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Research article

$p(x)$-stability of the Dirichlet problem for Poisson's equation with variable exponents

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Abstract: We investigate the structural stability of solutions to boundary value problems for the variable exponent $p(x)$-Laplacian. Stability questions for such problems under perturbations of the boundary operator, the differential operator, boundary data, or the domain have a long history and play a central role in the analysis of nonlinear partial differential equations (PDEs). In this work, we consider the Poisson boundary value problem with nonhomogeneous boundary conditions and study the behavior of its solutions under variations of the exponent functions $p(x)$. Our results extend the classical stability theorem of Lindqvist (1987), originally formulated for constant p, to the variable exponent setting. Moreover, our approach sharpens and generalizes the framework developed by Zhikov (2011), allowing for nonhomogeneous boundary data and providing stronger convergence results for the associated family of solutions. Specifically, it is shown that if the sequence $(p_j(x))$ increases uniformly to $p(x)$ in a bounded, smooth domain Ω, then the sequence (u_i) of solutions to the Dirichlet problem for the $p_i(x)$-Laplacian with fixed boundary datum φ converges (in a sense to be made precise) to the solution u_p of the Dirichlet problem for the $p(x)$-Laplacian with boundary datum φ. A similar result is proved for a decreasing sequence $p_j \searrow p$.

Keywords: Luxemburg norm; structural stability; modular vector space; $p(x)$-Laplacian; variable exponent spaces
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1. Introduction

Stability results are of paramount importance in the study of boundary value problems because they address the dependence of the solutions on the data. In the particular case of boundary value problems involving the p-Laplacian, there is special interest in the behavior of the solutions u_p with respect to variations of the parameter p. For p independent of the space variable, a stability result for the Poisson's problem with vanishing boundary value was studied in [16]. The corresponding generalization to variable p was carried out in [15]. This article is devoted to the discussion of stability for the solutions of the Poisson problem with Dirichlet boundary condition

$$\begin{cases} \Delta_{p(\cdot)}(w) = \operatorname{div}\left(|\nabla w|^{p(\cdot)-2}\nabla w\right) = f \in W^{-1,p(\cdot)}(\Omega) & \text{in } \Omega, \\ w|_{\partial\Omega} = \varphi \end{cases} \tag{1.1}$$

under perturbations of the parameter $p(\cdot)$. In (1.1), Ω is a bounded, smooth domain whose boundary and closure are denoted by $\partial\Omega$ and $\overline{\Omega}$, respectively, and $p = p(x)$ is a function on Ω.
The following notation is standard and will be employed throughout:

$$p_- = \inf_{\Omega} p(x), \;\; p_+ = \sup_{\Omega} p(x). \tag{1.2}$$

The central results presented in this work are Theorems 1.1 and 1.2.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded, smooth (at least C^1) domain with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Consider a non-decreasing sequence of functions $p_i : \Omega \to [1,\infty)$, $p_i \in C(\overline{\Omega})$ such that $p_i \to p$ uniformly in Ω, $1 < (p_1)_-$ and $p_+ < \infty$. Fix $\varphi \in W^{1,p(\cdot)}(\Omega)$ and $f \in \left(W_0^{1,(p_1)_-}(\Omega)\right)^*$. For each $i \in \mathbb{N}$ let $w_i \in W^{1,p_i(\cdot)}(\Omega)$ be the solution to the Poisson's problem (1.1) corresponding to $p = p_i$. Then*

- (i) *there exists $w \in W^{1,p(\cdot)}(\Omega)$ such that the sequence $(w_i)_{i\geq k}$ converges weakly to w in $W^{1,p_k(\cdot)}(\Omega)$, for each k,*
- (ii) *for each I, $\|w_i - w\|_{p_I(\cdot)} \to 0$ as $i \to \infty$,*
- (iii) $|||\nabla(w - w_i)|||_{p_i(\cdot)} \to 0$ *as* $i \to \infty$.

Moreover if the limit function p satisfies the log-Hölder condition (2.8), then the limit w belongs to $W^{1,p(\cdot)}(\Omega)$ and it is the unique solution of the problem (1.1).

Theorem 1.2. *Let Ω be as in Theorem 1.1 and let the sequence $(p_i) \in C(\overline{\Omega})$ decrease uniformly to p. Let $\varphi \in W^{1,p_1(\cdot)}(\Omega)$ and $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$. Denote by w and w_i the unique solution of the problem (1.1) with exponent $p(\cdot)$ and $p_i(\cdot)$, respectively. Assume $1 < p_-$, $p_+ < \infty$, and that for some $J \in \mathbb{N}$, it holds that $\int_\Omega |\nabla w|^{p_J(\cdot)}dx < \infty$. Then,*

$$\int_\Omega |\nabla(w_i - w)|^{p(\cdot)}dx \to 0 \text{ as } i \to \infty. \tag{1.3}$$

Theorem 1.1 should be compared to a related result in [17], in which the behavior of the solution u_p of (1.1) with $f = 0$, as $p \to \infty$ is characterized.

Variable exponent spaces can be traced back to Orlicz' 1931 paper [18] and since then, especially in the last few decades, they found their way into a variety of applications, such as the mathematical description of the hydrodynamics of electrorheological fluids [11], image restoration and super resolution [3–5], elasticity theory [21] and the mathematical setting of Lavrentiev's phenomenon [20].

2. Notation, terminology and known auxiliary results

The main results and definitions in this section have been dealt with in [7, 14]. In the sequel, $\Omega \subset \mathbb{R}^n$ will stand for a bounded, smooth domain with boundary $\partial\Omega$ and $p : \Omega \to (1, \infty)$ denotes a Borel-measurable function subject to the constraints

$$1 < p_- = \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) = p_+ < \infty. \tag{2.1}$$

The next definition concerns the variable exponent Lebesgue and Sobolev spaces and their corresponding Luxemburg norm.

Definition 2.1. *[7, 14]*

$$L^{p(\cdot)}(\Omega) = \left\{ f : \int_{\Omega} |\lambda f(x)|^{p(\cdot)}\, dx < \infty, \text{ for some } \lambda > 0 \right\}$$

and the Luxemburg norm is defined by

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} (|f(x)|/\lambda))^{p(\cdot)}\, dx \leq 1 \right\}.$$

It is a routine exercise to show that when p is constant on Ω, the above defined spaces coincide with the usual Lebesgue spaces.
If $p \leq q$ are measurable functions in Ω, the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous, that is there exists a positive constant $C(p, q)$ such that $\|u\|_{p(\cdot)} \leq C(p, q)\|u\|_{q(\cdot)}$ for any $u \in L^{q(\cdot)}(\Omega)$; see [14].
The following lemma is elementary, so its proof will be omitted; the reader is referred to [14] for the details.

Lemma 2.1. *For p satisfying (2.1) and $u \in L^{p(\cdot)}(\Omega)$, it holds that*

$$\min \left\{ \left(\int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p+}}, \left(\int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p-}} \right\} \leq \|u\|_{p(\cdot)} \leq \max \left\{ \left(\int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p+}}, \left(\int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p-}} \right\}. \tag{2.2}$$

The following is the variable exponent version of Hölder's inequality and will be used in the sequel. We refer the reader to [14] for the standard proof.

Lemma 2.2. *Let p and q be variable exponents satisfying (2.1) and $p(x)^{-1} + q(x)^{-1} = 1$ in Ω. Then, for $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, it holds that*

$$\int_{\Omega} |uv| dx \leq 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}. \tag{2.3}$$

As is expected, the generalized Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ introduced next are the natural habitat for the solutions of second order partial differential equations with non-standard growth. Specifically,

Definition 2.2. *[7, 14]*

$$W^{1,p(\cdot)}(\Omega) = \left\{ f : f \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla f| \in L^{p(\cdot)}(\Omega) \right\},$$

where $|\nabla f|$ stands for the Euclidean norm of ∇f and the Sobolev norm is defined as

$$\|f\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} + \||\nabla f|\|_{p(\cdot)}. \tag{2.4}$$

The Luxemburg norm closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ will be denoted as $W_0^{1,p(\cdot)}(\Omega)$.

The following theorem is well known; see [14] for an elementary proof.

Theorem 2.1. *[9, Proposition 2.5,(ii)], [14, Theorem 3.10 (vi)] Let $p \in C(\overline{\Omega})$ satisfy $p_- > 1$. Then the functional*

$$W_0^{1,p(\cdot)}(\Omega) \ni u \to \||\nabla u|\|_{p(\cdot)} \tag{2.5}$$

is a norm on $W_0^{1,p(\cdot)}(\Omega)$, equivalent to the Sobolev norm (2.4).

Remark 2.1. Notice that under the assumption $p \in C(\overline{\Omega})$ one has, for any $x \in \overline{\Omega}$

$$\lim_{\Omega \ni y \to x} p(y) = p(x). \tag{2.6}$$

Hence, p is $*$-continuous on $\overline{\Omega}$ ([14, p. 605]). In conjunction with [14, Theorem 3.10, (*iv*)], this observation yields the proof of Theorem 2.1.

In the sequel, it will be understood that any space $W_0^{1,q(\cdot)}(\Omega)$ is equipped with the norm $u \to \||\nabla u|\|_{q(\cdot)}$.

Theorem 2.2. *If $\Omega \subseteq \mathbb{R}^n$ is bounded, and $p \in C(\overline{\Omega})$ satisfies the bounds (2.1), then the embedding*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \tag{2.7}$$

is compact, and the space $W^{1,p(\cdot)}(\Omega)$ is uniformly convex (hence reflexive).

Proof. See [1, 14, 15]. □

The next theorem will play a pivotal role in the proof of Theorem 1.1.

Theorem 2.3. *Assume there exists $M > 0$ such that for all $x, y \in \overline{\Omega}$ it holds that*

$$|p(x) - p(y)| \leq M |\log |x - y||^{-1}. \tag{2.8}$$

Then $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Proof. See [10, Theorem 2.6]; see also [12]. □

As is customary, the dual space of a Banach space X will be denoted by X^*, and $\langle f, x\rangle$ will stand for the action of $f \in X^*$ on $x \in X$.
It is well known [14], that if $\Omega \subset \mathbb{R}^n$ is bounded, and $p \leq q$ in Ω, then $W^{1,q(\cdot)}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ with continuous embedding, and thus

$$\left(W^{1,p(\cdot)}(\Omega)\right)^* \hookrightarrow \left(W^{1,q(\cdot)}(\Omega)\right)^* \tag{2.9}$$

continuously.

Remark 2.2. Notice that as a consequence of the preceding discussion, if $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, and $(u_j) \subset W_0^{1,q(\cdot)}(\Omega)$ converges weakly to $u \in W_0^{1,q(\cdot)}(\Omega)$, then $\langle f, u_j - u\rangle \to 0$.

The following theorems are particular cases of [13, Theorems 3.2 and 3.3].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $p : \Omega \to \mathbb{R}$ be a continuous function subject to the constraints (2.1). Then, for any $\varphi \in W^{1,p(\cdot)}(\Omega)$ and $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, there exists a unique minimizer $u_p \in W_0^{1,p(\cdot)}(\Omega)$ to the Dirichlet integral*

$$W_0^{1,p(\cdot)}(\Omega) \ni u \to \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)} dx - \langle f, u\rangle. \tag{2.10}$$

Setting $w = \varphi - u$, the following immediate consequence of Theorem 2.4 is obtained.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $p : \Omega \to \mathbb{R}$ be a continuous function subject to the constraints (2.1). Then, for any $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ and any $\varphi \in W^{1,p(\cdot)}(\Omega)$, there exists a unique weak solution $w \in W^{1,p(\cdot)}(\Omega)$ to the Dirichlet problem*

$$\begin{cases} \Delta_{p(\cdot)} w = div\left(|\nabla w|^{p(\cdot)-2}\nabla w\right) = f & in\ \Omega, \\ w|_{\partial\Omega} = \varphi. \end{cases} \tag{2.11}$$

Specifically, w satisfies the identity

$$-\int_\Omega |\nabla w|^{p(x)-2}\, \nabla w(x) \nabla h(x)\, dx = \langle f, h\rangle \ \ for\ any\ h \in C_0^\infty(\Omega),$$

and $w - \varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.3. As Ω is assumed to be of class C^1, this definition is consistent with the classical pointwise definition; see [2, Théorème IX.17, p. 171].

Remark 2.4. As is apparent from theorems 2.4 and 2.5, the null function 0 is a solution of problem (2.11) if and only if $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ and $f = 0$.

The following lemma will be used in the sequel.

Lemma 2.3. *[8] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and p and q be real-valued-measurable functions on Ω with $p \le q \le p + \epsilon$ in Ω, with $0 < \epsilon < 1$. Then, for a measurable function $f : \Omega \to [-\infty, \infty]$ it holds*

$$\int_\Omega |f(x)|^{p(x)} dx \le \epsilon|\Omega| + \epsilon^{-\epsilon} \int_\Omega |f(x)|^{q(x)} dx. \tag{2.12}$$

Proof.

$$\begin{aligned} \int_\Omega |f(x)|^{p(x)} dx &= \int_{\{|f|<\epsilon\}} |f(x)|^{p(x)} dx + \int_{\{\epsilon<|f|<1\}} |f(x)|^{p(x)} dx + \int_{\{|f|>1\}} |f(x)|^{p(x)} dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The first integral, I_1, is clearly bounded by $\int_\Omega \epsilon^{p(x)}dx \leq \epsilon|\Omega|$. As to I_2, observe that

$$I_2 = \int_{\epsilon<|f|<1} |f(x)|^{p(x)-q(x)}|f(x)|^{q(x)}dx \leq \epsilon^{-\epsilon} \int_{\epsilon<|f|<1} |f(x)|^{q(x)}dx.$$

In conclusion,

$$\begin{aligned}\int_\Omega |f(x)|^{p(x)}dx &\leq \epsilon|\Omega| + \epsilon^{-\epsilon} \int_{\epsilon<|f|<1} |f(x)|^{q(x)}dx + \int_{|f|\geq 1} |f(x)|^{q(x)}dx \\ &\leq \epsilon|\Omega| + \epsilon^{-\epsilon} \int_\Omega |f(x)|^{q(x)}dx.\end{aligned}$$

□

Corollary 2.1. *Under the assumptions of the preceding theorem, for any $f \in L^q(\Omega)$, it holds that*

$$||g||_{p(\cdot)} \leq \left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right) ||g||_{q(\cdot)}. \tag{2.13}$$

Proof. Let $\lambda > 0$ be chosen so that $\int_\Omega \left|\frac{g(x)}{\lambda}\right|^{q(x)} dx \leq 1$. Estimate 2.12 holds that for $\frac{g}{\lambda}$ in the place of f, yielding

$$\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)^{-1} \int_\Omega \left|\frac{g(x)}{\lambda}\right|^{p(x)} dx \leq 1. \tag{2.14}$$

Because $\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)^{-1} < 1$,

$$\int_\Omega \left|\frac{g(x)}{\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)\lambda}\right|^{p(x)} dx \leq \left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)^{-1} \int_\Omega \left|\frac{g(x)}{\lambda}\right|^{p(x)} dx \leq 1. \tag{2.15}$$

Thus, $\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)\lambda \geq ||g||_{p(\cdot)}$, and (2.13) follows immediately. □

In particular, for $p \leq q \leq p + \epsilon$ in Ω and $f \in W^{1,q(\cdot)}(\Omega)$, it holds that

$$||f||_{1,p(\cdot)} \leq \left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right) ||f||_{1,q(\cdot)}, \tag{2.16}$$

and it follows that the norm of the embedding (2.9) is at most $\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\right)$.

Corollary 2.2. *Under the assumptions of Theorem 2.3,it holds that*

$$\int_\Omega \frac{|f(x)|^{p(x)}}{p(x)}dx \leq \epsilon|\Omega| + \epsilon^{-\epsilon}(1+\epsilon) \int_\Omega \frac{|f(x)|^{q(x)}}{q(x)}dx. \tag{2.17}$$

Proof. The proof follows from applying estimate (2.12) to the function $p^{-\frac{1}{p}}|f|$, defined on Ω and observing that $q(x)p(x)^{-\frac{q(x)}{p(x)}} < (1+\epsilon)$ on Ω. □

The following elementary, technical lemma will be singled out for use in the sequel.

Lemma 2.4. *Let* (p_i) $p_i \in C(\overline{\Omega})$, $p_i \to p$ *uniformly in* Ω, *and* $f \in L^{p(\cdot)}(\Omega)$, $f_i \in L^{p_i(\cdot)}(\Omega)$ *for each* $i \in \mathbb{N}$. *Assume that* $\int\limits_\Omega \frac{|f_i(x)|^{p_i(x)}}{p_i(x)}dx \to \int\limits_\Omega \frac{|f(x)|^{p(x)}}{p(x)}dx$ *as* $i \to \infty$. *Then* $\int\limits_\Omega |f_i(x)|^{p_i(x)}dx \to \int\limits_\Omega |f(x)|^{p(x)}dx$ *as* $i \to \infty$.

Proof.

$$\int\limits_\Omega |f_i(x)|^{p_i(x)}dx = \int\limits_\Omega |f_i|^{p_i(x)}\frac{p_i(x)}{p_i(x)}dx = \int\limits_\Omega |f_i(x)|^{p_i(x)}\frac{(p_i-p)(x)}{p_i(x)}dx + \int\limits_\Omega |f_i(x)|^{p_i(x)}\frac{p(x)}{p_i(x)}dx \tag{2.18}$$
$$= \int\limits_\Omega |f_i(x)|^{p_i(x)}\frac{p_i(x)-p(x)}{p_i(x)}dx + \int\limits_\Omega \left(\frac{|f_i(x)|^{p_i(x)}}{p_i(x)} - \frac{|f(x)|^{p(x)}}{p(x)}\right)p(x)\,dx + \int\limits_\Omega |f(x)|^{p(x)}\,dx.$$

□

The next theorem describes a fundamental geometric property of the functional

$$W_0^{1,p(\cdot)}(\Omega) \ni w \to F(u) = \rho_{p(\cdot)}(w) = \int\limits_\Omega \frac{|\nabla w(x)|^{p(x)}}{p(x)}dx, \tag{2.19}$$

where $|\nabla u|$ stands for the Euclidean norm of ∇u.

Theorem 2.6. *Given a domain* $\Omega \subset \mathbb{R}^n$ *and a function* $p : \Omega \to \mathbb{R}$ *with* $p \in C(\overline{\Omega})$ *and* $p_- = \inf\limits_\Omega p(x) > 1$, *the functional (2.19) is uniformly convex; this means that for any* $\varepsilon : 0 < \varepsilon$ *there exists* $\delta : 0 < \delta < 1$ *such that for any pair* $u, v \in W^{1,p(\cdot)}(\Omega)$

$$\rho_{p(\cdot)}\left(\frac{u-v}{2}\right) > \varepsilon\frac{\rho_{p(\cdot)}(u)+\rho_{p(\cdot)}(v)}{2} \Rightarrow \rho_{p(\cdot)}\left(\frac{u+v}{2}\right) < (1-\delta)\frac{\rho_{p(\cdot)}(u)+\rho_{p(\cdot)}(v)}{2}. \tag{2.20}$$

Proof. The proof is rather involved; the interested reader can find it in [1]. Notice that the boundedness of p is not needed here. □

3. Proof of Theorem 1.1

Consider a non-decreasing sequence (p_i) with $p_i \in C(\overline{\Omega})$ and $1 < p_i \to p$ uniformly in Ω. Fix $\varphi \in W^{1,p}(\Omega)$ and $f \in \left(W^{1,(p_1)_-}(\Omega)\right)^*$; it is assumed that $1 < (p_1)_- = \inf\limits_\Omega p_1$.
For each i, let u_i be the unique minimizer in $W_0^{1,p_i}(\Omega)$ of the functional

$$F_i(u) = \int\limits_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx - \langle f, u\rangle, \tag{3.1}$$

whose existence is guaranteed by Theorem 2.4. It has been shown in [13] that $w_i = \varphi - u_i$ is the unique solution of the Dirichlet problem (2.11) with $p = p_i$.
To facilitate the flow of ideas, the proof of Theorem 1.1 will be split into several lemmas.

Lemma 3.1. *Let* (u_i) *be the sequence of minimizers introduced above. Then, for any* $J \in \mathbb{N}$, *the sequence* $(u_i)_{i\geq J}$ *is bounded in* $W_0^{1,p_J(\cdot)}(\Omega)$.

Proof. Fix $\epsilon : 0 < \epsilon < e^{-1}$ and a natural number J such that $i \geq J \Rightarrow \|p_i - p\|_\infty < \frac{\epsilon}{2}$. Then, for $i \geq J$, one has $u_i \in W_0^{1,p_J(\cdot)}(\Omega)$. Assume first that $\int\limits_\Omega |\nabla u_i(x)|^{p_J(x)}dx \geq 1$. Then, on account of Lemma 2.1,

$$|||\nabla u_i|||_{p_J(\cdot)} \leq \left(\int\limits_\Omega |\nabla u_i(x)|^{p_J(x)}dx \right)^{\frac{1}{(p_J)_-}} \tag{3.2}$$

by virtue of Lemma 2.3. It follows that

$$\begin{aligned} \int\limits_\Omega |\nabla u_i(x)|^{p_J(x)}dx &\leq \|p_i - p_J\|_\infty |\Omega| + \|p_i - p_J\|_\infty^{-\|p_i - p_J\|_\infty} \int\limits_\Omega |\nabla u_i(x)|^{p_i(x)}dx \qquad (3.3)\\ &\leq \epsilon|\Omega| + \epsilon^{-\epsilon} \int\limits_\Omega 2^{p_i(x)} \left| \nabla \left(\frac{u_i - \varphi}{2} + \frac{\varphi}{2} \right)(x) \right|^{p_i(x)} dx \\ &\leq \epsilon|\Omega| + \epsilon^{-\epsilon} 2^{p_+ - 1} \left(\int\limits_\Omega p_i(x) \frac{|\nabla (u_i - \varphi)(x)|^{p_i(x)}}{p_i(x)} dx + \int\limits_\Omega |\nabla \varphi(x)|^{p_i(x)} dx \right) \\ &\leq \epsilon|\Omega| + \epsilon^{-\epsilon} 2^{p_+ - 1} p_+ \left(\int\limits_\Omega \frac{|\nabla (u_i - \varphi)(x)|^{p_i(x)}}{p_i(x)} dx - \langle f, u_i \rangle \right) \\ &+ \epsilon^{-\epsilon} 2^{p_+ - 1} \left(p_+ \langle f, u_i \rangle + \int\limits_\Omega |\nabla \varphi(x)|^{p_i(x)} dx \right). \end{aligned}$$

Due to the minimal character of u_i, one has

$$\begin{aligned} \left(\int\limits_\Omega \frac{|\nabla (u_i - \varphi)(x)|^{p_i(x)}}{p_i(x)} dx - \langle f, u_i \rangle \right) &\leq \int\limits_\Omega \frac{|\nabla \varphi(x)|^{p_i(x)}}{p_i(x)} dx \qquad (3.4)\\ &\leq \int\limits_\Omega |\nabla \varphi(x)|^{p_i(x)} dx \end{aligned}$$

and by virtue of Lemmas 2.3 and 2.1 and the choice of j, the latter is bounded above by

$$\left(\epsilon|\Omega| + \epsilon^{-\epsilon} \int\limits_\Omega |\nabla \varphi(x)|^{p(x)} dx \right) \leq \left(\epsilon|\Omega| + \epsilon^{-\epsilon} \|\nabla \varphi\|_p^\alpha \right), \tag{3.5}$$

where $\alpha = p_-$ if $\int\limits_\Omega |\nabla \varphi(x)|^{p(x)}dx \leq 1$ and $\alpha = p_+$ otherwise.

On the other hand, $f \in \left(W_0^{1,(p_1)_-}(\Omega) \right)^* \subset \left(W_0^{1,p_J}(\Omega) \right)^*$; also

$$|||\nabla u_i|||_{(p_1)_-} \leq C(J) |||\nabla u_i|||_{p_J}, \tag{3.6}$$

for a certain positive constant $C(J)$ independent of i. Thus,

$$\epsilon^{-\epsilon} 2^{p_+ - 1} p_+ |\langle f, u_i \rangle| \leq \epsilon^{-\epsilon} 2^{p_+ - 1} p_+ \|f\|_{\left(W_0^{1,(p_1)_-}(\Omega) \right)^*} |||\nabla u_i|||_{(p_1)_-} \tag{3.7}$$

$$\leq \epsilon^{-\epsilon}2^{p_+-1}p_+\|f\|_{\left(W_0^{1,(p_1)_-}(\Omega)\right)^*}C(J)|||\nabla u_i|||_{p_J(\cdot)} \tag{3.8}$$
$$\leq \left(\delta^{-1}\epsilon^{-\epsilon}2^{p_+-1}p_+\|f\|_{\left(W_0^{1,(p_1)_-}(\Omega)\right)^*}\right)^{\frac{(p_J)_-}{(p_J)_--1}}\frac{(p_J)_- - 1}{(p_J)_-}$$
$$+ \frac{1}{(p_J)_-}\left(\delta C(J)|||\nabla u_i|||_{p_J(\cdot)}\right)^{(p_J)_-}.$$

In particular, for $\delta = \left(\frac{(p_J)_-}{2}\right)^{\frac{1}{(p_J)_-}}(C(J))^{-1}$, it follows that

$$\epsilon^{-\epsilon}2^{p_+-1}p_+|\langle f, u_i\rangle| \leq A + \frac{1}{2}|||\nabla u_i|||_{p_J(\cdot)}^{(p_J)_-}, \tag{3.9}$$

where

$$A = \left(\delta^{-1}\epsilon^{-\epsilon}2^{p_+-1}p_+\|f\|_{\left(W_0^{1,(p_1)_-}(\Omega)\right)^*}\right)^{\frac{(p_J)_-}{(p_J)_--1}}\frac{(p_J)_- - 1}{(p_J)_-}.$$

Thus, (3.2), (3.3) and the estimates thereafter yield

$$\frac{1}{2}|||\nabla u_i|||_{p_J(\cdot)}^{(p_J)_-} \leq \epsilon|\Omega| + \epsilon^{-\epsilon}2^{p_+-1}p_+\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\|\nabla\varphi\|_{p(\cdot)}^{\alpha}\right) \tag{3.10}$$
$$+ A + \epsilon^{-\epsilon}2^{p_+-1}\left(\epsilon|\Omega| + \epsilon^{-\epsilon}\|\nabla\varphi\|_{p(\cdot)}^{\alpha}\right).$$

In conclusion, for any $j \geq J$,

$$|||\nabla u_i|||_{p_J(\cdot)} \leq \max\{1, 2B\}^{\frac{1}{(p_J)_-}}, \tag{3.11}$$

where B is the right hand side of (3.10), which does not depend on i.

It follows that the sequence (u_i) is uniformly bounded in $W_0^{1,p_J(\cdot)}(\Omega)$, as claimed. □

Lemma 3.2. *In the preceding notation, there is a subsequence of the sequence (u_i) that converges weakly in each $W_0^{1,p_j(\cdot)}(\Omega)$, to a function $u \in \bigcap_{j=1}^{\infty} W_0^{1,p_j(\cdot)}(\Omega)$.*

Proof. Fix $J \in \mathbb{N}$. Theorem 2.2 in conjunction with the theorem of Banach-Alaoglu yields the existence of a function $v_J \in W_0^{1,p_J(\cdot)}(\Omega)$ and a subsequence $(u_{J+k_j})_{j\geq 1}$ that converges to v_J weakly in $W_0^{1,p_J(\cdot)}(\Omega)$.

Applying the preceding reasoning to the sequence $(u_{J+k_j})_{j\geq m} \subset W_0^{1,p_{J+k_m}(\cdot)}(\Omega)$ for any $m > 1$ and denoting its weak limit by v_{J+m} it is immediate that $v_{J+m} = v_J$. Hence, the weak limit does not depend on J and it can be written $v_J = u$. One quickly obtains

$$u \in \bigcap_{i=1}^{\infty} W_0^{1,p_i(\cdot)}. \tag{3.12}$$

□

Remark 3.1. In the sequel, as is customary, the subsequence $(u_{J+k_j})_{j\geq 1}$ will be denoted by (u_j) and the subsequence (p_{J+k_j}) of the sequence (p_i) will be relabeled (p_j).

The next order of business is to prove that $u \in W^{1,p(\cdot)}(\Omega)$. To that effect, we start with the following assertion.

Lemma 3.3. *The weak limit function u whose existence is proved in Lemma 3.2 satisfies* $\int_\Omega |\nabla u(x)|^{p(x)}dx < \infty$.

Proof. Let $\eta > 0$ be arbitrary and fix $v \in W_0^{1,p(\cdot)}(\Omega)$. Choose $0 < \epsilon < e^{-1}$ and $0 < \delta < 1$ such that $\epsilon^{-\epsilon}(1+\epsilon) < (1+\delta)$ and

$$\max\left\{3\delta\int_\Omega \frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx, \epsilon|\Omega|\left(1+\epsilon^{-\epsilon}(1+\epsilon)\right), \delta|\langle f,v\rangle|, \delta|\langle f,u\rangle|\right\} < \frac{\eta}{5}, \tag{3.13}$$

and $k \in \mathbb{N}$ large enough so that $j \geq k \Rightarrow (1+\delta)|\langle f, u_j - u\rangle| < \frac{\eta}{5}$ and $\|p_j - p\|_\infty < \frac{\epsilon}{2}$. Because the functional

$$W_0^{1,p_k(\cdot)}(\Omega) \ni w \to \int_\Omega \frac{|\nabla(w-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \tag{3.14}$$

is weakly lower semicontinuous, it follows that

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \liminf_{j\geq k}\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx. \tag{3.15}$$

In what follows, let $r(\epsilon) = \epsilon^{-\epsilon}(1+\epsilon)$. Corollary 2.2 yields, for $j \geq k$,

$$\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \epsilon|\Omega| + r(\epsilon)\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx. \tag{3.16}$$

Because $v \in W_0^{1,p(\cdot)}(\Omega) \hookrightarrow W_0^{1,p_j(\cdot)}(\Omega)$, one has, by virtue of the minimizing property of u_j,

$$\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx - \langle f,u_j\rangle \leq \int_\Omega \frac{|\nabla(v-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx - \langle f,v\rangle \tag{3.17}$$

and invoking Lemma 2.3 again, one obtains

$$\int_\Omega \frac{|\nabla(v-\varphi)(x)|^{p_j(x)}}{p_(x)}dx \leq \epsilon|\Omega| + r(\epsilon)\int_\Omega \frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx. \tag{3.18}$$

In conclusion, for fixed k as above and $j \geq k$, on account of inequality (3.16), one has

$$\begin{aligned}\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx &\leq \epsilon|\Omega| + r(\epsilon)\left(\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx - \langle f,u_j\rangle\right)\\ &+ r(\epsilon)\langle f,u_j\rangle\\ &\leq \epsilon|\Omega| + r(\epsilon)\left(\int_\Omega |\frac{\nabla(v-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx - \langle f,v\rangle\right)\\ &+ r(\epsilon)\langle f,u_j\rangle\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&\leq \epsilon|\Omega| + r(\epsilon)\left(\epsilon|\Omega| + r(\epsilon)\int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle\right)\\
&+ r(\epsilon)\langle f, u_j\rangle\\
&\leq \epsilon|\Omega|(1 + r(\epsilon)) + r(\epsilon)\left(r(\epsilon)\int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle\right)\\
&+ r(\epsilon)\langle f, u_j\rangle\\
&\leq \frac{\eta}{5} + (1+\delta)^2\int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle - (r(\epsilon)-1)\langle f, v\rangle\\
&+ r(\epsilon)\langle f, u_j - u\rangle + r(\epsilon)\langle f, u\rangle\\
&\leq \frac{\eta}{5} + 3\delta\int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx + \int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle\\
&+ \delta|\langle f, v\rangle| + (1+\delta)|\langle f, u_j - u\rangle| + (r(\epsilon)-1)\langle f, u\rangle + \langle f, u\rangle\\
&\leq \int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle + \langle f, u\rangle + \eta.
\end{aligned}
$$

In all,

$$
\liminf_{j\geq k}\int\limits_{\Omega}\frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \int\limits_{\Omega} p^{-1}|\nabla(v-\varphi)(x)|^{p(x)}dx - \langle f, v\rangle + \langle f, u\rangle + \eta, \tag{3.20}
$$

that is, according to (3.15), for any $\eta > 0$,

$$
\int\limits_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle + \langle f, u\rangle + \eta. \tag{3.21}
$$

Consequently,

$$
\begin{aligned}
\int\limits_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx &\leq \liminf_{n\geq k}\int\limits_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p_n(x)}}{p_n(x)}dx && (3.22)\\
&\leq \int\limits_{\Omega}\frac{|\nabla(v-\varphi)(x)|^{p(x)}}{p(x)}dx - \langle f, v\rangle + \langle f, u\rangle + \eta. && (3.23)
\end{aligned}
$$

In particular, the preceding inequality yields

$$
\int\limits_{\Omega}\frac{|\nabla u|^{p(x)}}{p(x)}dx \leq 2^{p_+-1}\left(\int\limits_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx + \int\limits_{\Omega}\frac{|\nabla\varphi(x)|^{p(x)}}{p(x)}dx\right) < \infty. \tag{3.24}
$$

That is $|\nabla u| \in L^{p(\cdot)}(\Omega)$, as claimed. □

Remark 3.2. Because $p_j \to p$ pointwise, Lebesgue's dominated convergence yields

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx \to \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx \text{ as } j \to \infty \tag{3.25}$$

and on account of Lemma 2.4,

$$\int_\Omega |\nabla(u-\varphi)(x)|^{p_j(x)}dx \to \int_\Omega |\nabla(u-\varphi)(x)|^{p(x)}dx \text{ as } j \to \infty. \tag{3.26}$$

The next series of lemmas aims to improve the weak convergence statement $u_j \rightharpoonup u$ as $j \to \infty$.

Lemma 3.4. *If (u_j) is the sequence of minimizers defined in Lemma 3.2, then*

$$\int_\Omega |\nabla(u_j-\varphi)(x)|^{p_j(x)}dx \to \int_\Omega |\nabla(u-\varphi)(x)|^{p(x)}dx \text{ as } j \to \infty. \tag{3.27}$$

Proof. Fix ϵ, δ, η, and k as in the proof of Lemma 3.3. Observe that due to the minimal character of u_k,

$$\int_\Omega \frac{|\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx - \langle f, u_k\rangle \le \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx - \langle f, u\rangle. \tag{3.28}$$

Taking into consideration the fact that $u_k \to u$ weakly, it follows from the above that

$$\limsup_k \int_\Omega \frac{\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \le \limsup_k \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx, \tag{3.29}$$

whereas the fact that $|\nabla u| \in L^p(\Omega)$ coupled with a straightforward application of Lebesgue's dominated convergence theorem, yields

$$\lim_{k\to\infty} \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx = \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx. \tag{3.30}$$

On the other hand,

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \le \liminf_{j\ge k} \int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx. \tag{3.31}$$

In turn, for $j \ge k$, one has, owing to Corollary 2.2,

$$\begin{aligned}\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx &\le \epsilon|\Omega| + \epsilon^{-\epsilon}(1+\epsilon)\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx \\ &\le \eta + (1+\delta)\int_\Omega \frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx.\end{aligned} \tag{3.32}$$

Thus,

$$\liminf_{j\geq k}\int_{\Omega}\frac{|\nabla(u_j-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \eta+(1+\delta)\liminf_{j\geq k}\int_{\Omega}\frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx, \tag{3.33}$$

and it follows from (3.31) that

$$\begin{aligned}\lim_{k\to\infty}\int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx &= \int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx \\ &\leq \eta+(1+\delta)\liminf_{j}\int_{\Omega}\frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx.\end{aligned} \tag{3.34}$$

In all, from (3.29), (3.30), and (3.34), one deduces

$$\begin{aligned}\limsup_{k}\int_{\Omega}\frac{\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx &\leq \int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx \\ &\leq \eta+(1+\delta)\liminf_{j}\int_{\Omega}\frac{|\nabla(u_j-\varphi)(x)|^{p_j(x)}}{p_j(x)}dx,\end{aligned} \tag{3.35}$$

which in conjunction with Lemma 2.4 yields (3.27). □

The following, similar result can be proved along the same lines:

Lemma 3.5.

$$\int_{\Omega}\left|\nabla\left(\frac{u+u_j}{2}-\varphi\right)(x)\right|^{p_j(x)}dx \to \int_{\Omega}|\nabla(u-\varphi)(x)|^{p(x)}dx\, as\, j\to\infty. \tag{3.36}$$

Proof. Because $\frac{u+u_k}{2}\in W^{1,p_k}(\cdot)_0(\Omega)$, the minimality of u_k yields the inequality

$$\int_{\Omega}\frac{|\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx-\langle f,u_k\rangle \leq \int_{\Omega}\frac{\left|\nabla\left(\frac{u+u_k}{2}-\varphi\right)(x)\right|^{p_k(x)}}{p_k(x)}dx-\langle f,u\rangle. \tag{3.37}$$

Thus, owing to (3.35),

$$\begin{aligned}\int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx &= \limsup_{k}\int_{\Omega}\frac{|\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx \leq \limsup_{k}\int_{\Omega}\frac{\left|\nabla\left(\frac{u+u_k}{2}-\varphi\right)(x)\right|^{p_k(x)}}{p_k(x)}dx \\ &\leq \limsup_{k}\frac{1}{2}\left(\int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx+\int_{\Omega}\frac{|\nabla(u_k-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx\right) \\ &= \int_{\Omega}\frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx.\end{aligned} \tag{3.38}$$

The latter yields

$$\limsup_k \int_\Omega \frac{\left|\nabla\left(\frac{u+u_k}{2}-\varphi\right)(x)\right|^{p_k(x)}}{p_k(x)}dx = \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx. \tag{3.39}$$

It has been shown in the first part of the lemma that it holds

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx = \lim_{k\to\infty}\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx; \tag{3.40}$$

also

$$\begin{aligned}\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_k(x)}}{p_k(x)}dx &\le \liminf_{j\ge k}\int_\Omega \frac{\left|\nabla\left(\frac{u_j+u}{2}-\varphi\right)(x)\right|^{p_k(x)}}{p_k(x)}dx \\ &\le \eta + (1+\delta)\liminf_j \int_\Omega \frac{\left|\nabla\left(\frac{u_j+u}{2}-\varphi\right)(x)\right|^{p_j(x)}}{p_j(x)}dx.\end{aligned} \tag{3.41}$$

From (3.30) and (3.41) one concludes

$$\int_\Omega \frac{\left|\nabla\left(\frac{u_j+u}{2}-\varphi\right)(x)\right|^{p_j(x)}}{p_j(x)}dx \to \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx \text{ as } k\to\infty, \tag{3.42}$$

which yields Lemma 3.5 via Lemma 2.4. □

The following result yields a stronger convergence result for the sequence of minimizers.

Theorem 3.1. *Let (u_j) denote the subsequence introduced in Remark 3.1. Then it holds that*

$$\lim_{j\to\infty}\int_\Omega \left|\nabla(u-u_j)(x)\right|^{p_j(x)} dx = 0. \tag{3.43}$$

Proof. The proof of (3.43) uses the two preceding statements in conjunction with the uniform convexity property stated in Theorem 2.6.

First, observe that by Remark 3.2, one has

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx \to \int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p(x)}}{p(x)}dx. \tag{3.44}$$

On account of (3.35) and (3.44), it follows that there exists a constant $C(u,\varphi)$, depending only on u,φ, for which

$$\int_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx + \int_\Omega \frac{|\nabla(u_i-\varphi)(x)|^{p_i}}{p_i(x)}dx \le C(u,\varphi) = C. \tag{3.45}$$

Let $\delta > 0$ and assume that for a subsequence (p_{i_k}), it holds that $\int\limits_\Omega \left|\nabla(u_{j_k} - u)(x)\right|^{p_{j_k}(x)} dx > \delta$. Then,

$$2^{p_+} \int\limits_\Omega \left|\frac{\nabla(u_{j_k} - u)(x)}{2}\right|^{p_{j_k}(x)} dx > \delta \tag{3.46}$$

and

$$\begin{aligned}\int\limits_\Omega \left|\frac{\nabla(u_{j_k} - u)(x)}{2}\right|^{p_{j_k}(x)} dx &> 2^{-p_+}\delta = 2^{-p_+}\frac{\delta}{C}C \\ &\geq 2^{-p_+}\frac{\delta}{C}\left(\int\limits_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx + \int\limits_\Omega \frac{|\nabla(u_i-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx\right) \\ &= 2^{1-p_+}\frac{\delta}{C}\frac{\left(\int\limits_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx + \int\limits_\Omega \frac{|\nabla(u_i-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx\right)}{2}.\end{aligned} \tag{3.47}$$

On account of Theorem 2.6, it follows that there exists $\eta : 0 < \eta < 1$ for which

$$\int\limits_\Omega \frac{\left|\nabla\left(\frac{u+u_j}{2} - \varphi\right)(x)\right|^{p_i(x)}}{p_i(x)}dx \leq (1-\eta)\frac{\int\limits_\Omega \frac{|\nabla(u-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx + \int\limits_\Omega \frac{|\nabla(u_i-\varphi)(x)|^{p_i(x)}}{p_i(x)}dx}{2}. \tag{3.48}$$

Letting $i \to \infty$ in the last inequality leads to a contradiction. □

The final step is to verify that $u \in W_0^{1,p(\cdot)}(\Omega)$. To this end, it remains to show that $u \in L^{p(\cdot)}(\Omega)$ and that u can be approximated by $C_0^\infty(\Omega)$ functions in the norm $\|\cdot\|_{1,p(\cdot)}$. This is proved in the ensuing proposition:

Proposition 3.1. *The limit function u obtained in lemma 3.2 belongs to* $W_0^{1,p(\cdot)}(\Omega)$.

Proof. Observe that because $p_j \nearrow p$, for any $x \in \Omega$ with $p(x) < n$, one has

$$\frac{np_j(x)}{n - p_j(x)} \nearrow \frac{np(x)}{n - p(x)},$$

also,

$$\frac{np(x)}{n-p(x)} - p(x) = \frac{p^2(x)}{n-p(x)} \geq \frac{p_-^2}{n-p_-} > \frac{1}{n-1}.$$

Consequently, for large enough j,

$$\frac{np_j(x)}{n-p_j(x)} > \frac{np(x)}{n-p(x)} - \frac{1}{n-1} > p(x). \tag{3.49}$$

Let $0 < r < \frac{n^2}{p_+ + n}$. Set $\Omega_1 = \{x : 1 < p(x) < n\}$; $\Omega_2 = \{x : n - r < p(x) < p_+ + r\}$. Then, Ω_1 and Ω_2 are open, and $\Omega = \Omega_1 \cup \Omega_2$; let $\{\chi_1, \chi_2\}$ be a partition of unity subordinated to the cover $\{\Omega_1, \Omega_2\}$. It follows that $u\chi_k \in W_0^{1,p_j(\cdot)}(\Omega_k)$ for $k = 1, 2$ and all $j \in \mathbb{N}$, whence $u\chi_1 \in W_0^{1,p_j(\cdot)}(\Omega_1) \subseteq L^{\frac{np_j}{n-p_j}(\cdot)}(\Omega_1) \subseteq L^p(\Omega)$ for j chosen so that (3.49) holds. Similarly, the choice of r yields $n^2 - r(p + n) > n^2 - r(p_+ + n) > 0$. Thus, $n^2 - rn > rp$, and

$$u\chi_2 \in W_0^{1,p_j(\cdot)}(\Omega_2) \subseteq W_0^{1,n-r}(\Omega_2) \subseteq L^{\frac{n(n-r)}{r}}(\Omega_2) \subseteq L^{p(\cdot)}(\Omega). \tag{3.50}$$

In conclusion, $u = u\chi_1 + u\chi_2 \in L^{p(\cdot)}(\Omega)$, and it follows that

$$u \in W^{1,p(\cdot)}(\Omega) \bigcap \left(\bigcap_{j=1}^{\infty} W_0^{1,p_j(\cdot)}(\Omega) \right).$$

On account of Theorem 2.3, it must hold that $u \in W_0^{1,p(\cdot)}(\Omega)$, as claimed. □

Corollary 3.1. *The original sequence of minimizers alluded to in Lemma 3.1 converges weakly to $u \in W_0^{1,p(\cdot)}(\Omega)$ in every $W_0^{1,p_J(\cdot)}(\Omega)$, and Theorem 3.1 holds for the original sequence (u_i).*

Proof of Theorem 1.1. The proof of Theorem 1.1 follows by observing that for each i, $w_i = \varphi - u_i$ and $w = \varphi - u$.

Proof of Theorem 1.2. In this case, the sequence of minimizers (u_i) of the functionals

$$W_0^{1,p_i(\cdot)}(\Omega) \ni\to F_i(v) = \int_\Omega \frac{|\nabla(v - \varphi)(x)|^{p_i(x)}}{p_i(x)} dx - \langle f, v\rangle \tag{3.51}$$

is in fact bounded in $W_0^{1,p(\cdot)}(\Omega)$. The proof will be sketched, because it follows along the same lines as that of the boundedness of (u_i) in Theorem 1.1. Due to the assumption on φ and on the sequence (p_i) it is immediate from Lemma 2.3 that for i as large as necessary for $\|p - p_i\|_\infty < 1$,

$$\int_\Omega |\nabla\varphi(x)|^{p_i(x)} dx \le \|p_i - p_1\|_\infty |\Omega| + \|p_i - p_1\|_\infty^{-\|p_i - p_1\|_\infty} \int_\Omega |\nabla\varphi(x)|^{p_1(x)} dx. \tag{3.52}$$

Also,

$$\int_\Omega |\nabla u_i(x)|^{p(x)} dx \le \|p_i - p\|_\infty |\Omega| + \|p_i - p\|_\infty^{-\|p_i - p\|_\infty} \int_\Omega |\nabla u_i(x)|^{p_i(x)} dx. \tag{3.53}$$

Using the same ideas as in Lemma 3.1, it can be shown that the sequence $\left(\int_\Omega |\nabla u_i(x)|^{p_i(x)} dx \right)$ is bounded.
Hence (u_i) is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and therefore that there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ and a subsequence (u_j) of the sequence of minimizers (see Remark 3.1) such that $u_j \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$.
In this setting, one has

$$\int_\Omega |\nabla(u_j - \varphi)(x)|^{p(x)} dx \to \int_\Omega |\nabla(u - \varphi)(x)|^{p(x)} dx \text{ as } j \to \infty. \tag{3.54}$$

Indeed, owing to the weak lower semicontinuity of the minimal character of u_j and the integrability assumption on u, one has

$$\int_\Omega \frac{|\nabla(u - \varphi)(x)|^{p(x)}}{p(x)} dx \le \liminf_{j\ge J} \int_\Omega \frac{|\nabla(u_j - \varphi)(x)|^{p(x)}}{p(x)} dx, \tag{3.55}$$

and for any fixed $0 < \epsilon < e^{-1}$ and J so large that $j \geq J$ implies $\|p_j - p\|_\infty < \frac{\epsilon}{2}$, one has (Corollary 2.2)

$$\int_\Omega \frac{|\nabla(u_j - \varphi)(x)|^{p(x)}}{p(x)} dx \leq \epsilon|\Omega| + \epsilon^{-\epsilon}(1+\epsilon) \int_\Omega \frac{|\nabla(u_j - \varphi)(x)|^{p_j(x)}}{p_j(x)} dx \tag{3.56}$$
$$= \epsilon|\Omega| + \epsilon^{-\epsilon}(1+\epsilon)F_j(u_j) + \epsilon^{-\epsilon}(1+\epsilon)\langle f, u_j\rangle$$
$$\leq \epsilon|\Omega| + \epsilon^{-\epsilon}(1+\epsilon)F_j(u) + \epsilon^{-\epsilon}(1+\epsilon)\langle f, u_j\rangle.$$

Taking lim sup in (3.56) and observing that because $|\nabla u| \in L^{p_N}(\Omega)$ for some p_N,

$$\limsup_j F_j(u) = \limsup_j \left(\int_\Omega \frac{|\nabla(u - \varphi)(x)|^{p_j(x)}}{p_j(x)} dx - \langle f, u\rangle \right) = \int_\Omega \frac{|\nabla(u - \varphi)(x)|^{p(x)}}{p(x)} dx - \langle f, u\rangle, \tag{3.57}$$

and that $\limsup_j \langle f, u_j\rangle = \langle f, u\rangle$, it follows that

$$\limsup_j \int_\Omega \frac{|\nabla(u_j - \varphi)(x)|^{p(x)}}{p(x)} dx \leq \int_\Omega \frac{|\nabla(u - \varphi)(x)|^{p(x)}}{p(x)} dx; \tag{3.58}$$

coupled with (3.55) and using lemma 2.4, the latter yields (3.54), because p is bounded in Ω. Statement (1.3) in Theorem 1.2 follows from uniform convexity along the same lines as in the proof of Theorem 3.1.

4. Conclusions

In this article we have proved stability of the Poisson's problem for the $p(x)$-Laplacian with nonhomogeneous boundary conditions with respect to the variation of the exponent $p(x)$. Our result relies solely on modular uniform convexity, the variational structure of problem (2.11), and the high-precision embedding (2.1). This opens a promising line of work in the application of our techniques to study structural stability of other problems, such as the Neumann problem for Poisson's equation, the Poisson's problem for Kirchoff's operator, and the Poisson's problem for double-phase type operators [6].

Our work improves some of the results obtained by Zhikov in [19]. Specifically, in [19, Theorem 3.1] a sequence of exponent $p_\epsilon \to p$ a.e. is considered, subject to the condition

$$1 < \alpha \leq p_\epsilon(x) \leq \beta \tag{4.1}$$

and it is shown that the sequence of solutions of problem (2.11) with variable exponent $p_\epsilon(\cdot)$ and $\varphi = 0$ converges weakly in $W_0^{1,\alpha}(\Omega)$ to the solution u of (2.11) with $\varphi = 0$. This result is to be contrasted with our Theorem 1.1, which, though under stronger assumptions, includes arbitrary boundary datum and yields much stronger convergence results, because $W_0^{1,p(\cdot)}(\Omega) \subsetneq W_0^{1,\alpha}(\Omega)$.

On a final note, we point out that in principle there seems to be no obstacle for the application of our techniques to the [19, Thermistor problem (17.1)] to obtain stability with respect to the exponent of the Dirichlet problem with non-vanishing boundary value.

Author contributions

Both authors contributed to this paper equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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