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Theory article

Boundedness and continuity of local bilinear maximal commutators on Triebel–Lizorkin spaces and Besov spaces

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Abstract: In the present paper we investigated the mapping properties for the bilinear maximal commutator in the domain setting. Some new boundedness and continuity for the above operator on the Triebel–Lizorkin spaces and Besov spaces were established under suitable symbol function conditions. The main results essentially extend some known ones to the local setting.

Keywords: local bilinear maximal commutator; Triebel–Lizorkin space; Besov space; boundedness and continuity
Mathematics Subject Classification: 42B25, 46E35

1. Introduction

The primary purpose of this paper is to investigate the mapping properties for the local bilinear maximal commutators

$$\mathfrak{M}_{\Omega,b}(f,g)(x)=\sup_{0<r<\mathrm{dist}(x,\partial\Omega)}\frac{1}{|B(O,r)|}\int_{B(O,r)}|(b(x)-b(x+y))f(x+y)g(x-y)|dy,$$

on the Triebel–Lizorkin space $F_s^{p,q}(\Omega)$ and Besov space $B_s^{p,q}(\Omega)$, where Ω is a subdomain in $\mathbb{R}^n$. See Section 2 for the definitions of $F_s^{p,q}(\Omega)$ and $B_s^{p,q}(\Omega)$. This type of commutator was introduced by Wang and Liu [20] who established the boundedness properties of $\mathfrak{M}_{\Omega,b}$ on first-order Sobolev spaces. Here we shall establish the boundedness and continuity of $\mathfrak{M}_{\Omega,b}$ on $F_s^{p,q}(\Omega)$ and $B_s^{p,q}(\Omega)$. Our main results will extend the main results of [22] to the local setting. It should be pointed out that the commutativity with translations for maximal operators plays a key role in proving the boundedness and continuity of maximal operators on Triebel–Lizorkin spaces and Besov spaces. Since the local maximal operator lacks the commutativity with translations, it makes the study of the boundedness and continuity of local bilinear maximal commutators more complex.

Let us begin with a brief review of the research in the regularity theory of maximal operators.

1.1. Regularity of maximal operators

The regularity theory of maximal operators is an active area of current research. This topic originated with Kinnunen [5], who first established the $W^{1,p}(\mathbb{R}^n)$ $(1 < p < \infty)$ for the centered Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r, and $|B(x,r)|$ is the volume of $B(x,r)$. Later on, Kinnunen's result was extended to various settings, see [6] for the local case, [7] for the fractional case, and [1, 16] for the bilinear case. The continuity properties for the Hardy–Littlewood maximal operator and its various variants on Sobolev spaces can be found in [1, 12, 13]. An important extension of Sobolev regularity for maximal operators is to study their behaviors on other smooth function spaces. This direction was initiated by Korry [9], who concluded that M is bounded on inhomogeneous Triebel–Lizorkin spaces and inhomogeneous Besov spaces for $0 < s < 1$ and $1 < p, q < \infty$. As a direct consequence, it is valid that M is bounded on fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p < \infty$ (also see [8]). Subsequently, Luiro [13] established the continuity of $M : F_s^{p,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$. The continuity of $M : B_s^{p,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p, q < \infty$ can be found in [11].

It is worth noting that maximal operator M enjoys commutativity with translations, which plays a key role in proving the boundedness of maximal operators on first-order Sobolev spaces, Triebel–Lizorkin spaces, and Besov spaces (see [2, 5, 9]). However, the maximal operator in a local setting lacks commutativity with translations, which makes the investigation on the regularity of maximal operators in a local setting more interesting and challenging. In 1998, Kinnunen and Lindqvist [6] first studied the Sobolev regularity of local maximal operator

$$M_\Omega f(x) = \sup_{0<r<\mathrm{dist}(x,\partial\Omega)} \frac{1}{|B(O,r)|} \int_{B(O,r)} |f(x-y)|dy, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a subdomain of $\mathbb{R}^n$. They showed that $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded for all $1 < p < \infty$ (also see [2]). Later on, the above result was extended to the fractional variant (see [4, 18]) and to the multilinear variant (see [3, 16]). In [13], Luiro established the continuity of $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ for all $1 < p < \infty$. Luiro also studied the boundedness and continuity of M_Ω on Triebel–Lizorkin spaces (see Section 2 for its definition). Very recently, Liu, Liu and Wang [14] extended the above result to the multilinear setting.

Compared with the Hardy–Littlewood maximal operator, the bilinear maximal operator is more complex. Let Ω be a subdomain in $\mathbb{R}^n$. The local bilinear maximal operator is defined by

$$\mathfrak{M}_\Omega(f,g)(x) = \sup_{0<r<\mathrm{dist}(x,\partial\Omega)} \frac{1}{|B(O,r)|} \int_{B(O,r)} |f(x+y)g(x-y)|dy, \quad x \in \Omega,$$

where $O = (0,\dots,0) \in \mathbb{R}^n$. This type of maximal operator was introduced in [16] which studied the Sobolev regularity of $\mathfrak{M}_\Omega$. When $\Omega = \mathbb{R}^n$, $\mathfrak{M}_\Omega$ reduces to the usual bilinear maximal operator $\mathfrak{M}$, which originated from Calderón's work in 1964 when he posed whether the mapping $\mathfrak{M} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ is bounded. In Lacey's seminal work [10], Lacey addressed Calderón's conjecture by establishing the boundedness of $\mathfrak{M} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$ for $1 < p_1, p_2 < \infty$, $2/3 < p \le 1$, and $1/p_1 + 1/p_2 = 1/p$. It was pointed out in [1] that $\mathfrak{M}$ is bounded from $\mathfrak{M} : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p_1, p_2 < \infty$, $1 \le p < \infty$ and $1/p_1 + 1/p_2 = 1/p$. Based on the above Lebesgue boundedness, Carneiro and Moreira [1] proved that $\mathfrak{M} : W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded and continuous for $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. Subsequently, Liu, Liu and Zhang [15] established the boundedness and continuity of $\mathfrak{M}$ on $F_s^{p,q}(\mathbb{R}^n)$ and $B_s^{p,q}(\mathbb{R}^n)$. In the local setting, Liu, Wang and Xue [16] showed that $\mathfrak{M}_\Omega$ is bounded and continuous from $W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ to $W^{1,p}(\Omega)$, where $1 < p_1, p_2 < \infty$, $1 \le p < \infty$ and $1/p_1 + 1/p_2 = 1/p$.

1.2. Regularity of the bilinear maximal commutator

The regularity properties of maximal commutators have been studied by many authors. It should be pointed out that the commutators of bilinear operators were originally introduced by Pérez and Torres [17], who studied the boundedness for the commutators of the bilinear Calderón–Zygmund operator $[T, b]_i (i = 1, 2)$, where T is the bilinear Calderón–Zygmund operator. When $\Omega = \mathbb{R}^n$, the operator $\mathfrak{M}_{\Omega,b}$ reduces to the bilinear maximal commutator $\mathfrak{M}_b$. The bilinear maximal commutator $\mathfrak{M}_b$ was first introduced by Wang and Liu [21] in 2022 when they established the boundedness and continuity of $\mathfrak{M}_b$ on Triebel–Lizorkin spaces and Besov spaces under the condition that the symbol function b belongs to the Lipschitz space. Very recently, Wang and Liu [22] proved the following result.

Theorem A. ([22]) *Let* $0 < s < 1$, $1 < p_1, p_2, p_3, p < \infty$, *and* $\sum_{i=1}^3 \frac{1}{p_i} = \frac{1}{p}$.

(i) *Let* $p' < q < \infty$, $p_3^2 < p_2(p_1 + p_3)$ *and* $p_3^2 < p_1(p_2 + p_3)$. *If* $b \in F_s^{p_3,q}(\mathbb{R}^n)$, *then* $\mathfrak{M}_b : F_s^{p_1,q}(\mathbb{R}^n) \times F_s^{p_2,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ *is bounded and continuous.*

(ii) *Let* $b \in B_s^{p_3,q}(\mathbb{R}^n)$. *Then* $\mathfrak{M}_b : B_s^{p_1,q}(\mathbb{R}^n) \times B_s^{p_2,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ *is bounded and continuous.*

1.3. Motivation and main results

In this subsection we shall present the main motivations and results.

Based on Theorem A, a natural question is the following:

Question 1. What is the local case of Theorem A? More precisely, what conditions for p_1, p_2, p_3, p, q guarantee the boundedness and continuity of $\mathfrak{M}_{\Omega,b} : F_s^{p_1,q}(\Omega) \times F_s^{p_2,q}(\Omega) \to F_s^{p,q}(\Omega)$ when $b \in F_s^{p_3,q}(\Omega)$? What happens when we consider the Besov spaces?

This is the main motivation of this paper. Before establishing our main results, let us point out the following fact.

Remark 2. Let $1 < p_1, p_2, p_3, p < \infty$, $1/p = 1/p_1 + 1/p_2 + 1/p_3$, and $b \in L^{p_3}(\Omega)$. If $f \in L^{p_1}(\Omega)$ and $g \in L^{p_2}(\Omega)$, then we have

$$\|\mathfrak{M}_{\Omega,b}(f, g)\|_{L^p(\Omega)} \lesssim \|b\|_{L^{p_3}(\Omega)} \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_2}(\Omega)}.$$

By the above boundedness and the sublinearity of $\mathfrak{M}_{\Omega,b}$, we have that

$$\mathfrak{M}_{\Omega,b}: L^{p_1}(\Omega)\times L^{p_2}(\Omega)\to L^p(\Omega)$$

is continuous.

Our main results are the following:

Theorem 1. *Let* $1 < p_1, p_2, p_3, p < \infty$, $0 < s < 1$, $\sum_{i=1}^3 \frac{1}{p_i} = \frac{1}{p}$ *and* $\max\{p_1', p_2'\} < \min\{p, q\}$. *If* $b \in F_s^{p_3,q}(\Omega)$, *then the following mapping*

$$\mathfrak{M}_{\Omega,b}: F_s^{p_1,q}(\Omega)\times F_s^{p_2,q}(\Omega)\to F_s^{p,q}(\Omega)$$

is bounded and continuous. Moreover, there exists $C > 0$ *such that*

$$\|\mathfrak{M}_{\Omega,b}(f,g)\|_{F_s^{p,q}(\Omega)} \le C\|b\|_{F_s^{p_3,q}(\Omega)}\|f\|_{F_s^{p_1,q}(\Omega)}\|g\|_{F_s^{p_2,q}(\Omega)}. \tag{1.1}$$

Theorem 2. *Let* $1 < p_1, p_2, p_3, p, q < \infty$, $0 < s < 1$, *and* $\sum_{i=1}^3 \frac{1}{p_i} = \frac{1}{p}$. *If* $b \in B_s^{p_3,q}(\Omega)$, *then the following mapping*

$$\mathfrak{M}_{\Omega,b}: B_s^{p_1,q}(\Omega)\times B_s^{p_2,q}(\Omega)\to B_s^{p,q}(\Omega)$$

is bounded and continuous. Moreover, there exists $C > 0$ *such that*

$$\|\mathfrak{M}_{\Omega,b}(f,g)\|_{B_s^{p,q}(\Omega)} \le C\|b\|_{B_s^{p_3,q}(\Omega)}\|f\|_{B_s^{p_1,q}(\Omega)}\|g\|_{B_s^{p_2,q}(\Omega)}. \tag{1.2}$$

Remark 3. Theorems 1 and 2 can be regarded as a local variant of Theorem A. Compared with Theorem A, our main results and their proofs are more complex and refined. It should be pointed out that the estimate of difference for an objective function plays a key role in concluding the boundedness and continuity of the objective operator on Triebel–Lizorkin spaces and Besov spaces, both in the global and local cases. However, the difference estimates of the local bilinear maximal commutators are more detailed and complex than those of the global case.

This paper is organized as follows. Section 2 contains some properties of Triebel–Lizorkin spaces and Besov spaces as well as maximal functions, and some refined estimates of difference for extension functions of the local bilinear maximal commutator, which are the main ingredients of proving our main results. In Section 3 we present the proofs of Theorems 1 and 2. We would like to remark that the main ideas used to prove our main results are a combination of ideas and arguments from [13, 22]. The novelty is how to extend the results of [22] to the local setting.

Throughout this paper, for any $p \in [1,\infty]$, we denote by p' the dual exponent to p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. We set $p'=\infty$ when $p=1$ and $p'=1$ when $p=\infty$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but is independent of the essential variables. If there exists a constant $c>0$ depending only on ϑ such that $A \le cB$, we then write $A \lesssim_\vartheta B$; and if $A \lesssim_\vartheta B \lesssim_\vartheta A$, we then write $A \sim_\vartheta B$. For any x, $h \in \mathbb{R}^n$ and a function f defined on $\mathbb{R}^n$, we set $\Delta_h f(x) = f(x+h) - f(x)$.

2. Preliminaries

2.1. Triebel–Lizorkin spaces and Besov spaces

Let us start with the definition of local Triebel–Lizorkin spaces. This definition is due to Triebel. This is not the only way to define these spaces.

Definition 4. (Local Triebel–Lizorkin spaces, [13]). Let $1 < p, q < \infty$ and $0 < s < 1$. Let $F^{p,q}_s(\mathbb{R}^n)$ (resp., $\dot{F}^{p,q}_s(\mathbb{R}^n)$) be inhomogeneous (resp., homogeneous) Triebel–Lizorkin spaces. The inhomogeneous local Triebel–Lizorkin space $F^{p,q}_s(\Omega)$ is defined by

$$F^{p,q}_s(\Omega) = \{f|_\Omega : f \in F^{p,q}_s(\mathbb{R}^n)\}, \quad \|f\|_{F^{p,q}_s(\Omega)} = \inf\{\|g\|_{F^{p,q}_s(\mathbb{R}^n)} : g|_\Omega = f\}.$$

It is well known that (see [19]):

$$\dot{F}^{p,2}_0(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad 1 < p < \infty;$$

$$\|f\|_{F^{p,q}_s(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \ \ 1 < p, \ q < \infty, \ s > 0;$$
$$\|f\|_{F^{p,q}_{s_1}(\mathbb{R}^n)} \le \|f\|_{F^{p,q}_{s_2}(\mathbb{R}^n)}, \quad s_1 \le s_2, \ 1 < p, \ q < \infty;$$
$$\|f\|_{F^{p,q_2}_s(\mathbb{R}^n)} \le \|f\|_{F^{p,q_1}_s(\mathbb{R}^n)}, \quad s \in \mathbb{R}, \ 1 < p < \infty, \ 1 < q_1 \le q_2 < \infty. \tag{2.1}$$

Let $1 < p,q < \infty$, $0 < s < 1$, and $1 \le r < \min\{p,q\}$. Let $F_{p,q,r}(\mathbb{R}^n)$ be the set of all measurable functions $g : \mathbb{R}^n \times (0,1) \times B(O,1) \to \mathbb{R}$ satisfying

$$\|g\|_{F_{p,q,r}} := \Big(\int_{\mathbb{R}^n}\Big(\int_0^1\Big(\int_{B(O,1)} |g(x,t,h)|^r dh\Big)^{q/r}\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p} < \infty.$$

It was established in [19, p. 194] that

$$\|g\|_{F^{p,q}_s(\mathbb{R}^n)} \sim \|S_s g\|_{F_{p,q,r}} + \|g\|_{L^p(\mathbb{R}^n)}, \tag{2.2}$$

where S_s is defined by setting

$$S_s g(x,t,h) = \frac{|g(x+th) - g(x)|}{t^s}.$$

Next we present the definition of local Besov spaces.

Definition 5. (Local Besov spaces, [14]). Let $1 < p, q < \infty$ and $0 < s < 1$. Let $B^{p,q}_s(\mathbb{R}^n)$ (resp., $\dot{B}^{p,q}_s(\mathbb{R}^n)$) be inhomogeneous (resp., homogeneous) Besov spaces. The inhomogeneous local Besov space $B^{p,q}_s(\Omega)$ is given by

$$B^{p,q}_s(\Omega) = \{f|_\Omega : f \in B^{p,q}_s(\mathbb{R}^n)\}, \quad \|f\|_{B^{p,q}_s(\Omega)} = \inf\{\|g\|_{B^{p,q}_s(\mathbb{R}^n)} : g|_\Omega = f\}.$$

It is well known that (see [19]):

$$B^{p,q}_s(\mathbb{R}^n) \sim \dot{B}^{p,q}_s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \ \ 1 < p, \ q < \infty, \ s > 0;$$

$$\dot{B}^{p,p}_s(\mathbb{R}^n) = \dot{F}^{p,p}_s(\mathbb{R}^n), \ \ s \in \mathbb{R}, \ \ 1 < p < \infty;$$

$$\|f\|_{B^{p,q}_s(\mathbb{R}^n)} \sim \|f\|_{\dot{B}^{p,q}_s(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \ \ 1 < p, \ q < \infty, \ s > 0;$$

$$\|g\|_{B^{p,q}_{s_1}(\mathbb{R}^n)} \le \|g\|_{B^{p,q}_{s_2}(\mathbb{R}^n)}, \quad s_1 \le s_2,\ 1 < p,\ q < \infty;$$

$$\|g\|_{B^{p,q_2}_{s}(\mathbb{R}^n)} \le \|g\|_{B^{p,q_1}_{s}(\mathbb{R}^n)}, \quad s \in \mathbb{R},\ 1 < p < \infty,\ 1 < q_1 \le q_2 < \infty. \tag{2.3}$$

Let $1 < p, q < \infty$, $0 < s < 1$, and $1 \le r \le p$. Let $E_{p,q,r}(\mathbb{R}^n)$ be the set of all measurable functions $g : \mathbb{R}^n \times (0,1) \times B(O,1) \to \mathbb{R}$ satisfying

$$\|g\|_{E_{p,q,r}} := \Big(\int_0^1 \Big(\int_{\mathbb{R}^n} \Big(\int_{B(O,1)} |g(x,t,h)|^r dh \Big)^{p/r} dx \Big)^{q/p} \frac{dt}{t} \Big)^{1/q} < \infty.$$

It was pointed out in [14] that

$$\|g\|_{B^{p,q}_s(\mathbb{R}^n)} \sim \|S_s g\|_{E_{p,q,r}} + \|g\|_{L^p(\mathbb{R}^n)}.$$

When $r = p$, we denote $E_{p,q} = E_{p,q,r}$. Clearly,

$$\|g\|_{B^{p,q}_s(\mathbb{R}^n)} \sim \|S_s g\|_{E_{p,q}} + \|g\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

2.2. Preliminary lemmas

For $1 \le r < \infty$ we denote

$$M_r f(x) = (M|f|^r)^{1/r}(x), \quad x \in \mathbb{R}^n.$$

When $r = 1$ we denote $M_1 = M$. By Hölder's inequality, we have

$$\mathfrak{M}(f,g)(x) \le M_\tau f(x) M_{\tau'} g(x), \quad \forall \tau \in (1,\infty). \tag{2.5}$$

For convenience, we set

$$\rho_t u(x) = \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |u(z) - u(x)| dz, \quad x \in \mathbb{R}^n.$$

Denote

$$S_{t,s} u(x,t) = t^{-s} \rho_t u(x), \quad u \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

We have the following basic properties of M_r and $S_{t,s}$.

Lemma 3. *Let $1 < p,\ q < \infty$ and $1 \le \tau \le r < \min\{p,q\}$. Then*

(i) *For any $f \in L^p(\mathbb{R}^n)$ we have $\|M_\tau f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$.*

(ii) *For any $f \in F_{p,q,r}$ we have $\|M_\tau f\|_{F_{p,q,r}} \lesssim \|f\|_{F_{p,q,r}}$.*

(iii) *For any function $f \in L^p(\mathbb{R}^n, L^q((0,1), t^{-1}dt))$ we have*

$$\|M_\tau f\|_{F_{p,q,r}} \lesssim \|f\|_{L^p(\mathbb{R}^n, L^q((0,1),t^{-1}dt))}.$$

(iv) *For any $f \in E_{p,q}$ we have $\|M_\tau f\|_{E_{p,q}} \lesssim \|f\|_{E_{p,q}}$.*

(v) *For any function $f \in L^p(\mathbb{R}^n, L^q((0,1), t^{-1}dt))$, we have*

$$\|M_\tau f\|_{E_{p,q}} \lesssim \|f\|_{L^p(\mathbb{R}^n, L^q((0,1),t^{-1}dt))}.$$

(vi) *Let $0 < s < 1$. Then we have*

$$\|S_{t,s} f\|_{L^p(\mathbb{R}^n, L^q((0,1),t^{-1}dt))} \lesssim \min\{\|f\|_{F^{p,q}_s(\mathbb{R}^n)}, \|f\|_{B^{p,q}_s(\mathbb{R}^n)}\}.$$

Proof. Part (i) follows from the L^p bounds for the Hardy–Littlewood maximal operator. Part (ii) with $\tau = 1$ follows directly from the results in [9]. In view of part (ii) with $\tau = 1$, we have that for any $\tau > 1$,

$$\|M_\tau f\|_{F_{p,q,r}} \le \|M_1 f^\tau\|^{1/\tau}_{F_{p/\tau,q/\tau,r/\tau}} \lesssim \|f^\tau\|^{1/\tau}_{F_{p/\tau,q/\tau,r/\tau}} = \|f\|_{F_{p,q,r}}.$$

This yields part (ii). As proved in [13, p. 234], we have $\|M_1 f\|_{F_{p,q,r}} \lesssim \|f\|_{F_{p,q,r}}$. This implies that for any $\tau > 1$,

$$\|M_\tau f\|_{F_{p,q,r}} \le \|M_1 f^\tau\|^{1/\tau}_{F_{p/\tau,q/\tau,r/\tau}} \lesssim \|f^\tau\|^{1/\tau}_{L^{p/\tau}(\mathbb{R}^n,L^{q/\tau}((0,1),t^{-1}dt))} = \|f\|_{L^p(\mathbb{R}^n,L^q((0,1),t^{-1}dt))}.$$

Then part (iii) holds. We get by part (i) that $\|M_\tau f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$. This together with (2.4) implies that

$$\|M_\tau f\|_{E_{p,q}} \lesssim \Big(\int_0^1 \Big(\int_{B(O,1)}\int_{\mathbb{R}^n} |f(x,t,h)|^p dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q} = \|f\|_{E_{p,q}}.$$

This gives part (iv). We also note that

$$\Big(\int_0^1 \Big(\int_{B(O,1)}\int_{\mathbb{R}^n} |f(x,t)|^p dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q} \lesssim |B(O,1)|^{1/p}\|f\|_{L^p(\mathbb{R}^n,L^q((0,1),t^{-1}dt))}.$$

This proves part (v).

Next we prove part (vi). By some changes of variables, one gets

$$S_{t,s}f(x,t) = \frac{1}{|B(O,1)|}\int_{B(O,1)} |f(x+2th) - f(x)|t^{-s}dh.$$

Further we get by a change of variable that

$$\begin{aligned}
&\|S_{t,s}f\|_{L^p(\mathbb{R}^n,L^q((0,1),t^{-1}dt))}\\
&= \Big(\int_{\mathbb{R}^n}\Big(\int_0^1\Big(\frac{1}{|B(O,1)|}\int_{B(O,1)} |f(x+2th) - f(x)|t^{-s}dh\Big)^q\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p}\\
&\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_0^2\Big(\int_{B(O,1)} S_s f(x,t,h)dh\Big)^q\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p}\\
&\lesssim \|S_s f\|_{F_{p,q,1}} + \Big(\int_{\mathbb{R}^n}\Big(\int_1^2\Big(\int_{B(O,1)} S_s f(x,t,h)dh\Big)^q\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p}.
\end{aligned}$$

By a change of variable and Minkowski's inequality,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n}\Big(\int_1^2\Big(\int_{B(O,1)} S_s f(x,t,h)dh\Big)^q\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p}\\
&\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_1^2\Big(\int_{B(O,t)} |f(x+z)-f(x)|dz\Big)^q\frac{dt}{t}\Big)^{p/q}dx\Big)^{1/p}\\
&\lesssim \Big(\int_{\mathbb{R}^n}\Big(\int_{B(O,2)} |f(x+z)-f(x)|dz\Big)^p dx\Big)^{1/p}\\
&\lesssim \int_{B(O,2)}\Big(\int_{\mathbb{R}^n} |f(x+z)-f(x)|^p dx\Big)^{1/p}dz \lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This together with (2.2) implies that

$$\|S_{t,s}f\|_{L^p(\mathbb{R}^n,L^q((0,1),t^{-1}dt))} \lesssim \|f\|_{F^{p,q}_s(\mathbb{R}^n)}.$$

On the other hand, by a change of variable, Hölder's inequality, and applying Fubini's theorem,

$$\begin{aligned}
&\Big(\int_0^1\Big(\int_{\mathbb{R}^n}|S_{t,s}f(x,t)|^p dx\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}\\
&\le\Big(\int_0^1\Big(\int_{\mathbb{R}^n}\frac{1}{|B(O,1)|}\int_{B(O,1)}\frac{|f(x+2th)-f(x)|^p}{t^{sp}}dhdx\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}\\
&\lesssim\Big(\int_0^2\Big(\int_{B(O,1)}\int_{\mathbb{R}^n}\frac{|f(x+th)-f(x)|^p}{t^{sp}}dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}\\
&\le\|S_s f\|_{E_{p,q}}+\Big(\int_1^2\Big(\int_{B(O,1)}\int_{\mathbb{R}^n}|Sf(x,t,h)|^p dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}.
\end{aligned}$$

One can easily check that

$$\begin{aligned}
&\Big(\int_1^2\Big(\int_{B(O,1)}\int_{\mathbb{R}^n}|Sf(x,t,h)|^p dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}\\
&\le\Big(\int_1^2\Big(\int_{B(O,1)}\int_{\mathbb{R}^n}|f(x+th)-f(x)|^p dxdh\Big)^{q/p}\frac{dt}{t}\Big)^{1/q}\lesssim\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This together with (2.4) yields that

$$\|S_{t,s}f\|_{L^p(\mathbb{R}^n,L^q((0,1),t^{-1}dt))}\lesssim\|f\|_{B_s^{p,q}(\mathbb{R}^n)}.$$

So part (vi) holds. □

The following results play key roles in the continuity part of Theorems 1 and 2.

Lemma 4. *Let* $1\le p,q,r<\infty$ *and* F_j, Φ, Ψ_j *be mappings from* $\mathbb{R}^n\times(0,1)\times B(O,1)$ *to* $\mathbb{R}$ *such that*

$$|F_j(x,t,h)|\le\Phi(x,t,h)+\Psi_j(x,t,h),\quad\forall j\ge1\text{ and a.e. }(x,t,h)\in\mathbb{R}^n\times(0,1)\times B(O,1).$$

Then:

(i) *Let* $\|\Psi_j\|_{F_{p,q,r}}\to0$ *as* $j\to\infty$ *and* $\|\Phi\|_{F_{p,q,r}}<\infty$*. Moreover, for a.e.* $x\in\mathbb{R}^n$ *and* $t\in(0,1)$*, we have*

$$|\{h\in B(O,1):|F_j(x,t,h)|>\epsilon\}|\to0\ \text{ as }\ j\to\infty,\ \ \forall\epsilon>0.$$

Then we have $\|F_j\|_{F_{p,q,r}}\to0$ *as* $j\to\infty$.

(ii) *Let* $\|\Psi_j\|_{E_{p,q}}\to0$ *as* $j\to\infty$ *and* $\|\Phi\|_{E_{p,q}}<\infty$*. Moreover, for a.e.* $(x,t,h)\in\mathbb{R}^n\times(0,1)\times B(O,1)$ *we have* $F_j(x,t,h)\to0$ *as* $j\to\infty$*. Then we have* $\|F_j\|_{E_{p,q}}\to0$ *as* $j\to\infty$.

Proof. Part (i) was proved in [13]. Part (ii) can be proved by the dominated convergence theorem. The details are omitted. □

2.3. Difference estimates of extension functions

In this subsection we introduce some extension functions and establish some refined difference estimates. These are the main ingredients for proving our main results. In order to prove our main results, we need to introduce some auxiliary functions. Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $(f,g) \in L^1_{\rm loc}(\mathbb{R}^n)\times L^1_{\rm loc}(\mathbb{R}^n)$. We define the function $A_{b,f,g,x} : [0,\infty) \to \mathbb{R}$ by

$$A_{b,f,g,x}(r) = \begin{cases} 0, & \text{if } r=0;\\ \dfrac{1}{|B(O,r)|}\displaystyle\int_{B(O,r)} |(b(x)-b(x+y))f(x+y)g(x-y)|dy, & \text{if } r\in(0,\infty). \end{cases}$$

Lemma 5. *Let $f \in L^{p_1}_{\rm loc}(\mathbb{R}^n)$, $g \in L^{p_2}_{\rm loc}(\mathbb{R}^n)$, $b \in L^{p_3}_{\rm loc}(\mathbb{R}^n)$, where $1 < p_1, p_2, p_3 < \infty$ and $\sum_{i=1}^3 \frac{1}{p_i} \le 1$. Then we have:*

(i) *For any $x \in \mathbb{R}^n$, $A_{b,f,g,x}(r)$ is continuous on $(0,\infty)$.*

(ii) *For a.e. $x \in \mathbb{R}^n$, $A_{b,f,g,x}(r)$ is continuous at $r = 0$.*

Proof. Let $p_4 \in [1,\infty]$ be such that $\sum_{i=1}^4 \frac{1}{p_i} = 1$. Let $r \in (0,\infty)$ and $t \in (0,2r)$. By Hölder's inequality and the Lebesgue dominated convergence theorem, one gets

$$\begin{aligned}
&\Big|\int_{B(O,t)} |b(x)-b(x+y)||f(x+y)g(x-y)|dy - \int_{B(O,r)} |b(x)-b(x+y)||f(x+y)g(x-y)|dy\Big|\\
&\le \Big|\int_{B(O,2r)} |b(x)-b(x+y)||f(x+y)g(x-y)||\chi_{B(O,t)}(y)-\chi_{B(O,r)}(y)|dy\\
&\le |B(O,2r)|^{\frac{1}{p_4}}\|b(x)-b(x+\cdot)\|_{L^{p_3}(B(0,2r))}\|g(x-\cdot)\|_{L^{p_2}(B(O,2r))}\|f(x+\cdot)(\chi_{B(O,t)}-\chi_{B(O,r)})\|_{L^{p_1}(B(O,2r))}\\
&\to 0 \ \text{ as } t\to r.
\end{aligned}$$

This gives part (i).

On the other hand, we get by Hölder's inequality and the Lebesgue differentiation theorem that for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}
A_{b,f,g,x}(r) &= \frac{1}{|B(O,r)|}\int_{B(O,r)} |b(x)-b(x+y)||f(x+y)g(x-y)|dy\\
&\le \Big(\frac{1}{|B(O,r)|}\int_{B(O,r)} |b(x)-b(x+y)|^{p_3}dy\Big)^{1/p_3}\Big(\frac{1}{|B(O,r)|}\int_{B(O,r)} |f(x+y)|^{p_1}dy\Big)^{1/p_1}\\
&\quad\times\Big(\frac{1}{|B(O,r)|}\int_{B(O,r)} |g(x-y)|^{p_2}dy\Big)^{1/p_2} \to 0 \ \text{ as } \ r\to 0.
\end{aligned}$$

This gives part (ii). □

We now introduce the extension function of the local bilinear maximal commutator. This plays a key role in our proofs. Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $g \in L^1_{\rm loc}(\mathbb{R}^n)$, and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Define the extension function $T_{\Omega,b}(f,g)$ by

$$T_{\Omega,b}(f,g)(x) = \begin{cases} 0, & \text{if } x\in\Omega^c,\\ \mathfrak{M}_{\Omega,b}(f,g)(x), & \text{if } x\in\Omega. \end{cases} \tag{2.6}$$

We now establish the refined difference estimates for $T_{\Omega,b}(f,g)$.

Lemma 6. *Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $g \in L^1_{\rm loc}(\mathbb{R}^n)$, and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0, 1)$ and $h \in B(O, 1)$, we have*

$$\begin{aligned}
&|T_{\Omega,b}(f, g)(x + th) - T_{\Omega,b}(f, g)(x)|\\
&\le (|\Delta_{th}b(x)| + |b(x)|)(\mathfrak{M}(\Delta_{th}f, \Delta_{th}g)(x) + \mathfrak{M}(\Delta_{th}f, g)(x) + \mathfrak{M}(f, \Delta_{th}g)(x))\\
&\quad +\mathfrak{M}(b\Delta_{th}f, \Delta_{th}g)(x) + \mathfrak{M}(\Delta_{th}b\Delta_{th}f, \Delta_{th}g)(x) + \mathfrak{M}(b\Delta_{th}f, g)(x)\\
&\quad +\mathfrak{M}(\Delta_{th}b\Delta_{th}f, g)(x) + \mathfrak{M}(bf, \Delta_{th}g)(x) + \mathfrak{M}(\Delta_{th}bf, \Delta_{th}g)(x)\\
&\quad +|\Delta_{th}b(x)|\mathfrak{M}(f, g)(x) + \mathfrak{M}(\Delta_{th}bf, g)(x) + 2^{n+1}M_{\delta_1}f(x)M_{\delta_2}g(x)M_{\delta_3}\rho_t b(x)\\
&\quad +2^{n+1}(M_{\delta_4}f(x)M_{\delta_4'}\rho_t g(x)\rho_t b(x) + M_{\delta_5}f(x)M_{\delta_6}\rho_t g(x)M_{\delta_7}\rho_t b(x))\\
&\quad +2^{n+1}(M_{\delta_8}\rho_t f(x)M_{\delta_8'}g(x)\rho_t b(x) + M_{\delta_9}\rho_t f(x)M_{\delta_{10}}g(x)M_{\delta_{11}}\rho_t b(x))\\
&=: \Phi(x, t, h),
\end{aligned}$$

where $\delta_i \in (1, \infty)$, $1 \le i \le 11$, $\sum_{i=1}^{3} \frac{1}{\delta_i} = 1$, $\sum_{i=5}^{7} \frac{1}{\delta_i} = 1$ and $\sum_{i=9}^{11} \frac{1}{\delta_i} = 1$.

Proof. First, we prove the following:

Claim 1. Let x, y be the Lebesgue points of f and g in $\mathbb{R}^n$. There exist $r_1 \ge 0$ and $r_2 \ge 0$ such that $|r_1 - r_2| \le |x - y|$ and

$$|T_{\Omega,b}(f, g)(x) - T_{\Omega,b}(f, g)(y)| \le |A_{b,f,g,x}(r_1) - A_{b,f,g,y}(r_2)|. \tag{2.7}$$

We assume without loss of generality that $T_{\Omega,b}(f, g)(x) > T_{\Omega,b}(f, g)(y)$. For convenience, we set $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and

$$\mathcal{R}(f, g)(x) = \{r \in [0, \delta(x)] : T_{\Omega,b}(f, g)(x) = A_{b,f,g,x}(r)\}.$$

By Lemma 5 we see that for almost every $x \in \Omega$, the function $A_{b,f,g,x}(r)$ is continuous on $[0, \delta(x)]$. Thus, for a.e. $x \in \Omega$, the set $\mathcal{R}(f, g)(x)$ is non-empty. Let $r_1 \in \mathcal{R}(f, g)(x)$ and set $r_2 = \max\{0, r_1 - |x - y|\}$. Clearly, $|r_1 - r_2| \le |x - y|$. If $r_2 = 0$, then (2.7) is clear. If $r_2 > 0$, then $r_1 > |x - y|$ and $r_2 = r_1 - |x - y| \le \delta(x) - |x - y| \le \delta(y)$. In this case (2.7) holds since $T_{\Omega,b}(f, g)(y) \ge A_{b,f,g,y}(r_2)$.

In view of Claim 1, for the conclusion of Lemma 6, it suffices to show the following:

Claim 2. For a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0, 1)$ and $h \in B(O, 1)$, and for any r_1, $r_2 \ge 0$ and $|r_1 - r_2| \le t$, we have

$$|A_{b,f,g,x+th}(r_1) - A_{b,f,g,x}(r_2)| \le \Phi(x, t, h). \tag{2.8}$$

We now prove Claim 2. Let E be the set of all $x \in \mathbb{R}^n$ for which $A_{b,f,g,x}(r)$ is continuous on $[0, \infty)$. Invoking Lemma 5, we see that $|\mathbb{R}^n \setminus E| = 0$. Let x, $x + th \in E$ with $t \in (0, 1)$ and $h \in B(O, 1)$. Let r_1, $r_2 \ge 0$ with $|r_1 - r_2| \le t$. Without loss of generality we may assume r_1, $r_2 > 0$. Observe that

$$|A_{b,f,g,x+th}(r_1) - A_{b,f,g,x}(r_2)| \le |A_{b,f,g,x+th}(r_1) - A_{b,f,g,x}(r_1)| + |A_{b,f,g,x}(r_1) - A_{b,f,g,x}(r_2)|. \tag{2.9}$$

By some changes of variables, one gets

$$\begin{aligned}
|A_{b,f,g,x+th}(r_1) - A_{b,f,g,x}(r_1)| \le \frac{1}{|B(O, r_1)|} \int_{B(O,r_1)} &|(b(x + th) - b(x + y + th))f(x + y + th)g(x - y + th)\\
&-(b(x) - b(x + y))f(x + y)g(x - y)|dy\\
=:\ & I_1 + I_2,
\end{aligned}$$

where

$$I_1 := \frac{1}{|B(O,r_1)|}\int_{B(O,r_1)} |(b(x+th)-b(x+y+th))(f(x+y+th)g(x-y+th)-f(x+y)g(x-y))|dy,$$

$$I_2 := \frac{1}{|B(O,r_1)|}\int_{B(O,r_1)} (|\Delta_{th}b(x)|+|\Delta_{th}b(x+y)|)|f(x+y)g(x-y)|dy.$$

It is clear that

$$|b(x+th)-b(x+y+th)| \le |\Delta_{th}b(x)|+|b(x)|+|b(x+y)|+|\Delta_{th}b(x+y)|$$

and

$$\begin{aligned}&|f(x+y+th)g(x-y+th)-f(x+y)g(x-y)|\\ &\le |\Delta_{th}f(x+y)|(|\Delta_{th}g(x-y)|+|g(x-y)|)+|f(x+y)||\Delta_{th}g(x-y)|.\end{aligned}$$

Hence, we have

$$\begin{aligned}I_1 \le\ &(|\Delta_{th}b(x)|+|b(x)|)(\mathfrak{M}(\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}f,g)(x)+\mathfrak{M}(f,\Delta_{th}g)(x))\\ &+\mathfrak{M}(b\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}b\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(b\Delta_{th}f,g)(x)\\ &+\mathfrak{M}(\Delta_{th}b\Delta_{th}f,g)(x)+\mathfrak{M}(bf,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}bf,\Delta_{th}g)(x).\end{aligned}$$

We also note that

$$I_2 \le |\Delta_{th}b(x)|\mathfrak{M}(f,g)(x)+\mathfrak{M}(\Delta_{th}bf,g)(x).$$

It follows that

$$\begin{aligned}&|A_{b,f,g,x+th}(r_1)-A_{b,f,g,x}(r_1)|\\ &\le (|\Delta_{th}b(x)|+|b(x)|)(\mathfrak{M}(\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}f,g)(x)+\mathfrak{M}(f,\Delta_{th}g)(x))\\ &\quad+\mathfrak{M}(b\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}b\Delta_{th}f,\Delta_{th}g)(x)+\mathfrak{M}(b\Delta_{th}f,g)(x)\\ &\quad+\mathfrak{M}(\Delta_{th}b\Delta_{th}f,g)(x)+\mathfrak{M}(bf,\Delta_{th}g)(x)+\mathfrak{M}(\Delta_{th}bf,\Delta_{th}g)(x)\\ &\quad+|\Delta_{th}b(x)|\mathfrak{M}(f,g)(x)+\mathfrak{M}(\Delta_{th}bf,g)(x).\end{aligned}\tag{2.10}$$

It remains to estimate $|A_{b,f,g,x}(r_1)-A_{b,f,g,x}(r_2)|$. By some change of variables,

$$A_{b,f,g,x}(r_2)=\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big|\Big(b(x)-b\Big(x+\frac{r_2}{r_1}y\Big)\Big)f\Big(x+\frac{r_2}{r_1}y\Big)g\Big(x-\frac{r_2}{r_1}y\Big)\Big|dy.$$

Write

$$|A_{b,f,g,x}(r_1)-A_{b,f,g,x}(r_2)| \le I_{2,1}+I_{2,2}+I_{2,3},$$

where

$$I_{2,1} := \frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big|\Big(b\Big(x+\frac{r_2}{r_1}y\Big)-b(x+y)\Big)f(x+y)g(x-y)\Big|dy,$$

$$I_{2,2} := \frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big|\Big(b(x)-b\Big(x+\frac{r_2}{r_1}y\Big)\Big)f(x+y)\Big(g(x-y)-g\Big(x-\frac{r_2}{r_1}y\Big)\Big)\Big|dy,$$

$$I_{2,3} := \frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big|\Big(b(x)-b\Big(x+\frac{r_2}{r_1}y\Big)\Big)\Big(f(x+y)-f\Big(x+\frac{r_2}{r_1}y\Big)\Big)g\Big(x-\frac{r_2}{r_1}y\Big)\Big|dy.$$

It is not difficult to verify that when $|x-y|<t$,

$$|f(y)-f(x)|\le 2^n(\rho_t f(y)+\rho_t f(x)).$$

It follows that

$$\begin{aligned}I_{2,1}&\le \frac{2^n}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)+\rho_t b(x+y)\Big)|f(x+y)g(x-y)|dy\\&\le 2^n\mathfrak{M}(\rho_t bf,g)(x)+\frac{2^n}{|B(O,r_1)|}\int_{B(O,r_1)}\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)|f(x+y)g(x-y)|dy.\end{aligned}$$

Let $\delta_i\in(1,\infty),\ i=1,2,3$, and $\sum_{i=1}^{3}\frac{1}{\delta_i}=1$. By a change of variable, one gets

$$\begin{aligned}&\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)\Big)^{\delta_3}dy\Big)^{1/\delta_3}\\&\le\Big(\frac{1}{|B(O,r_2)|}\int_{B(O,r_2)}(\rho_t b(x+z))^{\delta_3}dz\Big)^{1/\delta_3}\le M_{\delta_3}\rho_t b(x).\end{aligned}$$

This together with Hölder's inequality implies that

$$\begin{aligned}&\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)|f(x+y)g(x-y)|dy\\&\le\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)\Big)^{\delta_3}dy\Big)^{1/\delta_3}\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}|f(x+y)|^{\delta_1}dy\Big)^{1/\delta_1}\\&\qquad\times\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}|g(x-y)|^{\delta_2}dy\Big)^{1/\delta_2}\\&\le M_{\delta_1}f(x)M_{\delta_2}g(x)M_{\delta_3}\rho_t b(x).\end{aligned}$$

This together with (2.5) and the fact that $M_{\delta_2'}\rho_t bf\le M_{\delta_1}fM_{\delta_3}\rho_t b$ implies that

$$I_{2,1}\le 2^{n+1}M_{\delta_1}f(x)M_{\delta_2}g(x)M_{\delta_3}\rho_t b(x).$$

Next we estimate $I_{2,2}$. We have

$$\begin{aligned}I_{2,2}&\le\frac{2^n}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t b(x)+\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)\Big)|f(x+y)|\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)dy\\&\le\frac{2^n\rho_t b(x)}{|B(O,r_1)|}\int_{B(O,r_1)}|f(x+y)|\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)dy\\&\qquad+\frac{2^n}{|B(O,r_1)|}\int_{B(O,r_1)}\rho_t b\Big(x+\frac{r_2}{r_1}y\Big)|f(x+y)|\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)dy.\end{aligned}$$

By Hölder's inequality, Minkowski's inequality, and some changes of variables, we have that for any $\delta_4\in(1,\infty)$,

$$\begin{aligned}&\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}|f(x+y)|\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)dy\\&\le\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}|f(x+y)|^{\delta_4}dy\Big)^{1/\delta_4}\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)^{\delta_4'}dy\Big)^{1/\delta_4'}\\&\le M_{\delta_4}f(x)\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}(\rho_t g(x-y))^{\delta_4'}dy\Big)^{1/\delta_4'}\\&\qquad+M_{\delta_4}f(x)\Big(\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)}\Big(\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)^{\delta_4'}dy\Big)^{1/\delta_4'}\\&\le 2M_{\delta_4}f(x)M_{\delta_4'}\rho_t g(x).\end{aligned}$$

Let $\delta_i \in (1,\infty)$, $i = 5,6,7$, and $\sum_{i=5}^{7} \frac{1}{\delta_i} = 1$. By Hölder's inequality, Minkowski's inequality, and some changes of variables again,

$$\frac{1}{|B(O,r_1)|}\int_{B(O,r_1)} \rho_t b\Big(x+\frac{r_2}{r_1}y\Big)|f(x+y)|\Big(\rho_t g(x-y)+\rho_t g\Big(x-\frac{r_2}{r_1}y\Big)\Big)dy \le 2M_{\delta_5}f(x)M_{\delta_6}\rho_t g(x)M_{\delta_7}\rho_t b(x).$$

It follows that

$$I_{2,2} \le 2^{n+1}(M_{\delta_4}f(x)M_{\delta_4'}\rho_t g(x)\rho_t b(x) + M_{\delta_5}f(x)M_{\delta_6}\rho_t g(x)M_{\delta_7}\rho_t b(x)).$$

Similarly we get

$$I_{2,3} \le 2^{n+1}(M_{\delta_8}\rho_t f(x)M_{\delta_8'}g(x)\rho_t b(x) + M_{\delta_9}\rho_t f(x)M_{\delta_{10}}g(x)M_{\delta_{11}}\rho_t b(x)),$$

where $\delta_i \in (1,\infty)$, $i = 8,9,10,11$, and $\sum_{i=9}^{11} \frac{1}{\delta_i} = 1$. Hence, we have

$$\begin{aligned}|A_{b,f,g,x}(r_1) - A_{b,f,g,x}(r_2)| \le\ & 2^{n+1}M_{\delta_1}f(x)M_{\delta_2}g(x)M_{\delta_3}\rho_t b(x)\\ &+2^{n+1}(M_{\delta_4}f(x)M_{\delta_4'}\rho_t g(x)\rho_t b(x) + M_{\delta_5}f(x)M_{\delta_6}\rho_t g(x)M_{\delta_7}\rho_t b(x))\\ &+2^{n+1}(M_{\delta_8}f(x)M_{\delta_8'}\rho_t g(x)\rho_t b(x) + M_{\delta_9}\rho_t f(x)M_{\delta_{10}}g(x)M_{\delta_{11}}\rho_t b(x)).\end{aligned}$$

This together with (2.9) and (2.10) yields (2.8). Then Lemma 6 is proved. □

3. Proof of Theorem 1

In this section we present the proofs of Theorems 1 and 2.

Proof of Theorem 1. We divide the proof of Theorem 1 into two steps.

Step 1. Proof of the boundedness part. Let $h_i \in F_s^{p_i,q}(\Omega)$ for $i = 1,2,3$. We want to show that

$$\|\mathfrak{M}_{\Omega,h_3}(h_1,h_2)\|_{F_s^{p,q}(\Omega)} \lesssim \|h_3\|_{F_s^{p_3,q}(\Omega)}\|h_1\|_{F_s^{p_1,q}(\Omega)}\|h_2\|_{F_s^{p_2,q}(\Omega)}. \tag{3.1}$$

Let $f \in F_s^{p_1,q}(\mathbb{R}^n)$, $g \in F_s^{p_2,q}(\mathbb{R}^n)$, and $b \in F_s^{p_3,q}(\mathbb{R}^n)$ satisfy $f|_\Omega = h_1$, $g|_\Omega = h_2$, and $b|_\Omega = h_3$. Let $T_{\Omega,b}(f,g)$ be defined as in (2.6). Since $\mathfrak{M}_{\Omega,h_3}(h_1,h_2) = T_{\Omega,b}(f,g)|_\Omega$, for (3.1) it is enough to prove that

$$\|T_{\Omega,b}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.2}$$

By Remark 2, we have

$$\|T_{\Omega,b}(f,g)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathfrak{M}_{\Omega,b}(f,g)\|_{L^p(\Omega)} \lesssim \|b\|_{L^{p_3}(\Omega)}\|f\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_2}(\Omega)} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{3.3}$$

In view of (2.2) and (3.3), for (3.2) it is enough to prove that

$$\|S_s(T_{\Omega,b}(f,g))\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.4}$$

By Lemma 6 and the definitions of S_s and $S_{t,s}$, we have that for a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0,1)$ and $h \in B(O,1)$,

$$S_s(T_{\Omega,b}(f,g))(x,t,h) \le 2^{n+1}\sum_{i=1}^{6} T_i(f,g)(x,t,h), \tag{3.5}$$

where

$$T_1(f,g) = |S_s b|\mathfrak{M}(f,g),$$

$$T_2(f,g) = |b|(\mathfrak{M}(S_s f, S_s g) + \mathfrak{M}(S_s f, g) + \mathfrak{M}(f, S_s g)),$$

$$T_3(f,g) = |S_s b|(\mathfrak{M}(S_s f, S_s g) + \mathfrak{M}(S_s f, g) + \mathfrak{M}(f, S_s g)),$$

$$T_4(f,g) = \mathfrak{M}(bS_s f, S_s g) + \mathfrak{M}(S_s b S_s f, S_s g) + \mathfrak{M}(bS_s f, g) + \mathfrak{M}(S_s b S_s f, g) \\ +\mathfrak{M}(bf, S_s g) + \mathfrak{M}(S_s bf, S_s g) + \mathfrak{M}(S_s bf, g),$$

$$T_5(f,g) = M_{\delta_1} f M_{\delta_1'} S_{t,s} g S_{t,s} b + M_{\delta_2} S_{t,s} f M_{\delta_2'} g S_{t,s} b,$$

$$T_6(f,g) = M_{\delta_3} f M_{\delta_4} g M_{\delta_5} S_{t,s} b + M_{\delta_6} f M_{\delta_7} S_{t,s} g M_{\delta_8} S_{t,s} b + M_{\delta_9} S_{t,s} f M_{\delta_{10}} g M_{\delta_{11}} S_{t,s} b.$$

Here $1 < \delta_i < \infty$, $1 \le i \le 11$, $\sum_{i=3}^{5} \frac{1}{\delta_i} = 1$, $\sum_{i=6}^{8} \frac{1}{\delta_i} = 1$, and $\sum_{i=9}^{11} \frac{1}{\delta_i} = 1$. Hence, by Minkowski's inequality and (3.5), one gets

$$\|S_s(T_{\Omega,b}(f,g))\|_{F_{p,q,r}} \le 2^{n+1} \sum_{i=1}^{6} \|T_i(f,g)\|_{F_{p,q,r}}. \tag{3.6}$$

In what follows, we set

$$\frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_3}, \quad \frac{1}{q_2} = \frac{1}{p_1} + \frac{1}{p_3}, \quad \frac{1}{q_3} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \max\{p_1', p_2'\} < r < \min\{p, q\}.$$

It is clear that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{p_3} + \frac{1}{q_3}, \quad p_1 > q_1', \ p_2 > q_2', \ p_3 > q_3'.$$

Next we estimate $\|T_i(f,g)\|_{F_{p,q,r}}$, $i = 1,2,3,4,5,6$, respectively.

Estimate for $\|T_1(f,g)\|_{F_{p,q,r}}$.

By Hölder's inequality, (2.2), and the L^p bounds of $\mathfrak{M}$, we have

$$\|T_1(f,g)\|_{F_{p,q,r}} \le \|S_s b\|_{F_{p_3,q,r}} \|\mathfrak{M}(f,g)\|_{L^{q_3}(\mathbb{R}^n)} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{3.7}$$

Estimate for $\|T_2(f,g)\|_{F_{p,q,r}}$.

By Hölder's inequality and Minkowski's inequality, one gets

$$\|T_2(f,g)\|_{F_{p,q,r}} \le \|b\|_{L^{p_3}(\mathbb{R}^n)} (\|\mathfrak{M}(S_s f, S_s g)\|_{F_{q_3,q,r}} + \|\mathfrak{M}(S_s f, g)\|_{F_{q_3,q,r}} + \|\mathfrak{M}(f, S_s g)\|_{F_{q_3,q,r}}).$$

Let $\frac{p_2}{q_3} < \tau_1' < \frac{rp_2}{q_3}$. Clearly, $\tau_1 < \frac{p_1}{q_3}$. By (2.5), (2.1), (2.2), and Lemma 3, we have

$$\begin{aligned} \|\mathfrak{M}(S_s f, S_s g)\|_{F_{q_3,q,r}} &\le \|M_{\tau_1} S_s f M_{\tau_1'} S_s g\|_{F_{q_3,q,r}} \\ &\le \|M_{\tau_1} S_s f\|_{F_{p_1, \frac{qp_1}{q_3}, \frac{rp_1}{q_3}}} \|M_{\tau_1'} S_s g\|_{F_{p_2, \frac{qp_2}{q_3}, \frac{rp_2}{q_3}}} \\ &\lesssim \|S_s f\|_{F_{p_1, \frac{qp_1}{q_3}, \frac{rp_1}{q_3}}} \|S_s g\|_{F_{p_2, \frac{qp_2}{q_3}, \frac{rp_2}{q_3}}} \\ &\lesssim \|f\|_{F_s^{p_1, \frac{qp_1}{q_3}}(\mathbb{R}^n)} \|g\|_{F_s^{p_2, \frac{qp_2}{q_3}}(\mathbb{R}^n)} \lesssim \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \end{aligned}$$

Let $\tau_2 \in (p_2', r)$. Then $\tau_2' < p_2$. By (2.5), (2.2), and Lemma 3, we have

$$\begin{aligned} \|\mathfrak{M}(S_s f, g)\|_{F_{q_3,q,r}} &\le \|M_{\tau_2} S_s f M_{\tau_2'} g\|_{F_{q_3,q,r}} \\ &\le \|M_{\tau_2} S_s f\|_{F_{p_1,q,r}} \|M_{\tau_2'} g\|_{L^{p_2}(\mathbb{R}^n)} \\ &\lesssim \|S_s f\|_{F_{p_1,q,r}} \|g\|_{L^{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}. \end{aligned}$$

Similarly one gets

$$\|\mathfrak{M}(f,S_sg)\|_{F_{q_3,q,r}}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

Thus, we obtain

$$\|T_2(f,g)\|_{F_{p,q,r}}\le\|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.8}$$

Estimate for $\|T_3(f,g)\|_{F_{p,q,r}}$.

By Hölder's inequality and Minkowski's inequality, one gets

$$\||S_sb|(\mathfrak{M}(S_sf,S_sg)\|_{F_{p,q,r}}\le\|S_sb\|_{F_{p_3,\frac{qp_3}{p},\frac{rp_3}{p}}}\|\mathfrak{M}(S_sf,S_sg)\|_{F_{q_3,\frac{qq_3}{p},\frac{rq_3}{p}}}.$$

In view of (2.1) and (2.2), we get

$$\|S_sb\|_{F_{p_3,\frac{qp_3}{p},\frac{rp_3}{p}}}\lesssim\|b\|_{F_s^{p_3,\frac{qp_3}{p}}(\mathbb{R}^n)}\lesssim\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}.$$

Let $\tau_3'\in(\frac{rp_2}{p},\min\{p_2,\frac{qp_2}{p}\})$ and $\alpha\in(\tau_3',\min\{p_2,\frac{qp_2}{p}\})$. Clearly, $\tau_3<\frac{p_1}{p}$. By (2.5) and Hölder's inequality, one gets

$$\|\mathfrak{M}(S_sf,S_sg)\|_{F_{q_3,\frac{qq_3}{p},\frac{rq_3}{p}}}\le\|M_{\tau_3}S_sfM_{\tau_3'}S_sg\|_{F_{q_3,\frac{qq_3}{p},\frac{rq_3}{p}}}\le\|M_{\tau_3}S_sf\|_{F_{p_1,\frac{qp_1}{p},\frac{rp_1}{p}}}\|M_{\tau_3'}S_sg\|_{F_{p_2,\frac{qp_2}{p},\frac{rp_2}{p}}}.$$

By (2.1), (2.2), Lemma 3, and Hölder's inequality, one gets

$$\|M_{\tau_3}S_sf\|_{F_{p_1,\frac{qp_1}{p},\frac{rp_1}{p}}}\lesssim\|S_sf\|_{F_{p_1,\frac{qp_1}{p},\frac{rp_1}{p}}}\lesssim\|f\|_{F_s^{p_1,\frac{qp_1}{p}}(\mathbb{R}^n)}\lesssim\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)},$$

$$\|M_{\tau_3'}S_sg\|_{F_{p_2,\frac{qp_2}{p},\frac{rp_2}{p}}}\lesssim\|M_{\tau_3'}S_sg\|_{F_{p_2,\frac{qp_2}{p},\alpha}}\lesssim\|S_sg\|_{F_{p_2,\frac{qp_2}{p},\frac{rp_2}{p}}}\lesssim\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

It follows that

$$\||S_sb|\mathfrak{M}(S_sf,S_sg)\|_{F_{p,q,r}}\lesssim\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

Let $\tau_4\in(p_2',r)$. Clearly, $\tau_4'<p_2$. By (2.5), Hölder's inequality and Lemma 3, one gets

$$\begin{aligned}\||S_sb|(\mathfrak{M}(S_sf,g)\|_{F_{p,q,r}}&\le\||S_sb|M_{\tau_4}S_sfM_{\tau_4'}g\|_{F_{p,q,r}}\\&\le\|M_{\tau_4'}g\|_{L^{p_2}(\mathbb{R}^n)}\|S_sb\|_{F_{p_3,\frac{qp_3}{q_2},\frac{rp_3}{q_2}}}\|M_{\tau_4}S_sf\|_{F_{p_1,\frac{qp_1}{q_2},\frac{rp_1}{q_2}}}\\&\lesssim\|g\|_{L^{p_2}(\mathbb{R}^n)}\|b\|_{F_s^{p_3,\frac{qp_3}{q_2}}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,\frac{qp_1}{q_2}}(\mathbb{R}^n)}\\&\lesssim\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.\end{aligned}$$

Similarly one gets

$$\||S_sb|\mathfrak{M}(S_sf,g)\|_{F_{p,q,r}}\lesssim\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

Thus, we have

$$\|T_3(f,g)\|_{F_{p,q,r}}\lesssim\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.9}$$

Estimate for $\|T_4(f,g)\|_{F_{p,q,r}}$.

Let $\tau_5'\in(\frac{p_2}{p},\frac{rp_2}{p})$ and $\tau_5<\frac{q_2}{p}$. By (2.5) and Hölder's inequality, one gets

$$\|\mathfrak{M}(bS_sf,S_sg)\|_{F_{p,q,r}}\le\|M_{\tau_5}(bS_sf)M_{\tau_5'}S_sg\|_{F_{p,q,r}}\le\|M_{\tau_5}(bS_sf)\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}}\|M_{\tau_5'}S_sg\|_{F_{p,\frac{qp_2}{p},\frac{rp_2}{p}}}.$$

In view of (2.1), (2.2), Lemma 3, and Hölder's inequality, we have

$$\|M_{\tau_5}(bS_sf)\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}} \lesssim \|bS_sf\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,\frac{qq_2}{p}}(\mathbb{R}^n)} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)},$$

$$\|M_{\tau_5'}S_sg\|_{F_{p,\frac{qp_2}{p},\frac{rp_2}{p}}} \lesssim \|S_sg\|_{F_{p,\frac{qp_2}{p},\frac{rp_2}{p}}} \lesssim \|g\|_{F_s^{p,\frac{qp_2}{p}}(\mathbb{R}^n)} \lesssim \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

It follows that

$$\|\mathfrak{M}(bS_sf, S_sg)\|_{F_{p,q,r}} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

Similarly one gets

$$\|\mathfrak{M}(S_sbf, S_sg)\|_{F_{p,q,r}} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

By (2.5) and Hölder's inequality again,

$$\|\mathfrak{M}(S_sbS_sf, S_sg)\|_{F_{p,q,r}} \le \|M_{\tau_5}(S_sbS_sf)M_{\tau_5'}S_sg\|_{F_{p,q,r}} \le \|M_{\tau_5}(S_sbS_sf)\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}}\|M_{\tau_5'}S_sg\|_{F_{p,\frac{qp_2}{p},\frac{rp_2}{p}}}.$$

By (2.1), (2.2), Lemma 3, and Hölder's inequality again,

$$\begin{aligned}\|M_{\tau_5}(S_sbS_sf)\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}} &\lesssim \|S_sbS_sf\|_{F_{q_2,\frac{qq_2}{p},\frac{rq_2}{p}}} \lesssim \|S_sb\|_{F_{p_3,\frac{qp_3}{p},\frac{rp_3}{p}}}\|S_sf\|_{F_{p_1,\frac{qp_1}{p},\frac{rp_1}{p}}}\\ &\lesssim \|b\|_{F_s^{p_3,\frac{qp_3}{p}}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,\frac{qq_2}{p}}(\mathbb{R}^n)} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}.\end{aligned}$$

Hence, we obtain

$$\|\mathfrak{M}(S_sbS_sf, S_sg)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

By (2.5) and Hölder's inequality again,

$$\|\mathfrak{M}(S_sbS_sf, g)\|_{F_{p,q,r}} \le \|M_{\tau_2}(S_sbS_sf)M_{\tau_2'}g\|_{F_{p,q,r}} \le \|M_{\tau_2}(S_sbS_sf)\|_{F_{q_2,q,r}}\|M_{\tau_2'}g\|_{L^{p_2}(\mathbb{R}^n)},$$

$$\|\mathfrak{M}(bS_sf, g)\|_{F_{p,q,r}} \le \|M_{\tau_2}(bS_sf)M_{\tau_2'}g\|_{F_{p,q,r}} \le \|M_{\tau_2}(bS_sf)\|_{F_{q_2,q,r}}\|M_{\tau_2'}g\|_{L^{p_2}(\mathbb{R}^n)}.$$

By (2.1), (2.2), Lemma 3, and Hölder's inequality again,

$$\begin{aligned}\|M_{\tau_2}(S_sbS_sf)\|_{F_{q_2,q,r}} &\lesssim \|S_sbS_sf\|_{F_{q_2,q,r}}\\ &\lesssim \|S_sb\|_{F_{p_3,\frac{qp_3}{q_2},\frac{rp_3}{q_2}}}\|S_sf\|_{F_{p_1,\frac{qp_1}{q_2},\frac{rp_1}{q_2}}}\\ &\lesssim \|b\|_{F_s^{p_3,\frac{qp_3}{p}}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,\frac{qq_2}{p}}(\mathbb{R}^n)} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)},\end{aligned}$$

$$\|M_{\tau_2}(bS_sf)\|_{F_{q_2,q,r}} \lesssim \|bS_sf\|_{F_{q_2,q,r}} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|S_sf\|_{F_{p_1,q,r}} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}.$$

The above estimates together with the trivial estimate $\|M_{\tau_2'}g\|_{L^{p_2}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p_2}(\mathbb{R}^n)}$ imply that

$$\|\mathfrak{M}(S_sbS_sf, g)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)},$$

$$\|\mathfrak{M}(bS_sf, g)\|_{F_{p,q,r}} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Similarly we obtain

$$\|\mathfrak{M}(S_sbf, g)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Let $q_2' < \tau_6' < \min\{p_2, q\}$. Then $\tau_6 < q_2$. Let $\alpha \in (\max\{\tau_6', r\}, \min\{p_2, q\})$. By (2.5), (2.1), (2.2), Lemma 3, and Hölder's inequality,

$$\begin{aligned}
\|\mathfrak{M}(bf, S_s g)\|_{F_{p,q,r}} &\le \|M_{\tau_6}(bf) M_{\tau_6'} S_s g\|_{F_{p,q,r}} \\
&\le \|M_{\tau_6}(bf)\|_{L^{q_2}(\mathbb{R}^n)} \|M_{\tau_6'} S_s g\|_{F_{p_2,q,r}} \\
&\lesssim \|bf\|_{L^{q_2}(\mathbb{R}^n)} \|M_{\tau_6'} S_s g\|_{F_{p_2,q,\alpha}} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|b\|_{L^{p_3}(\mathbb{R}^n)} \|S_s g\|_{F_{p_2,q,\alpha}} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|b\|_{L^{p_3}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, we obtain

$$\|T_4(f, g)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.10}$$

Estimate for $\|T_5(f, g)\|_{F_{p,q,r}}$.

Let $\delta_1' \in (p_1', r)$. Clearly, $\delta_1 < p_1$. By (2.1), Lemma 3, and Hölder's inequality,

$$\begin{aligned}
\|M_{\delta_1} f M_{\delta_1'} S_{t,s} g S_{t,s} b\|_{F_{p,q,r}} &\lesssim \|S_{t,s} b\|_{L^{p_3}(\mathbb{R}^n, L^{\frac{qp_3}{p}}((0,1), t^{-1}dt))} \|M_{\delta_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|M_{\delta_1'} S_{t,s} g\|_{F_{p_2, \frac{qq_3}{p}, \frac{rq_3}{p}}} \\
&\lesssim \|b\|_{F_s^{p_3, \frac{qp_3}{p}}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|S_{t,s} g\|_{L^{p_2}(\mathbb{R}^n, L^{\frac{qq_3}{p}}((0,1), t^{-1}dt))} \\
&\lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{F_s^{p_2, \frac{qq_3}{p}}(\mathbb{R}^n)} \\
&\lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly we obtain

$$\|M_{\delta_2} S_{t,s} f M_{\delta_2'} g S_{t,s} b\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Thus, we have

$$\|T_5(f, g)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)} \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.11}$$

Estimate for $\|T_6(f, g)\|_{F_{p,q,r}}$.

There exist $\delta_3 \in (1, p_1)$, $\delta_4 \in (1, p_2)$, and $q_3' < \delta_5 < \min\{p_3, q\}$ such that $\sum_{i=3}^{5} \frac{1}{\delta_i} = 1$. There exist $\delta_6 \in (1, p_1)$, $\delta_7 \in (1, \frac{rp_2}{q_1})$, and $\delta_8 \in (1, \frac{rp_3}{q_1})$ such that $\sum_{i=6}^{8} \frac{1}{\delta_i} = 1$. Let $\beta \in (\max\{\delta_5, r\}, \min\{p_3, q\})$. By (2.1), Lemmas 3 and 4, and Hölder's inequality,

$$\begin{aligned}
\|M_{\delta_3} f M_{\delta_4} g M_{\delta_5} S_{t,s} b\|_{F_{p,q,r}} &\le \|M_{\delta_3} f M_{\delta_4} g\|_{L^{q_3}(\mathbb{R}^n)} \|M_{\delta_5} S_{t,s} b\|_{F_{p_3,q,r}} \\
&\lesssim \|M_{\delta_3} f\|_{L^{p_1}(\mathbb{R}^n)} \|M_{\delta_4} g\|_{L^{p_2}(\mathbb{R}^n)} \|M_{\delta_5} S_{t,s} b\|_{F_{p_3,q,\beta}} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \|S_{t,s} b\|_{L^{p_3}(\mathbb{R}^n, L^q((0,1), t^{-1}dt))} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)},
\end{aligned}$$

$$\begin{aligned}
\|M_{\delta_6} f M_{\delta_7} S_{t,s} g M_{\delta_8} S_{t,s} b\|_{F_{p,q,r}} &\le \|M_{\delta_6} f\|_{L^{p_1}(\mathbb{R}^n)} \|M_{\delta_7} S_{t,s} g\|_{F_{p_2, \frac{qp_2}{q_1}, \frac{rp_2}{q_1}}} \|M_{\delta_8} S_{t,s} b\|_{F_{p_3, \frac{qp_3}{q_1}, \frac{rp_3}{q_1}}} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|S_{t,s} g\|_{L^{p_2}(\mathbb{R}^n, L^{\frac{qp_2}{q_1}}((0,1), t^{-1}dt))} \|S_{t,s} b\|_{L^{p_3}(\mathbb{R}^n, L^{\frac{qp_3}{q_1}}((0,1), t^{-1}dt))} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{F_s^{p_2, \frac{qp_2}{q_1}}(\mathbb{R}^n)} \|b\|_{F_s^{p_3, \frac{qp_3}{q_1}}(\mathbb{R}^n)} \\
&\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{F_s^{p_2,q}(\mathbb{R}^n)} \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly we get

$$\|M_{\delta_9} S_{t,s} f M_{\delta_{10}} g M_{\delta_{11}} S_{t,s} b\|_{F_{p,q,r}} \lesssim \|f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}.$$

It follows that

$$\|T_6(f,g)\|_{F_{p,q,r}} \lesssim \|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.12}$$

Combining (3.12) with (3.6)–(3.11) implies (3.4).

Step 2. Proof of the continuity part. Let $h_i \in F_s^{p_i,q}(\Omega)$ for $i = 1,2,3$. For $i = 1,2$, let $\{h_{i,j}\}_{j\geq 1} \subset F_s^{p_i,q}(\Omega)$ satisfy $h_{i,j} \to h_i$ in $F_s^{p_i,q}(\Omega)$ as $j \to \infty$. It suffices to show that

$$\|\mathfrak{M}_{\Omega,h_3}(h_{1,j},h_{2,j}) - \mathfrak{M}_{\Omega,h_3}(h_1,h_2)\|_{F_s^{p,q}(\Omega)} \to 0 \quad \text{as} \quad j\to\infty. \tag{3.13}$$

Let $f \in F_s^{p_1,q}(\mathbb{R}^n)$, $g \in F_s^{p_2,q}(\mathbb{R}^n)$, and $b \in F_s^{p_3,q}(\mathbb{R}^n)$ satisfy $f|_\Omega = h_1$, $g|_\Omega = h_2$, and $b|_\Omega = h_3$. Let $\tilde{h}_{1,j}$ be the extensions of the functions $h_1 - h_{1,j}$ with $\|\tilde{h}_{1,j}\|_{F_s^{p_1,q}(\mathbb{R}^n)} \to 0$ as $j \to \infty$. Let $\tilde{h}_{2,j}$ be the extensions of the functions $h_2 - h_{2,j}$ with $\|\tilde{h}_{2,j}\|_{F_s^{p_2,q}(\mathbb{R}^n)} \to 0$ as $j\to\infty$. It is clear that f_j (resp., g_j) is an extension of $h_{1,j}$ (resp., $h_{2,j}$) and $f - f_j = \tilde{h}_{1,j}$, $g - g_j = \tilde{h}_{2,j}$. Thus, we have $\|f_j - f\|_{F_s^{p_1,q}(\mathbb{R}^n)} \to 0$ and $\|g_j - g\|_{F_s^{p_2,q}(\mathbb{R}^n)} \to 0$ as $j\to\infty$. Observe that

$$T_{\Omega,b}(f_j,g_j)|_\Omega = \mathfrak{M}_{\Omega,h_3}(h_{1,j},h_{2,j}), \quad T_{\Omega,b}(f,g)|_\Omega = \mathfrak{M}_{\Omega,h_3}(h_1,h_2).$$

It follows that

$$(T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g))|_\Omega = \mathfrak{M}_{\Omega,h_3}(h_{1,j},h_{2,j}) - \mathfrak{M}_{\Omega,h_3}(h_1,h_2).$$

Thus, for (3.13) it suffices to prove that

$$\|T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j\to\infty. \tag{3.14}$$

It is not difficult to see that

$$|T_{\Omega,b}(f_j,g_j)(x) - T_{\Omega,b}(f,g)(x)| \leq |\mathfrak{M}_{\Omega,h_3}(h_{1,j},h_{2,j})(x) - \mathfrak{M}_{\Omega,h_3}(h_1,h_2)(x)|\chi_\Omega(x).$$

This together with Remark 2 implies that

$$\|T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g)\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j\to\infty. \tag{3.15}$$

In view of (2.2) and (3.15), for (3.14) it is enough to show that

$$\|S_s(T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g))\|_{F_{p,q,r}} \to 0 \quad \text{as} \quad i\to\infty. \tag{3.16}$$

Next we prove (3.16) by contradiction. Without loss of generality we may assume that there exists a constant $c > 0$ such that

$$\|S_s(T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g))\|_{F_{p,q,r}} > c, \quad \forall j \geq 1. \tag{3.17}$$

From the definitions of S_s and $S_{t,s}$, we have

$$|S_s(f_j) - S_s(f)| \leq S_s(f_j - f), \quad |S_{t,s}(f_j) - S_{t,s}(f)| \leq S_{t,s}(f_j - f). \tag{3.18}$$

Moreover, by the definition of M_τ, one can easily check that

$$M_\tau f_j \leq 2(M_\tau(f_j - f) + M_\tau f), \quad \forall \tau \in (1,\infty). \tag{3.19}$$

We get from (3.5) that for a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0, 1)$ and $h \in B(O, 1)$,

$$S_s(T_{\Omega,b}(f_j, g_j))(x, t, h) \le 2^{n+1} \sum_{i=1}^{6} T_i(f_j, g_j)(x, t, h). \tag{3.20}$$

By (3.5) and (3.20), we have that for a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0, 1)$ and $h \in B(O, 1)$,

$$|S_s(T_{\Omega,b}(f_j, g_j))(x, t, h) - S_s(T_{\Omega,b}(f,g))(x, t, h)| \le \Phi_j(x, t, h) + \Psi(x, t, h), \tag{3.21}$$

where

$$\Phi_j(x, t, h) = 2^{n+3} \sum_{i=1}^{6} (T_i(f_j - f, g)(x, t, h) + T_i(f_j - f, g_j - g)(x, t, h) + T_i(f, g_j - g)(x, t, h)),$$

$$\Psi(x, t, h) = 2^{n+4} \sum_{i=1}^{6} T_i(f, g)(x, t, h).$$

By Minkowski's inequality and (3.7)–(3.12), we have

$$\|\Phi_j\|_{F_{p,q,r}} \to 0 \quad \text{as} \quad j \to \infty, \quad \|\Psi\|_{F_{p,q,r}} < \infty. \tag{3.22}$$

Let $\lambda > 0$ and $(x, t) \in \mathbb{R}^n \times (0, 1)$. It is clear that

$$\begin{aligned} & S_s(T_{\Omega,b}(f_j, g_j) - T_{\Omega,b}(f, g))(x, t, h) \\ & \le t^{-s}(|T_{\Omega,b}(f_j, g_j)(x + th) - T_{\Omega,b}(f, g)(x + th)| + |T_{\Omega,b}(f_j, g_j)(x) - T_{\Omega,b}(f, g)(x)|). \end{aligned}$$

This together with (3.15) and Fubini's theorem implies that for any $t \in (0, 1)$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{B(O,1)} |S_s(T_{\Omega,b}(f_j, g_j) - T_{\Omega,b}(f, g))(x, t, h)|^p dh dx \\ & \lesssim t^{-sp} \|T_{\Omega,b}(f_j, g_j) - T_{\Omega,b}(f, g)\|^p_{L^p(\mathbb{R}^n)} \to 0 \ \text{ as } \ j \to \infty. \end{aligned}$$

Hence, for any $t \in (0, 1)$, there exists a subsequence $\{j_k\}_{k \ge 1}$ such that for a.e. $x \in \mathbb{R}^n$,

$$\int_{B(O,1)} |S_s(T_{\Omega,b}(f_{j_k}, g_{j_k}) - T_{\Omega,b}(f, g))(x, t, h)|^p dh \to 0 \ \text{ as } \ k \to \infty.$$

This together with Chebyshev's inequality implies that for any $t \in (0, 1)$ and a.e. $x \in \mathbb{R}^n$,

$$|\{h \in B(O, 1) : S_s(T_{\Omega,b}(f_{j_k}, g_{j_k}) - T_{\Omega,b}(f, g))(x, t, h) > \lambda\}| \to 0 \ \text{ as } \ k \to \infty. \tag{3.23}$$

By (3.21)–(3.23) and Lemma 4,

$$\|S_s(T_{\Omega,b}(f_{j_k}, g_{j_k}) - T_{\Omega,b}(f, g))\|_{F_{p,q,r}} \to 0 \ \text{ as } \ k \to \infty.$$

This leads to a contradiction with (3.17). The proof of Theorem 1 is now proved. □

Proof of Theorem 2. The proof of Theorem 2 will be divided into two parts.

Step 1. Proof of the boundedness part. Let $h_i \in B_s^{p_i,q}(\Omega)$ for $i = 1, 2, 3$. It suffices to show that

$$\|\mathfrak{M}_{\Omega,h_3}(h_1, h_2)\|_{B_s^{p,q}(\Omega)} \lesssim \|h_3\|_{B_s^{p_3,q}(\Omega)}\|h_1\|_{B_s^{p_1,q}(\Omega)}\|h_2\|_{B_s^{p_2,q}(\Omega)}. \tag{3.24}$$

Let $f \in B_s^{p_1,q}(\mathbb{R}^n)$, $g \in B_s^{p_2,q}(\mathbb{R}^n)$, and $b \in B_s^{p_3,q}(\mathbb{R}^n)$ satisfy $f|_\Omega = h_1$, $g|_\Omega = h_2$, and $b|_\Omega = h_3$. Clearly, $T_{\Omega,b}(f, g)|_\Omega = \mathfrak{M}_{\Omega,h_3}(h_1, h_2)$. Hence, for (3.24) it is enough to show that

$$\|T_{\Omega,b}(f, g)\|_{B_s^{p,q}(\mathbb{R}^n)} \lesssim \|b\|_{B_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.25}$$

By (3.3) and (2.4), for (3.25) it is enough to prove that

$$\|S_s(T_{\Omega,b}(f, g))\|_{E_{p,q}} \lesssim \|b\|_{B_s^{p_3,q}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}. \tag{3.26}$$

By Lemma 6 and the definitions of S_s and $S_{t,s}$, we have that for a.e. $x \in \mathbb{R}^n$ and $x + th \in \mathbb{R}^n$ with $t \in (0, 1)$ and $h \in B(O, 1)$,

$$S_s(T_{\Omega,b}(f, g))(x, t, h) \le 2^{n+1} \sum_{i=1}^{5} I_i(f, g)(x, t, h), \tag{3.27}$$

where

$$\begin{aligned}
I_1(f, g) &= |S_s b|\mathfrak{M}(f, g),\\
I_2(f, g) &= (|\Delta_{th} b| + |b|)(\mathfrak{M}(S_s f, \Delta_{th} g) + \mathfrak{M}(S_s f, g) + \mathfrak{M}(f, S_s g)),\\
I_3(f, g) &= \mathfrak{M}(b\Delta_{th} f, S_s g) + \mathfrak{M}(\Delta_{th} b\Delta_{th} f, S_s g) + \mathfrak{M}(bS_s f, g)\\
&\quad +\mathfrak{M}(\Delta_{th} bS_s f, g) + \mathfrak{M}(bf, S_s g) + \mathfrak{M}(\Delta_{th} bf, S_s g),\\
I_4(f, g) &= M_{\delta_1} f M_{\delta_1'}\rho_t g S_{t,s} b + M_{\delta_1}\rho_t f M_{\delta_1'} g S_{t,s} b,\\
I_5(f, g) &= M_{\delta_2} f M_{\delta_3} g M_{\delta_4} S_{t,s} b + M_{\delta_2} f M_{\delta_3}\rho_t g M_{\delta_4} S_{t,s} b + M_{\delta_2}\rho_t f M_{\delta_3} g M_{\delta_4} S_{t,s} b.
\end{aligned}$$

Here $1 < \delta_i < \infty$, $1 \le i \le 4$, $\sum_{i=2}^{4} \frac{1}{\delta_i} = 1$.

By (3.27) and Minkowski's inequality,

$$\|S_s(T_{\Omega,b}(f, g))\|_{E_{p,q}} \le 2^{n+1} \sum_{i=1}^{5} \|I_i(f, g)\|_{E_{p,q}}. \tag{3.28}$$

Let q_1, q_2, q_3 be given in the proof of Theorem 1. We now estimate $\|I_i(f, g)\|_{E_{p,q}}$, respectively.

Estimate for $\|I_1(f, g)\|_{E_{p,q}}$.

By (2.4), Hölder's inequality and the bounds for $\mathfrak{M}$, we have

$$\|I_1(f, g)\|_{E_{p,q}} \lesssim \|S_s b\|_{E_{p_3,q}}\|\mathfrak{M}(f, g)\|_{L^{q_3}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|b\|_{F_s^{p_3,q}(\mathbb{R}^n)}. \tag{3.29}$$

Estimate for $\|I_2(f, g)\|_{E_{p,q}}$.

By Minkowski's inequality and the boundedness for $\mathfrak{M}$, we have

$$\begin{aligned}
&\|\mathfrak{M}(S_s f, \Delta_{th} g) + \mathfrak{M}(S_s f, g) + \mathfrak{M}(f, S_s g)\|_{L^{q_3}(\mathbb{R}^n)}\\
&\le \|\mathfrak{M}(S_s f, \Delta_{th} g)\|_{L^{q_3}(\mathbb{R}^n)} + \|\mathfrak{M}(S_s f, g)\|_{L^{q_3}(\mathbb{R}^n)} + \|\mathfrak{M}(f, S_s g)\|_{L^{q_3}(\mathbb{R}^n)}\\
&\lesssim \|S_s f\|_{L^{p_1}(\mathbb{R}^n)}\|\Delta_{th} g\|_{L^{p_2}(\mathbb{R}^n)} + \|S_s f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|S_s g\|_{L^{p_2}(\mathbb{R}^n)}\\
&\lesssim \|S_s f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|S_s g\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

This together with (2.4), Hölder's inequality, and Minkowski's inequality implies that

$$\begin{aligned}\|I_2(f,g)\|_{E_{p,q}} &\lesssim (\|\Delta_{th}b\|_{L^{p_3}(\mathbb{R}^n)} + \|b\|_{L^{p_3}(\mathbb{R}^n)})\|\mathfrak{M}(S_sf,\Delta_{th}g) + \mathfrak{M}(S_sf,g) + \mathfrak{M}(f,S_sg)\|_{E_{q_3,q}} \\ &\lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}(\|S_sf\|_{E_{p_1,q}}\|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|S_sg\|_{E_{p_2,q}}) \\ &\lesssim \|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}\|b\|_{L^{p_3}(\mathbb{R}^n)}.\end{aligned} \tag{3.30}$$

Estimate for $\|I_3(f,g)\|_{E_{p,q}}$**.**

By the boundedness for $\mathfrak{M}$, Minkowski's inequality, and Hölder's inequality,

$$\begin{aligned}\|I_3(f,g)\|_{L^p(\mathbb{R}^n)} \le& \|\mathfrak{M}(b\Delta_{th}f,S_sg)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(\Delta_{th}b\Delta_{th}f,S_sg)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(bS_sf,g)\|_{L^p(\mathbb{R}^n)} \\ &+\|\mathfrak{M}(\Delta_{th}bS_sf,g)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(bf,S_sg)\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{M}(\Delta_{th}bf,S_sg)\|_{L^p(\mathbb{R}^n)} \\ \lesssim& (\|b\Delta_{th}f\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}b\Delta_{th}f\|_{L^{q_2}(\mathbb{R}^n)} + \|bf\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}bf\|_{L^{q_2}(\mathbb{R}^n)}) \\ &\times\|S_sg\|_{L^{p_2}(\mathbb{R}^n)} + (\|bS_sf\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}bS_sf\|_{L^{q_2}(\mathbb{R}^n)})\|g\|_{L^{p_2}(\mathbb{R}^n)}.\end{aligned}$$

Observe that

$$\begin{aligned}&\|b\Delta_{th}f\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}b\Delta_{th}f\|_{L^{q_2}(\mathbb{R}^n)} + \|bf\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}bf\|_{L^{q_2}(\mathbb{R}^n)} \\ &\le (\|b\|_{L^{p_3}(\mathbb{R}^n)} + \|\Delta_{th}b\|_{L^{p_3}(\mathbb{R}^n)})(\|\Delta_{th}f\|_{L^{p_1}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}) \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)},\end{aligned}$$

$$\begin{aligned}&\|bS_sf\|_{L^{q_2}(\mathbb{R}^n)} + \|\Delta_{th}bS_sf\|_{L^{q_2}(\mathbb{R}^n)} \\ &\le (\|b\|_{L^{p_3}(\mathbb{R}^n)} + \|\Delta_{th}b\|_{L^{p_3}(\mathbb{R}^n)})\|S_sf\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|S_sf\|_{L^{p_1}(\mathbb{R}^n)}.\end{aligned}$$

It follows that

$$\|I_3(f,g)\|_{L^p(\mathbb{R}^n)} \lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}(\|S_sf\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|S_sg\|_{L^{p_2}(\mathbb{R}^n)}).$$

This together with (2.4) implies that

$$\begin{aligned}\|I_3(f,g)\|_{E_{p,q}} &\lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}(\|S_sf\|_{E_{p_1,q}}\|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)}\|S_sg\|_{E_{p_2,q}}) \\ &\lesssim \|b\|_{L^{p_3}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.\end{aligned} \tag{3.31}$$

Estimate for $\|I_4(f,g)\|_{E_{p,q}}$**.**

Since $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_3} < 1$, then $p_2' < p_1$. We also note that $\rho_t u \le |u| + Mu$ for any function $u \in L^1_{\rm loc}(\mathbb{R}^n)$. Let $p_2' < \delta_1 < p_1$. By Hölder's inequality, one gets

$$\begin{aligned}\|M_{\delta_1}fM_{\delta_1'}\rho_t gS_{t,s}b\|_{L^p(\mathbb{R}^n)} &\le \|M_{\delta_1}f\|_{L^{p_1}(\mathbb{R}^n)}\|M_{\delta_1'}\rho_t g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|\rho_t g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)}.\end{aligned}$$

Similarly one gets

$$\|M_{\delta_2}\rho_t fM_{\delta_2'}gS_{t,s}b\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)}.$$

It follows that

$$\|I_4(f,g)\|_{E_{p,q}} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n,L^q((0,1),t^{-1}dt))} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|b\|_{B_s^{p_3,q}(\mathbb{R}^n)}. \tag{3.32}$$

Estimate for $\|I_5(f,g)\|_{E_{p,q}}$**.**

Since $\sum_{i=1}^{3}\frac{1}{p_i}=\frac{1}{p}<1$, there exist $\delta_2\in(1,p_1)$, $\delta_3\in(1,p_2)$, and $\delta_4\in(1,p_3)$ such that $\sum_{i=2}^{4}\frac{1}{\delta_i}=1$. By Hölder's inequality, one gets

$$\begin{aligned}\|M_{\delta_2}fM_{\delta_3}gM_{\delta_4}S_{t,s}b\|_{L^p(\mathbb{R}^n)}&\le\|M_{\delta_2}f\|_{L^{p_1}(\mathbb{R}^n)}\|M_{\delta_3}g\|_{L^{p_2}(\mathbb{R}^n)}\|M_{\delta_4}S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)}\\&\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)}.\end{aligned}$$

Similarly we obtain

$$\|M_{\delta_2}fM_{\delta_3}\rho_tgM_{\delta_4}S_{t,s}b\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)},$$

$$\|M_{\delta_2}\rho_tfM_{\delta_3}gM_{\delta_4}S_{t,s}b\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n)}.$$

These estimates together with (2.4) and Minkowski's inequality imply that

$$\|I_5(f,g)\|_{E_{p,q}}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|S_{t,s}b\|_{L^{p_3}(\mathbb{R}^n,L^q((0,1),t^{-1}dt))}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|b\|_{B_s^{p_3,q}(\mathbb{R}^n)}.\tag{3.33}$$

Then (3.26) follows from (3.28)–(3.33).

Step 2. Proof of the continuity part. Let $h_i\in B_s^{p_i,q}(\Omega)$ for $i=1,2,3$. For $i=1,2$, let $\{h_{i,j}\}_{j\ge1}\subset B_s^{p_i,q}(\Omega)$ satisfy $h_{i,j}\to h_i$ in $B_s^{p_i,q}(\Omega)$ as $j\to\infty$. It suffices to show that

$$\|\mathfrak{M}_{\Omega,b}(h_{1,j},h_{2,j})-\mathfrak{M}_{\Omega,b}(h_1,h_2)\|_{B_s^{p,q}(\Omega)}\to0\quad\text{as}\quad j\to\infty.\tag{3.34}$$

Let $f\in B_s^{p_1,q}(\mathbb{R}^n)$, $g\in B_s^{p_2,q}(\mathbb{R}^n)$, and $b\in B_s^{p_3,q}(\mathbb{R}^n)$ satisfy $f|_\Omega=h_1$, $g|_\Omega=h_2$, and $b|_\Omega=h_3$. By (3.3) and (2.4), for (3.34) it is enough to show that

$$\|S_s(T_{\Omega,b}(f_j,g_j)-T_{\Omega,b}(f,g))\|_{E_{p,q}}\to0\ \text{ as }\ j\to\infty.\tag{3.35}$$

Now we prove (3.35) by contradiction. We may assume without loss of generality that there exists a constant $c>0$ such that

$$\|S_s(T_{\Omega,b}(f_j,g_j)-T_{\Omega,b}(f,g))\|_{E_{p,q}}>c,\quad\forall j\ge1.\tag{3.36}$$

By Lemma 6, we have that for almost every $x\in\mathbb{R}^n$ and $x+th\in\mathbb{R}^n$ with $t\in(0,1)$ and $h\in B(O,1)$,

$$S_s(T_{\Omega,b}(f_j,g_j))(x,t,h)\le2^{n+1}\sum_{i=1}^{5}I_i(f_j,g_j)(x,t,h).\tag{3.37}$$

In view of (3.27) and (3.37), we have that for almost every $x\in\mathbb{R}^n$ and $x+th\in\mathbb{R}^n$ with $t\in(0,1)$ and $h\in B(O,1)$,

$$|S_s(T_{\Omega,b}(f_j,g_j))(x,t,h)-S_s(T_{\Omega,b}(f,g))(x,t,h)|\le\Gamma_j(x,t,h)+\Theta(x,t,h),\tag{3.38}$$

where

$$\Gamma_j(x,t,h)=2^{n+3}\sum_{i=1}^{5}(I_i(f_j-f,g_j-g)(x,t,h)+I_i(f_j-f,g)(x,t,h)+I_i(f,g_j-g)(x,t,h)),$$

$$\Theta(x,t,h) = 2^{n+5}\sum_{i=1}^{5} I_i(f,g)(x,t,h).$$

By Minkowski's inequality and (3.29)–(3.33), we have

$$\|\Gamma_j\|_{E_{p,q}} \to 0 \text{ as } j \to \infty, \quad \|\Theta\|_{E_{p,q}} < \infty.$$

On the other hand, it was proved in the proof of Theorem 1 that for any $t \in (0,1)$,

$$\|S_s(T_{\Omega,b}(f_j,g_j) - T_{\Omega,b}(f,g))\|_{L^p(\mathbb{R}^n\times B(O,1))} \to 0 \text{ as } j \to \infty. \tag{3.39}$$

Hence, there exists a subsequence $\{j_k\}_{k\geq 1}$ such that for any $t \in (0,1)$ and almost every $(x,h) \in \mathbb{R}^n \times B(O,1)$,

$$S_s(T_{\Omega,b}(f_{j_k},g_{j_k}) - T_{\Omega,b}(f,g))(x,t,h) \to 0 \text{ as } k \to \infty. \tag{3.40}$$

By (3.38)–(3.40) and Lemma 4, we have

$$\|S_s(T_{\Omega,b}(f_{j_k},g_{j_k}) - T_{\Omega,b}(f,g))\|_{E_{p,q}} \to 0 \text{ as } k \to \infty.$$

This contradicts with (3.36). So (3.35) holds. This completes the proof of Theorem 2. □

4. Conclusions

In the present paper we investigate the mapping properties of bilinear maximal commutator in the domain setting. We establish some new boundedness and continuity for the above operator on the Triebel–Lizorkin spaces and Besov spaces under suitable symbol function condition. The main results we obtain essentially extend some known ones to local setting.

Author contributions

F. Liu: Writing-review and editing, conceptualization; S. Liu: Writing-original draft, methodology; X. Zhang: Review and editing, conceptualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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