\varrho-1]} \mid \mathbf{x})\, \gamma^{[\varrho-1]}}\right\},$$

where the factor $\gamma^*/\gamma^{[\varrho-1]}$ accounts for the log–scale proposal (Jacobian).

Step 4 Draw $u \sim \text{Unif}(0, 1)$; if $u \le q$, set $\gamma^{[\varrho]} = \gamma^*$, otherwise, set $\gamma^{[\varrho]} = \gamma^{[\varrho-1]}$.

Step 5 For any $x \in (0, 1]$, record

$$R^{[\varrho]}(x) = \frac{\tilde{\gamma}(1-x)}{\tilde{\gamma}+x}, \qquad h^{[\varrho]}(x) = \frac{\tilde{\gamma}+1}{(\tilde{\gamma}+x)(1-x)}.$$

Step 6 Set $\varrho \leftarrow \varrho + 1$ and repeat Steps 2–5 for $\mathfrak{D}$ iterations (discard a burn–in if desired). Posterior means (Bayes estimates under SEL) are then approximated by Monte Carlo averages of the retained draws.

To reduce dependence on starting values and promote convergence, the first $\mathfrak{D}_0$ iterations are discarded as burn-in. The remaining draws $\{\varphi^{[\varrho]} : \varrho = \mathfrak{D}_0 + 1, \ldots, \mathfrak{D}\}$ are then used for Bayesian inference. Under SEL, the Bayes estimate of any scalar functional φ is approximated by the Monte Carlo average

$$\widehat{\varphi}_{\text{BS}} = \frac{1}{\mathfrak{D}^\star} \sum_{\varrho=\mathfrak{D}_0+1}^{\mathfrak{D}} \varphi^{[\varrho]}, \qquad \mathfrak{D}^\star = \mathfrak{D} - \mathfrak{D}_0.$$

For a $100(1-\alpha)\%$ highest posterior density (HPD) credible interval for φ, order the retained samples as $\gamma_{(1)} \le \cdots \le \gamma_{(\mathfrak{D}^\star)}$, set $h = \lfloor (1-\alpha)\mathfrak{D}^\star \rfloor$, and choose the start index

$$j^* = \underset{1 \le j \le \mathfrak{D}^\star - h}{\arg\min} \{\gamma_{(j+h)} - \gamma_{(j)}\}.$$

The HPD interval is then

$$(\gamma_{(j^*)},\ \gamma_{(j^*+h)}),$$

where $\lfloor \cdot \rfloor$ denotes the floor operator; see [21].

5. Bootstrap

The bootstrap is a resampling-based framework for statistical inference, widely employed to construct confidence intervals; see, for example, Efron [22]. In what follows, we adopt a *parametric* bootstrap to obtain confidence intervals for the one-parameter model, targeting the scalar parameter γ as well as the reliability $R(t)$ and hazard rate $h(t)$. Specifically, we consider two parametric bootstrap constructions: the percentile bootstrap (B-P) and the bootstrap-t (B-t) intervals.

5.1. Percentile bootstrap confidence interval

(1) Obtain the MLE or Bayesian estimate of the parameter γ, as well as the corresponding reliability $R(t)$ and hazard function $h(t)$.
(2) Generate a bootstrap sample using the fitted model and compute the bootstrap estimates, say γ^b, $R^b(t)$, and $h^b(t)$.
(3) Repeat Step (2) $\mathcal{B}$ times to obtain the sets

$$(\gamma^{b(1)}, \gamma^{b(2)}, \ldots, \gamma^{b(\mathcal{B})}), \quad (R^{b(1)}(t), R^{b(2)}(t), \ldots, R^{b(\mathcal{B})}(t)), \quad (h^{b(1)}(t), h^{b(2)}(t), \ldots, h^{b(\mathcal{B})}(t)).$$

(4) Arrange each set of bootstrap estimates in ascending order:

$$(\gamma^{b[1]}, \gamma^{b[2]}, \ldots, \gamma^{b[\mathcal{B}]}), \quad (R^{b[1]}(t), R^{b[2]}(t), \ldots, R^{b[\mathcal{B}]}(t)), \quad (h^{b[1]}(t), h^{b[2]}(t), \ldots, h^{b[\mathcal{B}]}(t)).$$

(5) The two–sided $100(1-\gamma)\%$ percentile bootstrap confidence intervals are then given by

$$\left[\gamma^{b\,[\mathcal{B}\frac{\gamma}{2}]},\ \gamma^{b\,[\mathcal{B}(1-\frac{\gamma}{2})]}\right], \quad \left[R^{b\,[\mathcal{B}\frac{\gamma}{2}]}(t),\ R^{b\,[\mathcal{B}(1-\frac{\gamma}{2})]}(t)\right], \quad \left[h^{b\,[\mathcal{B}\frac{\gamma}{2}]}(t),\ h^{b\,[\mathcal{B}(1-\frac{\gamma}{2})]}(t)\right].$$

5.2. Bootstrap-t confidence intervals

(1) Fit the one-parameter model and obtain the point estimates $\hat{\gamma}$, $\widehat{R}(t)$, and $\widehat{h}(t)$. Compute their asymptotic standard errors via the observed Fisher information (or an equivalent variance estimator): $\widehat{\mathrm{se}}(\hat{\gamma})$, $\widehat{\mathrm{se}}(\widehat{R}(t))$, and $\widehat{\mathrm{se}}(\widehat{h}(t))$.
(2) For $b = 1, \ldots, \mathcal{B}$, generate a bootstrap sample from the fitted model (using $\hat{\gamma}$), and recompute the estimates $\hat{\gamma}^{*(b)}$, $\widehat{R}^{*(b)}(t)$, and $\widehat{h}^{*(b)}(t)$. Using the Fisher information evaluated at the bootstrap fit, obtain $\widehat{\mathrm{se}}(\hat{\gamma}^{*(b)})$, $\widehat{\mathrm{se}}(\widehat{R}^{*(b)}(t))$, and $\widehat{\mathrm{se}}(\widehat{h}^{*(b)}(t))$.
(3) Form the bootstrap t-statistics:

$$T_{\gamma}^{(b)} = \frac{\hat{\gamma}^{*(b)} - \hat{\gamma}}{\widehat{\mathrm{se}}(\hat{\gamma}^{*(b)})}, \quad T_{R}^{(b)} = \frac{\widehat{R}^{*(b)}(t) - \widehat{R}(t)}{\widehat{\mathrm{se}}(\widehat{R}^{*(b)}(t))}, \quad T_{h}^{(b)} = \frac{\widehat{h}^{*(b)}(t) - \widehat{h}(t)}{\widehat{\mathrm{se}}(\widehat{h}^{*(b)}(t))}.$$

(4) Sort each set $\{T_{\gamma}^{(b)}\}$, $\{T_{R}^{(b)}\}$, and $\{T_{h}^{(b)}\}$ in ascending order to get $T_{\gamma}^{[1]} \le \cdots \le T_{\gamma}^{[\mathcal{B}]}$, $T_{R}^{[1]} \le \cdots \le T_{R}^{[\mathcal{B}]}$, $T_{h}^{[1]} \le \cdots \le T_{h}^{[\mathcal{B}]}$.
(5) Let $q_{\ell}^{\gamma} = T_{\gamma}^{[\mathcal{B}\ell]}$, $q_{\ell}^{R} = T_{R}^{[\mathcal{B}\ell]}$, and $q_{\ell}^{h} = T_{h}^{[\mathcal{B}\ell]}$ denote the empirical ℓ-quantiles (with $\ell \in (0,1)$). Then the two-sided $100(1-\alpha)\%$ bootstrap-t intervals are

$$\left[\hat{\gamma} - q_{1-\alpha/2}^{\gamma}\,\widehat{\mathrm{se}}(\hat{\gamma}),\ \hat{\gamma} - q_{\alpha/2}^{\gamma}\,\widehat{\mathrm{se}}(\hat{\gamma})\right], \quad \left[\widehat{R}(t) - q_{1-\alpha/2}^{R}\,\widehat{\mathrm{se}}(\widehat{R}(t)),\ \widehat{R}(t) - q_{\alpha/2}^{R}\,\widehat{\mathrm{se}}(\widehat{R}(t))\right],$$

$$\left[\widehat{h}(t) - q_{1-\alpha/2}^{h}\,\widehat{\mathrm{se}}(\widehat{h}(t)),\ \widehat{h}(t) - q_{\alpha/2}^{h}\,\widehat{\mathrm{se}}(\widehat{h}(t))\right].$$

6. Simulation study

This section evaluates the finite–sample performance of the proposed estimators for the model parameters γ as well as the reliability characteristics $R(t)$ and $h(t)$ under the UPHCT1 scheme with binomial removal. Unless otherwise stated, data are generated from the target distribution under prespecified UPHC schemes (n, m, r, T_1, T_2), and all computations are implemented in **R**. Point estimation is carried out via MLE and Bayesian estimation by non-informative and informative prior, and interval estimation uses asymptotic delta method standard errors for approximate confidence intervals (ACIs) and MCMC–based highest posterior density (HPD) intervals for the Bayesian procedures. The simulation settings are as follows:

- A total of 10000 Monte Carlo replications are performed for each design point from three different populations with different values of actual values of parameter, namely, Pop-1: $\gamma = 0.4$, Pop-2: $\gamma = 1.6$ and Pop-1: $\gamma = 3$.
- For Bayesian estimation, we run 12000 MCMC iterations with a burn-in of 2000, retaining 10000 post-burn samples for posterior summaries.
- All numerical experiments are conducted in the R environment. Optimization for ML and Bayesian uses standard routines (e.g., maxLik/Newton–Raphson), and MCMC diagnostics are checked to confirm mixing and stationarity.
- To assess small–sample variability of the classical estimators and to build nonparametric percentile intervals, bootstrap resampling with $B = 1000$ iterations is employed at each Monte Carlo run.
- At the mission time $x = s^\star$, the benchmark reliability $R(s^\star)$ and hazard rate $h(s^\star)$ are treated as known reference values; these serve as the ground truth for evaluating estimator accuracy.
- The sample sizes $n \in \{40, 100\}$, representing small, and large settings have been considered. For all populations, the threshold times are $t_1 \in \{0.55, 0.7\}$ and $t_2 \in \{0.8, 0.9\}$.
- To specify the numbers of recorded failures r and m in each life test, in the designs reported here, these are instantiated as follows: for $n = 40$, we take $m \in \{30, 35\}$ and $r \in \{25, 30\}$; for $n = 100$, we take $m \in \{70, 90\}$ and $r \in \{60, 80\}$.
- In this study, progressive removals are generated via a binomial scheme to determine the sequence $\{R_i\}_{i=1}^m$, using binomial parameter values $p \in \{0.3, 0.8\}$. In addition, censoring schemes are considered under binomial random removals. It is assumed that removals occur independently across units, with each unit being removed with a common probability p. Accordingly, the number of units withdrawn at each failure time is modeled by a binomial distribution, given by

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \qquad 0 \le r_1 \le n. \tag{6.1}$$

  Moreover,

$$P\left(R_j = r_j \,\middle|\, R_{(j-1)} = r_{(j-1)}, \ldots, R_1 = r_1\right) = \binom{n-m-\sum_{k=1}^{j-1} r_k}{r_j} p^{r_j}(1-p)^{n-m-\sum_{k=1}^{j} r_k}, \tag{6.2}$$

  where

$$0 \le r_j \le n - m - \sum_{k=1}^{j-1} r_k, \qquad j = 2, 3, \ldots, m-1.$$

- For each target $\varphi \in \{\gamma, R(t), h(t)\}$, we report the average relative absolute bias estimate (RAB), and MSE. Interval estimators are compared via average confidence (or credible) interval (ACI) and coverage percentage (CP).
- Unless otherwise specified, the confidence level for all confidence intervals is fixed at 95%.

UPHCT1 sample generation

Inputs: n (initial units), m (max recorded failures), r (Type-II target), k (guaranteed minimum), thresholds $0 < t_1 < t_2$, removals $\{R_i\}_{i=1}^{m}$, parameter $\gamma > 0$.
Outputs: sample $\{(x_i, R_i)\}_{i=1}^{m}$, survivors $r^\star$, stopping time T^*.

Steps

(1) Generate i.i.d. uniforms $\xi_1, \xi_2, \ldots, \xi_m \sim \mathcal{U}(0, 1)$.
(2) For $i = 1, 2, \ldots, m$ do:
  (a) $\zeta_i = \xi_i^{1/\left(i+\sum_{s=m-i+1}^{m} R_s\right)}$.
  (b) $u_i = 1 - \prod_{s=m-i+1}^{m} \zeta_s$ (so $0 < u_i < 1$).
  (c) Transform via the quantile $Q(u) = \gamma u/(\gamma + 1 - u)$:

$$x_i = Q(u_i) = \frac{\gamma\, u_i}{\gamma + 1 - u_i}.$$

(3) Count threshold crossings:

$$d_1 = \left|\{\, i \le m : x_i \le t_1 \,\}\right|, \qquad d_2 = \left|\{\, i \le m : x_i \le t_2 \,\}\right|.$$

(4) Let x_r, x_k, x_m denote the rth, kth, and mth order statistics of $\{x_i\}_{i=1}^{m}$.
(5) Compute the remaining survivors $r^\star$ using the appropriate case:

- If $x_r < x_m < t_1 < t_2$, then $r^\star = n - m - \sum_{i=1}^{m} R_i$.
- If $x_r < t_1 < x_m < t_2$ or $x_r < t_1 < t_2 < x_m$, then $r^\star = n - d_1 - \sum_{i=1}^{d_1} R_i$.
- If $t_1 < x_r < x_m < t_2$ or $t_1 < x_r < t_2 < x_m$, then $r^\star = n - k - \sum_{i=1}^{k} R_i$.
- If $t_1 < t_2 < x_r < x_m$, then $r^\star = n - d_2 - \sum_{i=1}^{d_2} R_i$.

(6) Determine the stopping time T^*:

- If $x_r < x_m < t_1 < t_2$, set $T^* = x_m$ (stop at the mth failure).
- If $x_r < t_1 < x_m < t_2$ or $x_r < t_1 < t_2 < x_m$, set $T^* = t_1$ (first threshold).
- If $t_1 < x_r < x_m < t_2$ or $t_1 < x_r < t_2 < x_m$, set $T^* = x_k$ (after at least k failures).
- If $t_1 < t_2 < x_r < x_m$, set $T^* = t_2$ (second threshold).

For each quantity of interest $\varphi \in \{\gamma, R(t), h(t)\}$, we report the Monte Carlo RAB along with its MSE, length of ACIs (LACI), length of credible confidence interval (LCCI), length of bootstrap-p (LBP), and length of bootstrap-t (LBT) are defined by

$$RAB\hat{\varphi}_\tau = \frac{1}{10000}\sum_{j=1}^{10000}\frac{|\hat{\varphi}_\tau^{(j)} - \varphi_\tau^{(j)}|}{\varphi_\tau^{(j)}}, \qquad \tau = 1, 2, 3,$$

$$\text{MSE}(\hat{\varphi}_\tau) = \frac{1}{2000}\sum_{j=1}^{2000}\left(\hat{\varphi}_\tau^{(j)} - \varphi_\tau\right)^2, \qquad \tau = 1, 2, 3,$$

where $\hat{\varphi}_\tau^{(j)}$ denotes the MLE or Bayesian estimate from the jth replication, with $\varphi_1 = \gamma$, $\varphi_2 = R(t)$, and $\varphi_3 = h(t)$.

To compare the $100(1-\alpha)\%$ interval estimators, we report the average confidence length (ACL) and the coverage percentage (CP) at the 95% level:

$$\text{ACL}_{\varphi_\tau}^{95\%} = \frac{1}{2000}\sum_{j=1}^{2000}\left(\mathcal{U}_{\hat{\varphi}_\tau^{(j)}} - \mathcal{L}_{\hat{\varphi}_\tau^{(j)}}\right), \qquad \tau = 1, 2, 3,$$

$$\text{CP}_{\varphi_\tau}^{95\%} = \frac{1}{2000}\sum_{j=1}^{2000}\mathbf{1}\left\{\mathcal{L}_{\hat{\varphi}_\tau^{(j)}} \le \varphi_\tau \le \mathcal{U}_{\hat{\varphi}_\tau^{(j)}}\right\}, \qquad \tau = 1, 2, 3,$$

where $\mathbf{1}\{\cdot\}$ is the indicator function, and $(\mathcal{L}_{\hat{\varphi}_\tau^{(j)}}, \mathcal{U}_{\hat{\varphi}_\tau^{(j)}})$ are the lower and upper bounds of the 95% asymptotic (or credible) interval for φ_τ in replication j.

All computations are carried out in *R*, using the coda package [23] for MCMC diagnostics and the maxLik package [16] for likelihood-based optimization.

Comments on simulation results

Tables 1–3 summarize the RAB, MSEs, LACI, LCCI, CP, LBP, and LBT for γ, $R(t)$, and $h(t)$, respectively, across representative UPHCT1 designs.

(1) Across designs, all procedures yield stable RAB with shrinking MSE and LCI as the effective information increases (larger n, and jointly larger m and r). This is consistent with asymptotic efficiency under UPHCT1 and mirrors findings in related UPHCT1 studies. (Good concentration and small bias as sample information grows.)
(2) MCMC–based Bayes estimates generally dominate their classical counterparts in RAB, MSE and LCI, especially when informative priors are used. In designs akin to those shown, MCMC with gamma informative prior typically edges MCMC with a non-informative prior, reflecting the informative prior robustness under censoring.
(3) ACIs built from MLE tend to be shorter for γ, $R(t)$, and $h(t)$, while HPD from Bayesian with an informative prior can be slightly tighter for γ, $R(t)$, and $h(t)$. HPD intervals constructed from the spacings–based posterior are usually the most efficient (shortest ACL) with near–nominal CP.
(4) Relaxing thresholds (increasing T_1 and/or T_2) improves identifiability for γ, which translates into lower RABs and MSEs and narrower intervals for $\gamma, R(t)$; $h(t)$ can be more sensitive near early times, but its precision also improves as effective failures increase.
(5) Across all designs, Bayesian estimators outperform MLE in small-to-moderate effective samples, yielding lower RABs and MSEs and shorter interval lengths while maintaining near-nominal coverage. As information grows, the gap narrows, but Bayesian remains slightly preferable.

(6) Between the two Bayesian implementations, LBT consistently exhibits smaller MSE and tighter HPD intervals than LBP, particularly under heavier censoring removals.
(7) Increasing the sample size n and the numbers of observed failures (m, r) increases the information content, leading to monotone improvements: RAB and MSE decrease, interval lengths contract, and coverage stabilizes around the nominal level for all parameters and for $R(t)$ and $h(t)$.
(8) Larger removal probability p reduces the effective sample size. Consequently, RABs and MSEs increase, intervals widen, and mild under-coverage can appear in the most severe settings. The deterioration is less pronounced for the better-calibrated Bayesian procedure (here, LBT).
(9) When credible prior information exists, a *concentrated gamma prior* is preferred: it lowers MSE and shortens HPD intervals without compromising coverage. If prior knowledge is weak or uncertain, a diffuse (non-informative) prior is safer to avoid prior-driven bias.
(10) Increasing the prior concentration shrinks the prior variance, producing more stable posteriors, smaller MSE, and shorter HPD intervals. Raising the shape alone pulls the posterior more strongly toward the prior mean.
(11) Use Bayesian estimation in this UPHCT1 setting; prefer the better-calibrated variant (LBT) when available, especially with an informative gamma prior. Performance improves with $n\uparrow$, $m\uparrow$, $r\uparrow$ and degrades as $p\uparrow$; therefore, designs should maximize observed failures and limit removals.

Table 1. Some statistical measures by simulation results when $\gamma = 0.4$.

$\gamma = 0.4$							MLE						Bayesian Non-informative						Bayesian Informative					
n	p	T_2	T_1	m	r		RAB	MSE	LACI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT
40	0.3	0.8	0.55	30	25	γ	0.1308	0.0282	0.6516	96.6%	0.0218	0.0217	0.4843	0.1362	1.0793	99.7%	0.0401	0.0400	0.1240	0.0193	0.3662	99.6%	0.0160	0.0159
						R	0.0733	0.0003	0.0897	98.5%	0.0028	0.0028	0.3238	0.0008	0.1076	99.8%	0.0040	0.0040	0.1637	0.0004	0.0682	99.4%	0.0029	0.0028
						h	0.0296	0.0117	2.1965	92.2%	0.0673	0.0681	0.0524	0.0266	2.0260	95.2%	0.0729	0.0730	0.0416	0.0054	1.8940	95.3%	0.0702	0.0704
				35	30	γ	0.1080	0.0262	0.5274	100.0%	0.0202	0.0171	0.4300	0.1068	1.0001	99.7%	0.0338	0.0350	0.1056	0.0141	0.3328	99.9%	0.0138	0.0142
						R	0.0608	0.0003	0.0705	99.8%	0.0023	0.0023	0.2424	0.0006	0.0871	99.5%	0.0030	0.0030	0.1033	0.0003	0.0469	99.7%	0.0020	0.0020
						h	0.0108	0.0108	1.7597	94.9%	0.0546	0.0544	0.0319	0.0225	1.4690	95.9%	0.0581	0.0582	0.0221	0.0042	1.2961	96.1%	0.0545	0.0545
	0.8			30	25	γ	0.1293	0.0322	0.6976	100.0%	0.0198	0.0190	0.4826	0.1355	1.0656	99.7%	0.0391	0.0394	0.1136	0.0183	0.3355	99.8%	0.0164	0.0163
						R	0.4697	0.0003	0.0746	99.8%	0.0024	0.0024	0.2753	0.0007	0.0920	99.1%	0.0032	0.0032	0.3642	0.0001	0.0522	99.5%	0.0020	0.0020
						h	0.4105	0.0120	1.9603	93.9%	0.0634	0.0622	0.3767	0.0258	1.6898	95.5%	0.0630	0.0626	0.3922	0.0046	1.4986	95.6%	0.0620	0.0619
				35	30	γ	0.1227	0.0203	0.5608	100.0%	0.0180	0.0180	0.4322	0.1129	0.9176	99.2%	0.0350	0.0345	0.1112	0.0180	0.3248	99.7%	0.0170	0.0172
						R	0.0852	0.0003	0.0687	99.6%	0.0021	0.0021	0.2243	0.0006	0.0814	98.2%	0.0029	0.0029	0.0867	0.0001	0.0468	99.1%	0.0018	0.0018
						h	0.0037	0.0114	1.7492	95.9%	0.0552	0.0561	0.0256	0.0226	1.3131	96.6%	0.0582	0.0580	0.0158	0.0045	1.2539	96.9%	0.0551	0.0550
	0.3	0.9	0.7	30	25	γ	0.0344	0.0270	0.6446	100.0%	0.0207	0.0201	0.4599	0.1293	1.0468	99.8%	0.0393	0.0390	0.1198	0.0183	0.3461	99.9%	0.0197	0.0191
						R	0.4195	0.0002	0.0920	100.0%	0.0027	0.0027	0.8714	0.0004	0.1099	99.6%	0.0038	0.0038	0.1626	0.0001	0.0579	99.9%	0.0032	0.0032
						h	0.1690	0.0090	9.2207	97.2%	0.2976	0.2932	0.1797	0.0248	6.5808	95.1%	0.2913	0.2943	0.0317	0.0063	6.5136	95.0%	0.2807	0.2790
				35	30	γ	0.0291	0.0238	0.5160	100.0%	0.0191	0.0162	0.4275	0.1027	0.9801	99.7%	0.0314	0.0341	0.1065	0.0132	0.3095	99.8%	0.0181	0.0180
						R	0.1865	0.0001	0.0784	100.0%	0.0024	0.0024	0.5552	0.0003	0.0861	99.7%	0.0032	0.0032	0.3645	0.0001	0.0718	99.8%	0.0028	0.0028
						h	0.0753	0.0083	8.4620	92.6%	0.2688	0.2635	0.0861	0.0222	6.3007	95.1%	0.2659	0.2678	0.0803	0.0051	6.1571	95.2%	0.2800	0.2756
	0.8			30	25	γ	0.0118	0.0315	0.6962	100.0%	0.0190	0.0189	0.4629	0.1272	0.9418	99.7%	0.0304	0.0324	0.0987	0.0123	0.3065	99.6%	0.0194	0.0192
						R	0.3339	0.0002	0.0842	100.0%	0.0028	0.0028	0.7933	0.0004	0.0952	99.6%	0.0036	0.0035	0.5455	0.0001	0.0749	99.7%	0.0030	0.0030
						h	0.1483	0.0102	8.9210	93.4%	0.2805	0.2816	0.1597	0.0266	6.5174	95.1%	0.2968	0.2930	0.1533	0.0062	6.4507	95.0%	0.2889	0.2845
				35	30	γ	0.0106	0.0182	0.4912	100.0%	0.0176	0.0174	0.4255	0.1046	0.9099	99.9%	0.0302	0.0329	0.1059	0.0147	0.2941	99.8%	0.0168	0.0168
						R	0.2408	0.0001	0.0721	100.0%	0.0021	0.0022	0.6455	0.0003	0.0834	99.3%	0.0030	0.0030	0.4300	0.0001	0.0625	99.5%	0.0024	0.0024
						h	0.1232	0.0089	8.1430	93.2%	0.2630	0.2626	0.1336	0.0224	6.1006	95.3%	0.2688	0.2659	0.1279	0.0048	5.8984	95.2%	0.2513	0.2538
100	0.3	0.8	0.55	70	60	γ	0.1281	0.0119	0.3196	99.9%	0.0102	0.0103	0.1384	0.0388	0.5883	99.7%	0.0235	0.0235	0.0881	0.0024	0.1630	99.7%	0.0061	0.0060
						R	0.0714	0.0002	0.0395	99.1%	0.0012	0.0012	0.0563	0.0002	0.0591	98.4%	0.0020	0.0020	0.0017	0.0003	0.0176	99.0%	0.0007	0.0007
						h	0.0093	0.0072	0.4035	99.4%	0.0130	0.0125	0.0046	0.0096	0.3809	96.0%	0.0156	0.0157	0.0004	0.0010	0.1128	95.4%	0.0114	0.0108
				90	80	γ	0.1020	0.0117	0.2836	99.3%	0.0090	0.0091	0.0269	0.0165	0.4278	99.1%	0.0162	0.0158	0.0231	0.0021	0.1291	99.6%	0.0056	0.0058
						R	0.0603	0.0002	0.0357	98.7%	0.0011	0.0011	0.0062	0.0002	0.0466	98.7%	0.0015	0.0015	0.0015	0.0002	0.0159	97.9%	0.0006	0.0006
						h	0.0091	0.0071	0.3805	99.8%	0.0126	0.0125	0.0046	0.0059	0.3725	98.7%	0.0143	0.0142	0.0004	0.0008	0.1167	98.4%	0.0249	0.0247
	0.8			70	60	γ	0.1200	0.0127	0.3107	99.7%	0.0098	0.0098	0.0888	0.0241	0.5096	99.6%	0.0190	0.0190	0.0128	0.0021	0.1446	99.6%	0.0053	0.0053
						R	0.0715	0.0002	0.0387	98.7%	0.0012	0.0013	0.0326	0.0002	0.0523	97.7%	0.0018	0.0017	0.0112	0.0000	0.0165	98.9%	0.0006	0.0006
						h	0.0113	0.0078	0.4623	99.4%	0.0155	0.0143	0.0020	0.0077	0.3434	97.3%	0.0168	0.0169	0.0012	0.0009	0.1105	96.0%	0.0139	0.0137
				90	80	γ	0.1120	0.0115	0.2767	99.5%	0.0092	0.0093	0.0291	0.0156	0.4320	99.5%	0.0158	0.0159	0.0113	0.0016	0.1232	99.7%	0.0048	0.0048
						R	0.0616	0.0002	0.0341	98.7%	0.0011	0.0011	0.0031	0.0001	0.0456	99.1%	0.0015	0.0015	0.0102	0.0000	0.0151	97.8%	0.0005	0.0005
						h	0.0031	0.0070	0.4239	99.8%	0.0147	0.0137	0.0019	0.0057	0.3350	98.5%	0.0158	0.0156	0.0010	0.0007	0.1155	98.6%	0.0163	0.0160
	0.3	0.9	0.7	70	60	γ	0.0326	0.0112	0.3042	99.9%	0.0101	0.0091	0.1284	0.0353	0.5421	99.8%	0.0215	0.0216	0.0708	0.0023	0.1523	99.8%	0.0104	0.0103
						R	0.0591	0.00004	0.0297	99.9%	0.0009	0.0009	0.0534	0.0001	0.0357	98.9%	0.0012	0.0012	0.0437	0.0000	0.0187	99.9%	0.0008	0.0008
						h	0.0913	0.0042	3.4735	93.8%	0.1164	0.1164	0.0840	0.0098	2.5298	95.1%	0.1075	0.1078	0.0229	0.0018	2.3759	95.0%	0.1086	0.1106
				90	80	γ	0.0284	0.0097	0.2636	99.8%	0.0088	0.0085	0.0194	0.0139	0.4056	99.3%	0.0142	0.0142	0.0226	0.0020	0.1205	99.4%	0.0081	0.0082
						R	0.0582	0.00003	0.0222	99.6%	0.0007	0.0007	0.0412	0.0001	0.0277	99.3%	0.0010	0.0010	0.0350	0.0000	0.0113	99.9%	0.0005	0.0005
						h	0.0053	0.0036	2.0390	97.7%	0.0651	0.0642	0.0109	0.0073	0.3816	95.2%	0.0683	0.0662	0.0082	0.0011	0.1645	95.3%	0.0675	0.0627
	0.8			70	60	γ	0.0080	0.0124	0.3091	100.0%	0.0089	0.0089	0.0729	0.0236	0.4716	99.8%	0.0183	0.0183	0.0119	0.0017	0.1421	99.9%	0.0093	0.0091
						R	0.1873	0.00004	0.0330	100.0%	0.0011	0.0011	0.0022	0.0001	0.0386	99.3%	0.0015	0.0015	0.0944	0.0000	0.0228	99.8%	0.0010	0.0010
						h	0.1437	0.0044	4.2172	92.7%	0.1266	0.1298	0.1356	0.0105	3.5033	95.0%	0.1332	0.1330	0.1398	0.0016	3.4283	95.0%	0.1339	0.1319
				90	80	γ	0.0080	0.0094	0.2778	99.8%	0.0085	0.0086	0.0283	0.0094	0.3576	99.7%	0.0122	0.0147	0.0104	0.0014	0.1210	99.8%	0.0078	0.0078
						R	0.0503	0.00003	0.0239	99.8%	0.0008	0.0008	0.0013	0.0001	0.0286	98.8%	0.0010	0.0010	0.0445	0.0000	0.0116	99.8%	0.0006	0.0006
						h	0.0091	0.0036	2.3764	96.4%	0.0748	0.0758	0.0146	0.0072	0.7190	95.0%	0.0766	0.0758	0.0121	0.0011	0.5614	95.0%	0.0746	0.0747

Table 2. Some statistical measures by simulation results when $\gamma = 1.6$.

$\gamma = 1.6$							MLE						Bayesian Non-informative						Bayesian Informative					
n	p	T_2	T_1	m	r		RAB	MSE	LACI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT
40	0.3	0.8	0.55	30	25	γ	0.3400	0.9953	3.2798	100.0%	0.1734	0.1735	0.6291	3.4928	5.4884	99.6%	0.1897	0.1883	0.1908	0.4949	1.9626	99.8%	0.0841	0.0840
						R	0.2470	0.0015	0.1000	97.9%	0.0031	0.0031	0.0491	0.0005	0.0880	96.0%	0.0029	0.0029	0.0497	0.0003	0.0974	96.7%	0.0019	0.0019
						h	0.1745	0.0589	3.0336	99.9%	0.1949	0.1956	0.0928	0.0222	2.2983	99.1%	0.0979	0.0978	0.0374	0.0040	2.2233	99.1%	0.0983	0.0969
				35	30	γ	0.3380	0.8376	2.6802	100.0%	0.1352	0.1341	0.5469	2.9835	5.1028	99.3%	0.1874	0.1842	0.1287	0.4969	1.6440	99.8%	0.0847	0.0844
						R	0.2285	0.0014	0.0938	98.1%	0.0030	0.0031	0.0047	0.0005	0.0799	95.5%	0.0023	0.0022	0.0471	0.0002	0.0922	98.0%	0.0026	0.0025
						h	0.1321	0.0557	3.0197	100.0%	0.1661	0.1662	0.0871	0.0206	2.1666	99.2%	0.0902	0.0817	0.0309	0.0037	2.1728	99.1%	0.0823	0.0877
	0.8			30	25	γ	0.3700	0.9529	3.0443	100.0%	0.0972	0.0978	0.5912	3.3258	5.2107	99.8%	0.1831	0.1878	0.1222	0.4493	1.8568	99.9%	0.0759	0.0742
						R	0.2665	0.0016	0.0983	98.7%	0.0030	0.0030	0.0363	0.0005	0.0908	95.7%	0.0033	0.0033	0.0153	0.0001	0.0822	96.7%	0.0024	0.0024
						h	0.0916	0.0628	4.6046	99.9%	0.1404	0.1406	0.0442	0.0214	2.6557	98.8%	0.1548	0.1556	0.0524	0.0037	2.7107	98.7%	0.1453	0.1401
				35	30	γ	0.3277	0.9009	3.0036	100.0%	0.0912	0.0811	0.5627	3.1545	5.0311	99.6%	0.1791	0.1592	0.1129	0.4264	1.7965	99.7%	0.0766	0.0765
						R	0.2487	0.0013	0.0910	97.9%	0.0029	0.0028	0.0090	0.0005	0.0901	95.4%	0.0031	0.0031	0.0106	0.0001	0.0794	96.7%	0.0029	0.0029
						h	0.0817	0.0551	3.9681	100.0%	0.1242	0.1262	0.0312	0.0202	2.0674	99.4%	0.1403	0.1355	0.0513	0.0034	2.6875	99.4%	0.2630	0.2602
	0.3	0.9	0.7	30	25	γ	0.0648	0.9717	3.1234	100.0%	0.1717	0.1613	0.6095	3.2944	5.1396	99.2%	0.1883	0.1884	0.1825	0.4114	1.8547	99.6%	0.1284	0.1250
						R	0.0751	0.0003	0.0739	100.0%	0.0024	0.0024	0.2032	0.0002	0.0602	99.1%	0.0023	0.0023	0.1129	0.0001	0.0430	99.5%	0.0022	0.0022
						h	0.0221	0.0334	4.4359	94.2%	0.1380	0.1377	0.0401	0.0208	3.7077	95.9%	0.1377	0.1372	0.0342	0.0062	3.5091	95.9%	0.1350	0.1344
				35	30	γ	0.0591	0.8153	2.4397	100.0%	0.1283	0.1284	0.5267	2.9054	4.9037	99.5%	0.1720	0.1699	0.1246	0.3973	1.5178	99.8%	0.1132	0.1123
						R	0.0706	0.0003	0.0666	99.8%	0.0022	0.0021	0.1705	0.0002	0.0584	99.2%	0.0020	0.0020	0.0932	0.0001	0.0417	98.5%	0.0018	0.0018
						h	0.0098	0.0305	4.3465	94.9%	0.1305	0.1245	0.0275	0.0203	3.1914	97.3%	0.1373	0.1238	0.0224	0.0056	3.4768	97.4%	0.1482	0.1480
	0.8			30	25	γ	0.1183	0.9285	3.0073	100.0%	0.0915	0.0890	0.4755	2.9721	4.5121	99.6%	0.1814	0.1811	0.1124	0.3923	1.7336	99.8%	0.1128	0.1111
						R	0.0900	0.0003	0.0758	99.6%	0.0025	0.0024	0.1925	0.0002	0.0713	98.7%	0.0023	0.0024	0.1155	0.0001	0.0495	99.3%	0.0019	0.0019
						h	0.0236	0.0367	4.9470	93.6%	0.1539	0.1517	0.0420	0.0209	3.9318	95.7%	0.1561	0.1549	0.0370	0.0058	3.6148	95.6%	0.1574	0.1567
				35	30	γ	0.1158	0.9145	2.9668	100.0%	0.0895	0.0809	0.5268	3.0623	4.8001	99.9%	0.1720	0.1372	0.1022	0.3290	1.6290	99.9%	0.1093	0.1100
						R	0.0901	0.0003	0.0635	99.7%	0.0021	0.0020	0.1758	0.0002	0.0545	98.8%	0.0019	0.0018	0.0915	0.0000	0.0364	98.7%	0.0016	0.0016
						h	0.0125	0.0315	3.6778	94.0%	0.1154	0.1139	0.0309	0.0202	3.0773	96.2%	0.1170	0.1166	0.0252	0.0046	2.8124	96.2%	0.1147	0.1141
100	0.3	0.8	0.55	70	60	γ	0.3354	0.6261	1.2250	100.0%	0.0384	0.0387	0.2427	1.7821	4.0156	99.9%	0.1589	0.1596	0.0053	0.1256	0.6687	99.6%	0.0494	0.0472
						R	0.1694	0.0012	0.0661	97.6%	0.0020	0.0020	0.0459	0.0004	0.0830	97.2%	0.0027	0.0026	0.0454	0.0000	0.0519	96.5%	0.0016	0.0016
						h	0.0939	0.0491	2.8698	100.0%	0.0894	0.0907	0.0567	0.0175	2.0016	99.9%	0.0971	0.0957	0.0358	0.0010	1.8928	99.8%	0.0968	0.0913
				90	80	γ	0.3042	0.5935	1.1440	100.0%	0.0346	0.0347	0.2122	1.6913	3.9182	99.6%	0.1469	0.1464	0.0004	0.0853	0.6073	99.8%	0.0396	0.0386
						R	0.1682	0.0009	0.0609	97.2%	0.0019	0.0019	0.0037	0.0004	0.0791	97.9%	0.0020	0.0019	0.0468	0.0000	0.0467	96.8%	0.0025	0.0024
						h	0.0046	0.0429	2.2126	100.0%	0.0708	0.0870	0.0310	0.0175	1.9593	99.9%	0.0900	0.0804	0.0308	0.0009	1.7804	99.8%	0.1692	0.1676
	0.8			70	60	γ	0.3245	0.6459	1.3847	100.0%	0.0420	0.0420	0.2583	1.9203	4.2647	99.4%	0.1631	0.1630	0.0016	0.0947	0.6819	99.2%	0.0397	0.0399
						R	0.2592	0.0013	0.0670	97.7%	0.0022	0.0021	0.0355	0.0004	0.0894	96.5%	0.0029	0.0029	0.0146	0.0000	0.0570	95.7%	0.0017	0.0017
						h	0.0910	0.0513	2.9989	100.0%	0.0917	0.0914	0.0407	0.0182	2.2696	99.9%	0.0963	0.0951	0.0407	0.0011	2.0970	99.8%	0.0944	0.0974
				90	80	γ	0.3004	0.6072	1.2564	100.0%	0.0378	0.0376	0.1569	1.4464	3.7892	99.8%	0.1400	0.1406	0.0010	0.0777	0.5927	99.8%	0.0355	0.0342
						R	0.2350	0.0010	0.0659	97.6%	0.0020	0.0020	0.0084	0.0004	0.0772	97.5%	0.0023	0.0021	0.0105	0.0000	0.0467	95.5%	0.0025	0.0025
						h	0.0722	0.0452	2.1277	100.0%	0.0816	0.0902	0.0309	0.0165	2.0293	99.9%	0.0617	0.0812	0.0419	0.0009	1.9823	99.9%	0.1663	0.1689
	0.3	0.9	0.7	70	60	γ	0.0523	0.5530	1.1517	100.0%	0.0378	0.0378	0.2255	1.6669	3.6107	99.6%	0.1473	0.1372	0.0011	0.1133	0.5659	99.7%	0.0688	0.0684
						R	0.0714	0.0002	0.0384	99.1%	0.0012	0.0012	0.0908	0.0001	0.0371	96.0%	0.0012	0.0012	0.0239	0.0000	0.0178	99.2%	0.0006	0.0006
						h	0.0099	0.0218	0.9053	99.7%	0.0324	0.0288	0.0059	0.0160	0.4177	97.5%	0.0311	0.0289	0.0013	0.0031	0.2006	95.7%	0.0293	0.0273
				90	80	γ	0.0502	0.4238	1.0924	100.0%	0.0306	0.0326	0.1946	1.5192	3.4419	99.9%	0.1358	0.1258	0.0004	0.0785	0.5295	99.9%	0.0629	0.0624
						R	0.0681	0.0002	0.0339	98.1%	0.0011	0.0011	0.0682	0.0001	0.0371	96.0%	0.0010	0.0010	0.0117	0.0000	0.0163	98.3%	0.0006	0.0006
						h	0.0081	0.0197	0.8934	99.9%	0.0095	0.0241	0.0016	0.0146	0.4056	98.8%	0.0264	0.0215	0.0012	0.0023	0.1984	98.1%	0.0422	0.0389
	0.8			70	60	γ	0.1025	0.6014	1.2592	100.0%	0.0386	0.0285	0.1954	1.7997	4.0800	99.4%	0.1580	0.1485	0.0015	0.0814	0.4589	99.8%	0.0779	0.0785
						R	0.0815	0.0002	0.0389	99.2%	0.0012	0.0012	0.0847	0.0001	0.0381	96.1%	0.0013	0.0013	0.0248	0.0000	0.0174	99.9%	0.0006	0.0006
						h	0.0087	0.0237	0.8764	99.0%	0.0271	0.0265	0.0073	0.0165	0.4360	97.2%	0.0319	0.0283	0.0032	0.0031	0.2048	95.1%	0.0270	0.0257
				90	80	γ	0.0992	0.4609	1.1092	100.0%	0.0271	0.0269	0.1486	1.3613	3.7407	99.7%	0.1063	0.0817	0.0009	0.0710	0.3164	99.7%	0.0613	0.0613
						R	0.0807	0.0002	0.0338	98.5%	0.0011	0.0011	0.0723	0.0001	0.0369	95.9%	0.0013	0.0013	0.0109	0.0000	0.0149	98.5%	0.0005	0.0006
						h	0.0071	0.0198	0.8051	99.8%	0.0252	0.0166	0.0012	0.0148	0.4347	98.9%	0.0275	0.0196	0.0030	0.0021	0.2049	97.9%	0.0741	0.0620

Table 3. Some statistical measures by simulation results when $\gamma = 3$.

$\gamma = 3$							MLE						Bayesian Non-informative						Bayesian Informative					
n	p	T_2	T_1	m	r		RAB	MSE	LACI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT	RAB	MSE	LCCI	CP	LBP	LBT
40	0.3	0.8	0.55	30	25	γ	0.4539	4.0708	5.8392	100.0%	0.1788	0.1812	0.6688	13.1059	10.3829	99.9%	0.3740	0.3632	0.1322	1.6268	3.7916	99.3%	0.1508	0.1495
						R	0.2953	0.0021	0.1091	97.9%	0.0037	0.0037	0.0406	0.0003	0.0948	95.3%	0.0032	0.0031	0.0501	0.0000	0.0913	95.7%	0.0026	0.0026
						h	0.1417	0.0863	5.3794	100.0%	0.1701	0.1731	0.0884	0.0126	3.8255	99.2%	0.1759	0.1746	0.0943	0.0020	3.8055	99.2%	0.1653	0.1665
				35	30	γ	0.4433	4.0192	5.1052	100.0%	0.1691	0.1792	0.6072	11.0159	9.7036	99.7%	0.3419	0.3407	0.1139	1.4971	3.0423	99.8%	0.1628	0.1574
						R	0.2833	0.0019	0.1047	97.4%	0.0033	0.0034	0.0392	0.0003	0.0911	95.2%	0.0031	0.0031	0.0501	0.0000	0.0875	97.0%	0.0033	0.0033
						h	0.1323	0.0806	5.2643	100.0%	0.1623	0.1723	0.0741	0.0113	3.5939	99.7%	0.1524	0.1525	0.0892	0.0020	3.7935	99.9%	0.2283	0.2272
	0.8			30	25	γ	0.4733	4.0225	5.5549	100.0%	0.1738	0.1746	0.6906	13.6622	10.6863	99.0%	0.3893	0.3914	0.1414	2.2912	4.0458	99.9%	0.1743	0.1753
						R	0.0759	0.0022	0.1114	98.2%	0.0035	0.0035	0.2993	0.0003	0.0933	95.6%	0.0034	0.0033	0.2590	0.0000	0.0935	96.5%	0.0028	0.0028
						h	0.1167	0.0906	5.3918	100.0%	0.1922	0.1891	0.1588	0.0135	3.2793	99.0%	0.1876	0.1839	0.1541	0.0021	3.8197	99.0%	0.1845	0.1761
				35	30	γ	0.4451	4.0410	5.1893	100.0%	0.1686	0.1692	0.6838	13.7295	10.3105	99.9%	0.3816	0.3828	0.1070	2.1831	3.9307	99.8%	0.1994	0.1958
						R	0.0691	0.0019	0.1097	98.2%	0.0033	0.0035	0.2573	0.0003	0.0891	95.5%	0.0032	0.0030	0.2250	0.0000	0.0909	95.7%	0.0037	0.0037
						h	0.0924	0.0822	4.9235	100.0%	0.1826	0.1625	0.1274	0.0121	3.1889	99.6%	0.1626	0.1826	0.1271	0.0019	3.2332	99.5%	0.2602	0.2601
	0.3	0.9	0.7	30	25	γ	0.1628	4.0077	5.5649	100.0%	0.1581	0.1595	0.5865	12.9979	10.0239	99.9%	0.3145	0.3041	0.1282	1.5237	3.0459	99.8%	0.2317	0.2288
						R	0.1268	0.0004	0.0721	99.3%	0.0022	0.0022	0.1155	0.0001	0.0566	99.5%	0.0021	0.0020	0.0666	0.0000	0.0367	99.1%	0.0018	0.0018
						h	0.0036	0.0474	4.0832	95.5%	0.1284	0.1295	0.0231	0.0117	3.1772	96.3%	0.1300	0.1301	0.0191	0.0036	3.0158	96.7%	0.1254	0.1253
				35	30	γ	0.1522	3.9133	5.0916	100.0%	0.1524	0.1424	0.5855	10.6491	9.5752	99.1%	0.3141	0.3208	0.1027	1.4026	2.8454	99.2%	0.2318	0.2267
						R	0.1145	0.0004	0.0593	98.4%	0.0019	0.0019	0.0896	0.0001	0.0329	98.1%	0.0015	0.0014	0.0408	0.0000	0.0257	97.7%	0.0014	0.0013
						h	0.0031	0.0417	3.3662	97.5%	0.1108	0.1109	0.0089	0.0117	2.2255	97.8%	0.1077	0.1078	0.0048	0.0034	2.2928	98.0%	0.1099	0.1074
	0.8			30	25	γ	0.2061	3.9567	5.0227	100.0%	0.1526	0.1503	0.5087	10.4346	10.3990	99.3%	0.3680	0.3732	0.1254	2.1492	2.6615	99.4%	0.2218	0.2172
						R	0.1298	0.0004	0.0688	99.3%	0.0023	0.0022	0.1184	0.0001	0.0576	99.0%	0.0020	0.0019	0.0645	0.00004	0.0401	99.5%	0.0017	0.0016
						h	0.0059	0.0452	3.7111	94.9%	0.1179	0.1150	0.0259	0.0117	3.4497	96.6%	0.1187	0.1189	0.0215	0.0034	3.2369	96.6%	0.1188	0.1185
				35	30	γ	0.2007	3.5429	4.9305	100.0%	0.1493	0.1423	0.6178	9.1560	10.0408	99.7%	0.3649	0.3678	0.0924	2.0508	3.4178	99.8%	0.2422	0.2417
						R	0.1207	0.0004	0.0611	99.3%	0.0020	0.0020	0.0183	0.0001	0.0371	98.3%	0.0014	0.0014	0.0213	0.00003	0.0302	97.8%	0.0012	0.0012
						h	0.0047	0.0416	3.3710	97.9%	0.1042	0.1039	0.0254	0.0108	3.1888	96.8%	0.1048	0.1045	0.0206	0.0031	3.1813	97.4%	0.1331	0.1313
100	0.3	0.8	0.55	70	60	γ	0.4036	2.9964	4.8516	100.0%	0.1587	0.1578	0.5820	11.3574	9.6886	99.5%	0.3556	0.3601	0.1073	1.4783	3.0267	99.7%	0.1475	0.1457
						R	0.1940	0.0005	0.0510	99.0%	0.0017	0.0017	0.0375	0.0001	0.0371	95.7%	0.0013	0.0013	0.0056	0.0000	0.0200	96.4%	0.0006	0.0006
						h	0.0261	0.0426	1.5486	99.9%	0.0503	0.0497	0.0024	0.0104	0.4181	99.4%	0.0553	0.0523	0.0056	0.0016	0.2455	99.5%	0.0484	0.0461
				90	80	γ	0.3943	2.8443	4.2250	100.0%	0.1299	0.1318	0.5108	10.6666	9.0810	99.9%	0.3254	0.3254	0.0663	0.8726	2.1039	99.7%	0.1082	0.1072
						R	0.0162	0.0004	0.0465	98.2%	0.0015	0.0015	0.0326	0.0001	0.0345	95.8%	0.0012	0.0011	0.0042	0.0000	0.0194	96.0%	0.0010	0.0010
						h	0.0190	0.0411	1.3991	100.0%	0.0426	0.0377	0.0017	0.0091	0.4076	99.9%	0.0443	0.0417	0.0049	0.0010	0.1730	99.9%	0.1380	0.1371
	0.8			70	60	γ	0.4273	2.9981	4.5656	100.0%	0.1416	0.1394	0.5857	11.6268	10.4425	99.2%	0.3646	0.3678	0.0928	1.3737	2.9333	99.8%	0.1450	0.1414
						R	0.0620	0.0005	0.0499	98.2%	0.0016	0.0016	0.0346	0.0001	0.0373	96.5%	0.0013	0.0013	0.0020	0.0000	0.0222	96.2%	0.0006	0.0006
						h	0.0287	0.0446	1.6360	100.0%	0.0518	0.0519	0.0042	0.0107	0.4406	99.9%	0.0546	0.0528	0.0076	0.0014	0.2429	99.8%	0.0523	0.0520
				90	80	γ	0.4055	2.7842	4.4790	100.0%	0.1245	0.1145	0.4924	9.5665	9.4259	99.5%	0.3340	0.3339	0.0910	1.2731	2.7032	99.6%	0.1574	0.1602
						R	0.0519	0.0004	0.0486	97.5%	0.0015	0.0014	0.0090	0.0001	0.0368	95.4%	0.0011	0.0011	0.0020	0.0000	0.0204	95.6%	0.0011	0.0011
						h	0.0268	0.0383	1.1619	100.0%	0.0414	0.0414	0.0041	0.0102	0.4084	99.9%	0.0413	0.0413	0.0049	0.0010	0.1944	99.6%	0.1374	0.1330
	0.3	0.9	0.7	70	60	γ	0.1348	2.8176	4.6683	100.0%	0.1484	0.1384	0.4634	3.5860	5.1506	98.9%	0.3047	0.3026	0.0913	1.3679	3.0369	99.8%	0.1626	0.1560
						R	0.0382	0.0003	0.0437	98.8%	0.0014	0.0014	0.1022	0.0001	0.0298	95.8%	0.0010	0.0010	0.0612	0.0000	0.0144	98.5%	0.0005	0.0005
						h	0.0031	0.0370	0.9011	100.0%	0.0300	0.0287	0.0214	0.0103	0.3399	99.5%	0.0282	0.0268	0.0134	0.0018	0.1743	96.8%	0.0253	0.0239
				90	80	γ	0.1131	2.7781	4.1225	100.0%	0.1163	0.1260	0.4590	2.8263	4.8812	99.7%	0.3010	0.3009	0.0589	0.7307	2.1008	99.7%	0.1347	0.1347
						R	0.0217	0.0003	0.0398	99.0%	0.0013	0.0013	0.0329	0.0001	0.0232	95.3%	0.0009	0.0009	0.0093	0.0000	0.0129	96.5%	0.0006	0.0006
						h	0.0027	0.0325	0.8702	100.0%	0.0288	0.0278	0.0080	0.0103	0.3010	99.8%	0.0193	0.0189	0.0041	0.0016	0.1594	99.9%	0.0901	0.0876
	0.8			70	60	γ	0.1934	2.9780	4.4200	100.0%	0.1373	0.1272	0.4687	4.0972	5.6152	99.8%	0.3081	0.3507	0.0810	1.3088	2.8664	99.9%	0.1709	0.1640
						R	0.1167	0.0003	0.0427	98.5%	0.0014	0.0014	0.0550	0.0001	0.0297	96.1%	0.0011	0.0011	0.0155	0.0000	0.0162	98.8%	0.0005	0.0005
						h	0.0051	0.0347	0.9357	99.4%	0.0283	0.0282	0.0039	0.0108	0.3579	99.0%	0.0289	0.0283	0.0006	0.0019	0.1942	96.9%	0.0258	0.0259
				90	80	γ	0.1892	2.5931	3.5596	100.0%	0.1037	0.1041	0.4565	3.0621	4.8896	99.8%	0.3006	0.3259	0.0913	1.1791	2.4907	99.2%	0.1633	0.1470
						R	0.1058	0.0003	0.0398	98.3%	0.0012	0.0012	0.0130	0.0001	0.0232	95.2%	0.0010	0.0009	0.0006	0.0000	0.0152	95.1%	0.0007	0.0007
						h	0.0041	0.0324	0.8354	100.0%	0.0211	0.0211	0.0033	0.0098	0.3254	99.9%	0.0211	0.0211	0.0005	0.0017	0.1325	99.9%	0.1025	0.1014

7. Real-life applications

To demonstrate the relevance and effectiveness of the proposed estimation techniques, three distinct real-world datasets from the engineering domain are examined. First, the data are validated using a time–temperature–transformation (TTT) plot and hazard plot, and then further examined by comparing results with a TTT plot, box plot, and violin plot for each dataset. Second, before deriving the theoretical results, we verify the proposed distribution of each data set and compare them with alternative unit models. To this end, we compute the MLEs of the parameters (with their standard errors, SEs) and evaluate the Kolmogorov–Smirnov (K–S) statistic with its (p)-value, along with the Aikake information criterion (AIC), the Bayesian information criterion (BIC), the Anderson–Darling (AD), and the Cramér–von Mises (CvM) goodness-of-fit measures to identify the best unit model. The proposed distribution has been compared by more alternative unit models as new unit-Lindley [24], Topp-Leone [25], Beta [26], unit Weibull [27], unit XLindley [28], unit-Gompertz [29], unit-Lindley [30], Alpha Power Topp-Leone [31], unit log–log [32], and unit Burr-XII [33].

Third, the estimators with the proposed distribution are presented through three graphical representations: the empirical versus estimated CDF, the histogram with the estimated PDF, and the P–P plot. Fourth, it is verified that these estimators possess a unique value, which corresponds to the maximum of the profile likelihood. Fifth, the parameters of the distribution are estimated using the different methods under study, based on the UPHCT1.

7.1. Tensile strength of polyester fibers

The first data set consists of 30 observations on the tensile strength of polyester fibers, originally reported by Mazucheli et al. [29]. The measured values are as follows: 0.023,0.032,0.081,0.054,0.069,0.094,0.105, 0.169,0.188,0.127,0.148, 0.216,0.255, 0.277,0.311,0.361,0.376,0.395, 0.432,0.463,0.481,0.519,0.529,0.567, 0.642, 0.674,0.752,0.823,0.887,0.926.

The TTT-plot (Figure 2, left) suggests an increasing failure rate, which is consistent with the estimated hazard rate curve (second) that shows a monotonic upward trend. The box plot (third) indicates that the data are moderately spread around the median without extreme outliers. The Violin plot (right) confirms the overall distributional shape and variability, highlighting a relatively symmetric spread around the center. Together, the four component plots of Figure 2 provide evidence that the proposed distribution is appropriate for modeling the given Data 1.

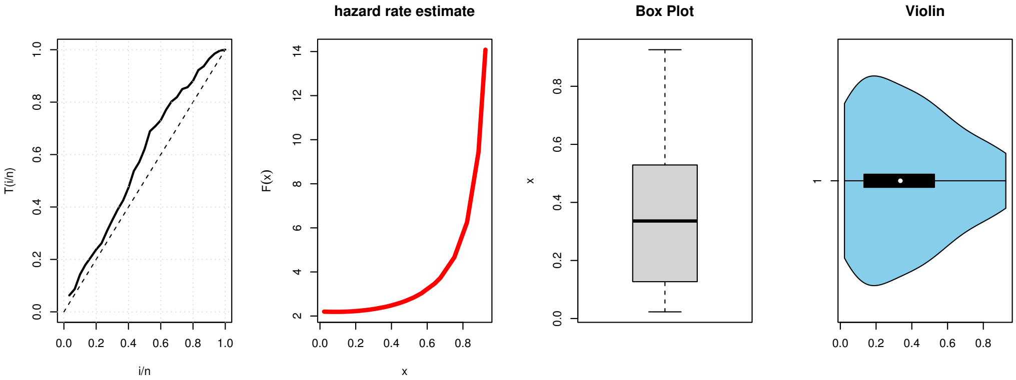

Figure 2. Graphical assessment of the fitted distribution using different diagnostic tools: Data 1.

In Figure 3, the empirical CDF (left) aligns closely with the estimated CDF, indicating a good fit of the proposed distribution. The histogram with the estimated PDF (center) shows that the theoretical curve successfully captures the overall shape of the data. The P–P plot (right) lies nearly along the diagonal line, further confirming that the fitted distribution provides an adequate representation of the observed sample.

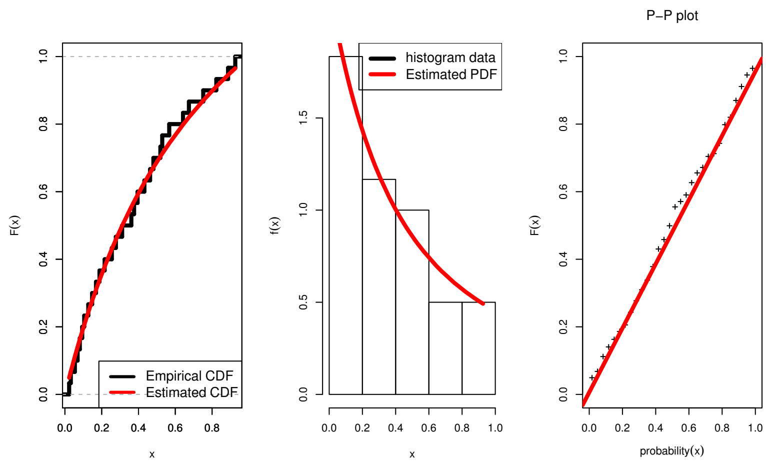

Figure 3. Graphical comparison between the empirical data and the fitted model: Data 1.

Across all diagnostics in Table 4, the proposed model provides the best overall fit. It has the smallest KSD (0.0570), lowest CvM (0.0146) and AD (0.1098), and the most favorable information criteria (AIC = -4.4408, BIC = -3.0396). The UW model is a close second in KSD (0.0576) but is clearly inferior in AIC and BIC. Models such as UXL and UL perform poorly, showing very large Kolmogorov–Smirnov-D values (KSD) values (0.2408 and 0.2722) and highly unfavorable AIC and BIC. PVKS values are near 1 for most models, but the sharper separation in KSD, CvM, AD, and ICs confirms that the proposed distribution offers the most adequate description of the data.

The profile likelihood in the Figure 4 curve (left) attains a clear and unique maximum, ensuring the existence and uniqueness of the MLEs for the parameter γ. The normalized profile likelihood (right) further supports the stability of the estimation, as the curve crosses the reference line at the identified estimate. These results confirm that the proposed distribution yields well-defined parameter estimates.

Table 4. MLE and different measures of statistics for unit models: Data 1.

		γ	θ	KSD	PVKS	CvM	AD	AIC	BIC
Target	Estimates	0.8279		0.0570	0.9999	0.0146	0.1098	-4.4408	-3.0396
	StEr	0.4698							
TL	Estimates	0.2963		0.3314	0.0019	0.1277	0.7879	21.7155	23.1167
	StEr	0.0388							
NUL	Estimates	1.1091		0.0665	0.9981	0.0189	0.1600	-3.8078	-2.4066
	StEr	0.2025							
Beta	Estimates	0.9666	0.2238	0.0669	0.9979	0.0184	0.1559	-2.6101	0.1923
	StEr	1.6205	0.4107						
UW	Estimates	0.5717	0.1320	0.0576	0.9998	0.0148	0.1100	-3.4345	-0.6321
	StEr	1.3690	0.2007						
UXL	Estimates	0.9127		0.2408	0.0513	0.1400	0.9397	86.4149	87.8161
	StEr	0.1313							
UG	Estimates	1.0383	0.7715	0.0733	0.9932	0.0184	0.1161	-3.8976	-1.0952
	StEr	0.4211	0.1919						
UL	Estimates	1.0504		0.2722	0.0187	0.1656	1.0870	20.1704	21.5716
	StEr	0.1455							
APTL	Estimates	115.1771	391.7070	0.0725	0.9941	0.0360	0.2897	-0.2175	2.5849
	StEr	0.2714	0.2196						
ULL	Estimates	1.0213	0.1580	0.0846	0.9704	0.0194	0.1221	-3.9134	-1.1110
	StEr	1.5245	0.1230						
UBXII	Estimates	1.0331	0.2060	0.0993	0.9008	0.0586	0.4419	1.9220	4.7244
	StEr	1.8465	0.3054						

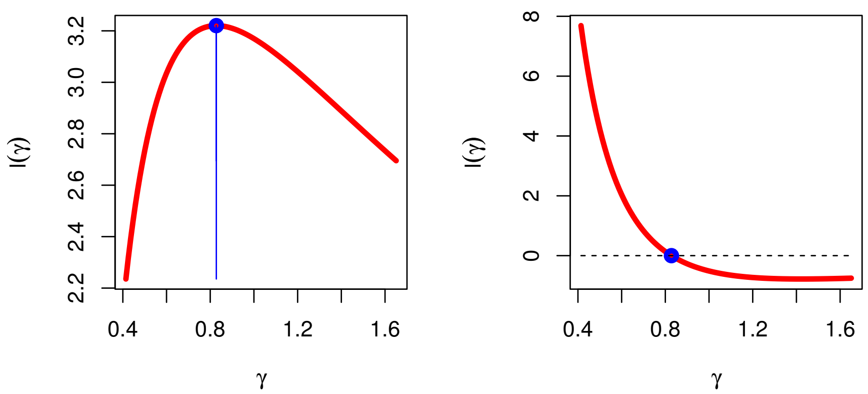

Figure 4. Profile likelihood plots for the parameter γ: Data 1.

Table 5 compares MLE and Bayesian estimators for Data 1 under the UPHCT1 scheme across combinations of (m, r, T_1, T_2), with reliability measures evaluated at time s^*. For each setting, the table reports the estimate of the parameter γ, its standard error SE(γ), the reliability $R(t)$, and the hazard rate $h(t)$. Two Bayesian strategies are used: Bayes1 adopts a non–informative prior, whereas Bayes2 employs an informative gamma prior whose hyperparameters are selected from the MLE output (elective hyperparameters method). Posterior summaries are obtained via MCMC with 12000 iterations, discarding the first 2000 as burn–in. Overall, MLE and Bayesian estimates are broadly consistent; however, the informative–prior scheme (Bayes2) typically yields more stable inference often reducing SE(γ) and producing smoother estimates of $R(t)$ and $h(t)$ at s^* while Bayes1 shows greater dispersion due to the diffuse prior.

Table 5. MLE and Bayesian estimation based on UPHCT1 for Data 1.

				p			0.3					0.8		
m	r	T1	T2		s^*	v	MLEr	bayes1	bayes2	s^*	v	MLEr	bayes1	bayes2
20	15	0.13225	0.336	γ	0.519	15	0.4203	0.9662	0.5276	0.567	15	0.3554	0.8553	0.4335
				SE(γ)			0.2223	0.7738	0.2087			0.1861	0.7307	0.1722
				R			0.2152	0.3129	0.2425			0.1668	0.2604	0.1876
				h			3.1436	2.7523	3.0345			3.3936	3.0126	3.3090
		0.436	0.7265	γ	0.7265	17	0.5600	1.5294	0.7351	0.7265	17	0.4821	1.2300	0.6277
				SE(γ)			0.3190	1.2155	0.3065			0.2706	1.0150	0.2586
				R			0.1191	0.1854	0.1376			0.1091	0.1719	0.1268
				h			4.4336	4.0996	4.3405			4.4837	4.1674	4.3948
26	20	0.13225	0.536	γ	0.536	20	0.4158	1.0617	0.4793	0.536	20	0.3071	0.6835	0.3529
				SE(γ)			0.2034	0.8762	0.1924			0.1394	0.5735	0.1255
				R			0.2027	0.3083	0.2190			0.1690	0.2601	0.1842
				h			3.2058	2.7811	3.1401			3.3412	2.9752	3.2801
		0.436	0.5265	γ	0.529	20	0.4158	1.0617	0.4793	0.567	20	0.3096	0.6668	0.3666
				SE(γ)			0.2034	0.8762	0.1924			0.1407	0.4931	0.1348
				R			0.2073	0.3144	0.2239			0.1529	0.2340	0.1700
				h			3.1816	2.7518	3.1149			3.4503	3.1200	3.3806

Figure 5 displays the trace of 10000 MCMC draws for γ using Data 1 with $m = 25$, $r = 20$, $T_1 = 0.436$, and $T_2 = 0.5265$ (see Table 5). The chain mixes well and exhibits clear convergence. The histogram in the right panel of Figure 5 indicates that the posterior distribution is approximately symmetric.

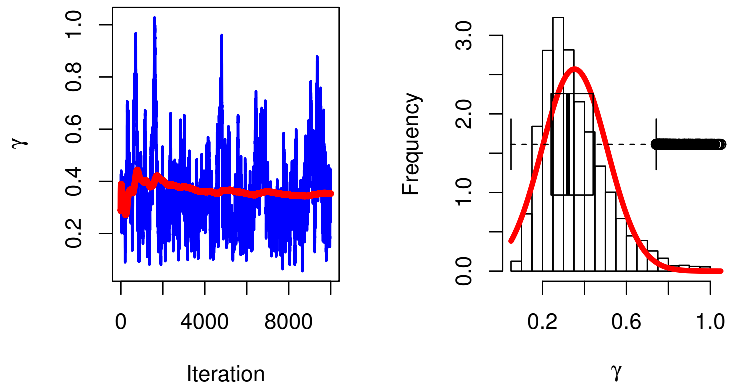

Figure 5. Trace plot and histograms for MCMC results of γ from Data 1 when $m = 25, r = 20, T_1 = 0.436$, and $T_2 = 0.5265$.

7.2. Aircraft air-conditioning failures

This dataset contains 30 failure times for an airplane's air-conditioning system, reported in Ref. [34]: 12, 120, 11, 23, 261, 87, 7, 120, 14, 62, 71, 11, 14, 47, 225, 71, 246, 21, 42, 20, 5, 3, 14, 11, 16, 90, 1, 16, 52, 95. For comparability on the unit interval, each value is rescaled by 265, yielding ([35]): 0.018867925, 0.045283019, 0.086792453, 0.026415094, 0.452830189, 0.984905660, 0.328301887, 0.052830189, 0.233962264, 0.196226415, 0.358490566, 0.267924528, 0.928301887, 0.177358491, 0.849056604, 0.079245283, 0.158490566, 0.075471698, 0.452830189, 0.052830189, 0.041509434, 0.052830189, 0.267924528, 0.041509434, 0.060377358, 0.041509434, 0.011320755, 0.339622642, 0.003773585, 0.060377358.

Figure 6 indicates an increasing failure rate the TTT plot lies above the diagonal, and the estimated hazard rises monotonically. The box plot shows moderate dispersion around the median without notable outliers, and the violin plot is roughly symmetric. Collectively, these diagnostics support the suitability of the proposed distribution for Data 2.

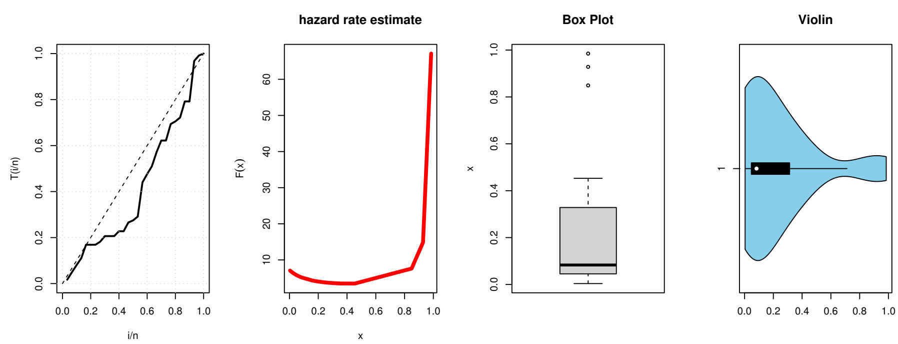

Figure 6. Graphical assessment of the fitted distribution using different diagnostic tools: Data 2.

For Dataset 2, Figure 7 shows strong agreement between the empirical and fitted CDFs. The fitted PDF traces the histogram well, capturing the sample's shape, and the P–P plot clusters along the 45° line. Collectively, these diagnostics indicate that the proposed distribution fits the data adequately.

The proposed model delivers the best overall fit in Table 6. It achieves the lowest KSD (0.1275), smallest CvM (0.1006) and AD (0.6139), and the most favorable information criteria (AIC = -33.01, BIC = -31.61). Its K–S p-value (PVKS = 0.7143) is comfortably high, indicating no evidence against the fit. Two competitors, UG and ULL, are somewhat close (e.g., KSD 0.1291 and 0.1510; AIC -32.19 and -30.98), but remain inferior on most metrics. Models such as UXL, UL, NUL, and APTL show clear lack of fit (large KSD, CvM, and AD and poor AIC and BIC). Overall, the metrics consistently favor the proposed distribution for Data 2.

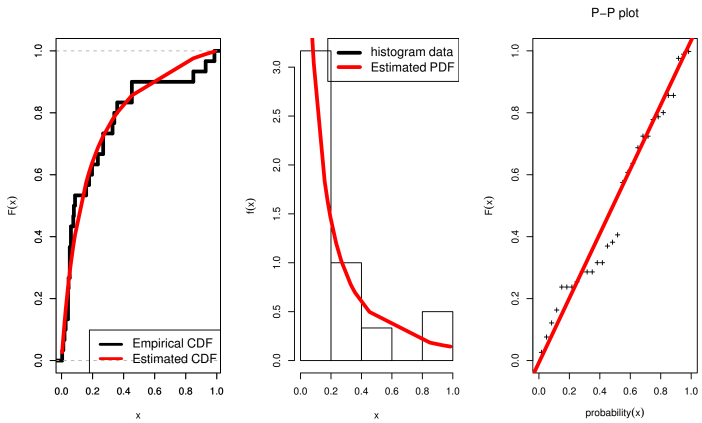

Figure 7. Graphical comparison between the empirical data and the fitted model: Data 2.

Table 6. MLE and different measures of statistics for unit models: Data 2.

		γ	θ	KSD	PVKS	CVM	AD	AIC	BIC
Proposed	Estimates	0.1616		0.1275	0.7143	0.1006	0.6139	-33.0141	-31.6129
	StEr	0.0597							
TL	Estimates	0.6017		0.1938	0.2097	0.2380	1.5103	-21.9604	-20.5592
	StEr	0.1098							
NUL	Estimates	0.0847		0.3690	0.0006	0.1097	0.7576	18.5980	19.9992
	StEr	0.0109							
Beta	Estimates	0.5141	0.1118	0.1958	0.2003	0.2173	1.3859	-22.4926	-19.6902
	StEr	1.3430	0.3643						
UW	Estimates	0.2787	0.0858	0.1742	0.3228	0.1593	1.0191	-26.3847	-23.5823
	StEr	1.4562	0.2262						
UXL	Estimates	0.5034		0.6821	0.0000	0.7261	3.9324	151.5946	152.9958
	StEr	0.0684							
UG	Estimates	0.4059	0.2491	0.1291	0.7127	0.1008	0.6214	-32.1936	-29.3912
	StEr	0.4686	0.1345						
UL	Estimates	0.5512		0.7201	0.0000	0.7809	4.1849	117.9145	119.3157
	StEr	0.0735							
APTL	Estimates	119.5303	215.1204	0.1814	0.2771	0.2924	1.8351	-16.4148	-13.6124
	StEr	0.1349	0.0592						
ULL	Estimates	1.1266	0.1739	0.1510	0.5006	0.1042	0.6532	-30.9827	-28.1803
	StEr	1.2791	0.0809						
UBXII	Estimates	0.6898	0.1537	0.2242	0.0979	0.4214	2.5452	-7.8299	-5.0275
	StEr	1.679494	0.31171						

Figure 8 shows a unimodal profile likelihood for γ, implying a unique MLE. The normalized profile likelihood intersects the reference level at the same point, indicating stable and identifiable estimation. Hence, the proposed distribution yields well-defined parameter estimates.

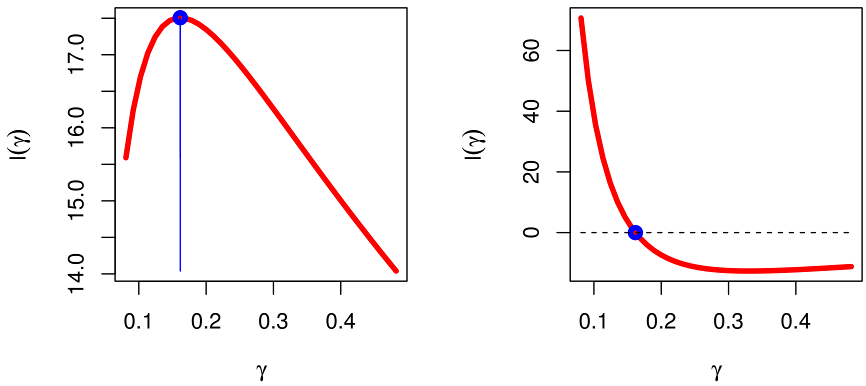

Figure 8. Profile likelihood plots for the parameter γ: Data 2.

Table 7 summarizes the MLE and Bayesian estimation results for Data 2 under the UPHCT1 model with different censoring thresholds (T_1, T_2) and sample settings (m, r). Overall, the results show that Bayesian estimation, particularly Bayes2, produces more stable and precise parameter estimates with smaller SE(γ) compared to both MLE and Bayes1. The reliability and hazard measures obtained from Bayes2 appear smoother and more consistent across censoring schemes, reflecting the advantage of incorporating informative prior knowledge. In contrast, Bayes1 exhibits slightly wider variation due to the absence of prior information. These findings highlight the practical benefit of the informative Bayesian approach for reliability assessment under hybrid censoring.

Table 7. MLE and Bayesian estimation based on UPHCT1 for Data 2.

				p			0.3					0.8		
m	r	T1	T2		s^*	v	MLEr	bayes1	bayes2	s^*	v	MLEr	bayes1	bayes2
20	15	0.04717	0.083019	γ	0.267925	15	0.0878	0.1553	0.0936	0.267925	15	0.1096	0.2112	0.1234
				SE(γ)			0.0345	0.1029	0.0326			0.0452	0.1414	0.0429
				R			0.1807	0.2687	0.1895			0.2125	0.3227	0.2309
				h			4.1770	3.7286	4.1320			4.0151	3.4530	3.9214
		0.083019	0.313208	γ	0.313208	16	0.0959	0.1561	0.1062	0.313208	16	0.1193	0.2145	0.1348
				SE(γ)			0.0374	0.1048	0.0340			0.0488	0.1416	0.0453
				R			0.1610	0.2284	0.1740			0.1895	0.2792	0.2067
				h			3.9005	3.5869	3.8401			3.7681	3.3509	3.6880
25	20	0.04717	0.083019	γ	0.358491	20	0.1117	0.1878	0.1222	0.339623	20	0.1118	0.1970	0.1230
				SE(γ)			0.0427	0.1224	0.0404			0.0440	0.1264	0.0425
				R			0.1524	0.2205	0.1630			0.1636	0.2424	0.1756
				h			3.6856	3.3894	3.6393			3.7293	3.3779	3.6759
		0.483019	0.713208	γ	0.713208	22	0.1364	0.2273	0.1484	0.713208	22	0.1380	0.2487	0.1528
				SE(γ)			0.0524	0.1461	0.0504			0.0543	0.1645	0.0507
				R			0.0461	0.0693	0.0494			0.0465	0.0742	0.0506
				h			4.6638	4.5501	4.6475			4.6617	4.5264	4.6416

Figure 9 presents the trace plot for the 10000 MCMC samples of γ obtained from Data 3 with $m = 25$, $r = 20$, $T_1 = 0.483019$, and $T_2 = 0.713208$ (see Table 7). The chain shows good mixing and clear convergence. The associated histograms (Figure 9) suggest that the resulting posterior distributions are close to symmetric.

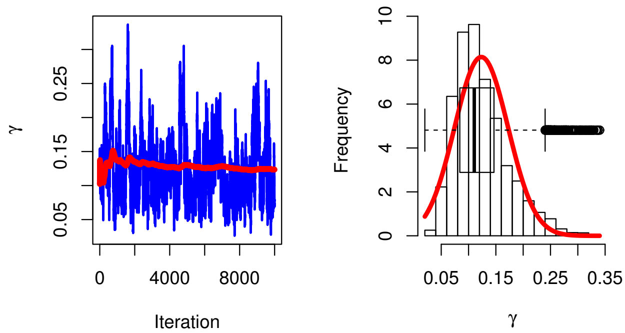

Figure 9. Trace plot and histograms for MCMC results of γ from Data 2 when $m = 25, r = 20, T_1 = 0.483019$, and $T_2 = 0.713208$.

7.3. Ordered failure times of 20 components

To illustrate the unit distribution, a data set reported by Nigm et al. [36] consisting of 20 ordered failure times for identical components has been analyzed. The observations are: 0.0009, 0.0040, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.1750, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.

Figure 10 points to an increasing failure rate: the TTT plot lies above the 45° line, and the estimated hazard is monotone increasing. The box plot shows moderate variability around the median with no marked outliers, while the violin plot is approximately symmetric. Taken together, these diagnostics affirm that the proposed distribution is suitable for Data Set 3.

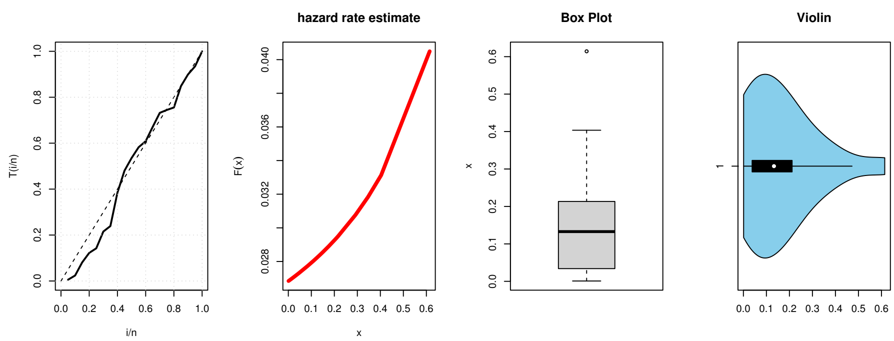

Figure 10. Graphical assessment of the fitted distribution using different diagnostic tools: Data 3.

In Figure 11, the empirical and fitted CDFs nearly coincide, the fitted PDF follows the histogram closely, and the P–P points lie close to the 45° line. Taken together, these diagnostics show that the proposed distribution provides an adequate fit for Data 3.

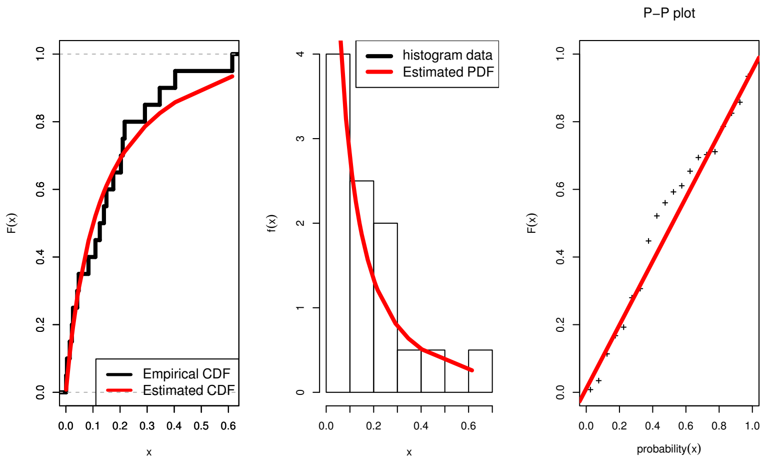

Figure 11. Graphical comparison between the empirical data and the fitted model: Data 3.

The proposed model delivers the best overall fit across all diagnostics for Data 3 in Table 8. It has the lowest KSD (0.1216), smallest CvM (0.0727) and AD (0.4165), the most negative AIC (-29.6330) and BIC (-28.6372), and the highest K-S p-value (PVKS = 0.8949), all indicating no evidence against the fit. The UW model is the closest competitor (KSD = 0.1319, PVKS = 0.8334) but remains inferior on information criteria and distance measures. Models such as TL, UL, UG, APTL, and especially NUL and UBXII, show noticeably poorer fit (larger KSD, CvM, and AD and less favorable AIC and BIC). Overall, the metrics consistently favor the proposed distribution for Data 3.

Figure 12 displays a single-peaked profile likelihood for γ, confirming a unique maximizer and thus a unique MLE. The normalized profile likelihood crosses the reference level at the same point, indicating stable and identifiable estimation. Therefore, the proposed distribution has well-defined parameter estimates.

Table 9 displays the estimation outcomes for Data 3 under the UPHCT1 setup across different censoring schemes. It is evident that the estimates of γ remain stable under both censoring thresholds, while the reliability $R(t)$ and hazard rate $h(t)$ respond sensitively to changes in s^*. For $p = 0.3$ and $p = 0.8$, the general trend shows that hazard rates increase with censoring severity, whereas the reliability values decline accordingly, reflecting the expected behavior of lifetime data. Moreover, the Bayesian estimates exhibit closer alignment with the likelihood-based results as sample size increases, suggesting improved efficiency under more informative data settings. This highlights the robustness of the proposed distribution in capturing the underlying reliability structure of Data 3, even under hybrid censoring conditions.

Figure 13 shows the trace plot of the 10000 MCMC draws for γ based on Data 3 with $m = 18$, $r = 13$, $T_1 = 0.1328$, and $T_2 = 0.211625$ (see Table 9). The chain mixes well and displays clear convergence. The corresponding histograms in Figure 13 indicate that the posterior distributions of the unknown parameters are approximately symmetric.

Table 8. MLE and different measures of statistics for unit models: Data 3.

		γ	θ	KSD	PVKS	CVM	AD	AIC	BIC
Proposed	Estimates	0.1265		0.1216	0.8949	0.0727	0.4165	-29.6330	-28.6372
	StEr	0.0545							
TL	Estimates	0.5113		0.1849	0.4480	0.0801	0.4754	-29.2334	-28.2376
	StEr	0.1143							
NUL	Estimates	0.0245		0.6330	0.0000	0.3693	2.1376	71.4040	72.3997
	StEr	0.0039							
UW	Estimates	0.1598	0.0710	0.1319	0.8334	0.0736	0.4330	-28.9150	-26.9235
	StEr	1.7271	0.2875						
UXL	Estimates	4.0928		0.1725	0.5348	0.0784	0.4702	-12.7986	-11.8028
	StEr	0.8619							
UG	Estimates	0.7743	0.5996	0.1494	0.7093	0.0911	0.5512	-25.5251	-23.5336
	StEr	0.2781	0.1204						
UL	Estimates	4.6365		0.1768	0.5037	0.0793	0.4718	-28.1389	-27.1432
	StEr	0.9012							
APTL	Estimates	100.1201	239.9820	0.1979	0.3651	0.0822	0.5649	-25.0943	-23.1029
	StEr	0.1211	0.0718						
ULL	Estimates	1.1712	0.2079	0.1545	0.6704	0.0945	0.5743	-26.1665	-24.1751
	StEr	1.2003	0.0809						
UBXII	Estimates	0.2783	0.1087	0.2274	0.2164	0.1163	0.7519	-24.6902	-22.6987
	StEr	4.3787	1.4950						

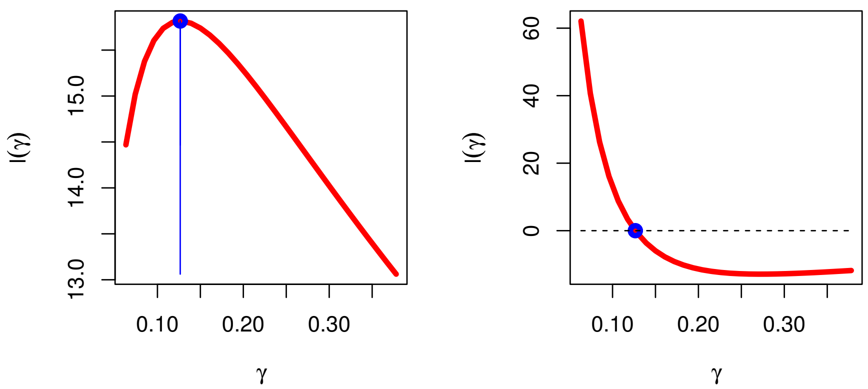

Figure 12. Profile likelihood plots for the parameter γ: Data 3.

Table 9. MLE and Bayesian estimation based on UPHCT1 for Data 3.

				p	0.3					0.8				
m	r	T1	T2		s^*	v	MLEr	bayes1	bayes2	s^*	v	MLEr	bayes1	bayes2
15	10	0.037875	0.1328	γ	0.1498	10	0.0469	0.0985	0.0556	0.1498	10	0.0446	0.0855	0.0537
				SE(γ)			0.0243	0.0748	0.0214			0.0237	0.0667	0.0228
				R			0.2029	0.3372	0.2300			0.1951	0.3089	0.2244
				h			6.2589	5.2041	6.0459			6.3196	5.4264	6.0900
		0.1328	0.211625	γ	0.211625	13	0.1012	0.1990	0.1092	0.211625	12	0.0620	0.1282	0.0706
				SE(γ)			0.0482	0.1384	0.0471			0.0311	0.0952	0.0293
				R			0.2550	0.3821	0.2684			0.1787	0.2974	0.1973
				h			4.4651	3.7038	4.3852			4.9228	4.2111	4.8113
18	13	0.04717	0.1328	γ	0.175	13	0.0715	0.1350	0.0781	0.2099	13	0.0660	0.1282	0.0740
				SE(γ)			0.0338	0.0948	0.0317			0.0327	0.0928	0.0301
				R			0.2393	0.3592	0.2547			0.1890	0.2996	0.2060
				h			5.2687	4.4383	5.1625			4.8904	4.2234	4.7875
		0.1328	0.211625	γ	0.211625	14	0.0690	0.1376	0.0752	0.211625	13	0.0660	0.1282	0.0740
				SE(γ)			0.0321	0.0981	0.0319			0.0327	0.0928	0.0306
				R			0.1939	0.3106	0.2066			0.1874	0.2974	0.2043
				h			4.8317	4.1321	4.7552			4.8707	4.2112	4.7690

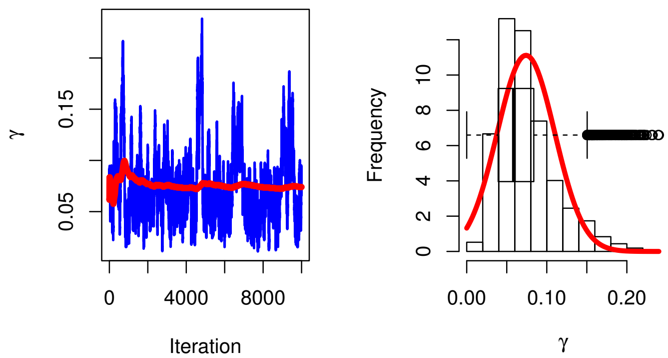

Figure 13. Trace plot and histograms for MCMC results of γ from Data 3 when $m = 18, r = 13, T_1 = 0.1328$, and $T_1 = 0.211625$.

8. Conclusions

This study has introduced, for the first time, a unified progressive hybrid censoring scheme with binomial random removals, thereby enhancing the flexibility of life-testing by accommodating random withdrawals while retaining informative failure observations. Within this framework, likelihood and Bayesian inferential procedures were developed for a one-parameter lifetime model, including estimation of the primary parameter as well as the associated reliability and hazard rate functions.

Comprehensive simulation experiments and real data applications indicate that Bayesian inference with gamma priors outperforms classical approaches, yielding smaller bias, reduced MSE, and tighter interval estimates with appropriate coverage. In addition, parametric bootstrap procedures were employed to construct interval estimates; across scenarios, the bootstrap-t intervals achieved coverage closer to the nominal level and shorter average lengths than the standard percentile bootstrap, indicating superior finite-sample performance. Furthermore, when benchmarked against ten competing unit distributions, including new unit-Lindley, Topp-Leone, Beta, unit-Weibull, unit-XLindley, unit-Gompertz, unit-Lindley, alpha power Topp-Leone, unit log-log, and unit Burr-XII, the proposed hone-parameter model provided superior overall fit and inferential accuracy under the proposed censoring scheme.

In conclusion, integrating binomial random removals into unified progressive hybrid censoring, together with Bayesian estimation under gamma priors offers a practical and powerful methodology for reliability analysis with censored lifetime data. Future research directions include extensions to multi-parameter families, stress-strength models, optimal design under cost and time constraints, and robustness studies under model misspecification.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. E. M. Almetwally, M. A. H. Sabry, R. Alharbi, D. Alnagar, S. A. M. Mubarak, E. H. Hafez, Marshall–Olkin alpha power extended Weibull distribution: different methods of estimation based on Type-I and Type II censoring, *Gazi Univ. J. Sci.*, **35** (2022), 293–312. https://doi.org/10.1155/2021/5533799
2. A. Childs, B. Chandrasekar, N. Balakrishnan, D. Kundu, Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Ann. Inst. Stat. Math.*, **55** (2003), 319–330. https://doi.org/10.1007/BF02530502
3. B. Chandrasekar, A. Childs, N. Balakrishnan, Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring, *Naval Research Logistics*, **51** (2004), 994–1004. https://doi.org/10.1002/nav.20038
4. N. Balakrishnan, A. Rasouli, N. S. Farsipour, Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution, *J. Stat. Comput. Simul.*, **78** (2008), 475–488. https://doi.org/10.1080/00949650601158336
5. N. Balakrishnan, E. Cramer, *The art of progressive censoring*, Statistics for Industry and Technology, Birkhäuser New York, 2014. https://doi.org/10.1007/978-0-8176-4807-7
6. J. Górny, E. Cramer, Modularization of hybrid censoring schemes and its application to unified progressive hybrid censoring, *Metrika*, **81** (2018), 173–210. https://doi.org/10.1007/s00184-017-0639-7

7. S. Dutta, S. Kayal, Estimation and prediction for Burr Type III distribution based on unified progressive hybrid censoring scheme, *J. Appl. Stat.*, **51** (2024), 1–33. https://doi.org/10.1080/02664763.2022.2113865

8. M. M. Yousef, A. S. Hassan, E. M. Almetwally, M. Nagy, A. H. Mansi, Utilizing unified progressive hybrid censored data in parametric inference under accelerated life tests, *Sci. Rep.*, **15** (2025), 32567. https://doi.org/10.1038/s41598-025-06927-5

9. S. Dutta, H. K. T. Ng, S. Kayal, Inference for Kumaraswamy-G family of distributions under unified progressive hybrid censoring with partially observed competing risks data, *Stat. Neerl.*, **78** (2024), 719–742. https://doi.org/10.1111/stan.12357

10. S. Anwar, S. A. Lone, A. Khan, S. Almutlak, Stress-strength reliability estimation for the inverted exponentiated Rayleigh distribution under unified progressive hybrid censoring with application, *Electron. Res. Arch.*, **31** (2023), 4011–4033. https://doi.org/10.3934/era.2023204

11. H. S. Mohammed, O. E. Abo-Kasem, A. Elshahhat, Optimum progressive data analysis and Bayesian inference for unified progressive hybrid INH censoring with applications to diamonds and gold, *Axioms*, **14** (2025), 559. https://doi.org/10.3390/axioms14080559

12. A. Prakash, A. Maurya, R. K. Maurya, Statistical and machine learning approach for progressive stress accelerated life testing under unified progressive hybrid censoring, *Qual. Reliab. Eng. Int.*, **41** (2025), 2463–2480. https://doi.org/10.1002/qre.3782

13. E. A. Elsherpieny, A. Abdel-Hakim, Statistical analysis of Alpha-Power exponential distribution using unified Hybrid censored data and its applications, *Comput. J. Math. Stat. Sci.*, **4** (2025), 283–315. https://doi.org/10.21608/cjmss.2025.346134.1094

14. R. Alotaibi, M. Nassar, A. Elshahhat, Analysis of unified hybrid Hjorth competing risk data and its application to multiple myeloma and electrical appliances, *J. Math.*, **2025** (2025), 5089157. https://doi.org/10.1155/jom/5089157

15. M. Muhammad, A new two parameter distribution with a finite support, *Appl. Math. Sci.*, **10** (2016), 1–12.

16. A. Henningsen, O. Toomet, maxLik: A package for maximum likelihood estimation in R, *Comput. Stat.*, **26** (2011), 443–458.

17. K. Krishnamoorthy, Y. Lin, Y. Xia, Confidence limits and prediction limits for a Weibull distribution based on the generalized variable approach, *J. Stat. Plan. Infer.*, **139** (2009), 2675–2684. https://doi.org/10.1016/j.jspi.2008.12.010

18. S. Dey, T. Dey, On progressively censored generalized inverted exponential distribution, *J. Appl. Stat.*, **41** (2014), 2557–2576. https://doi.org/10.1080/02664763.2014.922165

19. C. M. Carlo, *Markov chain Monte Carlo and Gibbs sampling*, Lecture notes for EEB, 2004, 1–24.

20. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.*, **21** (1953), 1087–1092. https://doi.org/10.1063/1.1699114

21. M. H. Chen, Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, *J. Comput. Graph. Stat.*, **8** (1999), 69–92. https://doi.org/10.2307/1390921

22. B. Efron, Bootstrap methods: another look at the jackknife, In: *Breakthroughs in statistics*, New York: Springer, 1992, 569–593. https://doi.org/10.1007/978-1-4612-4380-9_41
23. M. Plummer, N. Best, K. Cowles, K. Vines, *CODA: Convergence diagnosis and output analysis for MCMC*, **6** (2006), 7–11.
24. J. Mazucheli, S. R. Bapat, A. F. B. Menezes, A new one-parameter unit-Lindley distribution, *Chil. J. Stat.*, **11** (2020), 53–67.
25. A. A. Al-Babtain, I. Elbatal, C. Chesneau, M. Elgarhy, Sine Topp-Leone-G family of distributions: theory and applications, *Open Phys.*, **18** (2020), 574–593. https://doi.org/10.1515/phys-2020-0180
26. A. K. Gupta, S. Nadarajah, *Handbook of Beta distribution and its applications*, CRC Press, 2004. https://doi.org/10.1201/9781482276596
27. J. Mazucheli, A. F. B. Menezes, M. E. Ghitany, The unit-Weibull distribution and associated inference, *J. Appl. Probab. Stat.*, **13** (2018), 1–22.
28. A. M. Gemeay, M. I. A. Araibi, E. M. Almetwally, I. Elbatal, C. Tanış, L. P. Sapkota, et al., Modeling with the inverse unit new XLindley distribution: properties and real-world applications, *Eng. Rep.*, **7** (2025), e70354. https://doi.org/10.1002/eng2.70354
29. J. Mazucheli, A. F. Menezes, S. Dey, Unit-Gompertz distribution with applications, *Statistica*, **79** (2019), 25–43. https://doi.org/10.6092/issn.1973-2201/8497
30. J. Mazucheli, A. F. B. Menezes, S. Chakraborty, On the one parameter unit-Lindley distribution and its associated regression model for proportion data, *J. Appl. Stat.*, **46** (2019), 700–714. https://doi.org/10.1080/02664763.2018.1511774
31. L. Benkhelifa, Alpha power Topp-Leone Weibull distribution: properties, characterizations, regression modeling and applications, *J. Stat. Manage. Syst.*, **25** (2022), 1945–1970. https://doi.org/10.1080/09720510.2021.1995217
32. M. Ç. Korkmaz, Z. S. Korkmaz, The unit log–log distribution: a new unit distribution with alternative quantile regression modeling and educational measurements applications, *J. Appl. Stat.*, **50** (2023), 889–908. https://doi.org/10.1080/02664763.2021.2001442
33. M. Ç. Korkmaz, C. Chesneau, On the unit Burr-XII distribution with the quantile regression modeling and applications, *Comput. Appl. Math.*, **40** (2021), 29. https://doi.org/10.1007/s40314-021-01418-5
34. P. Lahiri, *Model selection*, IMS, 2001.
35. M. Elgarhy, A. Al Mutairi, A. S. Hassan, C. Chesneau, A. H. Abdel-Hamid, Bayesian and non-Bayesian estimations of truncated inverse power Lindley distribution under progressively type-II censored data with applications, *AIP Adv.*, **13** (2023), 095130. https://doi.org/10.1063/5.0172632
36. A. M. Nigm, E. K. Al-Hussaini, Z. F. Jaheen, Bayesian one-sample prediction of future observations under Pareto distribution, *Statistics*, **37** (2003), 527–536. https://doi.org/10.1080/0233188031000159883